1. (a) $y'' - 3y' + 2y = (1+x)e^{3x}$

e^{3x}는 homogeneous equation의 해가 아니다.

$\therefore y_p = (A_0 + A_1 x)e^{3x}$ (by guessing)
(or method of undetermined coefficients)

$y_p' = (3A_0 + A_1 + 3A_1 x)e^{3x}$

$y_p'' = (9A_0 + 6A_1 + 9A_1 x)e^{3x}$

$(1+x)e^{3x} = e^{3x}\left[(9A_0 + 6A_1 + 9A_1 x) - 3(3A_0 + A_1 + 3A_1 x) + 2(A_0 + A_1 x)\right]$

$= e^{3x}\left[(2A_0 + 3A_1) + 2A_1 x\right]$

$\therefore A_1 = \frac{1}{2}, \quad A_0 = -\frac{1}{4}$

$\therefore y_p(x) = \left(-\frac{1}{4} + \frac{1}{2}x\right)e^{3x}$

(b) $y'' + y = \sec x$,

$y'' + y = 0 \Rightarrow y_1 = \cos x \quad y_2 = \sin x$

$W[y_1, y_2] = y_1 y_2' - y_1' y_2 = 1$

$u_1' = -\sec x \sin x \qquad u_2' = \sec x \cos x$

$= -\tan x \qquad = 1$

$\therefore u_1 = \ln(\cos x) \qquad u_2 = x$

$\therefore y_p = u_1 y_1 + u_2 y_2$

$= \cos x \ln(\cos x) + x \sin x$, where $-\frac{\pi}{2} < x < \frac{\pi}{2}$

2. a. $\frac{d^4y}{dt^4} - 3\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} - \frac{dy}{dt} = 0$

$$\Rightarrow r^4 - 3r^3 + 3r^2 - r = r(r^3 - 3r^2 + 3r - 1)$$
$$= r(r-1)^3$$

$$r = 0,\ \underset{\sim}{1}$$

└> 3중근

$$\therefore y(t) = c_1 e^{0t} + (c_2 + c_3 t + c_4 t^2)e^t$$
$$= c_1 + (c_2 + c_3 t + c_4 t^2)e^t$$

b. $\frac{d^4y}{dt^4} + y = 0$

$$\Rightarrow r^4 + 1 = 0 \quad \Rightarrow r^4 = -1$$

$$-1 = e^{j\pi} = e^{j3\pi} = e^{j5\pi} = e^{jn\pi}$$

$$r_1^4 = -1 = e^{j\pi}$$
$$\therefore r_1 = e^{j\frac{\pi}{4}} = \frac{1}{\sqrt{2}}(1+j)$$

$$r_2^4 = -1 = e^{j3\pi}$$
$$\therefore r_2 = e^{j\frac{3\pi}{4}} = -\frac{1}{\sqrt{2}}(1-j)$$

$$r_3^4 = -1 = e^{j5\pi}$$
$$\therefore r_3 = e^{j\frac{5}{4}\pi} = -\frac{1}{\sqrt{2}}(1+j)$$

$$r_4^4 = -1 = e^{j7\pi}$$
$$\therefore r_4 = e^{j\frac{7}{4}\pi} = \frac{1}{\sqrt{2}}(1-j)$$

$\therefore$

r_1 과 r_4 는 complex conjugate

r_2 과 r_3 는 complex conjugate

$$\therefore\ y_1(t) = e^{\frac{1}{\sqrt{2}}t}\cos\frac{1}{\sqrt{2}}t \qquad y_2(t) = e^{-\frac{1}{\sqrt{2}}t}\cos\frac{1}{\sqrt{2}}t$$

$$y_3(t) = e^{-\frac{1}{\sqrt{2}}t}\sin\frac{1}{\sqrt{2}}t \qquad y_4(t) = e^{\frac{1}{\sqrt{2}}t}\sin\frac{1}{\sqrt{2}}t$$

$$\therefore\ y(t) = \cancel{C_1}\ e^{\frac{1}{\sqrt{2}}t}\left(c_1\cos\frac{1}{\sqrt{2}}t + c_2\sin\frac{1}{\sqrt{2}}t\right)$$
$$+ e^{-\frac{1}{\sqrt{2}}t}\left(c_3\cos\frac{1}{\sqrt{2}}t + c_4\sin\frac{1}{\sqrt{2}}t\right)$$

3. $y' = \begin{pmatrix} 1 & 12 \\ 3 & 1 \end{pmatrix} y, \quad y(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 12 \\ 3 & 1 \end{pmatrix} \Rightarrow \lambda_1 = 7, \quad \lambda_2 = -5$$

$\Downarrow$ eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\qquad$ $\Downarrow$ eigenvector $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$$\therefore\ y_1(t) = e^{7t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad y_2(t) = e^{-5t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\therefore\ y(t) = c_1 e^{7t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-5t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

초기조건 대입

$$c_1 = \frac{1}{2}, \quad c_2 = \frac{1}{2}$$

$$\therefore\ y(t) = \frac{1}{2}e^{7t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{1}{2}e^{-5t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{7t} - e^{-5t} \\ \frac{1}{2}e^{7t} + \frac{1}{2}e^{-5t} \end{pmatrix}$$

4\. (a) $y_1' = -2y_1 + 2y_2$

$y_2' = -2y_1 - 2y_2$

$$\frac{dy_2}{dy_1} = \frac{-2y_1 - 2y_2}{-2y_1 + 2y_2} \Rightarrow \text{임계점}: y_1 = 0,\ y_2 = 0$$

$$A = \begin{pmatrix} -2 & 2 \\ -2 & -2 \end{pmatrix} \Rightarrow \begin{aligned} p &= -2-2 = -4 \\ q &= \det A = 8 \\ \Delta &= p^2 - 4q = 16 - 4\cdot 8 = -16 \end{aligned}$$

$p \neq 0,\ \Delta < 0 \Rightarrow$ spiral point

$p < 0,\ q > 0 \Rightarrow$ stable point

(b) $y'' + \cos y = 0$

Let's $y_1 = y$, $y_2 = y_1'$ $\Rightarrow$ $y_1' = y_2$, $y_2'' = -\cos y_1$

$$\frac{dy_2}{dy_1} = \frac{-\cos y_1}{y_2} \quad \therefore \text{임계점} \left(\tfrac{1}{2}\pi + 2n\pi, 0\right), \left(-\tfrac{3}{2}\pi + 2n\pi, 0\right)$$

n은 정수.

i) 임계점 $\left(\frac{1}{2}\pi + 2n\pi, 0\right)$ 근처

Let's $x_1 = y_1 - \left(\frac{1}{2}\pi + 2n\pi\right)$, $x_2 = y_2$ $\Rightarrow$ $\cos y_1 = \cos\left(x_1 + \frac{\pi}{2} + 2n\pi\right) = -\sin x_1 \approx -x_1$

$$\therefore A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$p = 0$, $q = -1$ $\Rightarrow \therefore$ saddle point ($\because q < 0$)

ii) 임계점 $\left(\frac{3}{2}\pi + 2n\pi, 0\right)$ 근처

Let's $x_1 = y_1 - \left(\frac{3}{2}\pi + 2n\pi\right)$ $\Rightarrow \cos y_1 = \cos\left(x_1 + \frac{3}{2}\pi + 2n\pi\right)$

$x_2 = y_2$ $= \sin x_1 \approx x_1$

$$\therefore A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$p = 0, \quad q = 1 \quad \Rightarrow \therefore \text{center} \quad (\because q > 0,\ p = 0)$$

$\therefore$ $\left(\frac{1}{2}\pi + 2n\pi, 0\right)$: saddle point

$\left(\frac{3}{2}\pi + 2n\pi, 0\right)$: center

where n은 정수

5. $$y' = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix} y + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{2t}$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix}$$

$\lambda_1 = 1$, eigenvector : $\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$

λ_2 and λ_3 = $1 \pm j2$

i) $\lambda_1 = 1$

$$y_1 = e^{t} \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$$

ii) λ_2 와 λ_3 의 경우

λ_2 와 λ_3 는 complex conjugate.

$\therefore \lambda_2$ 만 고려

$\lambda_2 = 1 + j2$

eigenvector 를 구하면 $\begin{pmatrix} 0 \\ 1 \\ -j \end{pmatrix}$

$$\begin{pmatrix} 0 \\ 1 \\ -j \end{pmatrix} e^{(1+j2)t} = e^t(\cos 2t + j\sin 2t)\left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - j\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right]$$

$$= e^t\left[\cos 2t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \sin 2t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right]$$

$$+ je^t\left[\sin 2t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \cos 2t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right]$$

$$\therefore y_2 = e^t\begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix}, \quad y_3 = e^t\begin{pmatrix} 0 \\ \sin 2t \\ -\cos 2t \end{pmatrix}$$

$\therefore$ homogeneous equation 의 solution 은 다음과 같다.

$$y_h = e^t\left(C_1\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} + C_2\begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix} + C_3\begin{pmatrix} 0 \\ -\sin 2t \\ \cos 2t \end{pmatrix}\right)$$

particular solution 을 다음과 같다고 가정

$$y_p = be^{2t}$$

y_p 를 원식에 대입

$$2be^{2t} = Abe^{2t} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}e^{2t}$$

$$2b = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix} b + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\begin{pmatrix} 2b_1 - 1 \\ 2b_2 \\ 2b_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ 2b_1 + b_2 - 2b_3 \\ 3b_1 + 2b_2 + b_3 \end{pmatrix}$$

$$\therefore \; b_1 = 1$$
$$b_2 = -\frac{4}{5}$$
$$b_3 = \frac{7}{5}$$

$$\therefore \; y_p = \begin{pmatrix} 1 \\ -\frac{4}{5} \\ \frac{7}{5} \end{pmatrix} e^{2t}$$

$$\therefore \; y(t) = y_h(t) + y_p(t)$$

$$= e^{t}\left(c_1 \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ -\sin 2t \\ \cos 2t \end{pmatrix} \right)$$

$$+ \begin{pmatrix} 1 \\ -\frac{4}{5} \\ \frac{7}{5} \end{pmatrix} e^{2t}$$