- Assignments -

(Textbook Chapter & Number)

HW 1 : 2.56, 2.66, 2.67, 2.72, 2.77, 2.89

HW 2 : 4.25, 4.38, 4.40, 4.44, 4.46, 4.48, 4.51

HW 3 : 5.45, 5.51, 5.55, 5.57, 5.64, 5.66

HW 4 : 6.26, 6.27, 6.30, 6.34, 6.36, 6.38

HW 5 : 8.29, 8.31, 8.34, 8.36, 8.39, 8.49, 8.64

HW 6 : 10.5, 10.26, 10.31, 10.34, 10.35

Project 1

- Design an FIR filter that best approximates a square-root raised cosine filter with the following constraints

$$\left|H_d\left(e^{j2\pi f}\right)\right|^2 = \begin{cases} T, & 0 \le |f| \le \frac{1-\alpha}{2T} \\ \frac{T}{2}\left\{1+\cos\left(\frac{\pi T}{\alpha}\left(|f| - \frac{1-\alpha}{2T}\right)\right)\right\}, & \frac{1-\alpha}{2T} < |f| < \frac{1+\alpha}{2T} \\ 0, & \text{otherwise} \end{cases}$$

$$\alpha = 0.15 + 0.05 \times \text{Remainder(last two digits of ID number/4)}; \quad T = 1.0$$

  - Minimize the number of taps, where N is an odd number; $n_o = (N-1)/2$
  - Intersymbol interference (ISI)

$$ISI \triangleq 10\log_{10} \frac{\left|h[n_o]\right|^2}{\sum_{\substack{n=0 \\ n\neq n_o}}^{N-1} \left|h_d[n] - h[n]\right|^2} \ge \begin{cases} 40dB; & \alpha < 0.2 \\ 60dB; & \alpha > 0.2 \end{cases}$$

  - Stopband attenuation is the same as the ISI condition

- Project output due by May 28, 2008
  - Impulse and frequency response of the designed filter with desired one
  - Source program with flow chart; ISI value

2.56. (a)

$$\begin{aligned} y[n] &= h[n] * (e^{-j\omega_0 n}x[n]) \\ &= \sum_{k=-\infty}^{+\infty} e^{-j\omega_0 k}x[k]h[n-k]. \end{aligned}$$

Let $x[n] = ax_1[n] + bx_2[n]$, then:

$$\begin{aligned} y[n] &= h[n] * (e^{-j\omega_0 n}(ax_1[n] + bx_2[n])) \\ &= \sum_{k=-\infty}^{+\infty} e^{-j\omega_0 k}(ax_1[k] + bx_2[k])h[n-k] \\ &= a\sum_{k=-\infty}^{+\infty} e^{-j\omega_0 k}x_1[k]h[n-k] + b\sum_{k=-\infty}^{+\infty} e^{-j\omega_0 k}x_2[k]h[n-k] \\ &= ay_1[n] + by_2[n] \end{aligned}$$

where $y_1[n]$ and $y_2[n]$ are the responses to $x_1[n]$ and $x_2[n]$ respectively. We thus conclude that system S is linear.

(b) Let $x_2[n] = x[n-n_0]$, then:

$$\begin{aligned} y_2[n] &= h[n] * (e^{-j\omega_0 n}x_2[n]) \\ &= \sum_{k=-\infty}^{+\infty} e^{-j\omega_0 (n-k)}x_2[n-k]h[k] \\ &= \sum_{k=-\infty}^{+\infty} e^{-j\omega_0 (n-k)}x[n-n_0-k]h[k] \\ &\neq y[n-n_0]. \end{aligned}$$

We thus conclude that system S is not time invariant.

(c) Since the magnitude of $e^{-j\omega_0 n}$ is always bounded by 1 and $h[n]$ is stable, a bounded input $x[n]$ will always produce a bounded input to the stable LTI system and therefore the output $y[n]$ will be bounded. We thus conclude that system S is stable.

(d) We can rewrite $y[n]$ as:

$$\begin{aligned} y[n] &= h[n] * (e^{-j\omega_0 n}x[n]) \\ &= \sum_{k=-\infty}^{+\infty} e^{-j\omega_0 (n-k)}x[n-k]h[k] \\ &= \sum_{k=-\infty}^{+\infty} e^{-j\omega_0 n}e^{j\omega_0 k}x[n-k]h[k] \\ &= e^{-j\omega_0 n}\sum_{k=-\infty}^{+\infty} e^{j\omega_0 k}x[n-k]h[k]. \end{aligned}$$

System C should therefore be a multiplication by $e^{-j\omega_0 n}$.

2.66. (a)

$$\begin{aligned}
E(e^{j\omega}) &= H_1(e^{j\omega})X(e^{j\omega}) \\
F(e^{j\omega}) &= E(e^{-j\omega}) \\
&= H_1(e^{-j\omega})X(e^{-j\omega}) \\
G(e^{j\omega}) &= H_1(e^{j\omega})F(e^{j\omega}) \\
&= H_1(e^{j\omega})H_1(e^{-j\omega})X(e^{-j\omega}) \\
Y(e^{j\omega}) &= G(e^{-j\omega}) \\
&= H_1(e^{-j\omega})H_1(e^{j\omega})X(e^{j\omega}).
\end{aligned}$$

(b) Since:

$$Y(e^{j\omega}) = H_1(e^{-j\omega})H_1(e^{j\omega})X(e^{j\omega}),$$

We get:

$$H(e^{j\omega}) = H_1(e^{-j\omega})H_1(e^{j\omega}).$$

(c) Taking the inverse transform of $H(e^{j\omega})$, we get:

$$h[n] = h_1[-n] * h_1[n].$$

2.67. (a) Using the properties of the Fourier transform and the fact that $(-1)^n = e^{j\pi n}$, we get:

$$\begin{aligned}
V(e^{j\omega}) &= X(e^{j(\omega+\pi)}) \\
W(e^{j\omega}) &= H_1(e^{j\omega})V(e^{j\omega}) \\
&= H_1(e^{j\omega})X(e^{j(\omega+\pi})) \\
Y(e^{j\omega}) &= W(e^{j(\omega-\pi)}) \\
&= H_1(e^{j(\omega-\pi)})X(e^{j\omega})
\end{aligned}$$

$H(e^{j\omega})$ is thus given by:

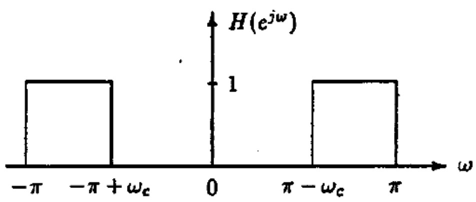
$$H(e^{j\omega}) = H_1(e^{j(\omega-\pi)}).$$

(b)

$$H(e^{j\omega}) = H_1(e^{j(\omega-\pi)}).$$

With the given choice of $H_1(e^{j\omega})$,

$$H(e^{j\omega}) = \begin{cases} 0 , & |\omega| < \pi - \omega_c \\ 1 , & \pi - \omega_c < |\omega| \le \pi. \end{cases}$$



2.72. The analysis equation for the Fourier transform:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

(a) The Fourier transform of $x^*[n]$,

$$\begin{aligned}\sum_{n=-\infty}^{\infty} x^*[n]e^{-j\omega n} &= \left(\sum_{n=-\infty}^{\infty} x[n]e^{j\omega n}\right)^* \\ &= X^*(e^{-j\omega n}).\end{aligned}$$

(b) The Fourier transform of $x^*[-n]$,

$$\begin{aligned}\sum_{n=-\infty}^{\infty} x^*[-n]e^{-j\omega n} &= \sum_{l=-\infty}^{\infty} x^*[l]e^{j\omega l} \\ &= \left(\sum_{l=-\infty}^{\infty} x[l]e^{-j\omega l}\right)^* \\ &= X^*(e^{j\omega}).\end{aligned}$$

2.77. (a) The Fourier transform of $y^*[-n]$ is $Y^*(e^{j\omega})$, and $X(e^{j\omega})Y(e^{j\omega})$ forms a transform pair with $x[n] * y[n]$. So

$$G(e^{j\omega}) = X(e^{j\omega})Y^*(e^{j\omega})$$

and

$$g[n] = x[n] * y^*[-n]$$

form a transform pair.

(b)

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})e^{j\omega n}d\omega &= \sum_{n=-\infty}^{\infty} \left(x[n] * y^*[-n]\right) e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x[k]y^*[k-n]e^{-j\omega n}\end{aligned}$$

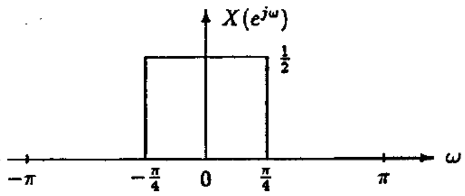
for $n = 0$:

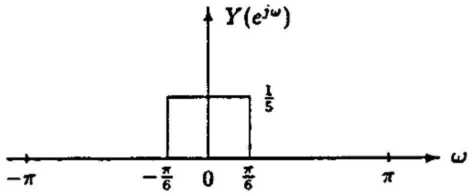
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(e^{j\omega})Y^*(e^{j\omega})d\omega = \sum_{k=-\infty}^{\infty} x[k]y^*[k]$$

(c) Using the result from part (b):

$$\begin{aligned}x[n] &= \frac{\sin(\pi n/4)}{2\pi n} \\ y^*[n] &= \frac{\sin(\pi n/6)}{5\pi n}\end{aligned}$$

We recognize each sequence to be a pulse in the frequency domain:





Substituting into Eq. (P2.77-1):

$$\begin{aligned}\sum_{n=-\infty}^{\infty} x[n]y^*[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega \\ &= \frac{1}{2\pi}\left[(\frac{1}{2})(\frac{1}{5})(\frac{2\pi}{6})\right] \\ &= \frac{1}{60}\end{aligned}$$

2.89. **(a)**

$$E\{x[n]x[n]\} = \phi_{xx}[0].$$

(b)

$$\begin{aligned}\Phi_{xx}(e^{j\omega}) &= X(e^{j\omega})X^*(e^{j\omega}) \\ &= W(e^{j\omega})H(e^{j\omega})W^*(e^{j\omega})H^*(e^{j\omega}) \\ &= \Phi_{ww}(e^{j\omega})|H(e^{j\omega})|^2 \\ &= \sigma_w^2\frac{1}{1-\cos(\omega)+1/4}.\end{aligned}$$

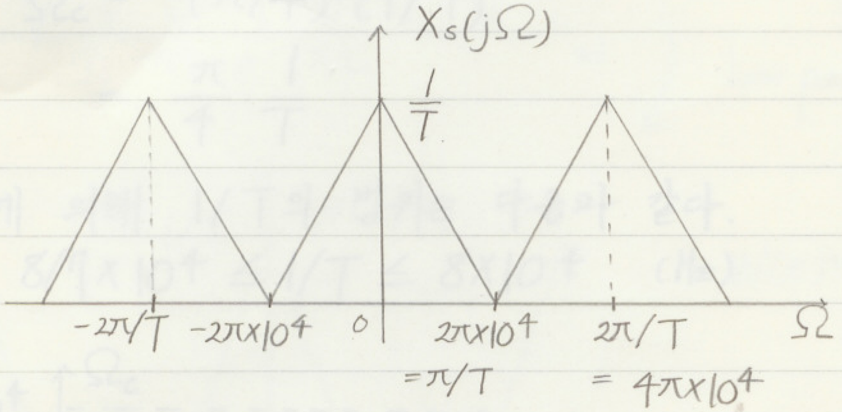
(c)

$$\begin{aligned}\phi_{xx}[n] &= \phi_{ww}[n] * h[n] * h[-n] \\ &= \sigma_w^2\left(\left(\frac{1}{2}\right)^n u[n] * \left(\frac{1}{2}\right)^{-n} u[-n]\right) \\ &= \sigma_w^2\phi_{hh}[n].\end{aligned}$$

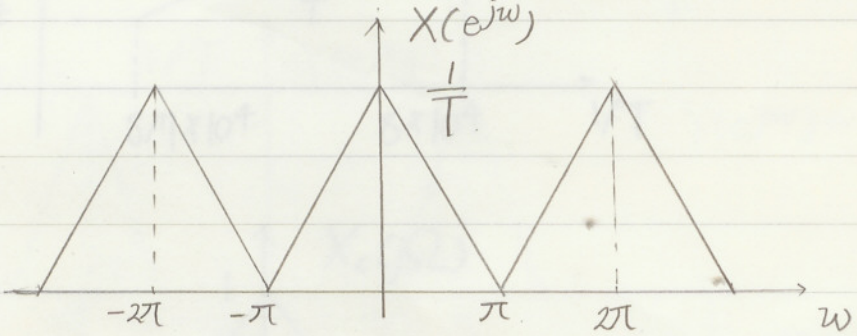
4.25 1

(a)

$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$



$$X(e^{j\omega}) = X_s(j(\omega/T))$$



(b) discrete-time filter $H(e^{j\omega})$ 의 pass band $|\omega| \le \pi/4$ 내에서 aliasing 이 일어나지 않아야 한다. 즉,

$$2\pi - (2\pi\times10^4)\cdot T \ge \pi/4$$

$$\Rightarrow \quad T \le \frac{2\pi - \pi/4}{2\pi\times10^4} = \frac{7}{8}\times10^{-4} \ (\text{sec})$$

또, 주어진 filter 가 Low pass filter로 의미가 있으려면,

$$(2\pi\times10^4)\cdot T \ge \pi/4$$

$$\Rightarrow \quad T \ge (1/8)\times10^{-4} \ (\text{sec})$$

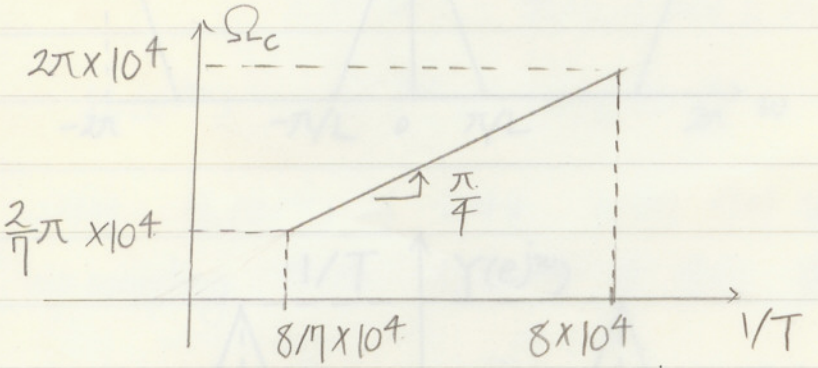
$\therefore \quad 1/8 \times 10^{-4} \le T \le 7/8 \times 10^{-4}$ (sec)

(c)

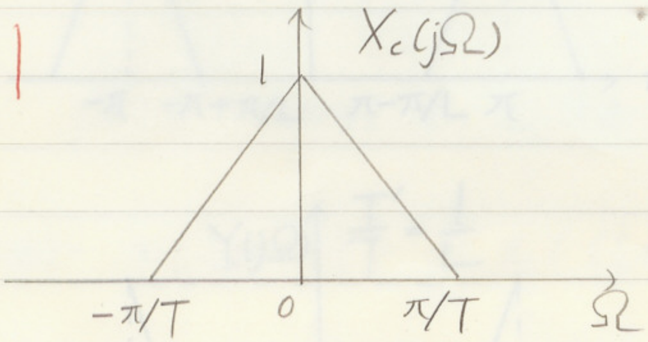
$$\Omega_c = (\pi/4)\cdot(1/T)$$
$$= \frac{\pi}{4}\cdot\frac{1}{T}$$

(b)에 의해 1/T의 범위는 다음과 같다.

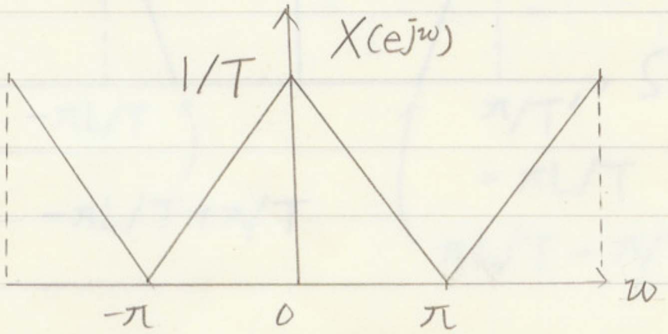
$8/7 \times 10^{4} \le 1/T \le 8 \times 10^{4}$ (Hz)



4.38

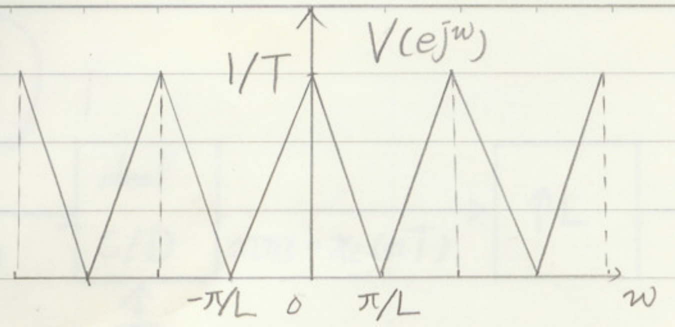


sampling, T



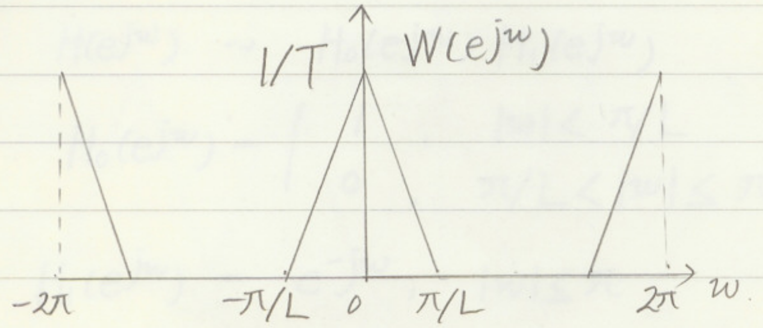
$$X(e^{jw}) = \frac{1}{T}\cdot X_c\left(j\frac{w}{T}\right), \quad |w| < \pi$$

expansion, L



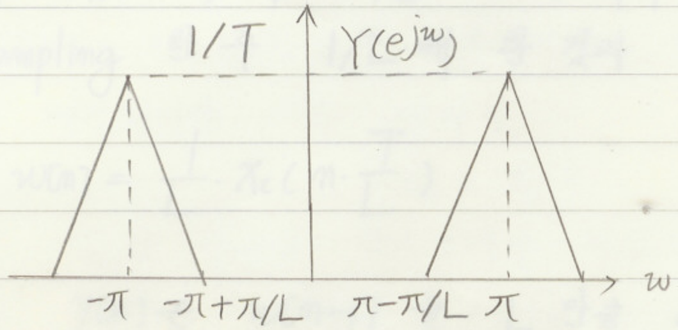
$$V(e^{jw}) = X(e^{jwL})$$

Low pass filtering, $w_c = \pi/L$.



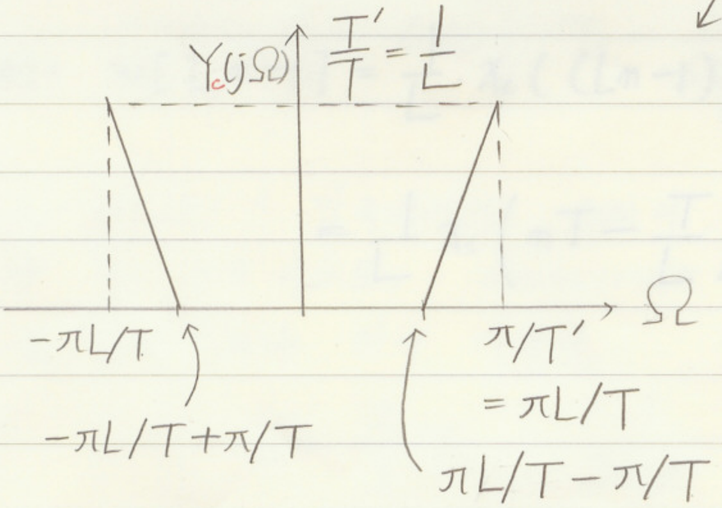
$$W(e^{jw}) = X(e^{jwL}) \cdot H(e^{jw})$$

$\times\ e^{j\pi n}$



$$Y(e^{jw}) = W(e^{j(w-\pi)})$$

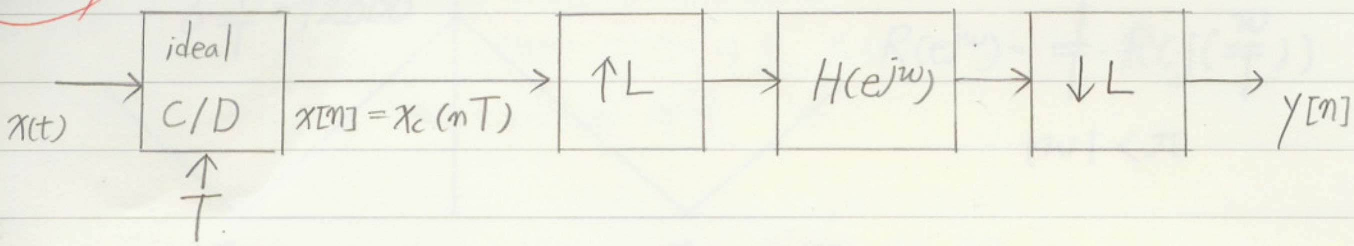
reconstruct, $T' = T/L$



$$Y(j\Omega) = T' \cdot Y(e^{j\Omega T'}), \quad |\Omega| < \pi/T'$$

4.40

1



$$H(e^{jw}) \rightsquigarrow H_0(e^{jw})\, H_1(e^{jw})$$

$$H_0(e^{jw}) = \begin{cases} 1, & |w| < \pi/L \\ 0, & \pi/L < |w| \le \pi. \end{cases}$$

$$H_1(e^{jw}) = e^{-jw}, \quad |w| \le \pi$$

그러면 $H_0(e^{jw})$ 의 출력을 $w[n]$ 이라 할 때, $w[n]$은 $x[n]$을 L배 up sampling 한 후 $1/L$ 배 한 것과 같다. 즉,

$$w[n] = \frac{1}{L} \cdot x_c\left(n \cdot \frac{T}{L}\right)$$

한편, $y[n]$은 $w[n-1]$ 을 L 만큼 down sampling 한 것과 같으므로,

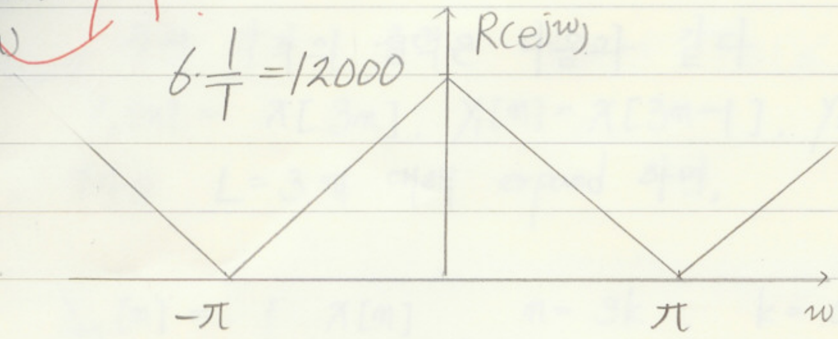
$$y[n] = w[Ln-1] = \frac{1}{L} \cdot x_c\left((Ln-1)\frac{T}{L}\right)$$

$$= \frac{1}{L} x_c\left(nT - \frac{T}{L}\right)$$

4.44 1

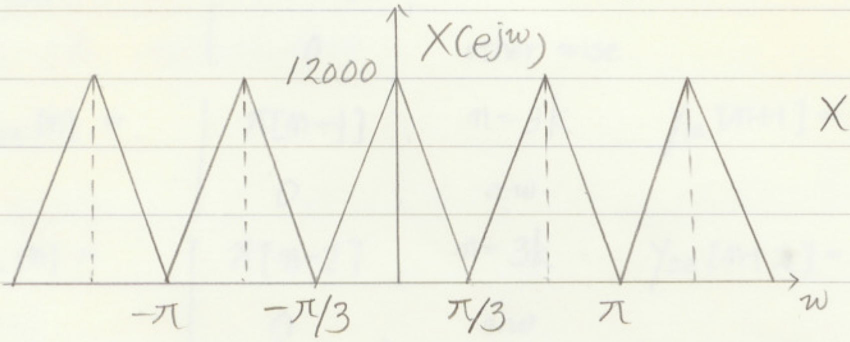
(a)

$6 \cdot \frac{1}{T} = 12000$



$R(e^{jw}) = \frac{1}{T} \cdot R(j(\frac{w}{T}))$

$|w| < \pi$



$X(e^{jw}) = R(e^{jw3})$

(b) $y[n] = \alpha r_c(nT_2)$ 가 만족되려면 $X(e^{jw})$ 에서 $\pi/3 \le |w| \le \pi$ 부분이 제거되어야 한다. 따라서,

$$w_0 = \pi/3$$

한편, T_2는 다음을 만족시켜야 하므로,

$$(\pi/3) \cdot (1/T_2) = 2\pi \cdot 1000$$

$$\Rightarrow \frac{1}{T_2} = 6000, \quad T_2 = \frac{1}{6000}$$

(c) $s_c(t) = \beta r_c(t)$ 가 만족되려면, $s[n]$이 $r_c(t)$의 sampling 한 신호에 scale 만 다른 경우 이므로, reconstruct 를 위한 T_3과 sampling period 를 일치 시켜야 한다. 따라서,

$$T_3 = \frac{2}{3} T_1 = \frac{1}{3000}$$

또는, $S(e^{jw})$ 와 $S(j\Omega)$ 의 관계로 부터,

$$\frac{2}{3}\pi \cdot \frac{1}{T_3} = \frac{\pi}{T_1} \rightarrow T_3 = \frac{2}{3} T_1 = \frac{1}{3000}$$

4.46. 1

(a). 우선 각각의 출력은 다음과 같다.

$y_0[n] = x[3n]$, $y_1[n] = x[3n-1]$, $y_2[n] = x[3n-2]$

각각을 $L=3$ 에 대해 expand 하면,

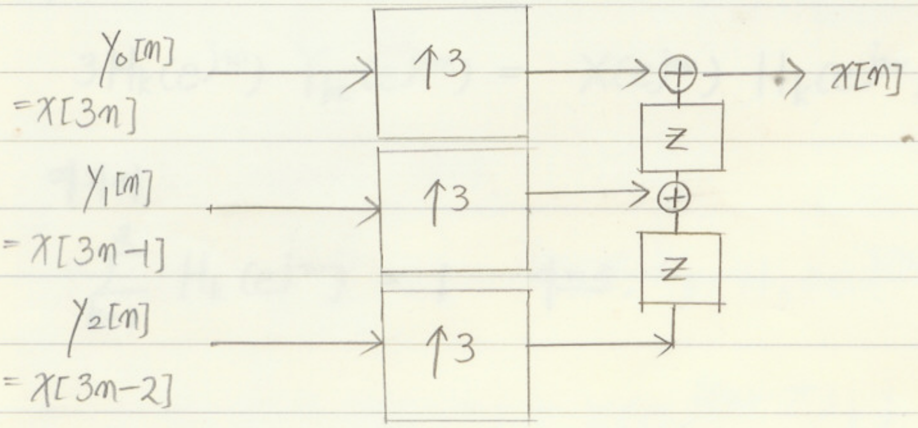
$$y_{0e}[n] = \begin{cases} x[n], & n = 3k, \quad k = 0, \pm1, \pm2, \cdots \\ 0, & \text{other wise} \end{cases}$$

$$y_{1e}[n] = \begin{cases} x[n-1], & n=3k \\ 0, & \text{o.w} \end{cases} \qquad y_{1e}[n+1] = \begin{cases} x[n], & n=3k-1 \\ 0, & \text{o.w} \end{cases}$$

$$y_{2e}[n] = \begin{cases} x[n-2], & n=3k \\ 0, & \text{o.w.} \end{cases} \qquad y_{2e}[n+2] = \begin{cases} x[n], & n=3k-2 \\ 0, & \text{o.w.} \end{cases}$$

따라서,

$$x[n] = y_{0e}[n] + y_{1e}[n+1] + y_{2e}[n+2]$$



Yes

$$x[n] = \begin{cases} y_0[n/3], & n = 3k \\ y_1[(n+1)/3], & n = 3k-1 \\ y_2[(n+2)/3], & n = 3k-2 \end{cases}$$

(b)

$$H_k(e^{jw}) = \begin{cases} 1, & k\pi/3 \le |w| \le (k+1)\pi/3, \quad k=0,1,2 \\ 0, & o.w. \end{cases}$$

이면,

$$Y_k(e^{jw}) = \frac{1}{3}\sum_{i=0}^{2} X(e^{j(w-2\pi i)/3}) \cdot H_k(e^{j(w-2\pi i)/3}).$$

L=3 으로 expand 하면,

$$Y_{ke}(e^{jw}) = \frac{1}{3}\sum_{i=0}^{2} X(e^{j(w-\frac{2}{3}\pi i)}) H_k(e^{j(w-\frac{2}{3}\pi i)})$$

그런데,

$$H_k(e^{jw}) \cdot H_k(e^{j(w-\frac{2}{3}\pi i)}) = \begin{cases} H_k(e^{jw}) & i=0 \\ 0 & o.w. \end{cases}$$

이므로,

$$3H_k(e^{jw}) \cdot Y_{ke}(e^{jw}) = X(e^{jw}) \cdot H_k(e^{jw}), \quad k=0,1,2$$

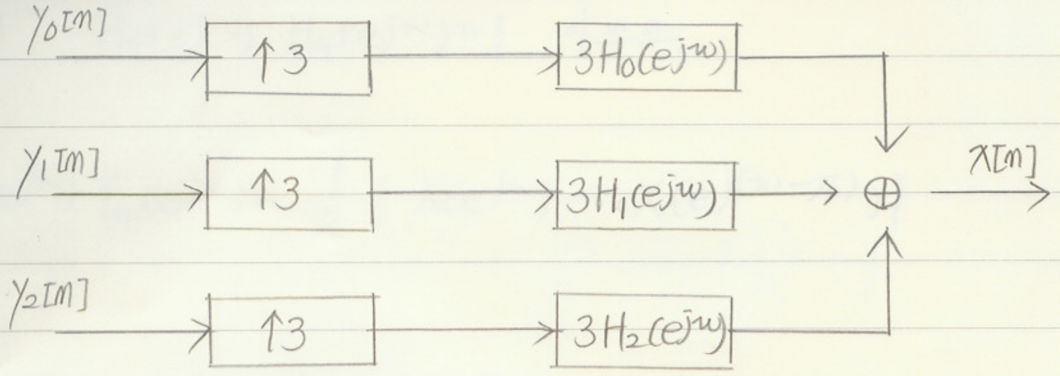
여기서,

$\sum_{k=0}^{2} H_k(e^{jw}) = 1$ 이므로,

$$\sum_{k=0}^{2} 3H_k(e^{jw}) Y_{ke}(e^{jw}) = X(e^{jw}) \cdot \{H_0(e^{jw}) + H_1(e^{jw}) + H_2(e^{jw})\}$$
$$= X(e^{jw})$$

를 얻는다.

이를 회로도로 나타내면 다음과 같다.



(c)

$$H_3(e^{jw}) = 1, \qquad H_4(e^{jw}) = \begin{cases} 1, & 0 \le w < \pi \\ -1, & -\pi \le w < 0 \end{cases}$$

이면,

$$Y_3(e^{jw}) = \frac{1}{2}\sum_{i=0}^{1} X(e^{j(w-2\pi i)/2}) \cdot H_3(e^{j(w-2\pi i)/2})$$

$$Y_4(e^{jw}) = \frac{1}{2}\sum_{i=0}^{1} X(e^{j(w-2\pi i)/2}) H_4(e^{j(w-2\pi i)/2})$$

$L=2$로 expand 하면,

$$Y_3(e^{jw}) = \frac{1}{2}\sum_{i=0}^{1} X(e^{j(w-\pi i)}) \cdot H_3(e^{j(w-\pi i)})$$

$$= \frac{1}{2} \cdot \{ X(e^{jw}) + X(e^{j(w-\pi)}) \}$$

$$Y_4(e^{jw}) = \frac{1}{2}\sum_{i=0}^{1} X(e^{j(w-\pi i)}) \cdot H_4(e^{j(w-\pi i)})$$

$$= \frac{1}{2} \cdot H_4(e^{jw}) \cdot \{ X(e^{jw}) - X(e^{j(w-\pi)}) \}$$

$$(\because H_4(e^{j(w-\pi)}) = -H_4(e^{jw}))$$

그런데. $H_4(e^{j\omega}) \cdot H_4(e^{j\omega}) = 1$ 이므로,

$$H_4(e^{j\omega})Y_4(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) - X(e^{j(\omega-\pi)})\}$$

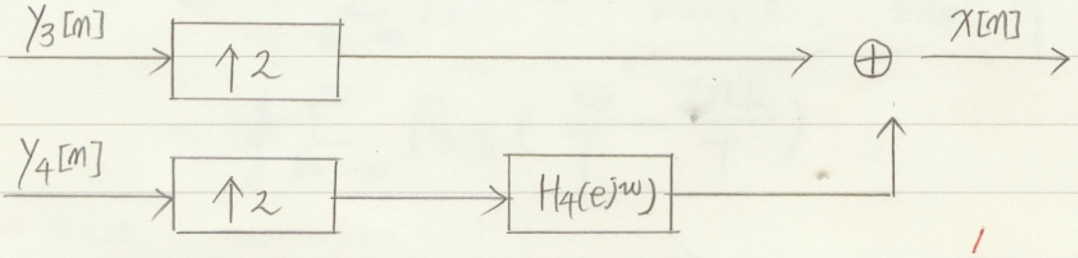
따라서.

$$Y_3(e^{j\omega}) + H_4(e^{j\omega})Y_4(e^{j\omega})$$

$$= \frac{1}{2}\{X(e^{j\omega}) + X(e^{j(\omega-\pi)}) + X(e^{j\omega}) - X(e^{j(\omega-\pi)})\}$$

$$= \frac{1}{2} \cdot 2 \cdot X(e^{j\omega}) = X(e^{j\omega})$$

를 얻는다. 이를 회로도로 나타내면 다음과 같다.



4.48. 1

(a)

$$\phi_{x_c x_c}(\tau) = E(x_c(t)\, x_c^*(t+\tau))$$

$$\leftrightarrow P_{x_c x_c}(\Omega) = \int_{-\infty}^{\infty} \phi_{x_c x_c}(\tau) e^{-j\Omega\tau}\, d\tau$$

이면,

$$\phi_{xx}[m] = E[x^*[n+m]\, x[n]] = E[x_c^*((n+m)T) \cdot x(nT)]$$

$$= \phi_{x_c x_c}(mT)$$

한편,

$$P_{x_c x_c}(\Omega) = \Pi(\Omega / 2\Omega_0)$$

$$\phi_{x_c x_c}(\tau) = \mathcal{F}^{-1}\{P_{x_c x_c}(\Omega)\} = \frac{\Omega_0}{\pi} \operatorname{sinc}\left(\frac{\Omega_0}{\pi} \cdot t\right)$$

$$= \frac{\sin \Omega_0 t}{\pi t}$$

따라서,

$$\phi_{xx}[m] = \frac{\sin \Omega_0 m T}{\pi \cdot m T} = \phi_{x_c x_c}(mT)$$

(b)

$$P_{xx}(w) = \frac{1}{T} \sum_{k=-\infty}^{\infty} P_{x_c x_c}\left(\frac{w}{T} - k\Omega_s\right), \quad \Omega_s = \frac{2\pi}{T}$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} P_{x_c x_c}\left(\frac{w}{T} - \frac{2\pi k}{T}\right)$$

이므로,

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} \Pi\left((w - 2\pi k) / 2\Omega_0 T\right)$$

따라서

$$2\pi k = 2\Omega_0 T \rightarrow T = \frac{\pi}{\Omega_0} \cdot k, \quad k = 1, 2, 3, \cdots$$

이면,

$$P_{xx}(w) = \frac{1}{T} k = const$$

가 된다.

$$\left(\because P_{xx}(w) = \frac{1}{T} \sum_{k=0}^{M-1} \sum_{r=-\infty}^{\infty} \Pi\left((w - 2\pi(Mr + k)) / 2\Omega_0 T\right)\right)$$

(c) $P_{xx}(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} \Lambda((\omega - 2\pi k) / \Omega_0 T)$ $\quad (\because P_{x_c x_c}(\Omega) = \Lambda(\Omega/\Omega_0))$

이므로,

$$= \frac{1}{T} \sum_{k=0}^{M-1} \sum_{r=-\infty}^{\infty} \Lambda((\omega - 2\pi(Mr+k)) / \Omega_0 T)$$

따라서,

$$2\pi k = \Omega_0 T \rightarrow T = \frac{2\pi}{\Omega_0} k \quad , \quad k = 1, 2, 3, \cdots$$

이면,

$$P_{xx}(\omega) = \frac{1}{T} k = const$$

가 된다.

(d) $\phi_{xx}[m] = K\delta[m]$ 이 되어야 한다

예를 들어 (b)의 경우, $\leftarrow$ (b) 번의 다른 풀이

$\phi_{xx}[m] = \frac{\sin \Omega_0 mT}{\pi m T}$ 이므로,

$\Omega_0 T = k\pi \rightarrow T = \frac{\pi}{\Omega_0} k$ 로 동일한 조건을 얻는다

이를 continuous time domain 상에서의 제약으로 나타내면,

$\phi_{xx}[m] = \phi_{x_c x_c}(mT)$ 이므로,

$\phi_{x_c x_c}(mT) = 0$, $m \neq 0$ 이 만족되어야 한다

* (c)의 경우, 이 방법으로 풀면, $\leftarrow$ (c) 번의 다른 풀이

$$\phi_{xx}[m] = \frac{\Omega_0}{2\pi} \operatorname{sinc}^2\left(\frac{\Omega_0}{2\pi} mT\right) = \frac{\Omega_0}{2\pi}\left(\frac{\sin(\Omega_0 mT/2)}{\Omega_0 mT/2}\right)^2, \quad T = \frac{2\pi}{\Omega_0} \cdot k$$

4.51

$x_i[n] = x_c\left(n\frac{T}{L}\right)$ 이므로, (Fig P4.51-1의 ideal LPF gain = L 가정)

$\phi_1[n] = E[x_i^*[m+n]\, x_i[m]] = E\left[x_c^*\left((m+n)\frac{T}{L}\right)\cdot x_c\left(m\frac{T}{L}\right)\right]$

$= \phi_{x_c x_c}\left(n\frac{T}{L}\right)$

반면,

$\phi_2[n] = E[x^*[m+n]\, x[m]] = E[x_c^*((m+n)T)\cdot x_c(mT)] = \phi_{x_c x_c}(nT)$ 이므로,

$\phi_2[n] = \phi_1[Ln]$

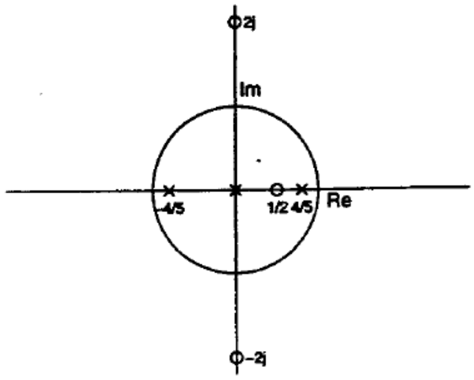
따라서 $\phi_2[n]$을 upsampling 하면 $\phi_1[n]$을 얻을 수 있고, 이를 위해서는 $H_2(e^{j\omega})$가

$$H_2(e^{j\omega}) = \begin{cases} L & |\omega| < \pi/L \\ 0 & \text{o.w} \end{cases} \quad \text{(ideal LPF)}$$

가 되어야 한다.

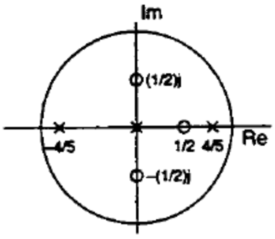
5.45.

$$H(z) = \frac{(1-0.5z^{-1})(1+2jz^{-1})(1-2jz^{-1})}{(1-0.8z^{-1})(1+0.8z^{-1})}$$

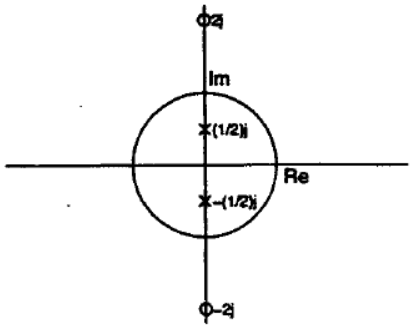


(a) A minimum phase system has all poles and zeros inside $|z| = 1$

$$H_1(z) = \frac{(1-0.5z^{-1})(1+\frac{1}{4}z^{-2})}{(1-0.64z^{-2})}$$

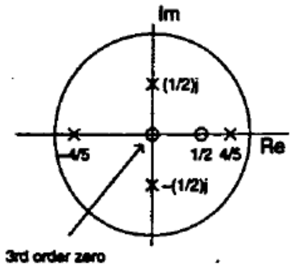


$$H_{ap}(z) = \frac{(1+4z^{-2})}{(1+\frac{1}{4}z^{-2})}$$

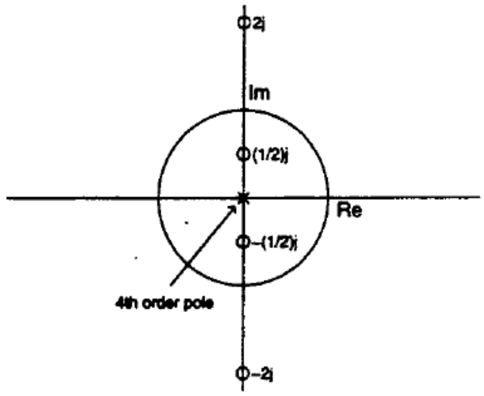


(b) A generalized linear phase system has zeros and poles at $z = 1, -1, 0$ or ∞ or in conjugate reciprocal pairs.

$$H_2(z) = \frac{(1 - 0.5z^{-1})}{(1 - 0.64z^{-2})(1 + \frac{1}{4}z^{-2})}$$



$$H_{lin}(z) = (1 + \tfrac{1}{4}z^{-2})(1 + 4z^{-2})$$



5.51. *False.* Let $h[n]$ equal

$$h[n] = \frac{\sin \omega_c(n - 4.3)}{\pi(n - 4.3)} \longleftrightarrow H(e^{j\omega}) = \begin{cases} e^{-4.3\omega}, & |\omega| < \omega_c \\ 0, & \text{otherwise} \end{cases}$$

Proof: Although the group delay is constant (grd $[H(e^{j\omega})] = 4.3$) the resulting M is not an integer.

$$\begin{aligned} h[n] &= \pm h[M - n] \\ H(e^{j\omega}) &= \pm e^{jM\omega} H(e^{-j\omega}) \\ e^{-j4.3\omega} &= \pm e^{j(M+4.3)\omega}, \qquad |\omega| < \omega_c \\ M &= -8.6 \end{aligned}$$

5.55.

- Since $x[n]$ is real the poles & zeros come in complex conjugate pairs.
- From (1) we know there are no poles except at zero or infinity.
- From (3) and the fact that $x[n]$ is finite we know that the signal has generalized linear phase.
- From (3) and (4) we have $\alpha = 2$. This and the fact that there are no poles in the finite plane except the five at zero (deduced from (1) and (2)) tells us the form of $X(z)$ must be

$$X(z) = x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3} + x[4]z^{-4} + x[5]z^{-5}$$

The phase changes by π at $\omega = 0$ and π so there must be a zero on the unit circle at $z = \pm 1$. The zero at $z = 1$ tells us $\sum x[n] = 0$. The zero at $z = -1$ tells us $\sum (-1)^n x[n] = 0$.
We can also conclude $x[n]$ must be a Type III filter since the length of $x[n]$ is odd and there is a zero at both $z = \pm 1$. $x[n]$ must therefore be antisymmetric around $n = 2$ and $x[2] = 0$.

- From (5) and Parseval's theorem we have $\sum |x[n]|^2 = 28$.
- From (6)

$$\begin{aligned} y[0] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\omega})d\omega = 4 \\ &= x[n] * u[n]\,|_{n=0} = x[-1] + x[0] \end{aligned}$$

$$\begin{aligned} y[1] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\omega})e^{j\omega}d\omega = 6 \\ &= x[n] * u[n]\,|_{n=1} = x[-1] + x[0] + x[1] \end{aligned}$$

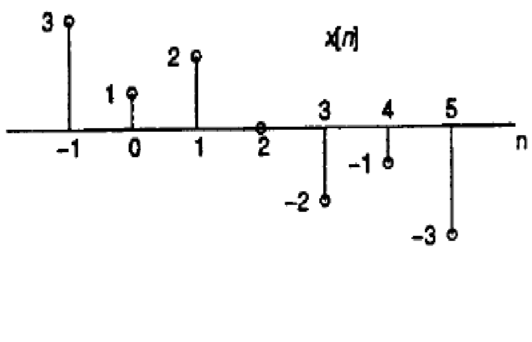
- The conclusion from (7) that $\sum (-1)^n x[n] = 0$ we already derived earlier.
- Since the DTFT $\{x_e[n]\} = \mathcal{R}e\{X(e^{j\omega})\}$ we have

$$\begin{aligned} \frac{x[5] + x[-5]}{2} &= -\frac{3}{2} \\ x[5] &= -3 + x[-5] \\ x[5] &= -3 \end{aligned}$$

Summarizing the above we have the following (dependent) equations

(1) $x[-1] + x[0] + x[1] + x[2] + x[3] + x[4] + x[5] = 0$
(2) $-x[-1] + x[0] - x[1] + x[2] - x[3] + x[4] - x[5] = 0$
(3) $x[2] = 0$
(4) $x[-1] = -x[5]$
(5) $x[0] = -x[4]$
(6) $x[1] = -x[3]$
(7) $x[-1]^2 + x[0]^2 + x[1]^2 + x[2]^2 + x[3]^2 + x[4]^2 + x[5]^2 = 28$
(8) $x[-1] + x[0] = 4$
(9) $x[-1] + x[0] + x[1] = 6$
(10) $x[5] = -3$

$x[n]$ is easily obtained from solving the equations in the following order: (3),(10),(4),(8),(5),(9), and (6).



5.57. (a)

$$x[n] = s[n]\cos\omega_0 n = \frac{1}{2}s[n]e^{j\omega_0 n} + \frac{1}{2}s[n]e^{-j\omega_0 n}$$

$$X(e^{j\omega}) = \frac{1}{2}S(e^{j(\omega-\omega_0)}) + \frac{1}{2}S(e^{j(\omega+\omega_0)})$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \frac{1}{2}e^{-j\phi_0}S(e^{j(\omega-\omega_0)}) + \frac{1}{2}e^{j\phi_0}S(e^{j(\omega+\omega_0)})$$

$$\begin{aligned} y[n] &= \frac{1}{2}s[n]e^{j(\omega_0 n-\phi_0)} + \frac{1}{2}s[n]e^{-j(\omega_0 n-\phi_0)} \\ &= s[n]\cos(\omega_0 n - \phi_0) \end{aligned}$$

(b) This time,

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \frac{1}{2}e^{-j\phi_0}e^{-j\omega n_d}S(e^{j(\omega-\omega_0)}) + \frac{1}{2}e^{j\phi_0}e^{-j\omega n_d}S(e^{j(\omega+\omega_0)})$$

$$\begin{aligned} y[n] &= \delta[n-n_d] * \left(\frac{1}{2}s[n]e^{j(\omega_0 n-\phi_0)} + \frac{1}{2}s[n]e^{-j(\omega_0 n-\phi_0)}\right) \\ &= \delta[n-n_d] * s[n]\cos(\omega_0 n - \phi_0) \\ &= s[n-n_d]\cos(\omega_0 n - \omega_0 n_d - \phi_0) \end{aligned}$$

Therefore, if $\phi_1 = \phi_0 + \omega_0 n_d$ then

$$y[n] = s[n-n_d]\cos(\omega_0 n - \phi_1)$$

for narrowband $s[n]$.

(c)

$$\begin{aligned} \tau_{gr} &= -\frac{d}{d\omega}arg[H(e^{j\omega})] = -\frac{d}{d\omega}[-\phi_0 - \omega n_d] = n_d \\ \tau_{ph} &= -\frac{1}{\omega}arg[H(e^{j\omega})] = -\frac{1}{\omega}[-\phi_0 - \omega n_d] = \frac{\phi_0}{\omega} - n_d \end{aligned}$$

$$y[n] = s[n - \tau_{gr}(\omega_0)]\cos[\omega_0(n - \tau_{ph}(\omega_0))]$$

(d) The effect would be the same as the following:

(i) Bandlimit interpolate the composite signal to a C–T signal with some rate T.

(ii) Delay the envelope by $T \cdot \tau_{gr}$, and delay the carrier by $T \cdot \tau_{ph}$.

(iii) Sample to a D–T signal at rate T

5.64. (a) We desire $|H(z)H_c(z)| = 1$, where $H_c(z)$ is stable and causal and $H(z)$ is not minimum phase. So,

$$|H_{ap}(z)H_{min}(z)H_c(z)| = 1$$

Since $|H_{ap}(z)| = 1$, we want

$$|H_{min}(z)H_c(z)| = 1$$

This means we have

$$H_c(z) = \frac{1}{H_{min}(z)}$$

which will be stable and causal since all the zeros of $H_{min}(z)$, which become the poles of $H_c(z)$, are inside the unit circle.

(b) Since

$$H_c(z) = \frac{1}{H_{min}(z)}$$

We have

$$G(z) = H_{ap}(z)$$

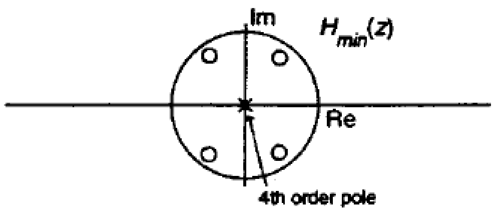
(c)

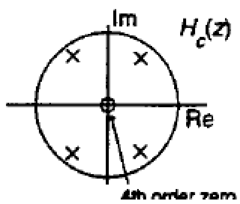
$$H(z) = (1 - 0.8e^{j0.3\pi}z^{-1})(1 - 0.8e^{-j0.3\pi}z^{-1})(1 - 1.2e^{j0.7\pi}z^{-1})(1 - 1.2e^{-j0.7\pi}z^{-1})$$

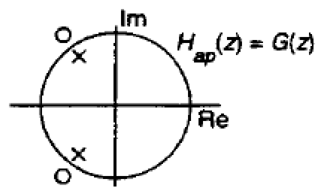
$$H_{min}(z) = (1.44)(1 - 0.8e^{j0.3\pi}z^{-1})(1 - 0.8e^{-j0.3\pi}z^{-1})(1 - (5/6)e^{j0.7\pi}z^{-1})(1 - (5/6)e^{-j0.7\pi}z^{-1})$$

$$H_c(z) = \frac{1}{(1.44)(1 - 0.8e^{j0.3\pi}z^{-1})(1 - 0.8e^{-j0.3\pi}z^{-1})(1 - (5/6)e^{j0.7\pi}z^{-1})(1 - (5/6)e^{-j0.7\pi}z^{-1})}$$

$$G(z) = H_{ap}(z) = \frac{(z^{-1} - (5/6)e^{-j0.7\pi})(z^{-1} - (5/6)e^{j0.7\pi})}{(1 - (5/6)e^{j0.7\pi}z^{-1})(1 - (5/6)e^{-j0.7\pi}z^{-1})}$$







5.66. (a) We use the allpass principle and place a pole at $z = z_k$ and a zero at $z = \frac{1}{z_k^*}$.

$$\begin{aligned} H(z) &= H_{min}(z)\frac{z^{-1} - z_k^*}{1 - z_k z^{-1}} \\ &= Q(z)(z^{-1} - z_k^*) \end{aligned}$$

(b)

$$H(z) = Q(z)z^{-1} - z_k^* Q(z)$$

$$h[n] = q[n-1] - z_k^* q[n]$$

$$H_{min}(z) = Q(z) - z_k Q(z) z^{-1}$$

$$h_{min}[n] = q[n] - z_k q[n-1]$$

(c)

$$\begin{aligned} \varepsilon &= \sum_{m=0}^{n} |h_{min}[m]|^2 - \sum_{m=0}^{n} |h[m]|^2 \\ &= \sum_{m=0}^{n} (|q[m]|^2 - z_k q[m-1] q^*[m] - z_k^* q^*[m-1] q[m] + |z_k|^2 |q[m-1]|^2) \\ &\quad - \sum_{m=0}^{n} (|q[m-1]|^2 - z_k^* q^*[m-1] q[m] - z_k q[m-1] q^*[m] + |z_k|^2 |q[m]|^2) \\ &= (1 - |z_k|^2) \sum_{m=0}^{n} (|q[m]|^2 - |q[m-1]|^2) \\ &= (1 - |z_k|^2)|q[n]|^2 \end{aligned}$$

(d)

$$\varepsilon = (1 - |z_k|^2)|q[n]|^2 \geq 0 \quad \forall n \text{ since } |z_k| < 1$$

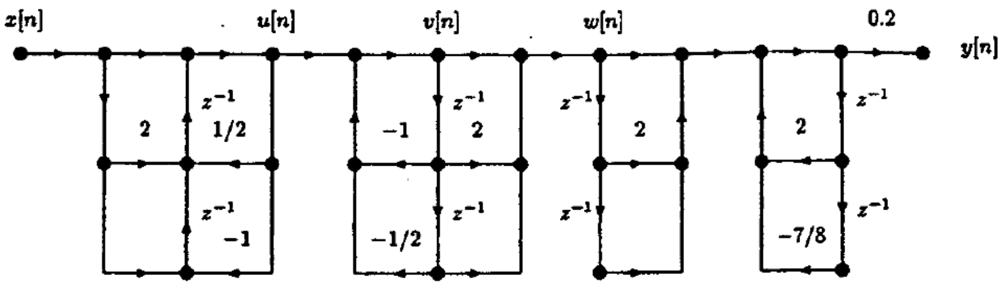
Then

$$\sum_{m=0}^{n} |h_{min}[m]|^2 - \sum_{m=0}^{n} |h[m]|^2 \geq 0$$

$$\sum_{m=0}^{n} |h[m]|^2 \leq \sum_{m=0}^{n} |h_{min}[m]|^2 \quad \forall n$$

6.26. (a) We can rearrange $H(z)$ this way:

$$H(z) = \frac{(1+z^{-1})^2}{1-\frac{1}{2}z^{-1}+z^{-2}} \cdot \frac{(1+z^{-1})^2}{1+z^{-1}+\frac{1}{2}z^{-2}} \cdot (1+z^{-1})^2 \cdot \frac{1}{1-2z^{-1}+\frac{7}{8}z^{-2}} \cdot 0.2$$

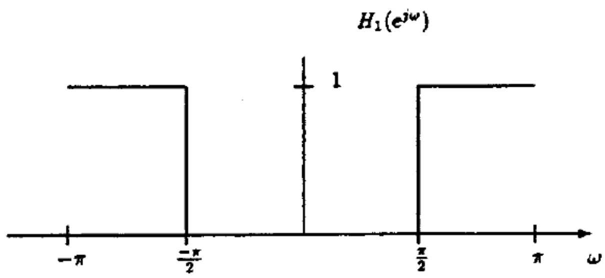


The solution is not unique; the order of the denominator 2nd-order sections may be rearranged.

(b)

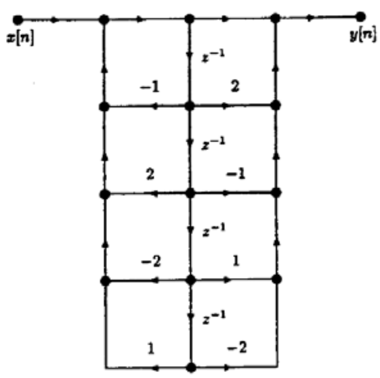
$$\begin{aligned}
u[n] &= x[n] + 2x[n-1] + x[n-2] + \frac{1}{2}u[n-1] - u[n-2] \\
v[n] &= u[n] - v[n-1] - \frac{1}{2}v[n-2] \\
w[n] &= v[n] + 2v[n-1] + v[n-2] \\
y[n] &= w[n] + 2w[n-1] + w[n-2] + 2y[n-1] - \frac{7}{8}y[n-2].
\end{aligned}$$

6.27. (a) $H_1(e^{j\omega}) = H(e^{j(\omega+\pi)})$.



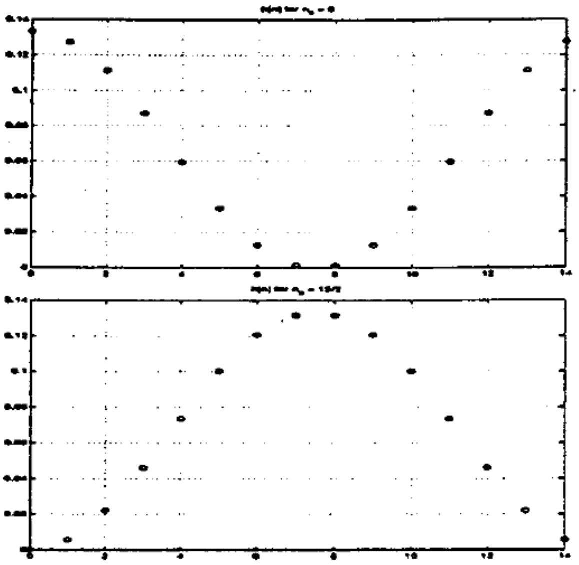
(b) For $H_1(z) = H(-z)$, replace each z^{-1} by $-z^{-1}$. Alternatively, replace each coefficient of an odd-delayed variable by its negative.

(c)



6.30.

(a)



(b)

$$
\begin{aligned}
H(z) &= \frac{1}{15}\sum_{n=0}^{14}\left[1+\cos\left(\frac{2\pi}{15}(n-n_0)\right)\right]z^{-n} \\
&= \frac{1}{15}\sum_{n=0}^{14}z^{-n} + \frac{1}{15}\sum_{n=0}^{14}\frac{1}{2}\left[e^{j\frac{2\pi}{15}(n-n_0)} + e^{-j\frac{2\pi}{15}(n-n_0)}\right]z^{-n} \\
&= \frac{1}{15}\frac{1-z^{-15}}{1-z^{-1}} + \frac{1}{15}\frac{1}{2}\frac{e^{-j\frac{2\pi}{15}n_0}[1-(e^{j\frac{2\pi}{15}}z^{-1})^{15}]}{1-e^{j\frac{2\pi}{15}}z^{-1}} \\
&\quad + \frac{1}{15}\frac{1}{2}\frac{e^{j\frac{2\pi}{15}n_0}[1-(e^{-j\frac{2\pi}{15}}z^{-1})^{15}]}{1-e^{-j\frac{2\pi}{15}}z^{-1}} \\
&= \frac{1}{15}(1-z^{-15})\left[\frac{1}{1-z^{-1}} + \frac{\frac{1}{2}e^{-j\frac{2\pi}{15}n_0}}{1-e^{j\frac{2\pi}{15}}z^{-1}} + \right. \\
&\quad \left.\frac{\frac{1}{2}e^{j\frac{2\pi}{15}n_0}}{1-e^{-j\frac{2\pi}{15}}z^{-1}}\right].
\end{aligned}
$$

(c)

$$
H(e^{j\omega}) = \frac{1}{15}e^{-j7\omega}\left[\frac{\sin((15\omega)/2)}{\sin(\omega/2)} - \frac{1}{2}\frac{e^{-j\frac{\pi}{15}}\sin((15\omega)/2)}{\sin((\omega-\frac{2\pi}{15})/2)} - \frac{1}{2}\frac{e^{j\frac{\pi}{15}}\sin((15\omega)/2)}{\sin((\omega+(2\pi)/15)/2)}\right].
$$

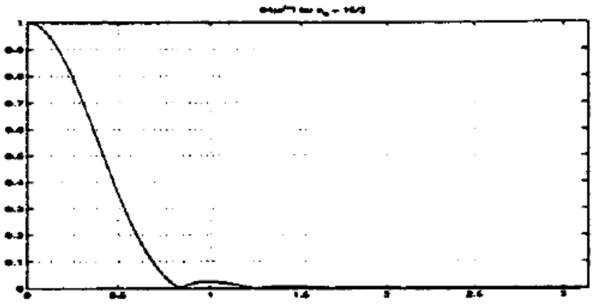
$$
H(e^{j\omega}) = \frac{1-e^{-j15\omega}}{15}\left[\frac{1}{1-e^{-j\omega}} + \frac{\frac{1}{2}e^{-j\frac{2\pi n_0}{15}}}{1-e^{j\frac{2\pi}{15}}e^{-j\omega}} + \frac{\frac{1}{2}e^{j\frac{2\pi n_0}{15}}}{1-e^{-j\frac{2\pi}{15}}e^{-j\omega}}\right]
$$

When $n_0 = 15/2$, .

$$
H(e^{j\omega}) = \frac{1}{15}\left[\frac{e^{j\frac{\omega}{2}}(1-e^{-j15\omega})}{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}} - \frac{\frac{1}{2}e^{j\frac{\omega-(2\pi/15)}{2}}(1-e^{-j15\omega})}{e^{j\frac{\omega-(2\pi/15)}{2}} - e^{-j\frac{\omega-(2\pi/15)}{2}}} - \right.
$$

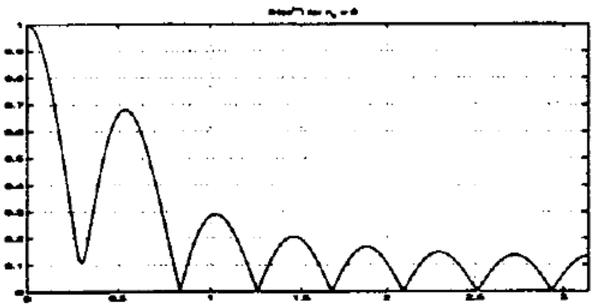
$$\left.\frac{\frac{1}{2}e^{j\frac{\omega+(2\pi/15)}{2}}(1-e^{-j15\omega})}{e^{j\frac{\omega+(2\pi/15)}{2}}-e^{-j\frac{\omega+(2\pi/15)}{2}}}\right]$$

$$\begin{aligned} &= \frac{1}{15}\left[\frac{e^{-j\omega 7}(e^{j\omega\frac{15}{2}}-e^{-j\omega\frac{15}{2}})}{2j\sin\frac{\omega}{2}} - \right.\\ &\quad \frac{\frac{1}{2}e^{-j\omega 7}e^{-j\frac{\pi}{15}}(e^{j\omega\frac{15}{2}}-e^{-j\omega\frac{15}{2}})}{2j\sin\left(\frac{\omega-(2\pi/15)}{2}\right)} - \\ &\quad \left.\frac{\frac{1}{2}e^{-j\omega 7}e^{j\frac{\pi}{15}}(e^{j\frac{15}{2}\omega}-e^{-j\frac{15}{2}\omega})}{2j\sin\left(\frac{\omega+(2\pi/15)}{2}\right)}\right] \\ &= \frac{e^{-j\omega 7}}{15}\left[\frac{\sin(15\omega/2)}{\sin(\omega/2)} - \frac{\frac{1}{2}e^{-j\frac{\pi}{15}}\sin(15\omega/2)}{\sin\left(\frac{\omega-(2\pi/15)}{2}\right)} - \right.\\ &\quad \left.\frac{\frac{1}{2}e^{j\frac{\pi}{15}}\sin(15\omega/2)}{\sin\left(\frac{\omega+(2\pi/15)}{2}\right)}\right] \end{aligned}$$



When $n_0 = 0$,

$$\begin{aligned} H(e^{j\omega}) &= \frac{e^{-j\omega 7}}{15}\left[\frac{\sin(15\omega/2)}{\sin(\omega/2)} + \frac{\frac{1}{2}e^{-j\frac{\pi}{15}}\sin(15\omega/2)}{\sin\left(\frac{\omega-(2\pi/15)}{2}\right)} + \right.\\ &\quad \left.\frac{\frac{1}{2}e^{j\frac{\pi}{15}}\sin(15\omega/2)}{\sin\left(\frac{\omega+(2\pi/15)}{2}\right)}\right] \end{aligned}$$



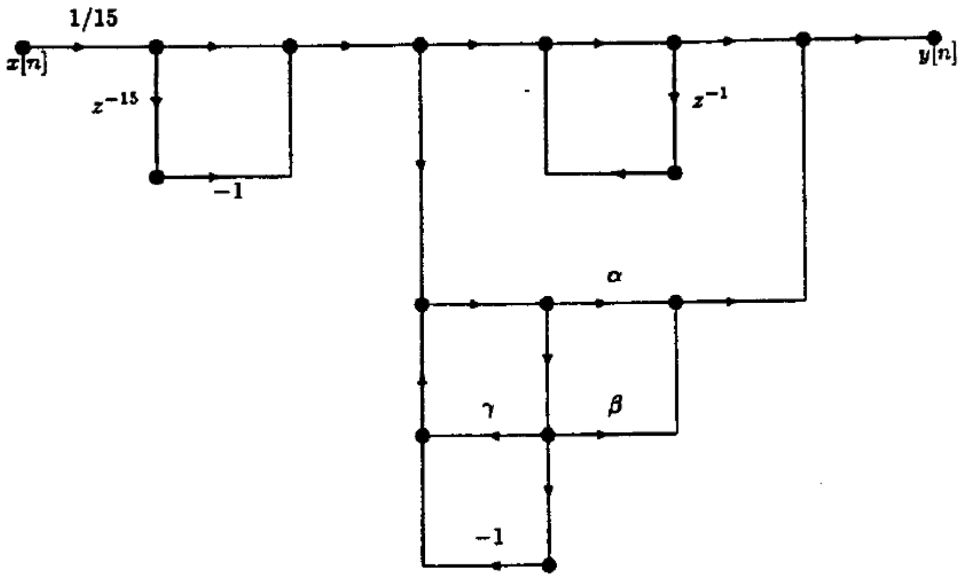
The system will have generalized linear phase if the impulse response has even symmetry (note it cannot have odd symmetry), or alternatively, if the frequency response can be expressed as:

$$H(e^{j\omega}) = e^{-j\omega 7}A_e(e^{j\omega})$$

where $A_e(e^{j\omega})$ is a real, even, periodic function in ω. We thus conclude that the system will have generalized linear phase for $n_0 = \frac{15}{2}k$, where k is an odd integer.

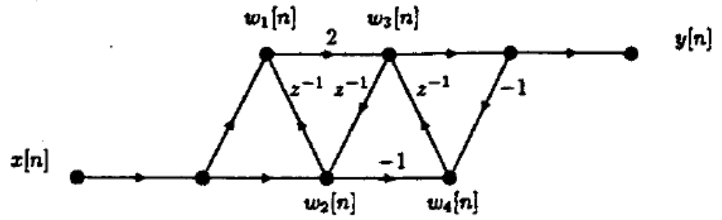
(d) Rewrite $H(z)$ as

$$H(z) = \frac{1 - z^{-15}}{15}\left[\frac{1}{1 - z^{-1}} + \frac{\cos\frac{2\pi n_0}{15} - \cos\left(\frac{2\pi}{15} + \frac{2\pi n_0}{15}\right)z^{-1}}{1 - 2\cos\frac{2\pi}{15}z^{-1} + z^{-2}}\right].$$



where $\alpha = \cos(2\pi n_0/15)$, $\beta = -\cos(2\pi(n_0+1)/15)$, and $\gamma = 2\cos(2\pi/15)$.

6.34. (a) We have:



First, we find the system function, we have:

$$
\begin{aligned}
(1)\ w_1[n] &= x[n] + w_2[n-1] \\
(2)\ w_2[n] &= x[n] + w_3[n-1] \\
(3)\ w_3[n] &= 2w_1[n] + w_4[n-1] \\
(4)\quad y[n] &= w_3[n] \\
(5)\ w_4[n] &= -y[n] - w_2[n]
\end{aligned}
$$

Taking the Z-transform of the above equations and combining terms, we get:

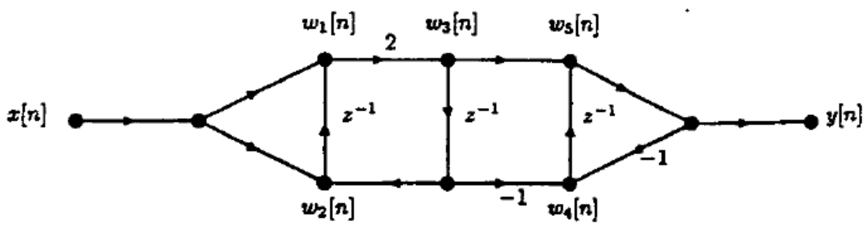
$$(1 - z^{-1})Y(z) + z^{-1}Y(z) = (2 + z^{-1})X(z).$$

The system function is thus given by:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2 + z^{-1}}{1 + z^{-1} - z^{-2}}$$

Since the system function is second order (highest order term is z^{-2}), we should be able to implement this system using only 2 delays, this can be done with a direct form II implementation. Therefore, the minimum number of delays required to implement an equivalent system is 2.

(b) Now we have:



Let's find the transfer function, we have:

$$\begin{aligned}
(1)\ w_1[n] &= x[n] + w_2[n-1] \\
(2)\ w_2[n] &= x[n] + w_3[n-1] \\
(3)\ w_3[n] &= 2w_1[n] \\
(4)\ w_4[n] &= -w_3[n-1] - y[n] \\
(5)\ w_5[n] &= w_3[n] + w_4[n-1] \\
(6)\quad y[n] &= w_5[n]
\end{aligned}$$

Taking the Z-transform of the above equations and combining terms, we get:

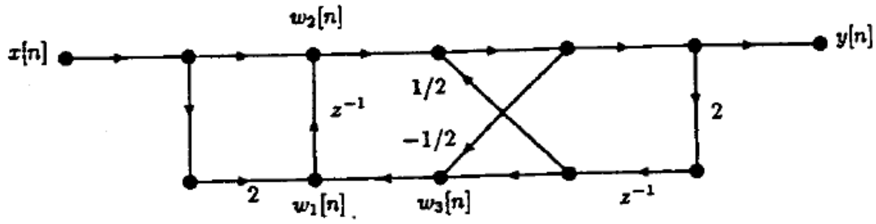
$$(1+z^{-1})Y(z) = \frac{(1-z^{-2})(2+2z^{-1})}{1-2z^{-2}}X(z).$$

The system function is thus given by:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2(1+z^{-1})(1-z^{-1})}{1-2z^{-2}}.$$

Since the transfer function is not the same as the one in part a, we conclude that system B does not represent the same input-output relationship as system A. This should not be surprising since in system B we added two unidirectional wires and therefore changed the input-output relationship.

6.36. (a) Transpose = reverse arrows direction and reverse the input/output, we get:



(b) From part (a), we have:

$$(1)\ w_1[n] = 2x[n] + w_3[n]$$
$$(2)\ w_2[n] = x[n] + w_1[n-1]$$
$$(3)\ w_3[n] = -\frac{1}{2}y[n] + 2y[n-1]$$
$$(4)\ y[n] = w_2[n] + y[n-1]$$

Taking the Z−transform of the above equations, substituting and rearranging terms, we get:

$$(1 - \frac{1}{2}z^{-1} - 2z^{-2})Y(z) = (2z^{-1} + 1)X(z).$$

Finally, inverse Z−transforming, we get the following difference equation:

$$y[n] - \frac{1}{2}y[n-1] - 2y[n-2] = x[n] + 2x[n-1].$$

(c) From part (b), the system function is given by:

$$H(z) = \frac{1 + 2z^{-1}}{1 - \frac{1}{2}z^{-1} - 2z^{-2}}.$$

It has poles at

$$z = -\frac{8}{1-\sqrt{33}} \text{ and } z = -\frac{8}{1+\sqrt{33}}$$

which are outside the unit circle, therefore the system is NOT BIBO stable.

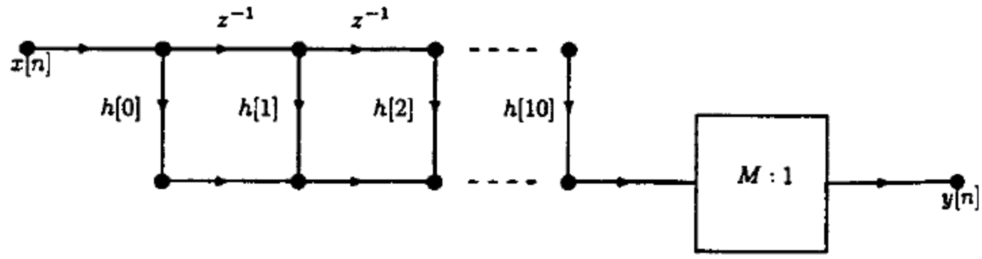
(d)

$$y[2] = x[2] + 2x[1] + \tfrac{1}{2}y[1] + 2y[0]$$
$$y[0] = x[0] = 1$$
$$y[1] = x[1] + 2x[0] + \tfrac{1}{2}y[0] = \tfrac{1}{2} + 2 + \tfrac{1}{2} = 3$$

Therefore,

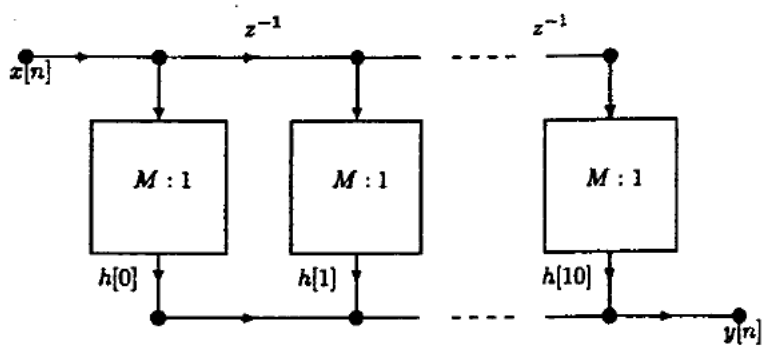
$$y[2] = \frac{1}{4} + 1 + \frac{3}{2} + 2 = \frac{19}{4}.$$

6.38. (a)



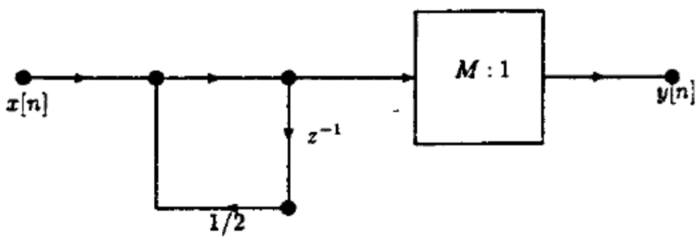
$M(N+1)$ multiplies per output sample; MN adds per output sample.

(b)



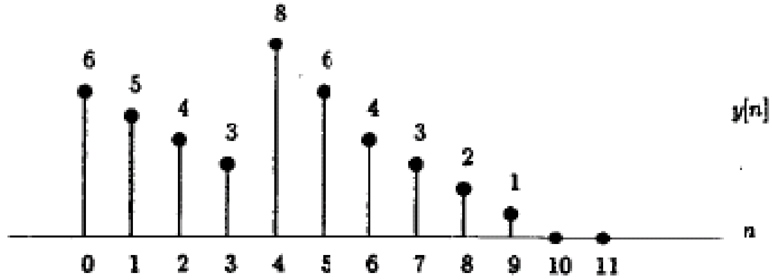
$N+1$ multiplies per output sample; N adds per output sample. The number of computations has been reduced by a factor of M in both adds and multiplies.

(c)



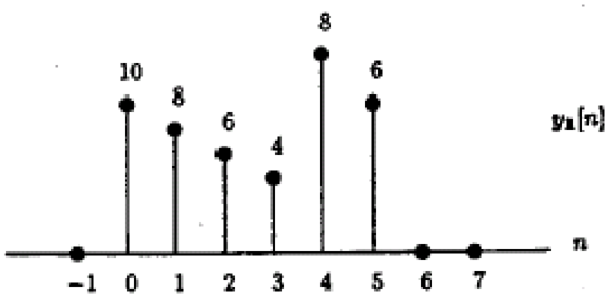
The total computation can not be reduced because to compute the value of any given output sample, the previous output value must be known.

8.29. Circular convolution equals linear convolution plus aliasing. First, we find $y[n] = x_1[n] * x_2[n]$:



Note that $y[n]$ is a ten point sequence ($N = 6 + 5 - 1$).

(a) For $N = 6$, the last four non-zero point ($6 \leq n \leq 9$) will alias to the first four points, giving us $y_1[n] = x_1[n] \circledast_6 x_2[n]$



(b) For $N = 10, N \geq 6 + 5 - 1$, so no aliasing occurs, and circular convolution is identical to linear convolution.

8.31. We have a 10-point sequence, $x[n]$. We want a modified sequence, $x_1[n]$, such that the 10-pt. DFT of $x_1[n]$ corresponds to

$$X_1[k] = X(z)|_{z=\frac{1}{2}e^{j((2\pi k/10)+(\pi/10))}}$$

Recall the definition of the Z-transform of $x[n]$:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Since $x[n]$ is of finite duration ($N = 10$), we assume:

$$x[n] = \begin{cases} \text{nonzero}, & 0 \leq n \leq 9 \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$X(z) = \sum_{n=0}^{9} x[n] z^{-n}$$

Substituting in $z = \frac{1}{2} e^{j[(2\pi k/10)+(\pi/10)]}$:

$$X(z)\big|_{z=\frac{1}{2}e^{j[(2\pi k/10)+(\pi/10)]}} = \sum_{n=0}^{9} x[n] \left(\frac{1}{2} e^{j[(2\pi k/10)+(\pi/10)]} \right)^{-n}$$

We seek the signal $x_1[n]$, whose 10-pt. DFT is equivalent to the above expression. Recall the analysis equation for the DFT:

$$X_1[k] = \sum_{n=0}^{9} x_1[n] W_{10}^{kn}, \quad 0 \leq k \leq 9$$

Since $W_{10}^{kn} = e^{-j(2\pi/10)kn}$, by comparison

$$x_1[n] = x[n] \left(\frac{1}{2} e^{j(\pi/10)} \right)^{-n}$$

8.34. (a) The DFT of the even part of a real sequence:
If $x[n]$ is of length N, then $x_e[n]$ is of length $2N-1$:

$$x_e[n] = \begin{cases} \frac{x[n]+x[-n]}{2}, & (-N+1) \le n \le (N-1) \\ 0 & \text{otherwise} \end{cases}$$

$$X_e[k] = \sum_{n=-N+1}^{N-1} \left(\frac{x[n]+x[-n]}{2}\right) W_{2N-1}^{kn}, \quad (-N+1) \le k \le (N-1)$$

$$= \sum_{n=-N+1}^{0} \frac{x[-n]}{2} W_{2N-1}^{kn} + \sum_{n=0}^{N-1} \frac{x[n]}{2} W_{2N-1}^{kn}$$

Let $m = -n$,

$$X_e[k] = \sum_{n=0}^{N-1} \frac{x[n]}{2} W_{2N-1}^{-kn} + \sum_{n=0}^{N-1} \frac{x[n]}{2} W_{2N-1}^{kn}$$

$$X_e[k] = \sum_{n=0}^{N-1} x[n] \cos\left(\frac{2\pi kn}{2N-1}\right)$$

Recall

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad 0 \le k \le (N-1)$$

and

$$Re\{X[k]\} = \sum_{n=0}^{N-1} x[n] \cos\left(\frac{2\pi kn}{N}\right)$$

So: $\text{DFT}\{x_e[n]\} \neq Re\{X[k]\}$

(b)

$$Re\{X[k]\} = \frac{X[k]+X^*[k]}{2}$$

$$= \frac{1}{2}\sum_{n=0}^{N-1} x[n] W_N^{kn} + \frac{1}{2}\sum_{n=0}^{N-1} x[n] W_N^{-kn}$$

$$= \frac{1}{2}\sum_{n=0}^{N-1} (x[n] + x[N-n]) W_N^{kn}$$

So,

$$Re\{X[k]\} = \text{DFT}\left\{\frac{1}{2}(x[n] + x[N-n])\right\}$$

8.36. We have the finite-length sequence:

$$x[n] = 2\delta[n] + \delta[n-1] + \delta[n-3]$$

(i) Suppose we perform the 5-pt DFT:

$$X[k] = 2 + W_5^k + W_5^{3k}, \quad 0 \le k \le 5$$

where $W_5^k = e^{-j(\frac{2\pi}{5})k}$.

(ii) Now, we square the DFT of $x[n]$:

$$
\begin{aligned}
Y[k] &= X^2[k] \\
&= 2 + 2W_5^k + 2W_5^{3k} \\
&\quad + 2W_5^k + W_5^{2k} + W_5^{5k} \\
&\quad + 2W_5^{3k} + W_5^{4k} + W_5^{6k}, \qquad 0 \le k \le 5
\end{aligned}
$$

Using the fact $W_5^{5k} = W_5^0 = 1$ and $W_5^{6k} = W_5^k$

$$Y[k] = 3 + 5W_5^k + W_5^{2k} + 4W_5^{3k} + W_5^{4k}, \qquad 0 \le k \le 5$$

(a) By inspection,

$$y[n] = 3\delta[n] + 5\delta[n-1] + \delta[n-2] + 4\delta[n-3] + \delta[n-4], \qquad 0 \le n \le 5$$

(b) This procedure performs the autocorrelation of a real sequence. Using the properties of the DFT, an alternative method may be achieved with convolution:

$$y[n] = \text{IDFT}\{X^2[k]\} = x[n] * x[n]$$

The IDFT and DFT suggest that the convolution is circular. Hence, to ensure there is no aliasing, the size of the DFT must be $N \ge 2M - 1$ where M is the length of $x[n]$. Since $M = 3, N \ge 5$.

8.39. We have two 100-pt sequences which are nonzero for the interval $0 \le n \le 99$.

If $x_1[n]$ is nonzero for $10 \le n \le 39$ only, the linear convolution

$$x_1[n] * x_2[n]$$

is a sequence of length $40 + 100 - 1 = 139$, which is nonzero for the range $10 \le n \le 139$.

A 100-pt circular convolution is equivalent to the linear convolution with the first 40 points aliased by the values in the range $100 \le n \le 139$.

Therefore, the 100-pt circular convolution will be equivalent to the linear convolution only in the range $40 \le n \le 99$.

8.49. $x_2[n]$ is $x_1[n]$ time aliased to have only N samples. Since

$$x_1[n] = (\frac{1}{3})^n u[n],$$

We get:

$$x_2[n] = \begin{cases} \frac{(\frac{1}{3})^n}{1-(\frac{1}{3})^N} & , \quad n = 0, \ldots, N-1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

8.64. (a) The Z-transform of $h[n]$

$$\begin{aligned} H(z) &= \sum_{n=-\infty}^{\infty} h[n]z^{-n} \\ H(z) &= 1 - \frac{1}{2}z^{-n_0} \end{aligned}$$

The N-pt DFT of $h[n]$: $(N = 4n_0)$

$$\begin{aligned} H[k] &= \sum_{n=0}^{4n_0-1} h[n]W_{4n}^{kn_0}, \quad 0 \le k \le (4n_0 - 1) \\ &= 1 - \frac{1}{2}W_{4n_0}^{kn_0} \\ H[k] &= 1 - \frac{1}{2}e^{-j(\pi/2)k} \end{aligned}$$

(b)

$$\begin{aligned} H_i(z) &= \frac{1}{1 + 1/2z^{-n_0}}, \quad |z| > \left(\frac{1}{2}\right)^{-n_0} \text{ for causality} \\ h_i[n] &= \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{n/n_0} \delta[n - kn_0] \end{aligned}$$

The filter is IIR.

(c)

$$G[k] = \frac{1}{H[k]} = \frac{1}{1 - e^{-j(\pi/2)k}}, \quad 0 \le k \le (4n_0 - 1)$$

The impulse response, $g[n]$, is just $h_i[n]$ time-aliased by $4n_0$ points:

$$\begin{aligned} g[n] &= \left(1 + \frac{1}{16} + \frac{1}{256} + \cdots\right)\delta[n] + \left(\frac{1}{2} + \frac{1}{32} + \frac{1}{512} + \cdots\right)\delta[n - n_0] \\ &+ \left(\frac{1}{4} + \frac{1}{64} + \frac{1}{1024} + \cdots\right)\delta[n - 2n_0] + \left(\frac{1}{8} + \frac{1}{128} + \frac{1}{2048} + \cdots\right)\delta[n - 3n_0] \\ g[n] &= \frac{16}{15}\delta[n] + \frac{8}{15}\delta[n - n_0] + \frac{4}{15}\delta[n - 2n_0] + \frac{2}{15}\delta[n - 3n_0] \end{aligned}$$

(d) Indeed,

$$G[k]H[k] = 1, \quad 0 \le k \le (4n_0 - 1)$$

However, this relationship is only true at $4n_0$ distinct frequencies. This fact does not imply that for all ω:

$$G(e^{j\omega})H(e^{j\omega}) = 1$$

(e)

$$\begin{aligned} y[n] &= g[n] * h[n] \\ &= \frac{16}{15}\delta[n] + \frac{8}{15}\delta[n - n_0] + \frac{4}{15}\delta[n - 2n_0] + \frac{2}{15}\delta[n - 3n_0] - \frac{8}{15}\delta[n - n_0] \\ &\quad - \frac{4}{15}\delta[n - 2n_0] - \frac{2}{15}\delta[n - 3n_0] - \frac{1}{15}\delta[n - 4n_0] \\ y[n] &= \frac{16}{15}\delta[n] - \frac{1}{15}\delta[n - 4n_0] \end{aligned}$$

10.5. (a) After windowing, we have

$$\begin{aligned} x[n] &= \cos(\Omega_0 T n) \\ &= \frac{1}{2}\left[e^{j\Omega_0 T n} + e^{-j\Omega_0 T n}\right] \\ &= \frac{1}{2}\left[e^{j\frac{2\pi}{N}(\frac{N\Omega_0 T}{2\pi})n} + e^{-j\frac{2\pi}{N}(\frac{N\Omega_0 T}{2\pi})n}\right] \end{aligned}$$

for $n = 0, \ldots, N-1$ and $x[n] = 0$ outside this range. Using the DFT properties we get

$$X[k] = \frac{N}{2}\delta[((k - \tfrac{N\Omega_0 T}{2\pi}))_N] + \frac{N}{2}\delta[((k + \tfrac{N\Omega_0 T}{2\pi}))_N]$$

If we choose

$$T = \frac{2\pi}{N\Omega_0}k_0$$

then

$$X[k] = \frac{N}{2}\delta[k - k_0] + \frac{N}{2}\delta[k - (N - k_0)],$$

which is nonzero for $X[k_0]$ and $X[N - k_0]$, but zero everywhere else.

(b) *No*, the choice for T is not unique since we can choose the integer k_0.

10.26. Plugging in the relation for $c_{vv}[m]$ into the equation for $I(\omega)$ gives

$$\begin{aligned} I(\omega) &= \frac{1}{LU}\sum_{m=-(L-1)}^{L-1}\left[\sum_{n=0}^{L-1} v[n]v[n+m]\right]e^{-j\omega m} \\ &= \frac{1}{LU}\sum_{n=0}^{L-1} v[n]\sum_{m=-(L-1)}^{L-1} v[n+m]e^{-j\omega m} \end{aligned}$$

Let $\ell = n + m$ in the second summation. This gives

$$\begin{aligned} I(\omega) &= \frac{1}{LU}\sum_{n=0}^{L-1} v[n]\sum_{\ell=n-(L-1)}^{n+(L-1)} v[\ell]e^{-j\omega(\ell-n)} \\ &= \frac{1}{LU}\sum_{n=0}^{L-1} v[n]e^{j\omega n}\sum_{\ell=n-(L-1)}^{n+(L-1)} v[\ell]e^{-j\omega\ell} \end{aligned}$$

Note that for all values of $0 \le n \le L-1$, the second summation will be over *all* non-zero values of $v[\ell]$ in the range $0 \le \ell \le L-1$. As a result,

$$\begin{aligned} I(\omega) &= \frac{1}{LU}\sum_{n=0}^{L-1} v[n]e^{j\omega n}\sum_{\ell=0}^{L-1} v[\ell]e^{-j\omega\ell} \\ &= \frac{1}{LU}V^*(e^{j\omega})V(e^{j\omega}) \\ &= \frac{1}{LU}|V(e^{j\omega})|^2 \end{aligned}$$

Note that in this analysis, we have assumed that $v[n]$ is a real sequence.

10.31. (a) Sampling the continuous-time input signal

$$x(t) = e^{j(3\pi/8)10^4 t}$$

with a sampling period $T = 10^{-4}$ yields a discrete-time signal

$$x[n] = x(nT) = e^{j3\pi n/8}$$

In order for $X_w[k]$ to be nonzero at exactly one value of k, it is necessary for the frequency of the complex exponential of $x[n]$ to correspond to that of a DFT coefficent, $w_k = 2\pi k/N$. Thus,

$$\begin{aligned} \frac{3\pi}{8} &= \frac{2\pi k}{N} \\ N &= \frac{16k}{3} \end{aligned}$$

The smallest value of k for which N is an integer is $k = 3$. Thus, the smallest value of N such that $X_w[k]$ is nonzero at exactly one value of k is

$$N = 16$$

(b) The rectangular windows, $w_1[n]$ and $w_2[n]$, differ only in their lengths. $w_1[n]$ has length 32, and $w_2[n]$ has length 8. Recall that compared to that of a longer window, the Fourier transform of a shorter window has a larger mainlobe width and higher sidelobes. Since the DFT is a sampled version of the Fourier transform, we might try to look for these features in the two plots. We notice that the second plot, Figure P10.31-3, appears to have a larger mainlobe width and higher sidelobes. As a result, we conclude that Figure P10.31-2 corresponds to $w_1[n]$, and P10.31-3 corresponds to $w_2[n]$.

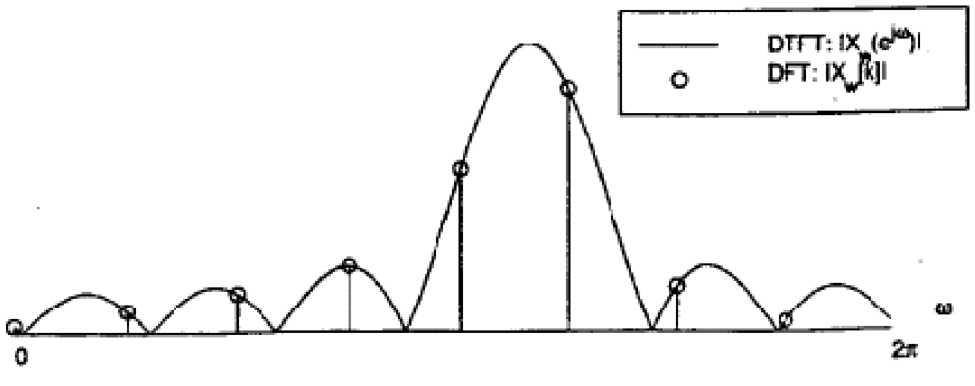
(c) A simple technique to estimate the value of ω_0 is to find the value of k at which the peak of $|X_w[k]|$ occurs. Then, the estimate, is

$$\hat{\omega}_0 = \frac{2\pi k}{N}$$

The corresponding value of $\hat{\Omega}_0$ is

$$\hat{\Omega}_0 = \frac{2\pi k}{NT}$$

This estimate is not exact, since the peak of the Fourier transform magnitude $|X_w(e^{j\omega})|$ might occur between two values of the DFT magnitude $|X_w[k]|$, as shown below.



The maximum possible error, $\Omega_{\text{max error}}$, of the frequency estimate is one half of the frequency resolution of the DFT.

$$\begin{aligned} \Omega_{\text{max error}} &= \frac{1}{2}\frac{2\pi}{NT} \\ &= \frac{\pi}{NT} \end{aligned}$$

For the system parameters of $N = 32$, and $T = 10^{-4}$, this is

$$\Omega_{\text{max error}} = 982 \text{ rad/s}$$

10.34. In this problem, we are given

- $x[n] = A\cos(\omega_0 n + \theta) + e[n]$
- θ is a uniform random variable on 0 to 2π
- $e[n]$ is an independent, zero mean random variable

(a) Computing the autocorrelation function,

$$\begin{aligned}
\phi_{xx}[m] &= \mathcal{E}\{x[n]x[n+m]\} \\
&= \mathcal{E}\{(A\cos(\omega_0 n + \theta) + e[n])(A\cos(\omega_0(n+m)+\theta) + e[n+m])\} \\
&= \mathcal{E}\{A^2\cos(\omega_0 n+\theta)\cos(\omega_0(n+m)+\theta)\} \\
&\quad + \mathcal{E}\{Ae[n]\cos(\omega_0(n+m)+\theta)\} + \mathcal{E}\{Ae[n+m]\cos(\omega_0 n+\theta)\} \\
&\quad + \mathcal{E}\{e[n]e[n+m]\} \\
&= A^2\mathcal{E}\{\cos(\omega_0 n+\theta)\cos(\omega_0(n+m)+\theta)\} \\
&\quad + A\mathcal{E}\{e[n]\}\mathcal{E}\{\cos(\omega_0(n+m)+\theta)\} + A\mathcal{E}\{e[n+m]\}\mathcal{E}\{\cos(\omega_0 n+\theta)\} \\
&\quad + \mathcal{E}\{e[n]e[n+m]\}
\end{aligned}$$

First, note that

$$\cos(a)\cos(b) = \frac{1}{2}\cos(a+b) + \frac{1}{2}\cos(a-b)$$

Therefore, the first term can be re-expressed as

$$A^2\mathcal{E}\left\{\frac{1}{2}\cos(2\omega_0 n + \omega_0 m + 2\theta) + \frac{1}{2}\cos(\omega_0 m)\right\}$$

Next, note that

$$\mathcal{E}\{e[n]\} = 0$$

As a result, the two middle terms drop out. Finally, note that since $e[n]$ is a sequence of zero-mean variables that are uncorrelated with each other,

$$\mathcal{E}\{e[n]e[n+m]\} = \sigma_e^2\delta[m], \qquad \text{where } \sigma_e^2 = \mathcal{E}\{e^2[n]\}$$

Putting this together, we get

$$\phi_{xx}[m] = A^2\mathcal{E}\left\{\frac{1}{2}\cos(2\omega_0 n + \omega_0 m + 2\theta) + \frac{1}{2}\cos(\omega_0 m)\right\} + \sigma_e^2\delta[m]$$

Since $\frac{1}{2\pi}\int_0^{2\pi}\cos(2\omega_0 n + \omega_0 m + 2\theta)d\theta = 0$, we have

$$\phi_{xx}[m] = \frac{A^2}{2}\cos(\omega_0 m) + \sigma_e^2\delta[m]$$

(b) Since the Fourier transform of $\cos(\omega_0 m)$ is $\pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$ for $|\omega| \le \pi$,

$$\Phi_{xx}(e^{j\omega}) = P_{xx}(\omega) = \frac{A^2\pi}{2}[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)] + \sigma_e^2$$

10.35. (a) Plugging in the equation

$$I[k] = I(\omega_k) = \frac{1}{L}|V[k]|^2$$

into the relation

$$\text{var}[I(\omega)] \simeq P_{xx}^2(\omega)$$

we find that

$$\begin{aligned}
\text{var}\left[\frac{1}{L}|V[k]|^2\right] &\simeq P_{xx}^2(\omega) \\
\text{var}\left[|V[k]|^2\right] &\simeq L^2 P_{xx}^2(\omega)
\end{aligned}$$

This equation can be used to find the approximate variance of $|X[k]|^2$. We substitute the signal $X[k]$ for $V[k]$, the DFT length N for L, and use the power spectrum

$$P_{xx}(w) = \sigma_x^2$$

This gives

$$\text{var}\left[|X[k]|^2\right] = N^2\sigma_x^4$$

(b) The cross-correlation is found below.

$$\begin{aligned}
\mathcal{E}\{X[k]X^*[r]\} &= \sum_{n_1=0}^{N-1}\sum_{n_2=0}^{N-1}\mathcal{E}\{x[n_1]x[n_2]\}\,W_N^{kn_1}W_N^{-rn_2} \\
&= \sum_{n_1=0}^{N-1}\sum_{n_2=0}^{N-1}\sigma_x^2\delta[n_1-n_2]W_N^{kn_1}W_N^{-rn_2} \\
&= \sum_{n=0}^{N-1}\sigma_x^2 W_N^{(k-r)n} \\
&= \sigma_x^2\left[\frac{1-W_N^{N(k-r)}}{1-W_N^{(k-r)}}\right] \\
&= N\sigma_x^2\delta[k-r]
\end{aligned}$$

Note that the cross-correlation is zero everywhere except when $k = r$. This is what one would expect for white noise, since samples for which $k \neq r$ are completely uncorrelated.

DSP PROJECT

- SRRC(Square Root Raised Cosine) Filter Design -

1. Specification

(1) Ideal SRRC (Square Root Raised Cosine)

- 이상적인 SRRC 필터의 주파수 응답은 다음과 같은 특성을 만족시켜야 한다. 아래의 주파수 특성을 충족하는 SRRC 필터의 계수는 무한한 길이를 가지므로 실제 설계에서는 일종의 vestigial LPF 로 생각하고 이에 가장 가깝도록 설계하였다. (w_p , w_s 기준)

$$\text{Ideal SRRC : } h_d[n] \leftrightarrow H_d(e^{j2\pi f})$$

$$\left|H_d\left(e^{j2\pi f}\right)\right|^2 = \begin{cases} T & 0 \le |f| \le \dfrac{1-\alpha}{2T} \\ \dfrac{T}{2}\left(1+\cos\dfrac{\pi|f|+\alpha-1/2T}{2\alpha}\right) & \dfrac{1-\alpha}{2T} \le |f| \le \dfrac{1+\alpha}{2T} \\ 0 & \text{otherwise} \end{cases}$$

$$\alpha = 0.15 + 0.05 \times \text{Remainder(last two digits of ID numbet/4)};\ \ T = 1.0$$

(2) SBATT (Stop Band Attenuation)

- roll-off factor 와 oversampling factor 을 다음과 같이 설정하였다.

$$\text{roll-off factor : } \alpha = 0.2 \ (\because\ 73 \bmod 4 = 1)$$

$$\text{oversampling factor : } L = 2$$

$$\begin{array}{lcc} \text{pass band :} & 0 \le |w| \le \dfrac{\pi(1-\alpha)}{L} & 0 \le |f| \le \dfrac{(1-\alpha)}{2L} \\ \text{transition :} & \dfrac{\pi(1-\alpha)}{L} \le |w| \le \dfrac{\pi(1+\alpha)}{L} & \dfrac{(1-\alpha)}{2L} \le |f| \le \dfrac{(1+\alpha)}{2L} \\ \text{stop band :} & \dfrac{\pi(1+\alpha)}{L} \le |w| \le \pi & \dfrac{(1+\alpha)}{2L} \le |f| \le 0.5 \end{array}$$

$$\text{SBATT : 40dB}$$

(3) ISI (Inter Symbol Interference)

- ISI 는 1) 각 항들의 power 비율을 살펴보는 경우와 2) 각 항들의 magnitude 비율을 살펴보는 경우의 두 가지 형태에 대해 살펴보았다. 아래는 SRRC 필터와 RC 필터를 아래의 수식과 같이 설정하였을 때, ISI 의 수식을 나타낸 것이다.

$$\text{SRRC}: \qquad h_t[n] \qquad n = 0, 1, \cdots, N-1$$

$$\text{RC}: \quad h[n] = \sum_{k=0}^{n} h_t[k] h_t[k+N-1-n] \quad n = 0, 1, \cdots, 2N-2$$

$$\text{oversampling factor} : L$$

(a) Type1 – power

$$\text{ISI1} = \frac{\left|h[N-1]\right|^2}{\sum_{j=1}^{\left\lfloor \frac{N-1}{L} \right\rfloor} \left|h[N-1 \pm jL]\right|^2} = \frac{\left(\sum_{k=0}^{N-1} h_t^2[k]\right)^2}{\sum_{j=1}^{\left\lfloor \frac{N-1}{L} \right\rfloor} 2\left|\sum_{k=0}^{N-1-jT} h_t[k] h_t[k+jL]\right|^2}$$

(b) Type2 – magnitude

$$\text{ISI2} = \frac{\sum_{j=1}^{\left\lfloor \frac{N-1}{L} \right\rfloor} \left|h[N-1 \pm jL]\right|}{\left|h[N-1]\right|} = \frac{\sum_{j=1}^{\left\lfloor \frac{N-1}{L} \right\rfloor} 2\left|\sum_{k=0}^{N-1-jT} h_t[k] h_t[k+jL]\right|}{\sum_{k=0}^{N-1} h_t^2[k]}$$

(c) Constraint

$$10\log_{10}(\text{ISI1}) \geq 40\text{dB}$$

$$20\log_{10}(\text{ISI2}) \leq -40\text{dB}$$

2. Algorithm [1]

(1) SM-NLMS (Set Membership Normalized Least Mean Square)

- SM-NLMS 는 stop band 을 세부 구간으로 나누어 각 지점에서의 이상적인 주파수 특성과 실제 필터의 주파수 특성의 오차를 구한 뒤 이를 토대로 필터 계수를 보정해 나가는 알고리즘이다. 자세한 알고리즘은 다음과 같다.

우선 충분히 큰 K 값에 대해 SRRC 디지털 필터의 K-point DFT 을 구한다.

$$H_t(k) = \sum_{n=0}^{N-1} h_t[n] e^{-j\frac{2\pi nk}{K}}$$

여기서 다음을 정의하고,

$$\mathbf{x}_k = \begin{pmatrix} 1 \\ e^{-j\frac{2\pi k}{K}} \\ \vdots \\ e^{-j\frac{2\pi(N-1)k}{K}} \end{pmatrix} \quad \text{and} \quad \theta = \begin{pmatrix} h_t(0) \\ h_t(1) \\ \vdots \\ h_t(N-1) \end{pmatrix}$$

이상적인 필터의 $2\pi k / K$ 위치에서의 주파수 응답을 d_k, 실제 필터의 주파수 응답 $\theta^T \mathbf{x}_k$ 과의 오차의 상한을 γ_k 이라고 하면, 필터 설계의 문제는 다음을 만족시키는 필터계수 θ 을 찾는 문제로 표현된다.

$$\left| d_k - \theta^T \mathbf{x}_k \right| \le \gamma_k$$

이 때 필터계수 θ는 다음의 해 집합에 속하게 된다.

$$\Theta \triangleq \left\{ \theta \in R^N : \ \left| d_k - \theta^T \mathbf{x}_k \right| \le \gamma_k, \ \forall k \right\}$$

Real parameter 에 대한 SM-NLMS 에 대해 Chia-Yu YAO [1]는 다음과 같이 필터 계수를 수정해 나갈 경우 θ가 해 집합 Θ 내의 특정 값으로 수렴함을 증명하였다.

$$\hat{\theta}_k = \theta_{k-1} + \beta_k \frac{\mathrm{Re}\left\{ \delta_k \mathbf{x}^*_k \right\}}{N}$$

$$\text{where} \quad \begin{aligned} \delta_k &= d_k - \hat{\theta}^T_{k-1} \mathbf{x}_k \\ \beta_k &= \begin{cases} 1 - \dfrac{\gamma_k}{|\delta_k|} & \text{if } |\delta_k| > \gamma_k \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(2) Gradient descent (Steepest descent)

- ISI 는 앞서 살펴본 바와 같이 필터 계수들에 대한 다변함수 형태로 표현된다. 여기서는 보다 일반적인 ISI 수식으로 Type2 ISI 을 이용하기로 한다.

$$\mathrm{ISI2} = \frac{\sum_{j=1}^{\left\lfloor \frac{N-1}{L} \right\rfloor} \left| h[N-1 \pm jL] \right|}{\left| h[N-1] \right|} = \frac{\sum_{j=1}^{\left\lfloor \frac{N-1}{L} \right\rfloor} 2 \left| \sum_{k=0}^{N-1-jT} h_t[k] h_t[k+jL] \right|}{\sum_{k=0}^{N-1} h_t^2[k]}$$

Chia-Yu YAO [1]는 ISI 의 numerator 의 절대값을 제곱으로 바꾸어 다음과 같이 표현한 뒤, 이를 바탕으로 Gradient 을 계산하였다.

$$E=\sum_{j=1}^{\left\lfloor \frac{N-1}{L} \right\rfloor} 2\left(\sum_{k=0}^{N-1-jT} h_t[k]h_t[k+jL]\right)^2$$

이 때의 Gradient 는 다음과 같이 계산할 수 있다.

$$\frac{\partial E}{\partial h_t(i)}=4\sum_{j=1}^{\left\lfloor \frac{N-1}{L} \right\rfloor}\left(\sum_{k=0}^{N-1-jT} h_t[k]h_t[k+jL]\right)\times\left[h_t[i+jL]\prod_{i,(0,N-1-jL)}+h_t[i-jL]\prod_{i,(jL,N-1)}\right]$$

$$\text{where}\quad \prod_{i,(a,b)}=\begin{cases}1 & a\le i\le b\\ 0 & \text{otherwise}\end{cases}$$

$$\nabla_\theta E=\begin{pmatrix}\frac{\partial E}{\partial h_t[0]} & \frac{\partial E}{\partial h_t[1]} & \cdots & \frac{\partial E}{\partial h_t[N-1]}\end{pmatrix}^T$$

이제 ISI 조건을 충족시키기 위한 Gradient descent 과정은 다음과 같다.

If $-\nabla_\theta E^T(\hat{\theta}_k-\hat{\theta}_{k-1})\ge 0$

$$\hat{\theta}_k, new=\hat{\theta}_k-\mu\nabla_\theta E$$

else

$$\mathbf{b}=\frac{\hat{\theta}_k-\hat{\theta}_{k-1}}{\left\|\hat{\theta}_k-\hat{\theta}_{k-1}\right\|}$$

$$\mathbf{p}=\nabla_\theta E-(\nabla_\theta E^T\mathbf{b})\mathbf{b}\qquad \text{where}\quad \mu=\begin{cases}\varepsilon\left(1-\frac{\gamma_{\mathrm{ISI}}}{\mathrm{ISI}}\right) & \text{if ISI}>\gamma_{\mathrm{ISI}}\\ 0 & \textit{otherwise}\end{cases},\ \varepsilon=0.5$$

$$\hat{\theta}_k, new=\hat{\theta}_k-\mu\mathbf{p}$$

여기서 $\nabla_\theta E$ 가 실제 ISI 의 Gradient 을 대변할 수 있으려면, ISI 의 denominator 가 고정되어 있어야 하는데, 이 부분이 항상 만족된다고 보기 어려우므로 ISI 의 실제 Gradient 을 이용하여 설계하였다. 또한 Chia-Yu YAO [1]은 Gradient descent 의 방법을 적용할 때, 필터 계수가 이전의 진행 방향과 반대 방향으로 변화하는 경우에 이전 방향에 수직한 방향으로 변하게 하여 급격한 변화를 막고자 하였는데, 이는 수렴 속도를 빠르게 하기 위한 것으로 보인다.

ISI 의 실제 Gradient 를 이용하는 경우에는 계산 량이 증가하여 이 부분을 제외하였다. 한편, ISI 의 Gradient 는 다음과 같이 계산할 수 있다.

$$\frac{\partial F}{\partial h_t(i)}=\frac{\sum_{j=1}^{\left\lfloor \frac{N-1}{L} \right\rfloor} 2\left((A\ge 0)-(A<0)\right)\left\{\frac{\partial A}{\partial h_t(i)}\times B-A\times\frac{\partial B}{\partial h_t(i)}\right\}}{B^2}$$

$$\text{where}\quad \begin{cases} A = \sum_{k=0}^{N-1-jT} h_t[k]h_t[k+jL] & \dfrac{\partial A}{\partial h_t(i)} = h_t[i+jL]\prod_{i,(0,N-1-jL)} + h_t[i-jL]\prod_{i,(jL,N-1)} \\ B = \sum_{k=0}^{N-1} h_t^2[k] & \dfrac{\partial B}{\partial h_t(i)} = 2h_t[i] \end{cases}$$

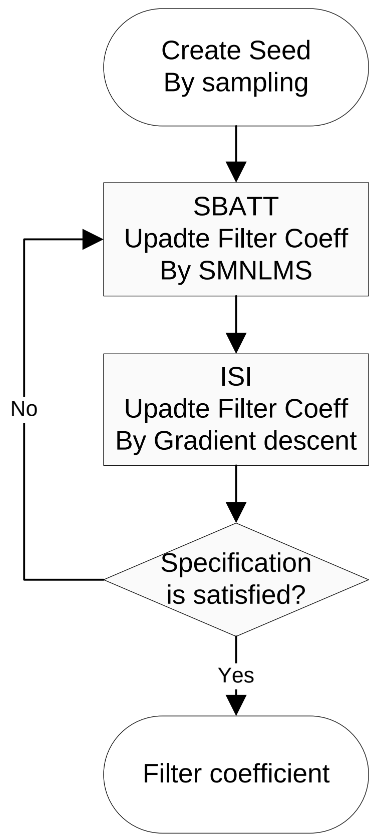
$$\nabla_\theta F = \begin{pmatrix} \dfrac{\partial F}{\partial h_t[0]} & \dfrac{\partial F}{\partial h_t[1]} & \cdots & \dfrac{\partial F}{\partial h_t[N-1]} \end{pmatrix}^T$$

따라서 ISI 조건을 충족시키기 위한 Gradient descent 과정을 다시 쓰면 다음과 같다.

$$\hat{\theta}_k, new = \hat{\theta}_k - \mu \nabla_\theta F$$

이 때 Gradient descent 가 적용될 수 있도록 $\nabla_\theta F$ 의 magnitude 가 10^{-3} 이 되도록 조정하였다.

(3) Flow chart

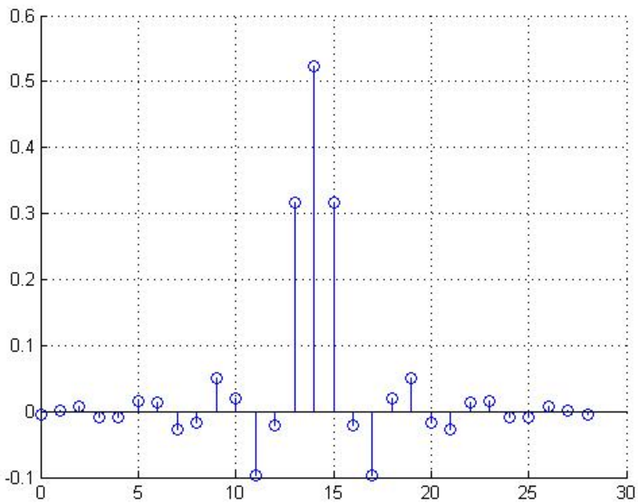


- 전체 알고리즘은 SMNLMS 와 Gradient descent 을 반복 수행하여 주어진 Specification 을 만족시키는 필터 계수를 찾아낸다. 이때 초기의 필터 계수는 이상적인 SRRC 필터의 time domain 에서의 함수를 oversample 하여 얻는다. 또한 초기의 필터 탭 수는 SRRC 필터를 일종의 LPF 으로 간주하여 SBATT 조건을 Kaiser window 에 적용하여 설정한 뒤, 주어진 알고리즘 내에서 수렴하지 않으면 탭 수를 늘려나갔다.

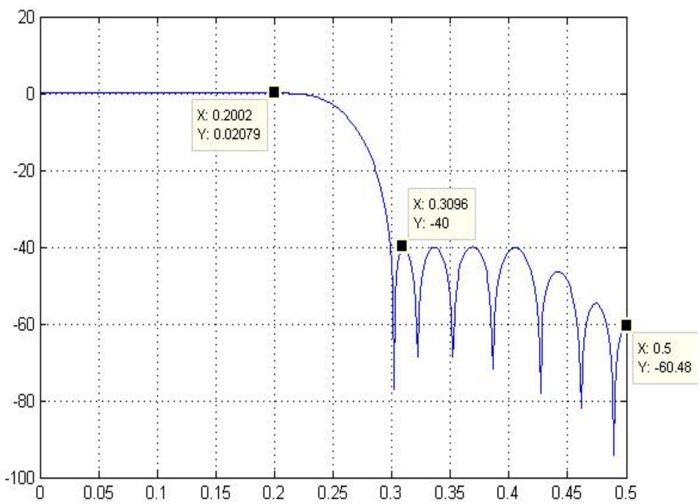
3. Implementation

(1) Time domain $h_t[n]$

$h_t[n]$ = [-0.0046126 0.0020771 0.0075704 -0.0079469 -0.0096055 0.014835 0.013353 -0.027437 -0.016986 0.049967 0.019475 -0.097481 -0.021343 0.31622 0.52382 0.31622 -0.021343 -0.097481 0.019475 0.049967 -0.016986 -0.027437 0.013353 0.014835 -0.0096055 -0.0079469 0.0075704 0.0020771 -0.0046126]; % 29tap; n = 0, 1, 2, … , 28



(2) Frequency domain $H(e^{jw})$



(3) ISI

(a) Type1 – power

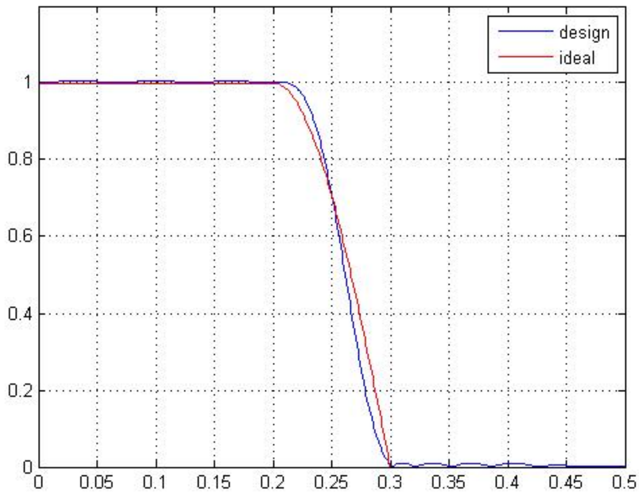
10log10(ISI1) ~ 48.2530dB

(b) Type2 – magnitude

20log10(ISI2) ~ -39.9954dB

- 시뮬레이션 상에서는 Type1:48.25312810880618, Type2: -40.00000000000309 로 계산되었으나, long 타입으로 필터 계수를 표현하여 다소의 오차가 발생하였다.

(4) Comparison with ideal SRRC



-이상적인 SRRC 필터와 설계한 SRRC 필터를 Linear scale 에서 비교하여 보았을 때, pass band 에서는 두 필터의 주파수 응답이 거의 동일하였다. 그러나 이상적인 필터의 경우 ISI 가 0 인 반면, 설계한 필터에서는 ISI 을 0.01 으로 조정하였기 때문에 transition 구간에서 두 필터의 주파수 응답의 차이가 발생하였다. 또 이상적인 필터의 stop band 구간에서는 주파수 응답이 zero 인데 설계한 필터에서는 0.01 으로 조정하여 stop band 구간에서 ripple 이 발생하였다.

4. Conclusion

- 주어진 Specification 에 맞는 SRRC 필터를 설계할 수 있었다. 단, 이 알고리즘을 적용하는 경우 SBATT 조건과 ISI 조건을 동시에 충족하는 해가 항상 존재한다고 보장할 수 없기 때문에 적절한 탭 수를 선정하는 것이 매우 중요하며, 또한 SM-NLMS 의 기법을 적용할 경우는 해가 존재한다면 반드시 수렴하지만, Gradient descent 의 기법을 적용하는 경우는 필터 계수를 변화시키는 벡터의 크기가 충분히 작아야만 수렴할 것을 기대할 수 있다. 또 SM-NLMS 알고리즘을 적용할 때, stop band 만 고려하기 때문에 pass band 의 특성은 초기의 입력 값에 크게

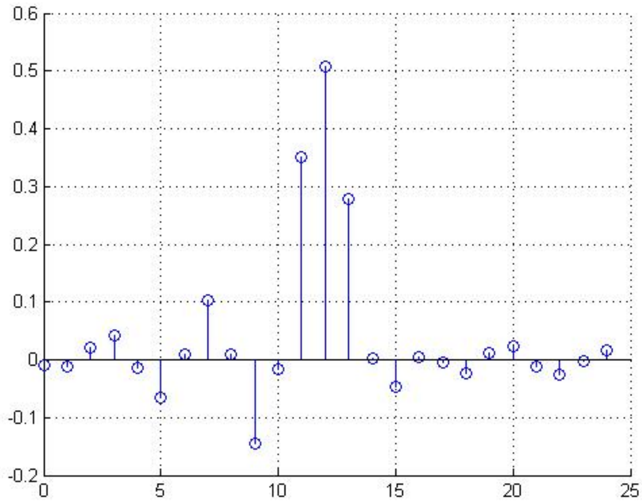
의존하며, 때때로 그 크기가 1 과 동떨어지게 변화할 수 있다. 제시한 code 에서는 이 부분을 보정하기 위해 stop band 조건을 토대로 필터 계수를 조정하고 난 후에 전체 필터 계수를 dc gain 로 나누어 pass band 에서의 크기가 1 이 넘지 않게 하였다.

5. 부록 : non-linear 25tap; 27tap design

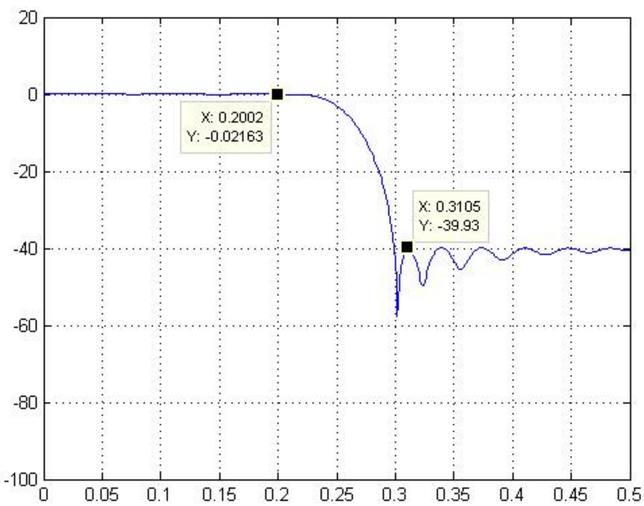
- 25tap non-linear SRRC filter

(1) Time domain $h_t[n]$

$h_t[n]$ = [-0.010123 -0.012731 0.020329 0.041089 -0.013816 -0.06507 0.0093173 0.10236 0.0090194 -0.14551 -0.016895 0.35159 0.50735 0.27856 0.0032718 -0.046286 0.0052712 -0.0052148 -0.022399 0.010725 0.023673 -0.0112 -0.02576 -0.0029318 0.01538];



(2) Frequency domain $H(e^{jw})$



(3) ISI

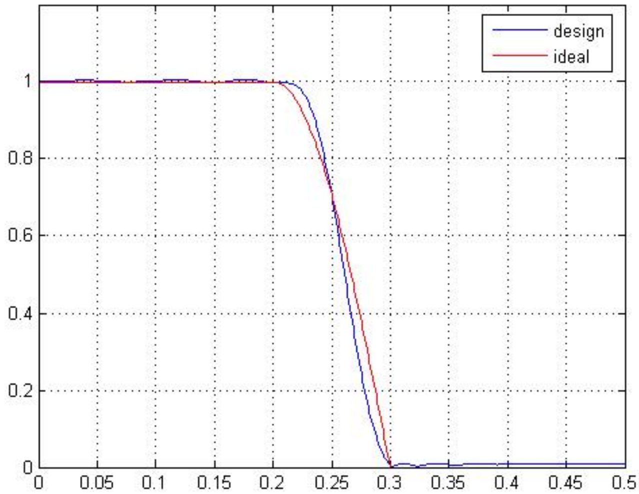
(a) Type1 – power

$$10\log_{10}(ISI1) \sim 46.3605dB$$

(b) Type2 – magnitude

$$20\log_{10}(ISI2) \sim -36.4671dB$$

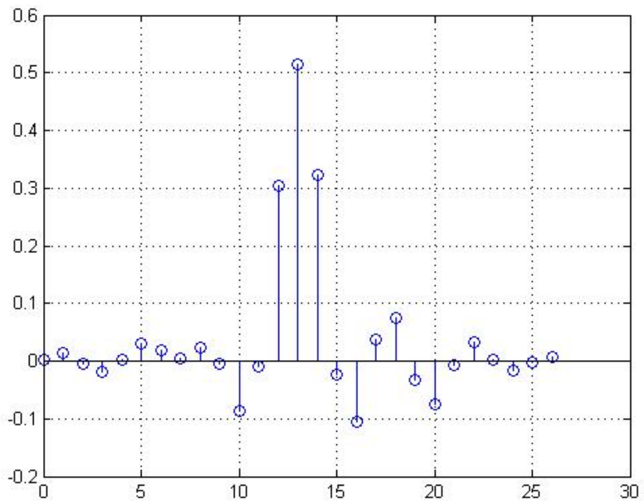
(4) Comparison with ideal SRRC



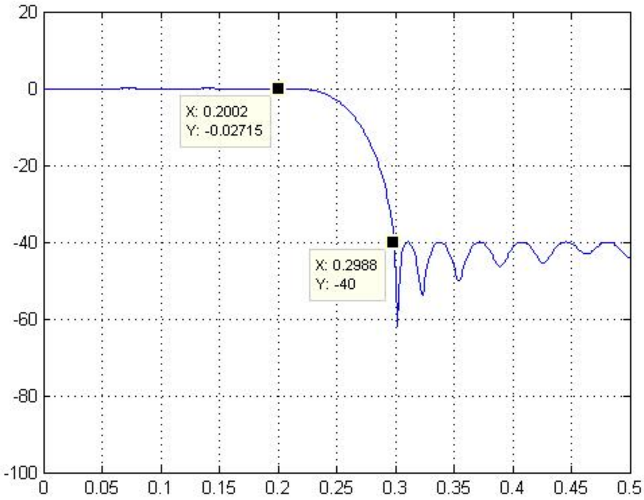
- 27tap non-linear SRRC filter

(1) Time domain $h_t[n]$

$h_t[n]$ = [0.0016132 0.013867 -0.0040627 -0.018395 0.0015166 0.030279 0.01797 0.0041324 0.024013 -0.0043216 -0.08644 -0.009577 0.30354 0.51365 0.32261 -0.023721 -0.10468 0.037601 0.074876 -0.033884 -0.075063 -0.0061851 0.032182 0.0017666 -0.01751 -0.0020767 0.0063007];



(2) Frequency domain $H(e^{jw})$



(3) ISI

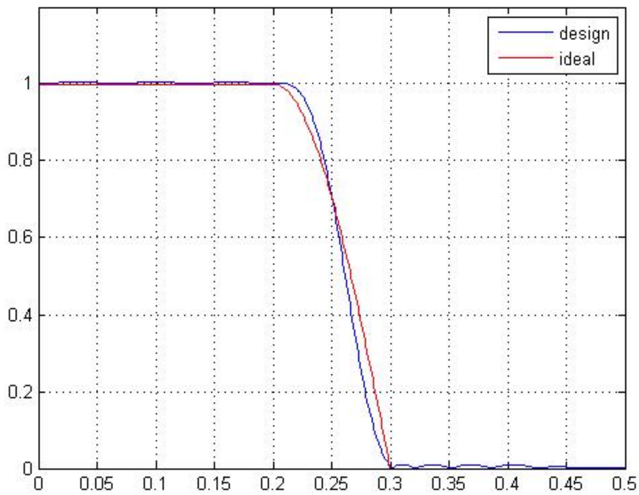
(a) Type1 – power

10log10(ISI1) ~ 45.3677dB

(b) Type2 – magnitude

20log10(ISI2) ~ -39.9942dB

(4) Comparison with ideal SRRC



6. Reference

[1] Chia-Yu YAO, The Design of Square-Root-Raised-Cosine FIR Filters by an Iterative Technique, IEEE TRANS. FUNSAMENTALS, VOL.E90-A, N0.1 JANUARY 2007

[2] Alan V. Oppenheim, Ronald W. Schafer and John R. Buck, DISCRETE-TIME SIGNAL PROCESSING, Second edition, PRETICE HALL

7. MATLAB code

```
% SMNLMS.m
clc
close all
clear all

% System parameter
a = 0.2; % roll off factor
T = 1; %symbol rate
L = 2; % oversampling factor
K = 2^10;

gamma = 0.01; %SBATT cond
gammaISI = 0.01; %ISI cond

% Seed
% suqare root raised cosine : sqrrc
N = 29;
n = 0:N-1;
hsqrrc = (T/L)*sqrrc((n-(N-1)/2)*T/L,a,T);

theta = hsqrrc;

cnt = 0;
beta = 1;
mu = 1;

for i = 1:500
% SBATT
while (beta ~= 0) | (mu ~= 0)
    cnt = cnt + 1;
    if cnt>20000, break, end
    for k = ceil(K*(1+a)/(2*L))-1:floor(K/2);
        theta_bp = theta;
        % updating for reducing SBATT
        delta = 0 - theta*transpose(x(k,K,N));
```

```
            if abs(delta)>gamma % delta
                beta = 1-gamma/abs(delta);
            else
                beta = 0;
            end
            % update theta
            theta = theta + beta*real(delta*conj(x(k,K,N)))/N;

            % ISI
            ep = 0.5;
            if ISI_OG(theta,L) > gammaISI
                mu = ep*(1-gammaISI/ISI_OG(theta,L));
            else
                mu = 0; % no update
            end
            % update theta
            theta = theta-mu*dF(theta,L);
end
    theta = theta/(theta*transpose(x(1,K,N))); % consider stop band
    if isnan(ISI_OG(theta,L))
        error('NaN')
    end
    beta
    mu
end
beta = 1;
mu = 1;
end
```

```
% dF.m
function y = dF(h,L)
N = length(h);
temp = 0;
for k = 1:N
    temp = temp + h(k)^2;
end
denum = temp^2;
for i = 0:N-1
```

```
        num(i+1) = 0;
        for j = 1:floor((N-1)/L)
            if (i>=0 & i<= N-1-j*L) & (i>=j*L & i<= N-1)
                temp1 = h(i+1+j*L)+h(i+1-j*L);
            elseif i>=0 & i<= N-1-j*L
                temp1 = h(i+1+j*L);
            elseif i>=j*L & i<= N-1
                temp1 = h(i+1-j*L);
            end
            temp2 = 0;
            for k=1:(N-j*L)
                temp2 = temp2 + h(k)*h(k+j*L);
            end
            num(i+1) = num(i+1) + 2*(temp*temp1-2*h(i+1)*temp2)*((temp2>=0)-(temp2<0));
        end
end
y = num/denum;
y = (10^(-3))*y/norm(y);
end
```

```
% test_Thete.m
clc
close all
clear all
theta = [-0.0046126 0.0020771 0.0075704 -0.0079469 -0.0096055 0.014835 0.013353 -0.027437 -0.016986 0.049967 0.019475 -0.097481 -0.021343 0.31622 0.52382 0.31622 -0.021343 -0.097481 0.019475 0.049967 -0.016986 -0.027437 0.013353 0.014835 -0.0096055 -0.0079469 0.0075704 0.0020771 -0.0046126];

N = length(theta);
L = 2;
K = 1024;
theta = theta/(theta*transpose(x(1,K,N)));
[H,w]=freqz(theta,K);
figure(1)
plot(w/2/pi, 20*log10(abs(H)))
grid on
axis([0 0.5 -100 20])
```

```
20*log10(ISI_OG(theta,L))
-10*log10(ISI(theta,L))

figure(2)
n2 = 0:length(theta)-1;
stem(n2, theta)
grid on
```