Homework set 1 selected solution

P. 2-1 A rhombus is an equilateral parallelogram. Denote two neighboring sides of a rhombus by vectors A and B

a) Verify that the two diagonals are **A + B** and **A – B**.

From vector addition by the head-to tail rule

$\vec{A}+\vec{B}=\vec{C}$ : major diagonal

$\vec{A}-\vec{B}=\vec{D}$ : minor diagonal

b) Obtain a formula for $\sin(\alpha-\beta)$ by taking the vector product $\mathbf{a}_B \times \mathbf{a}_A$.

$$\begin{aligned}\vec{C}\cdot\vec{D} &= \left(\vec{A}+\vec{B}\right)\cdot\left(\vec{A}-\vec{B}\right) \\ &= \vec{A}\cdot\vec{A}-\vec{A}\cdot\vec{B}+\vec{B}\cdot\vec{A}-\vec{B}\cdot\vec{B} \\ &= A^2-B^2=0 \quad (\because A=B)\end{aligned}$$

$$\Rightarrow \vec{C}\perp\vec{D}$$

P. 2-4 Let unit vectors $\mathbf{a}_A$ and $\mathbf{a}_B$ denote the directions of vectors A and B in the xy-plane that make angles α and β, respectively, with the x-axis.

a) Obtain a formula for the expansion of the cosine of the difference of two angles, $\cos(\alpha-\beta)$. By taking the scalar product $\mathbf{a}_A \cdot \mathbf{a}_B$.

$$\begin{aligned}\cos\left(\alpha-\beta\right) &= \vec{a}_A\cdot\vec{a}_B \\ &= \left(\hat{x}\cos\alpha+\hat{y}\sin\alpha\right)\cdot\left(\hat{x}\cos\beta+\hat{y}\sin\beta\right) \\ &= \cos\alpha\cos\beta\left(\hat{x}\cdot\hat{x}\right)+\cos\alpha\sin\beta\left(\hat{x}\cdot\hat{y}\right) \\ &= \sin\alpha\cos\beta\left(\hat{y}\cdot\hat{x}\right)+\sin\alpha\sin\beta\left(\hat{y}\cdot\hat{y}\right) \\ &= \cos\alpha\cos\beta+\sin\alpha\sin\beta\end{aligned}$$

b) Obtain a formula for $\sin(\alpha-\beta)$ by taking the vector product $\mathbf{a}_B \times \mathbf{a}_A$.

$$\begin{aligned}-\hat{z}\sin\left(\alpha-\beta\right) &= \vec{a}_A\times\vec{a}_B = \left(\hat{x}\cos\alpha+\hat{y}\sin\alpha\right)\times\left(\hat{x}\cos\beta+\hat{y}\sin\beta\right) \\ &= \cos\alpha\cos\beta(\hat{x}\times\hat{x})+\cos\alpha\sin\beta(\hat{x}\times\hat{y})+\sin\alpha\cos\beta(\hat{y}\times\hat{x})+\sin\alpha\sin\beta(\hat{y}\times\hat{y}) \\ &= -\hat{z}\left(\sin\alpha\cos\beta-\cos\alpha\sin\beta\right)\end{aligned}$$

$$\Rightarrow \sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

P. 2-7 Given vector $\vec{A}=\hat{x}5-\hat{y}2+\hat{z}$ , find the expression of

a) a unit vector $\mathbf{a}_B$ such that $\mathbf{a}_B \parallel \mathbf{A}$, and

Let $\vec{a}_B = \hat{x}B_x + \hat{y}B_y + \hat{z}B_z$ , then $B_x^2 + B_y^2 + B_z^2 = 1 \cdots(1)$ $(\because unit\ vector)$

$$\vec{a}_B \parallel \vec{A} \Rightarrow \vec{a}_B \times \vec{A} = 0 \Rightarrow \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ B_x & B_y & B_z \\ 5 & -2 & 1 \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} \quad\quad B_y + 2B_z = 0 & \cdots(2) \\ -B_x \quad\quad + 5B_z = 0 & \cdots(3) \\ -2B_x - 5B_y \quad\quad = 0 & \cdots(4) \end{cases}$$

Solving Eqs. (1)~(4),

$$B_x = \frac{5}{\sqrt{30}}, B_y = \frac{2}{\sqrt{30}}, B_z = \frac{1}{\sqrt{30}}$$

$$\Rightarrow \vec{a}_B = \frac{1}{\sqrt{30}}\left(\hat{x}5 - \hat{y}2 + \hat{z}\right)$$

b) a unit vector $\mathbf{a}_C$ in the xy-plane such that $\mathbf{a}_C \perp \mathbf{A}$.

Let $\vec{a}_c = \hat{x}C_x + \hat{y}C_y$ ($\because \vec{a}_c$ is in xy-plane -> no $\hat{z}$ component)

then $C_x^2 + C_y^2 = 1 \cdots(5)$

$$\vec{a}_c \perp \vec{A} \Rightarrow \vec{a}_c \cdot \vec{A} = 0 \Rightarrow (\hat{x}C_x + \hat{y}C_y)\cdot(\hat{x}5 - \hat{y}2 + \hat{z}) = 0$$

$$\Rightarrow 5C_x - 2C_y = 0 \cdots (6)$$

Solving (5)(6), $C_x = \dfrac{2}{\sqrt{29}}$ , $C_y = \dfrac{5}{\sqrt{29}}$

$$\Rightarrow \vec{a}_c = \frac{1}{\sqrt{29}}(\hat{x}2 + 5\hat{y})$$

6) By using the representations of summation convention and Levi-Civita symbol, prove that the scalar triple product $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ can be calculated by the following determinant in Cartesian coordinate system.

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} \mathbf{A}_x & \mathbf{A}_y & \mathbf{A}_z \\ \mathbf{B}_x & \mathbf{B}_y & \mathbf{B}_z \\ \mathbf{C}_x & \mathbf{C}_y & \mathbf{C}_z \end{vmatrix}$$

$$\begin{aligned} \vec{A}\cdot(\vec{B}\times\vec{C}) &= A_i(\vec{B}\times\vec{C})_i \\ &= A_i\varepsilon_{ijk}B_jC_k \\ &= \varepsilon_{ijk}A_iB_jC_k \qquad\qquad (i,j,k = x,y,z) \\ &= \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \end{aligned}$$