Homework set 2 selected solution

P. 2-18 Given a scalar field $V = 2xy - yz + xz$

a) Find the vector representing the direction and the magnitude of the maximum rate of increase of V at point $P(2,-1,0)$, and

find ∇V at $P(2,-1,0)$

$$\nabla V = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \right) V = \hat{x}(2y+z) + \hat{y}(2x-z) + \hat{z}(x-y)$$

$$Magnitude = |\nabla V| \, at \, P(2,-1.0) = \sqrt{(\nabla V)_x^2 + (\nabla V)_y^2 + (\nabla V)_z^2}$$

$$= \sqrt{(-2)^2 + 4^2 + (2+1)^2} = \sqrt{29}$$

b) Find the rate of increase of V at point $P(2,-1,0)$ in the direction toward the point $Q(0,2,6)$.

Find $(\nabla V) \cdot \vec{a}_{PQ}$ at $P(2,-1,0)$ and $Q(0,2,6)$

Vector from P to Q = $\overrightarrow{PQ} = \hat{x}(-2) + \hat{y}3 + \hat{z}6$

Unit vector from P to Q = $\vec{a}_{PQ} = \dfrac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \dfrac{\overrightarrow{PQ}}{\sqrt{(-2)^2 + 3^2 + 6^2}} = \dfrac{1}{7}(-\hat{x}2 + \hat{y}3 + \hat{z}6)$

$\therefore (\nabla V) \cdot \vec{a}_{PQ}$ at $P(2,-1,0)$ $= [\hat{x}(-2) + \hat{y}4 + \hat{z}3] \cdot \dfrac{1}{7}(-\hat{x}2 + \hat{y}3 + \hat{z}6)$

$$= \frac{1}{7}(4 + 12 + 18) = \frac{34}{7}$$

P. 2-20 Find the divergence of the following radial fields:

Find the divergence in spherical coordinates

a) $f_1(\vec{R}) = \hat{R}R^n \equiv \vec{A}$

$$\nabla \cdot \vec{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R) = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 R^n) = (n+2)R^{n-1}$$

b) $f_2(\vec{R}) = \hat{R}\dfrac{k}{R^2} \equiv \vec{B}$

$$\nabla \cdot \vec{B} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 B_R) = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{k}{R^2} \right) = 0$$