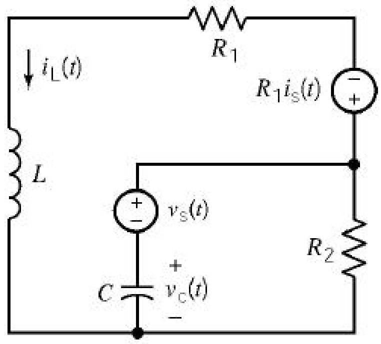
P9.3-3

After the switch closes, a source transformation gives:



KCL:

$$i_L(t)+C\frac{dv_c(t)}{dt}+\frac{v_s(t)+v_c(t)}{R_2}=0$$

KVL:

$$R_1 i_s(t)+R_1 i_L(t)+L\frac{di_L(t)}{dt}-v_c(t)-v_s(t)=0$$

$$v_c(t)=R_1 i_s(t)+R_1 i_L(t)+L\frac{di_L(t)}{dt}-v_s(t)$$

Differentiating

$$\frac{d\,v_c(t)}{dt}=R_1\frac{d\,i_s(t)}{dt}+R_1\frac{d\,i_L(t)}{dt}+L\frac{d^2 i_L(t)}{dt^2}-\frac{d\,v_s(t)}{dt}$$

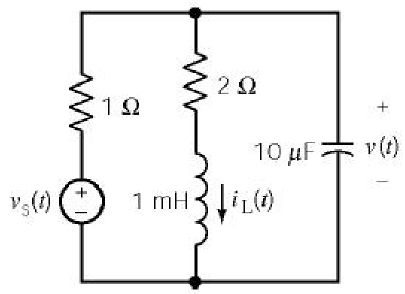
Then

$$i_L(t)+C\left(R_1\frac{d\,i_s(t)}{dt}+R_1\frac{d\,i_L(t)}{dt}+L\frac{d^2 i_L(t)}{dt^2}-\frac{d\,v_s(t)}{dt}\right)+\frac{v_s(t)}{R_2}$$
$$+\frac{1}{R_2}\left(R_1 i_s(t)+R_1 i_L(t)+L\frac{di_L(t)}{dt}-v_s(t)\right)=0$$

Solving for $i_L(t)$:

$$\frac{\mathrm{d}^2 i_L(t)}{\mathrm{d}t^2}+\left[\frac{R_1}{L}+\frac{1}{R_2 C}\right]\frac{di_L(t)}{dt}+\left[\frac{R_1}{LR_2C}+\frac{1}{LC}\right]i_L(t)=\frac{-R_1}{LCR_2}i_s(t)-\frac{R_1}{L}\frac{di_s(t)}{dt}+\frac{1}{L}\frac{dv_s(t)}{dt}$$

P9.4-3



KCL: $\dfrac{v(t)-v_s(t)}{1}+i_L(t)+\left(10\times10^{-6}\right)\dfrac{dv(t)}{dt}=0$

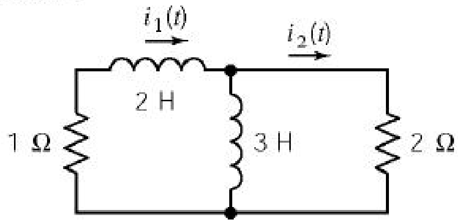
KVL: $v(t)=\;2i_L(t)+\left(1\times10^{-3}\right)\dfrac{di_L(t)}{dt}$

$$0=2\,i_L(t)+\left(1\times10^{-3}\right)\frac{di_L(t)}{dt}-v_s(t)+i_L(t)+\left(10\times10^{-6}\right)(2)\frac{di_L(t)}{dt}+\left(10\times10^{-6}\right)\left(10^{-3}\right)\frac{d^2 i_L(t)}{dt}$$

$$v_s(t)=3i_L(t)+0.00102\frac{di_L(t)}{dt}+10^{-8}\frac{d^2 i_L(t)}{dt^2}\quad\Rightarrow\quad\frac{d^2 i_L(t)}{dt}+102000\frac{di_L(t)}{dt}+3\times10^8 i_L(t)\;=\;10^8\;v_s(t)$$

$$s^2+102000s+3\times10^8\;=\;0\quad\Rightarrow\quad s_1=3031,\;\;s_2=-98969$$

P9.5-3



$$\text{KVL}: i_1 + 5\frac{di_1(t)}{dt} - 3\frac{di_2(t)}{dt} = 0 \qquad (1)$$

$$\text{KVL}: -3\frac{di_1(t)}{dt} + 3\frac{di_2(t)}{dt} + 2i_2(t) = 0 \qquad (2)$$

Using the operator $s = \dfrac{d}{dt}$, the KVL equations are

$$\left.\begin{array}{l}(1+5s)i_1 + (-3s)i_2 = 0\\(-3s)i_1 + (3s+2)i_2 = 0\end{array}\right\} \Rightarrow (1+5s)i_1 - (3s)\frac{3s}{3s+2}i_1 = 0 \Rightarrow \left[(1+5s)(3s+2) - (3s)^2\right]i_1 = 0$$

The characteristic equation is $(1+5s)(3s+2) - 9s^2 = 6s^2 + 13s + 2 = 0 \Rightarrow s_{1,2} = -\dfrac{1}{6}, -2$

The currents are $i_1(t) = Ae^{-t/6} + Be^{-2t}$ and $i_2(t) = Ce^{-t/6} + De^{-2t}$, where the constants A, B, C and D must be evaluated using the initial conditions. Using the given initial values of the currents gives

$$i_1(0) = 11 = A + B \quad \text{and} \quad i_2(0) = 11 = C + D$$

Let $t = 0$ in the KCL equations (1) and (2) to get

$$\frac{di_1(0)}{dt} = -\frac{33}{2} = -\frac{A}{6} - 2B \quad \text{and} \quad \frac{di_2(0)}{dt} = -\frac{143}{6} = -\frac{C}{6} - D$$

So $A = 3$, $B = 8$, $C = -1$ and $D = 12$. Finally,

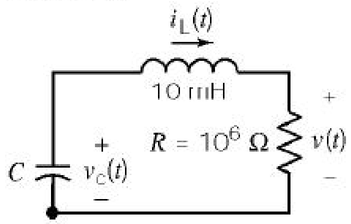
$$\underline{i_1(t) = 3e^{-t/6} + 8e^{-2t} \text{ A}} \quad \text{and} \quad \underline{i_2(t) = -e^{-t/6} + 12e^{-2t} \text{ A}}$$

P9.6-3

Assume that the circuit is at steady state before $t = 0$. The initial conditions are

$$v_c(0^-) = 10^4 \text{ V} \;\; \& \;\; i_L(0^-) = 0 \text{ A}$$

After $t = 0$



KVL: $\quad -v_c(t) + .01\dfrac{di_L(t)}{dt} + 10^6 i_L(t) = 0 \qquad (1)$

KCL:

$$i_L(t) = -C\frac{dv_C(t)}{dt} = -C\left[.01\frac{d^2 i_L(t)}{dt^2} + 10^6\frac{di_L(t)}{dt}\right] \qquad (2)$$

$$\therefore \quad 0.01\,C\frac{d^2 i_L(t)}{dt^2} + 10^6 C\frac{di_L(t)}{dt} + i_L(t) = 0$$

The characteristic equation is: $(0.01\,C)s^2 + (10^6 C)s + 1 = 0$

The natural frequencies are: $s_{1,2} = \dfrac{-10^6 C \pm \sqrt{(10^6 C)^2 - 4(0.01C)}}{2(0.01C)}$

For critically-damped response: $10^{12}C^2 - .04C = 0 \;\;\Rightarrow\;\; C = 0.04$ pF so $s_{1,2} = -5\times10^7, -5\times10^7$.

The natural response is of the form: $i_L(t) = A_1 e^{-5\times10^7 t} + A_2\, t\, e^{-5\times10^7 t}$

Now from (1) $\Rightarrow \;\; \dfrac{di_L}{dt}(0^+) \;=\; 100\left[v_c(0^+) - 10^6 i_L(0^+)\right] \;=\; 10^6 \text{ A/s}$

So $i_L(0) = 0 = A_1$ and $\dfrac{di_L(0)}{dt} = 10^6 \;=\; A_2 \quad \therefore\;\; i_L(t) = \; 10^6\, t\, e^{-5\times10^7 t}$ A

Now $\underline{v(t) = 10^6 i_L(t) = 10^{12}\, t\, e^{-5\times10^7 t} \text{ V}}$

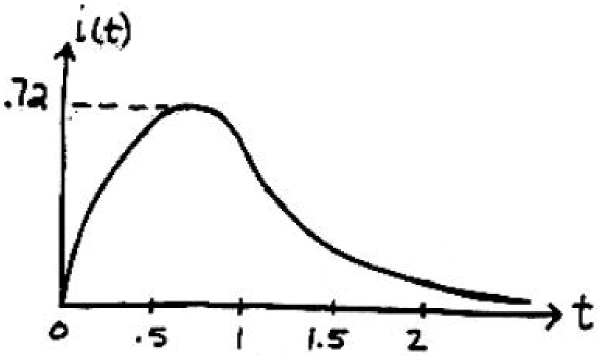
P9.6-5

After t=0, using KVL yields:

$$\frac{di(t)}{dt}+Ri(t)+\underbrace{2+4\int_0^t i(\tau)d\tau}_{v(t)} = 6 \qquad (1)$$

Take the derivative with respect to t:

$$\frac{d^2i(t)}{dt^2}+R\frac{di(t)}{dt}+4i(t) = 0$$



The characteristic equation is $s^2+Rs+4 = 0$

Let R=4 for critical damping $\Rightarrow$ $(s+2)^2 = 0$

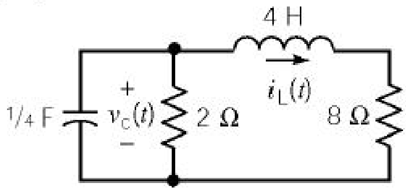
So the natural response is $i(t) = A\,t\,e^{-2t}+B\,e^{-2t}$

$$i(0) = 0 \Rightarrow B = 0 \text{ and } \frac{di(0)}{dt}=4-R(i(0))=4-R(0)= 4= A$$

$\therefore$ $\underline{i(t) = 4\,t\,e^{-2t}\text{A}}$

P9.7-3

After $t=0$



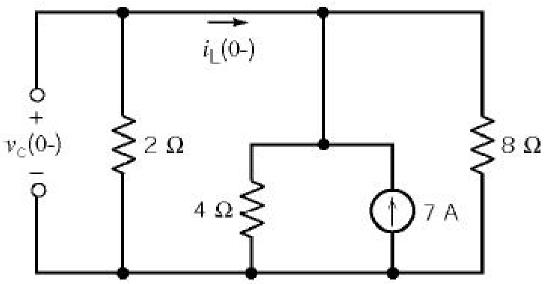
$$\text{KCL}: \frac{1}{4}\frac{dv_c(t)}{dt}+\frac{v_c(t)}{2}+i_L(t)=0 \qquad (1)$$

$$\text{KVL}: v_c(t)=\frac{4\,di_L(t)}{dt}+8\,i_L(t) \qquad (2)$$

Characteristic Equation: $\dfrac{d^2i_L(t)}{dt^2}+4\dfrac{di_L(t)}{dt}+5\,i_L(t)=0 \Rightarrow s^2+4s+5 = 0 \Rightarrow s_{1,2}=-2\pm i$

Natural Response: $i_L(t)= e^{-2t}\left[A_1\cos\,t+A_2\sin\,t\right]$

Before $t = 0$



$$\frac{v_c(0^-)}{2} = 7\left(\frac{4\|8}{4\|8+2}\right) \Rightarrow v_c(0^+) = v_c(0^-) = 8 \text{ V}$$

$$i_L(0^+) = i_L(0^-) = -\frac{8}{2} = -4 \text{ A}$$

$$\frac{di_L(0^+)}{dt} = \frac{v_c(0^+)}{4} - 2i_L(0^+) = \frac{8}{4} - 2(-4) = 10 \ \frac{\text{A}}{\text{s}}$$

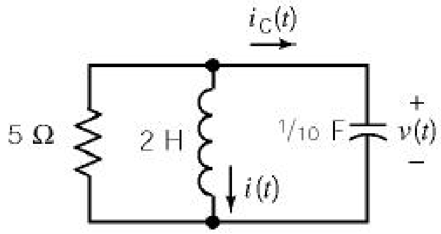
$$i_L(0^+) = -4 = A_1$$

$$\frac{di_L(0^+)}{dt} = 10 = -2A_1 + A_2 \Rightarrow A_2 = 2$$

$$\therefore \underline{i_L(t) = e^{-2t}\left[-4\cos t + 2\sin t\right] \text{ A}}$$

P9.7-5

After $t = 0$



The characteristic equation is:

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0 \text{ or } s^2 + 2s + 5 = 0$$

The natural frequencies are: $s_{1,2} = -1 \pm j2$

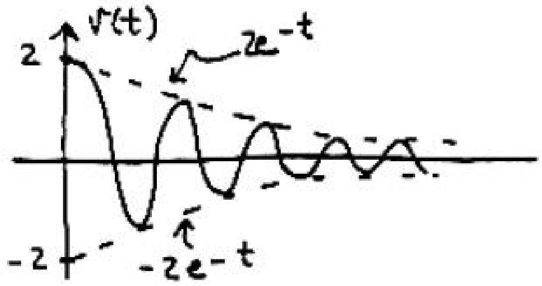
The natural response is of the form:

$$v(t) = e^{-t}\left[B_1 \cos 2t + B_2 \sin 2t\right]$$

$v(0^+) = 2 = B_1$. From KCL, $i_c\left(0^+\right) = -\frac{v\left(0^+\right)}{5} - i\left(0^+\right) = -\frac{2}{5} - \frac{1}{10} = -\frac{1}{2}\ \frac{\text{V}}{\text{s}}$ so

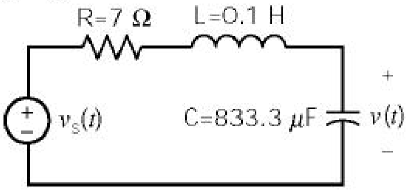
$$\frac{dv\left(0^+\right)}{dt} = 10\left(-\frac{1}{2}\right) = -B_1 + 2B_2 \Rightarrow B_2 = -\frac{3}{2}$$

Finally, $\underline{v\left(t\right) = 2e^{-t}\cos 2t - \frac{3}{2}e^{-t}\sin 2t \text{ V}}$ $\quad t \geq 0$



P9.8-2

After $t = 0$



$$\frac{d^2v(t)}{dt}+\frac{R}{L}\frac{dv(t)}{dt}+\frac{1}{LC}v(t)=\frac{v_s(t)}{LC}$$

$$\frac{d^2v(t)}{dt}+70\frac{dv(t)}{dt}+12000v(t)=12000v_s(t)$$

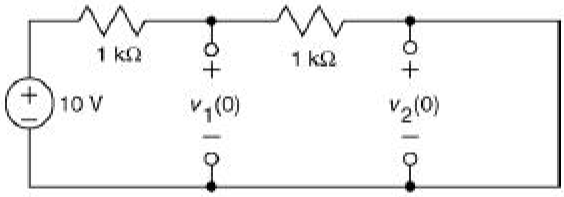
(a) Try a forced response of the form $v_f(t)=A$. Substituting into the differential equations gives $0+0+12000A=24000 \quad \Rightarrow A=2$. Therefore $v_f(t)=2\ \text{V}$.

(b) Try a forced response of the form $v_f(t)=A+B\,t$. Substituting into the differential equations gives $70A+12000A\,t+12000B=2400\,t$. Therefore $A=0.2$ and $B=\dfrac{-70A}{12000}=-1.167\times10^{-3}$. Finally $\underline{v_f(t)=\left(-1.167\times10^{-3}\right)t+0.2\ \text{V}}$.

(c) Try a forced response of the form $v_f(t)=A\,e^{-30t}$. Substituting into the differential equations gives $900\,Ae^{-30t}-2100\,Ae^{-30t}+12000\,Ae^{-30t}=12000\,e^{-30t}$. Therefore $A=\dfrac{12000}{10800}=1.11$. Finally $\underline{v_f(t)=1.11e^{-250t}\ \text{V}}$.

P9.9-3

First, find the steady state response for t < 0. The input is constant so the capacitors will act like an open circuits at steady state.



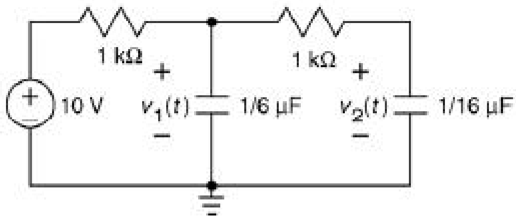
$$v_1(0) = \frac{1000}{1000+1000}(10) = 5 \text{ V}$$

and

$$v_2(0) = 0 \text{ V}$$

For $t > 0$,

Node equations:



$$\frac{v_1 - 10}{1000} + \left(\frac{1}{6}\times10^{-6}\right)\frac{d}{dt}v_1 + \frac{v_1 - v_2}{1000} = 0$$

$$\Rightarrow 2\,v_1 + \left(\frac{1}{6}\times10^{-3}\right)\frac{d}{dt}v_1 - 10 = v_2$$

$$\frac{v_1 - v_2}{1000} = \left(\frac{1}{16}\times10^{-6}\right)\frac{d}{dt}v_2$$

$$\Rightarrow \quad v_1 - v_2 = \left(\frac{1}{16}\times10^{-3}\right)\frac{d}{dt}v_2$$

After some algebra:

$$\frac{d^2}{dt^2}v_1 + \left(2.8\times10^4\right)\frac{d}{dt}v_1 + \left(9.6\times10^7\right)v_1 = 9.6\times10^8$$

The forced response will be a constant, $v_f = B$ so

$$\frac{d^2}{dt^2}B + \left(2.8\times10^4\right)\frac{d}{dt}B + \left(9.6\times10^7\right)B = 9.6\times10^8 \quad \Rightarrow \quad B = 10 \text{ V}.$$

To find the natural response, consider the characteristic equation:

$$s^2 + \left(2.8\times10^4\right)s + \left(9.6\times10^7\right) = 0 \quad \Rightarrow \quad s_{1,2} = -4\times10^3, -2.4\times10^4$$

The natural response is

$$v_n = A_1\,e^{-4\times10^3\,t} + A_2\,e^{-2.4\times10^4\,t}$$

so

$$v_1(t) = A_1\,e^{-4\times10^3\,t} + A_2\,e^{-2.4\times10^4\,t} + 10$$

At $t = 0$

$$5 = v_1(0) = A_1\, e^{-4\times10^3(0)} + A_2\, e^{-2.4\times10^4(0)} + 10 = A_1 + A_2 + 10 \qquad (1)$$

Next

$$2\,v_1 + \left(\frac{1}{6}\times10^{-3}\right)\frac{d}{dt}v_1 - 10 = v_2 \quad \Rightarrow \quad \frac{d}{dt}v_1 = -12000v_1 + 6000\,v_2 + 6\times10^4$$

At $t = 0$

$$\frac{d}{dt}v_1(0) = -12000v_1(0) + 6000\,v_2(0) + 6\times10^4 = -12000(5) + 6000\,(0) + 6\times10^4 = 0$$

so

$$\frac{d}{dt}v_1(t) = A_1\left(-4\times10^3\right)e^{-4\times10^3 t} + A_2\left(-2.4\times10^4\right)e^{-2.4\times10^4 t}$$

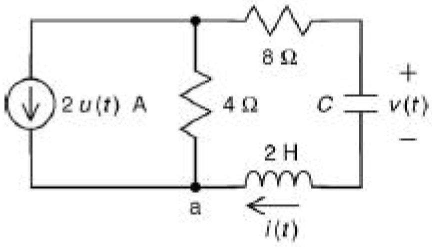
At $t = 0+$

$$0 = \frac{d}{dt}v_1(0) = A_1\left(-4\times10^3\right)e^{-4\times10^3(0)} + A_2\left(-2.4\times10^4\right)e^{-2.4\times10^4(0)} = A_1\left(-4\times10^3\right) + A_2\left(-2.4\times10^4\right)$$

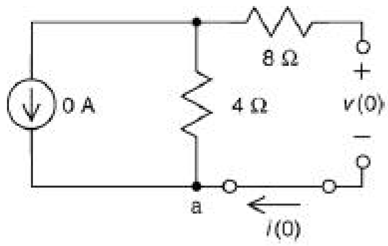
so $A_1 = -6$ and $A_2 = 1$. Finally

$$v_1(t) = 10 + e^{-2.4\times10^4 t} - 6\,e^{-4\times10^3 t} \text{ V for } t > 0$$

P9.9-7



First, find the steady-state response for $t < 0$. The input is constant so the capacitor will act like an open circuit at steady state, and the inductor will act like a short circuit.

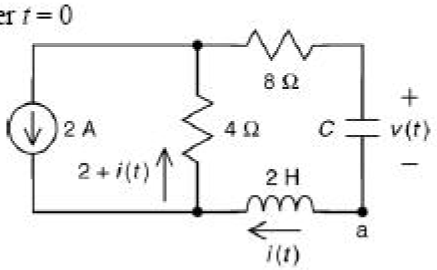


and

$i(0) = 0$ A

$v(0) = 0$ V

After $t = 0$



Apply KCL at node a: $C \frac{d}{dt} v = i$

Apply KVL to the right mesh:

$$8\,i + v + 2\frac{d}{dt} i + 4(2+i) = 0$$

$$12\,i + v + 2\frac{d}{dt} i = -8$$

After some algebra: $$\frac{d^2}{dt^2} v + (6)\frac{d}{dt} v + \left(\frac{1}{2\,C}\right) v = -\frac{4}{C}$$

The forced response will be a constant, $v_f = B$ so

$$\frac{d^2}{dt^2} B + (6)\frac{d}{dt} B + \left(\frac{1}{2\,C}\right) B = -\frac{4}{C} \quad \Rightarrow \quad B = -8 \text{ V}$$

(a)
When $C = 1/18$ F the differential equation is $\frac{d^2}{dt^2} v + (6)\frac{d}{dt} v + (9) v = -72$.

The characteristic equation is $s^2 + 6\,s + 9 = 0 \quad \Rightarrow \quad s_{1,2} = -3, -3$

Then $v(t) = (A_1 + A_2\, t) e^{-3t} - 8$.

Using the initial conditions:

$$0 = v(0) = \left(A_1 + A_2(0)\right)e^0 - 8 \quad \Rightarrow \quad A_1 = 8$$

$$0 = i(0) = C\frac{d}{dt}v(0) = C\left[-3\left(A_1 + A_2(0)\right)e^0 + A_2\, e^0\right] \quad \Rightarrow \quad A_2 = 3\,A_1 = 24$$

So

$$v(t) = (8 + 24t)e^{-3t} - 8 \text{ V} \quad \text{for } t > 0$$

(b)

When $C = 1/10$ F the differential equation is $\frac{d^2}{dt^2}v + (6)\frac{d}{dt}v + (5)v = -40$

The characteristic equation is $s^2 + 6\,s + 5 = 0 \quad \Rightarrow \quad s_{1,2} = -1, -5$

Then $v(t) = A_1\, e^{-t} + A_2\, e^{-5\,t} - 8$.

Using the initial conditions:

$$\left.\begin{array}{l} 0 = v(0) = A_1\, e^0 + A_2\, e^0 - 8 \quad \Rightarrow \quad A_1 + A_2 = 8 \\ 0 = \frac{d}{dt}v(0) = -A_1\, e^0 - 5\,A_2\, e^0 \quad \Rightarrow \quad -A_1 - 5A_2 = 0 \end{array}\right\} \quad \Rightarrow \quad A_1 = 10 \text{ and } A_2 = -2$$

So

$$v(t) = 10\,e^{-t} - 2\,e^{-5\,t} - 8 \text{ V} \quad \text{for } t > 0$$

(c)

When $C = 1/20$ F the differential equation is $\frac{d^2}{dt^2}v + (6)\frac{d}{dt}v + (10)v = -80$

The characteristic equation is $s^2 + 6\,s + 10 = 0 \quad \Rightarrow \quad s_{1,2} = -3 \pm j$

Then $v(t) = e^{-3\,t}\left(A_1 \cos t + A_2 \sin t\right) - 8$.

Using the initial conditions:

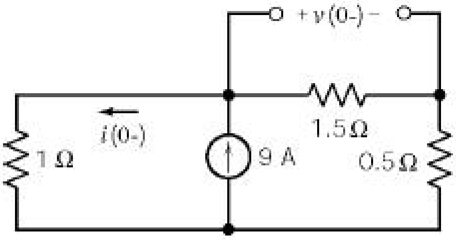
$$0 = v(0) = e^0\left(A_1 \cos 0 + A_2 \sin 0\right) - 8 \quad \Rightarrow \quad A_1 = 8$$

$$0 = \frac{d}{dt}v(0) = -3\,e^0\left(A_1 \cos 0 + A_2 \sin 0\right) + e^0\left(-A_1 \sin 0 + A_2 \cos 0\right) \quad \Rightarrow \quad A_2 = 3\,A_1 = 24$$

So

$$v(t) = e^{-3\,t}\left(8 \cos t + 24 \sin t\right) - 8 \text{ V} \quad \text{for } t > 0$$

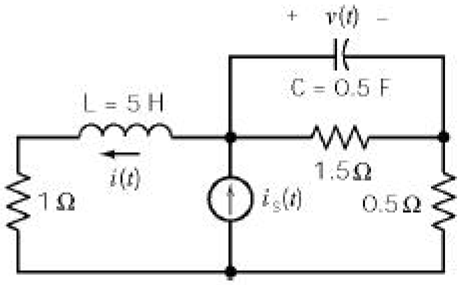
Assume that the circuit is at steady before $t = 0$.



$$i(0^+) = i(0^-) = \frac{2}{2+1}\times 9 = 6 \text{ A}$$

$$v(0^+)=v(0^-) = \frac{1}{2+1}\times 9\times 1.5 = 4.5 \text{ V}$$

After $t = 0$:



Apply KCL at the top node of the current source to get

$$i(t) + 0.5\frac{dv(t)}{dt} + \frac{v(t)}{1.5} = i_s(t) \qquad (1)$$

Apply KVL and KCL to get

$$v(t) + \left[0.5\frac{dv(t)}{dt} + \frac{v(t)}{1.5}\right]0.5 = \frac{5di(t)}{dt} + i(t) \qquad (2)$$

Solving for $i(t)$ in (1) and plugging into (2) yields

$$\frac{d^2v(t)}{dt^2} + \frac{49}{30}\frac{dv(t)}{dt} + \frac{4}{5}v(t) = \frac{2}{5}i_s(t) + 2\frac{di_s(t)}{dt} \quad \text{where } i_s(t) = 9 + 3e^{-2t} \text{ A}$$

Using the operator $s = \frac{d}{dt}$, the characteristic equation is $s^2 + \frac{49}{30}s + \frac{4}{5} = 0$ and the characteristic roots are $s_{1,2} = -.817 \pm j.365$. The natural response has the form

$$v_n(t) = e^{-0.817t}\left[A_1\cos(0.365t) + A_2\sin(0.365t)\right]$$

Try a forced response of the form $v_f(t) = B_0 + B_1e^{-2t}$. Substituting into the differential equations gives $B_0 = 4.5$ and $B_1 = -7.04$. The complete response has the form

$$v(t) = e^{-.817t}\left[A_1\cos(0.365t) + A_2\sin(0.365t)\right] + 4.5 - 7.04e^{-2t}$$

Next, consider the initial conditions:

$$v(0) = 4.5 = A_1 + 4.5 - 7.04 \quad \Rightarrow \quad A_1 = 7.04$$

$$\frac{d\,v(0)}{dt} = 2\,i_s(0) - 2\,i(0) - \frac{4}{3}v(0) \;=\; 2(9+3) - 2(6) - \frac{4}{3}(4.5) \;=\; 6$$

$$6 = \frac{d\,v(0)}{dt} = -0.817A_1 + 0.365\,A_2 + 14.08 \;\Rightarrow A_2 = -6.38$$

So the voltage is given by

$$v(t) \;=\; e^{-0.817t}\left[7.04\cos(0.365\,t) + A_2\sin(0.365\,t)\;\right] + 4.5 - 7.04\,e^{-2t}$$

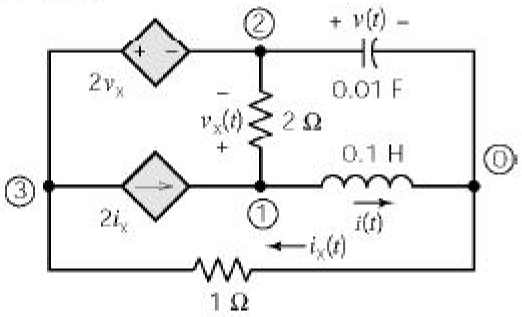
Next the current given by

$$i(t) \;=\; i_s(t) \;-\frac{v(t)}{1.5} - 0.5\,\frac{d\,v(t)}{dt}$$

Finally

$$i(t) \;= \underline{e^{-0.817t}\left[2.37\cos(0.365t) \;+\; 7.14\sin(0.365t)\right] \;+\; 6 \;+ 0.65e^{-2t}\ \mathrm{A}}$$

P9.10-4



(Encircled numbers are node numbers.)

Apply KCL to the supernode corresponding to the dependent voltage source to get

$$i_x(t)-2i_x(t)-0.01\frac{dv(t)}{dt}+\frac{v_x(t)}{2}=0$$

Apply KCL at node 1 to get

$$i(t)-2i_x(t)+\frac{v_x(t)}{2}=0$$

Apply KVL to the top-right mesh to get

$$v_x(t)+v(t)-0.1\frac{di(t)}{dt}=0$$

Apply KVL to the outside loop to get $i_x(t)=-2\,v_x(t)-v(t)$.

Eliminate $i_x(t)$ to get

$$\frac{5}{2}v_x(t)+v(t)-0.01\frac{dv(t)}{dt}=0$$
$$i(t)+\frac{9}{2}v_x(t)+2v(t)=0$$
$$v_x(t)=-v(t)+0.01\frac{d\,i(t)}{dt}$$

Then eliminate $v_x(t)$ to get

$$-1.5\,v(t)-0.01\frac{dv(t)}{dt}+0.25\frac{di(t)}{dt}=0$$
$$-2.5\,v(t)+i(t)+0.45\frac{di(t)}{dt}=0$$

Using the operator $s=\frac{d}{dt}$ we have

$$(-1.5-.01s)v(t)+(.25s)\,i(t)\ =0$$
$$(-2.5)v(t)+(1+.45s)\,i(y)\ =0$$

The characteristic equation is $s^2+13.33\,s\,+333.33\,=0$. The natural frequencies are $s_1,\,s_2\,=-6.67\pm j\,17$. The natural response has the form $v_n(t)=\left[A\cos 17t\,+B\sin 17t\right]\,e^{-6.67t}$. The forced response is $v_f(t)=0$. The complete response has the form $v(t)=\left[A\cos 17t\,+B\sin 17t\right]\,e^{-6.67t}$.

The given initial conditions are $i(0)=0$ and $v(0)=10$ V. Then

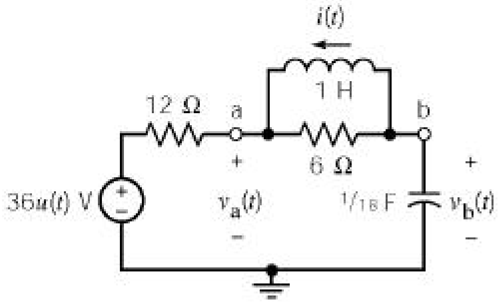
$$v(0)=10=A \quad \text{and} \quad \frac{dv(0)}{dt}=-111=-6.67A+17B \Rightarrow B=-2.6$$

Finally $\underline{i(t)=[3.27\ \sin\ 17\,t]\,e^{-6.67\,t}\ \text{A}}$.

P9.11-4

Before $t=0$ the voltage source voltage is 0 V so $v_b(0+)=v_b(0-)=0$ V and $i(0+)=i(0-)=0$ A. Apply KCL at node a to get

$$\frac{v_a(0+)-36}{12}-i\ (0+)+\frac{v_a(0+)-v_b\ (0+)}{6}=0 \ \Rightarrow\ v_a(0+)+2v_a(0+)=36 \ \Rightarrow\ v_a(0)\ =12\ \text{V}$$



After t = 0 the node equations are:

$$-\frac{v_a(t)-v_s(t)}{12}+\frac{1}{L}\int_0^t\left(v_b(\tau)-v_a(\tau)\right)d\tau+\frac{v_b(t)-v_a(t)}{6}=0$$

$$C\frac{d\,v_b(t)}{d\,t}+\frac{v_b(t)-v_a(t)}{6}+\frac{1}{L}\int_0^t\left(v_b(\tau)-v_a(\tau)\right)d\tau=0$$

Using the operator $s=\dfrac{d}{dt}$ we have

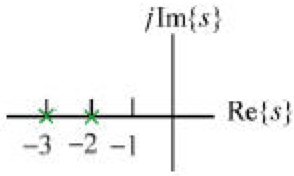
$$\left(\frac{1}{12}+\frac{1}{6}+\frac{1}{s}\right)v_a(t)+\left(-\frac{1}{6}-\frac{1}{s}\right)v_b(t)=\frac{v_s(t)}{12}$$

$$\left(-\frac{1}{6}-\frac{1}{s}\right)v_a(t)+\left(\frac{1}{18}s+\frac{1}{6}+\frac{1}{s}\right)v_b(t)=0$$

Using Cramer's rule

$$(s^2+5s+6)\, v_b(t) = (s+6)\, v_s(t) = (s+6)\,(36)$$

The characteristic equation is $s^2+5s+6=0$. The natural frequencies are $s_{1,2}=-2,-3$. The natural response has the form $v_n(t) = A_1 e^{-2t} + A_2 e^{-3t}$. Try $v_f(t) = B$ as the forced response. Substituting into the differential equation gives $B=36$ so $v_f(t) = 36$ V. The complete response has the form $v_b(t) = A_1 e^{-2t} + A_2 e^{-3t} + 36$.



Next

$$v_b(0^+) = 36 + A_1 + A_2$$
$$\frac{dv_b}{dt}(0^+) = -2A_1 - 3A_2$$

Apply KCL at node a to get

$$\frac{1}{18}\frac{dv_b(t)}{dt} + \frac{v_b(t)-v_a(t)}{6} + i(t) = 0$$

At $t = 0+$

$$\frac{1}{18}(-2A_1 - 3A_2) = \frac{1}{18}\frac{d\,v_b(0^+)}{dt} = \frac{v_a(0^+)-v_b(0^+)}{6} - i(0^+) = \frac{12-0}{6} - 0 = 2$$

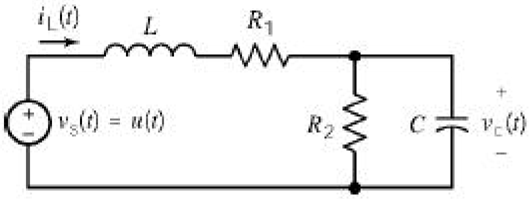
So

$$\left.\begin{array}{l} 0 = v_b(0^+) = 36 + A_1 + A_2 \\ \frac{1}{18}(-2A_1 - 3A_2) = 2 \end{array}\right\} \Rightarrow \quad A_1 = -72,\ A_2 = 36$$

Finally

$$\underline{v_b(t) = 36 - 72e^{-2t} + 36e^{-3t} \text{ V for } t \geq 0}$$

DP 9-1



When the circuit reaches steady state after $t = 0$, the capacitor acts like an open circuit and the inductor acts like a short circuit. Under these conditions

$$v_C(\infty) = \frac{R_2}{R_1 + R_2} 1$$

The specifications require that $v_C(\infty) = \frac{1}{2}$ so

$$\frac{1}{2} = \frac{R_2}{R_1 + R_2} \quad \Rightarrow \quad R_1 = R_2$$

Next, represent the circuit by a 2nd order differential equation:

KCL at the top node of R_2 gives: $$\frac{v_C(t)}{R_2} + C\frac{d}{dt}v_C(t) = i_L(t)$$

KVL around the outside loop gives: $$v_s(t) = L\frac{d}{dt}i_L(t) + R_1\, i_L(t) + v_C(t)$$

Use the substitution method to get

$$v_s(t) = L\frac{d}{dt}\left(\frac{v_C(t)}{R_2} + C\frac{d}{dt}v_C(t)\right) + R_1\left(\frac{v_C(t)}{R_2} + C\frac{d}{dt}v_C(t)\right) + v_C(t)$$

$$= LC\frac{d^2}{dt^2}v_C(t) + \left(\frac{L}{R_2} + R_1 C\right)\frac{d}{dt}v_C(t) + \left(1 + \frac{R_1}{R_2}\right)v_C(t)$$

The characteristic equation is

$$s^2 + \left(\frac{1}{R_2 C} + \frac{R_1}{L}\right)s + \left(\frac{1+\frac{R_1}{R_2}}{LC}\right) = s^2 + 6s + 8 = (s+2)(s+4) = 0$$

Equating coefficients of like powers of s:

$$\frac{1}{R_2 C} + \frac{R_1}{L} = 6 \quad \text{and} \quad \frac{1+\frac{R_1}{R_2}}{LC} = 8$$

Using $R_1 = R_2 = R$ gives

$$\frac{1}{RC} + \frac{R}{L} = 6 \quad \Rightarrow \quad \frac{1}{LC} = 4$$

These equations do not have a unique solution. Try $C = 1$ F. Then $L = \frac{1}{4}$ H and

$$\frac{1}{R} + 4R = 6 \quad \Rightarrow \quad R^2 - \frac{3}{2}R + \frac{1}{4} = 0 \quad \Rightarrow \quad R = 1.309\,\Omega \text{ or } R = 0.191\,\Omega$$

Pick $R = 1.309\ \Omega$. Then

$$v_c(t) = \frac{1}{2} + A_1 e^{-2t} + A_2 e^{-4t} \text{ V}$$

$$i_L(t) = \frac{v_C(t)}{1.309} + \frac{d}{dt}v_C(t) = -1.236\,A_1 e^{-2t} - 3.236\,A_2 e^{-4t} + 0.3819$$

At t = 0^+

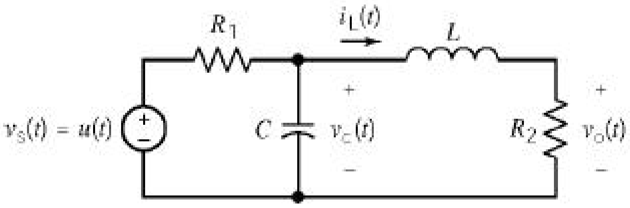
$$0 = v_c(0^+) = A_1 + A_2 + 0.5$$

$$0 = i_L(0^+) = -1.236\,A_1 - 3.236\,A_2 + 0.3819$$

Solving these equations gives $A_1 = -1$ and $A_2 = 0.5$, so

$$v_c(t) = \frac{1}{2} - e^{-2t} + \frac{1}{2}e^{-4t} \text{ V}$$

DP 9-5



When the circuit reaches steady state after $t = 0$, the capacitor acts like an open circuit and the inductor acts like a short circuit. Under these conditions

$$v_C(\infty)=\frac{R_2}{R_1+R_2}1,\quad i_L(\infty)=\frac{1}{R_1+R_2}\quad \text{and}\quad v_o(\infty)=\frac{R_2}{R_1+R_2}1$$

The specifications require that $v_o(\infty)=\frac{1}{2}$ so

$$\frac{1}{2}=\frac{R_2}{R_1+R_2}\quad\Rightarrow\quad R_1=R_2$$

Next, represent the circuit by a 2nd order differential equation:

KVL around the right-hand mesh gives: $\quad v_C(t)=L\frac{d}{dt}i_L(t)+R_2\,i_L(t)$

KCL at the top node of the capacitor gives: $\quad \frac{v_s(t)-v_C(t)}{R_1}-C\frac{d}{dt}v_C(t)=i_L(t)$

Use the substitution method to get

$$\begin{aligned}v_s(t)&=R_1\,C\frac{d}{dt}\left(L\frac{d}{dt}i_L(t)+R_2\,i_L(t)\right)+\left(L\frac{d}{dt}i_L(t)+R_2\,i_L(t)\right)+R_1\,i_L(t)\\&=R_1LC\frac{d^2}{dt^2}i_L(t)+\left(L+R_1\,R_2\,C\right)\frac{d}{dt}i_L(t)+\left(R_1+R_2\right)i_L(t)\end{aligned}$$

Using $i_L(t)=\frac{v_o(t)}{R_2}$ gives

$$v_s(t)=\frac{R_1}{R_2}LC\frac{d^2}{dt^2}v_o(t)+\left(\frac{L}{R_2}+R_1\,C\right)\frac{d}{dt}v_o(t)+\left(\frac{R_1+R_2}{R_2}\right)v_o(t)$$

The characteristic equation is

$$s^2+\left(\frac{1}{R_1\,C}+\frac{R_2}{L}\right)s+\left(\frac{1+\frac{R_2}{R_1}}{LC}\right)=s^2+6s+8=(s+2)(s+4)=0$$

Equating coefficients of like powers of s:

$$\frac{1}{R_1\,C}+\frac{R_2}{L}=6\quad \text{and}\quad \frac{1+\frac{R_2}{R_1}}{LC}=8$$

Using $R_1=R_2=R$ gives

$$\frac{1}{R\,C}+\frac{R}{L}=6\quad\Rightarrow\quad \frac{1}{LC}=4$$

These equations do not have a unique solution. Try C = 1 F. Then $L=\frac{1}{4}$ H and

$$\frac{1}{R}+4R=6 \quad \Rightarrow \quad R^2-\frac{3}{2}R+\frac{1}{4}=0 \quad \Rightarrow \quad R=1.309\,\Omega \text{ or } R=0.191\,\Omega$$

Pick R = 1.309 Ω. Then

$$v_o(t)=\frac{1}{2}+A_1\,e^{-2t}+A_2\,e^{-4t}\ \text{V}$$

$$i_L(t)=\frac{v_o(t)}{1.309}=\frac{1}{2.618}+\frac{A_1}{1.309}\,e^{-2t}+\frac{A_2}{1.309}\,e^{-4t}\ \text{V}$$

$$v_C(t)=1.309\,i_L(t)+\frac{1}{4}\frac{d}{dt}i_L(t)=\frac{1}{2}+0.6167\,A_1\,e^{-2t}+0.2361\,A_2\,e^{-4t}$$

At t = 0^+

$$0=i_L(0^+)=\frac{1}{2.618}+\frac{A_1}{1.309}+\frac{A_2}{1.309}$$

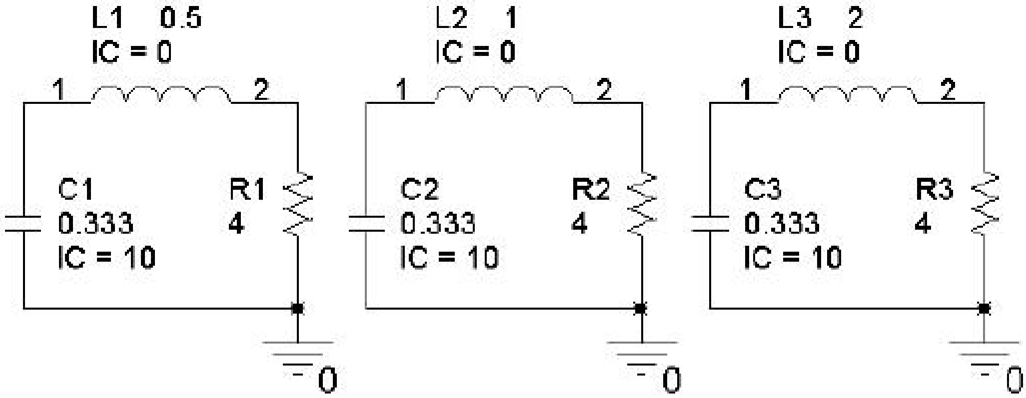
$$0=v_C(0^+)=\frac{1}{2}+0.6167\,A_1+0.2361\,A_2$$

Solving these equations gives A_1 = -1 and A_2 = 0.5, so

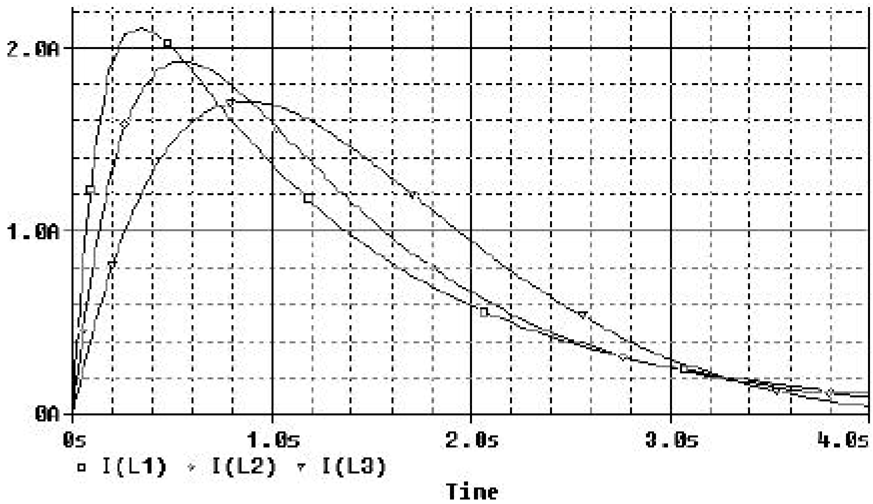
$$v_o(t)=\frac{1}{2}-e^{-2t}+\frac{1}{2}e^{-4t}\ \text{V}$$

DP 9-9

Let's simulate the three copies of the circuit simultaneously. Each copy uses a different value of the inductance.



The PSpice transient response shows that when $L = 1$ H the inductor current has its maximum at approximately t=0.5 s.



Consequently, we choose $\underline{L = 1 \text{ H.}}$