L-Moments

originally made by

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Definitions: Product-Moments

Mean, measure of location or center

$$\mu_x = E[\, X \,]$$

Variance, measure of spread, or dispersion

$$\sigma_x^2 = E[\, (X - \mu_x)^2]$$

Coef. of Skewness, measure of asymmetry

$$\gamma_x = E[\, (X - \mu_x)^3] \,/ \sigma_x^3$$

Product Moment–Estimators

$$\bar{X} = \sum_{i=1}^{n} X_i / n$$

$$S^2 = \frac{1}{(n-1)} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

$$G = \frac{n}{(n-1)(n-2)} \sum_{i=1}^{n} (X_i - \bar{X})^3 / S^3$$

Conventional Moment Ratios

Conventional descriptions of shape are

Coefficient of Variation, CV: σ / μ

Coefficients of skewness, γ: $E[(X-\mu)^3]/\sigma^3$

Coefficients of kurtosis, κ: $E[(X-\mu)^4]/\sigma^4$

Moments for Distributions

Distribution	Moments
Uniform	$\mu = (b+a)/2;\ \sigma^2 = (b-a)^2/12,\ \gamma = 0$ $f_X(x) = 1/(b-a)$ for $a < x < b$
Exponential	$\mu = \xi + \beta;\ \sigma^2 = \beta^2;\ \gamma = 2$ $f_X(x) = \exp\{-(x-\xi)/\beta\}/\beta,\ \xi <$
Normal	$\mu;\ \sigma^2;\ \gamma = 0$ $f_X(x) = \exp\{-0.5(x-\mu)^2/\sigma^2\}/ \quad (2\pi\sigma^2)$
Gumbel	$\alpha = \xi + 0.5772/\alpha$ $\sigma^2 = 1.645/\alpha^2;\ \gamma = 1.1396$ $F_X(x) = \exp\{-\exp[-\alpha(x-\xi)]\}\ -\infty < x < \infty$

Concerns: Product-Moments

Sample estimates are imprecise and their large bias depends upon

- Sample size
- Underlying distribution

Bounds on sample estimates

{if estimators in S uses (n-1) and in skew estimator uses n/[(n-1)(n-2)] }

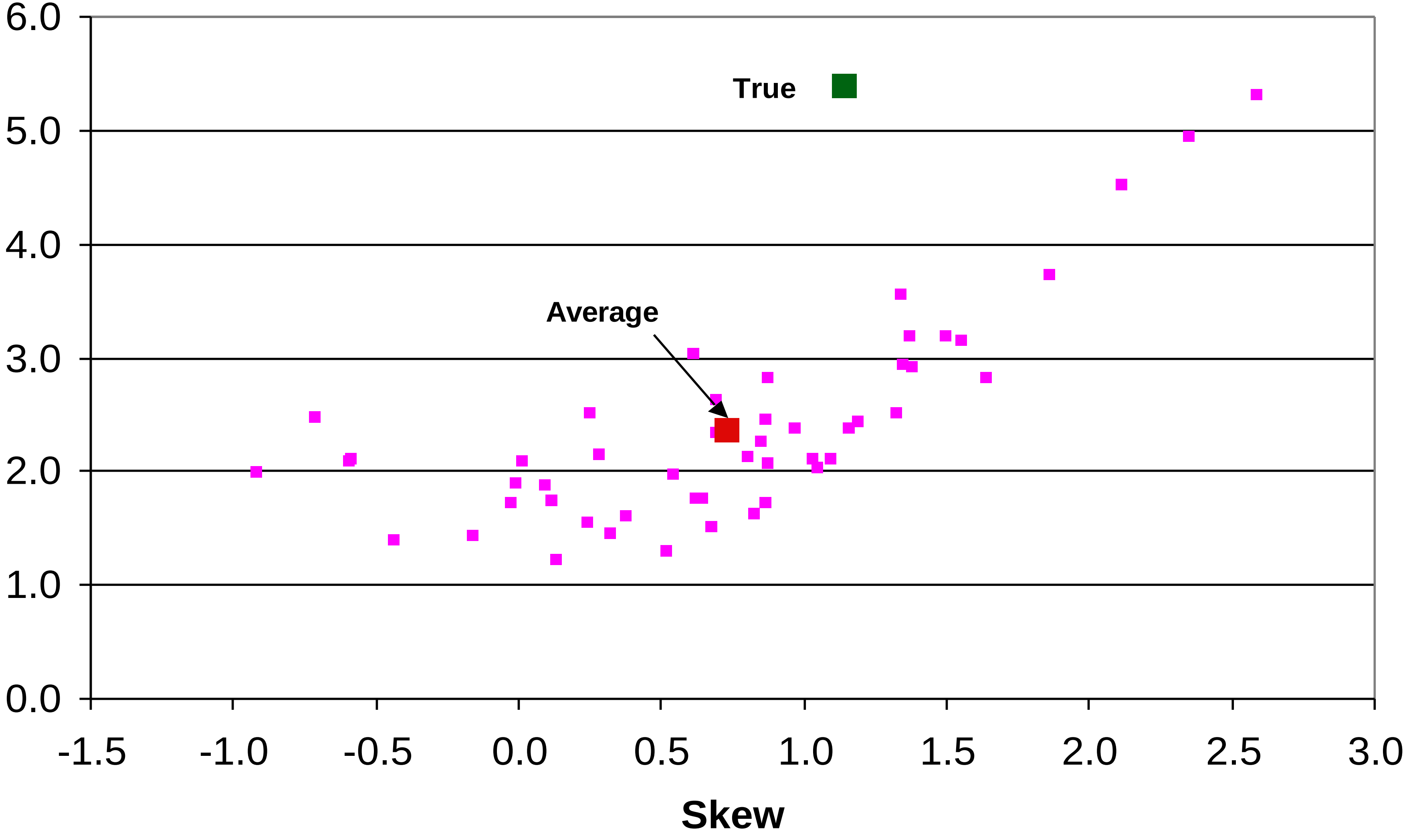
$$|CV| \leq n^{0.5}$$

$$|CS| \leq n^{0.5}$$

Bound on CV assumes observations must be positive.

Product-Moment Skew-Kurtosis estimators: n=10

Samples drawn from a Gumbel distribution.



L-Moments

An alternative to product moments

L-Moments: an alternative

L-moments can summarize data as do conventional moments. However, their estimators are linear combinations of the ordered observations.

Because L-moments avoid squaring and cubing the data, their estimators do not suffer from the severe bias problems encountered with product moments.

L-Moments: an alternative

Let $X_{(i|n)}$ be ith largest obs. in sample of size n.

Measure of Scale

expected difference between largest and smallest observations in a sample of size 2:

$$\lambda_2 = (1/2)\ E[\, X_{(2|2)} - X_{(1|2)}\,]$$

Measure of Asymmetry

$$\lambda_3 = (1/3)\ E[\, X_{(3|3)} - 2\, X_{(2|3)} + X_{(1|3)}\,]$$

where $\lambda_3 > 0$ for positively skewed dists.

L-Moments: an alternative

Measure of Kurtosis

$$\lambda_4 = (1/4)\, E[\, X_{(4|4)} - 3\, X_{(3|4)} - 3\, X_{(2|4)} + X_{(1|4)}\,]$$

For highly kurtotic distributions, λ_4 large.
For the uniform distribution $\lambda_4 = 0$.

L-kurtosis λ_4 can be written

$$\lambda_4 = (1/4)\, \{E[\, X_{(4|4)} - X_{(1|4)}\,] - 3\, E[\, X_{(3|4)} - X_{(2|4)}\,]\}$$

Dimensionless L-moment ratios

L-moment Coefficient of variation (L-CV):

$$\tau_2 = \lambda_2/\lambda_1 = \lambda_2/\mu$$

L-moment coef. of skew (L-Skewness)

$$\tau_3 = \lambda_3/\lambda_2$$

L-moment coef. of kurtosis (L-Kurtosis)

$$\tau_4 = \lambda_4/\lambda_2$$

(Note: Hosking calls L-CV τ instead of τ_2.)

Values of L-Moments for Several Distributions

Distribution	L-Moments
Uniform	$\lambda_2 = (b-a)/6;\ \tau_3 = 0;\ \tau_4 = 0$
Exponential	$\lambda_2 = \beta/2;\ \tau_3 = 1/3;\ \tau_4 = 1/6$
Normal	$\lambda_2 = \sigma/\mathrm{sqrt}[\pi];$ $\tau_3 = 0;\ \tau_4 = 0.1226$
Gumbel	$\lambda_2 = \alpha \ln(2);$ $\tau_3 = 0.1699;\ \tau_4 = 0.1504$

Values of L-Moments for Several Distributions

GEV:

$$F[x] = \exp\{ - [1 - (\kappa / \alpha) (x - \xi)]^{1/\kappa} \}$$

$$\lambda_2 = \alpha (1 - 2^{-\kappa}) \Gamma(1+\kappa) / \kappa$$

$$\tau_3 = 2 (1 - 3^{-\kappa})/(1 - 2^{-\kappa}) - 3$$

$$\tau_4 = \{1 - 5\ 4^{-\kappa}) + 10\ 3^{-\kappa}) - 6\ 2^{-\kappa})\} / (1 - 2^{-\kappa})$$

Generalized Pareto (GP):

$$F[x] = 1 - [1 - (\kappa / \alpha) (x - \xi)]^{1/\kappa}$$

$$\lambda_2 = \alpha / [(1 + \kappa)(2 + \kappa)$$

$$\tau_3 = (1 - \kappa)/(3 + \kappa)$$

$$\tau_4 = (1 - \kappa)(2 - \kappa)/[(3 + \kappa)(4 + \kappa)]$$

(From Hosking, 1990)

Probability Weighted Moments (PWMs)

An vehicle for computing L-moments. Actually PWMs were developed first, but now have been replaced in parameter estimation by L-moments.

Probability Weighted Moments

PWMs are used to estimate L-moments.

Define: $F(X)$ = CDF for X

r^{th} order PWM is: $\beta_r = E\{ X [F(X)]^r \}$

Instead of taking expectation of X to a power to calculate variance or skew, PWMs are expectation of X times powers of F(X).

For $r = 0$, β_0 is just the population mean $E[X]$.

Probability Weighted Moments

Estimation of PWMs: Because $(r+1)\,\beta_r$ is expected value of **largest** observation in a sample of size $(r+1)$, can use ordered sample values $X_{(i)}$,

$$X_{(1)} \leq \ldots \leq X_{(n)}$$

to compute sample estimator:

$$\hat{\beta}_r = \frac{1}{n} \sum_{i=r}^{n} \binom{i-1}{r} X_{(i)} \Big/ \binom{n-1}{r+1} = \frac{1}{r+1} \sum_{i=r}^{n} \binom{i-1}{r} X_{(i)} \Big/ \binom{n}{r+1}$$

Formulas for PWMs

More simply for r = 0, 1, 2, formulas are

$$b_o = \bar{X}$$

$$b_1 = \frac{1}{n(n-1)} \sum_{j=2}^{n} (j-1) X_{(j)}$$

$$b_2 = \frac{1}{n(n-1)(n-2)} \sum_{j=3}^{n} (j-1)(j-2) X_{(j)}$$

BACKGROUND: PWM expectation

To show that $(1+r)\,\beta_r = E\{\, X_{(r+1|r+1)} \,\}$, start with

$\Pr\{\, X_{(r+1|r+1)} \le x \,\} = [\, F(x) \,]^{r+1}$

Probability density function for $X_{(r+1|r+1)}$ is:

$$f_{X(r+1|r+1)}(x) = (r+1)\,[F(X)]^r f(x)$$

Hence

$$E\{X_{(r+1|r+1)}\} = \int x \{\, (r+1)\,[F(X)]^r f(x) \,\}\, dx$$
$$= (1+r)\,\beta_r$$

Computation of L-moments

Can use relationships with PWMs to compute L-moments: its convenient.

$$\lambda_1 = \beta_0$$

$$\lambda_2 = 2\,\beta_1 - \beta_0$$

$$\lambda_3 = 6\,\beta_2 - 6\,\beta_1 + \beta_0$$

$$\lambda_4 = 20\,\beta_3 - 30\,\beta_2 + 12\,\beta_1 - \beta_0$$

Definitions of Dimensionless Product-Moment and L-Moment Ratios

Name	Denoted	Definition
	Product-Moment Ratios	
Coef. of Variation	CV_x	σ_x/μ_x
Coef. of Skewness	γ_x	$E[(X - \mu_x)^3] / \sigma_x^3$
Coef. of Kurtosis	κ_x	$E[(X - \mu_x)^4] / \sigma_x^4$
	L-Moment Ratios	
L-Coef. of Variation*	L-CV	$\tau_2 = \lambda_2/\lambda_1$
L-Coef. of Skewness	L-Skewness	$\tau_3 = \lambda_3/\lambda_2$
L-Coef. of Kurtosis	L-Kurtosis	$\tau_4 = \lambda_4/\lambda_2$

* Hosking and Wallis (1997) uses τ instead of τ_2 to represent the L-CV ratio.

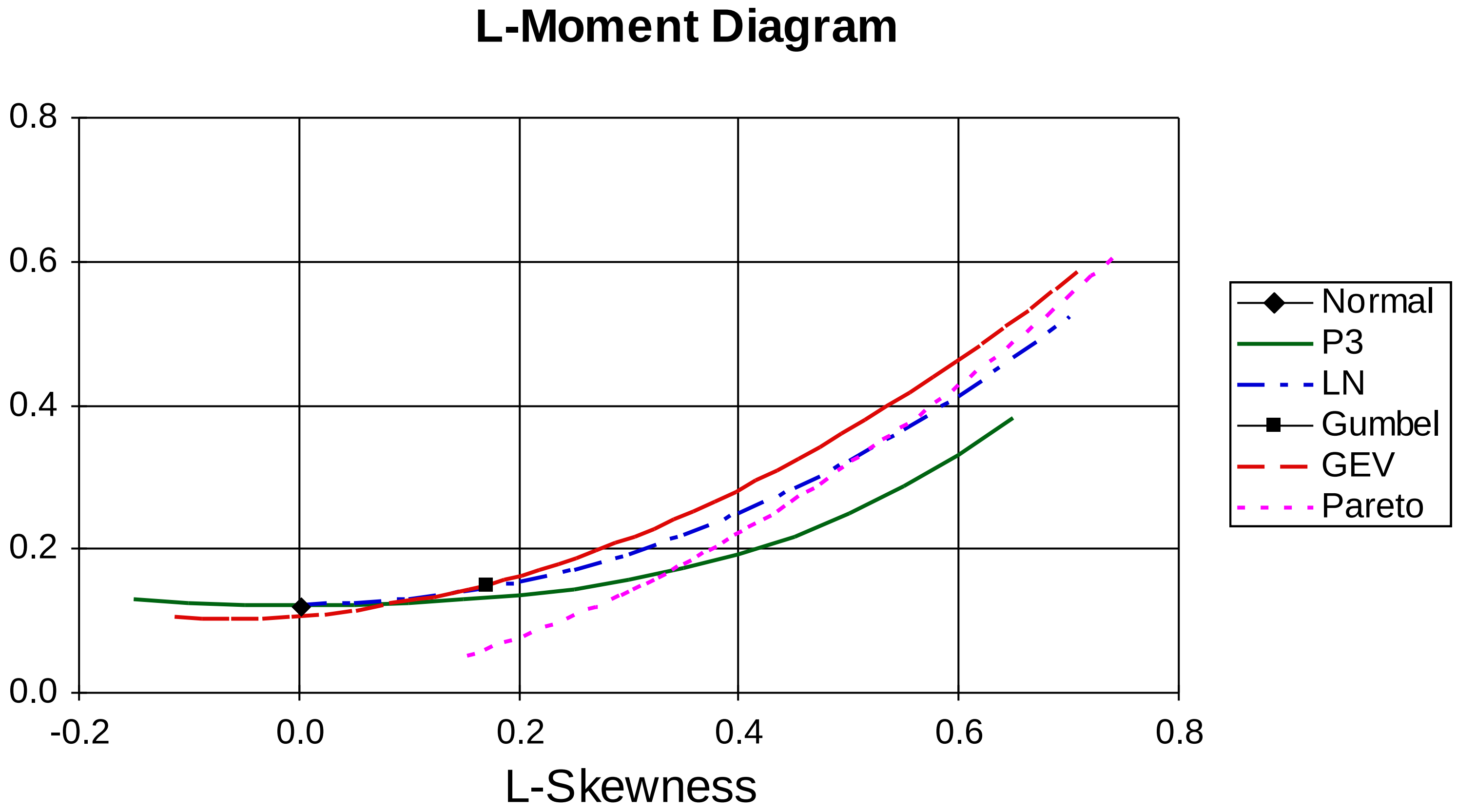
Distribution Selection

L-moments work well for selection of a family of distributions (lognormal, Gumbel, Pearson type 3 …) to describe different phenomena.

L-Moment Diagrams

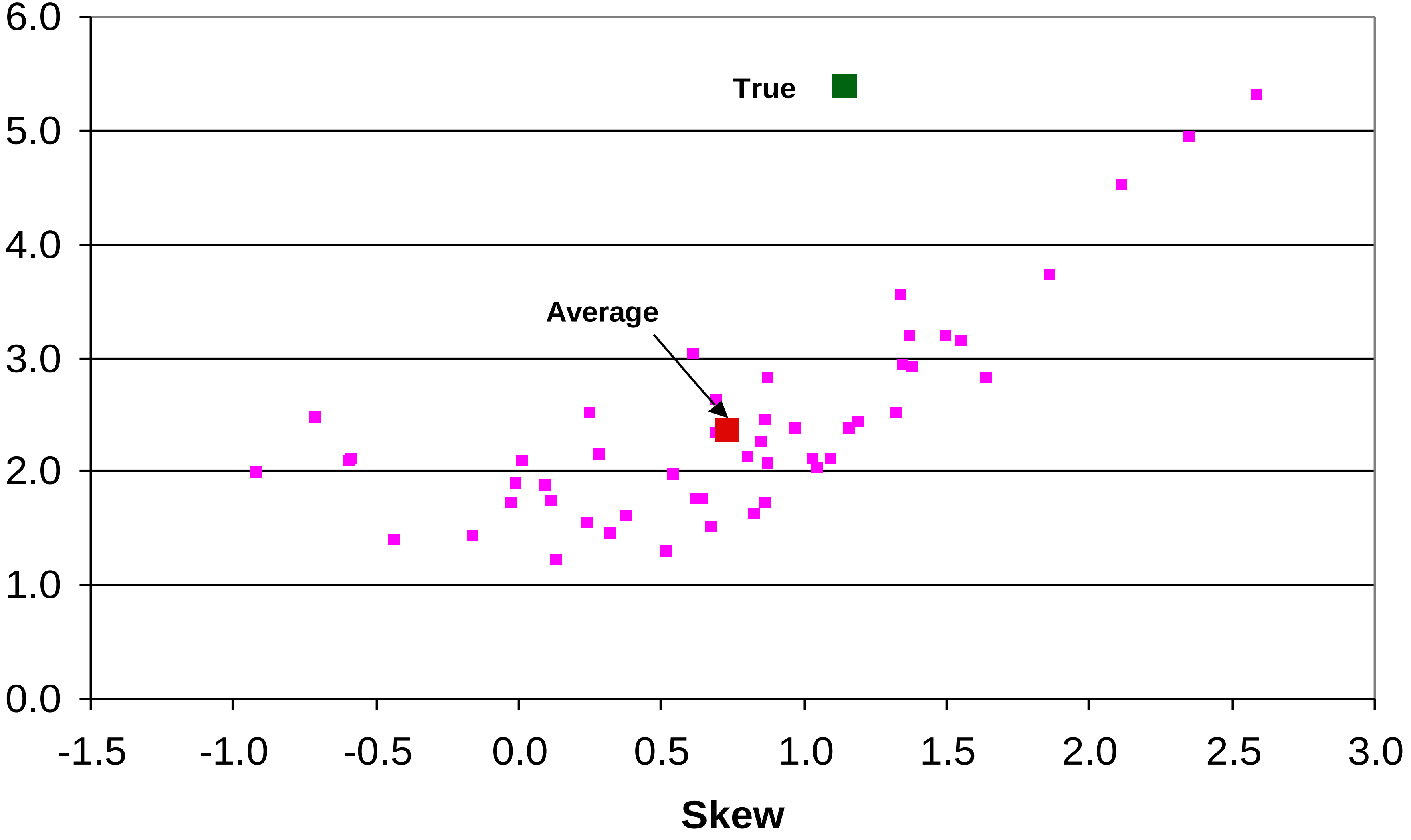
- Used to choose among alternative distributions to describe floods, water quality, wind speeds or rainfall depths at different locations.
- Plot of τ_3 versus $\tau_2 = \lambda_2/\lambda_1$ when choosing among 2-parameter distributions
- Plot τ_4 versus τ_3 when choosing among 3-parameter distributions.

Relationships for L-moments



Product-Moment Skew-Kurtosis estimators: n=10

Samples drawn from a Gumbel distribution.



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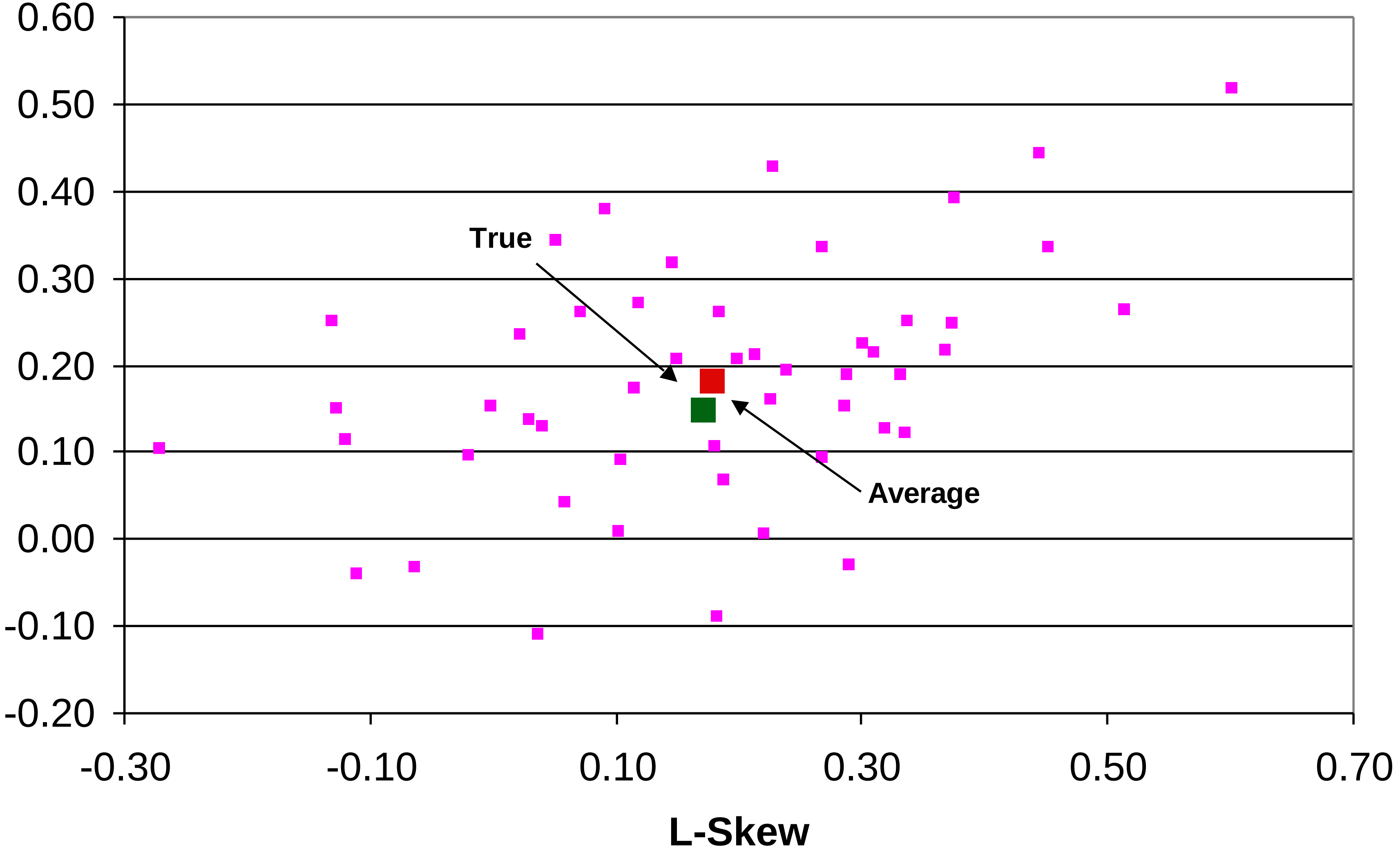
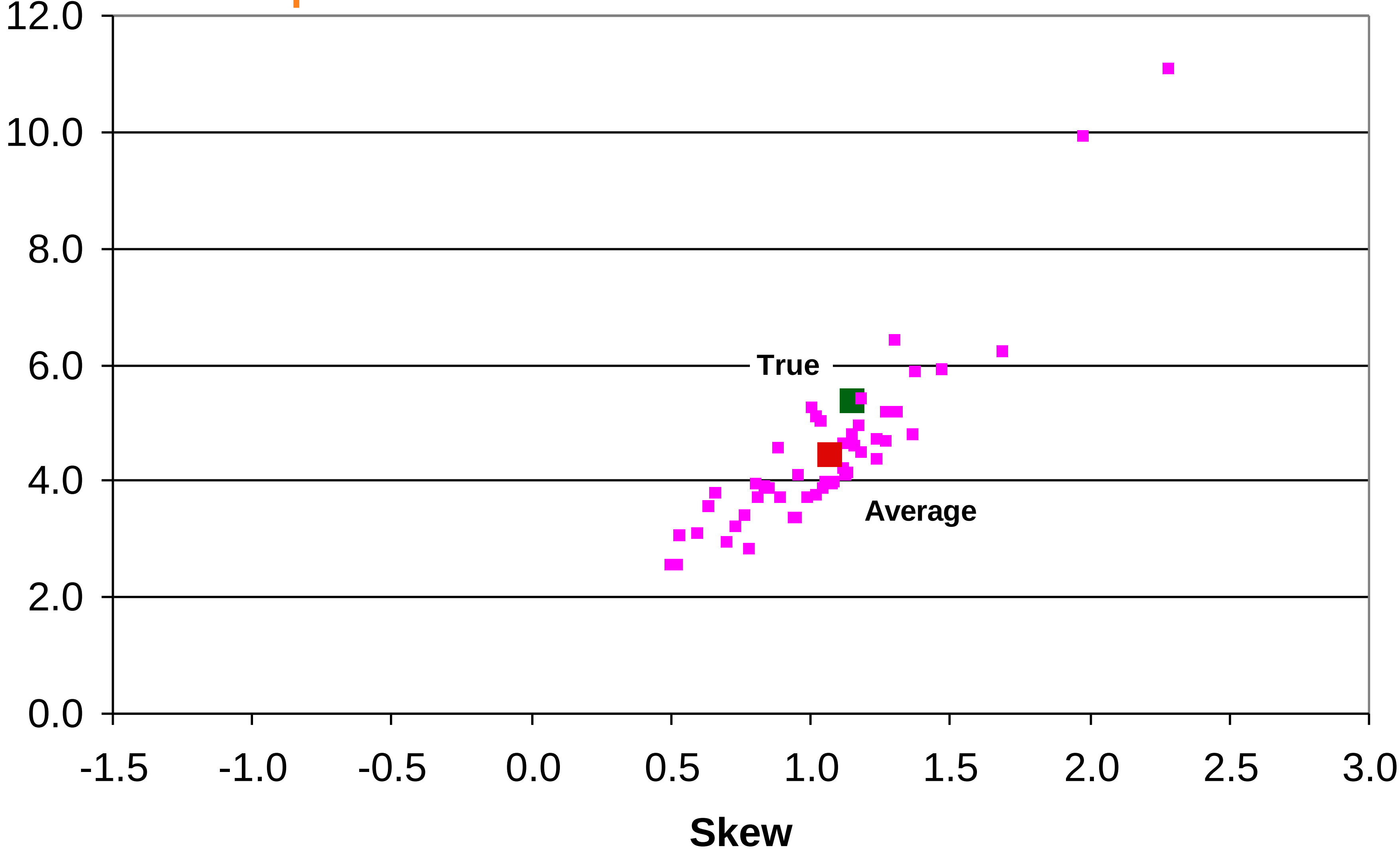


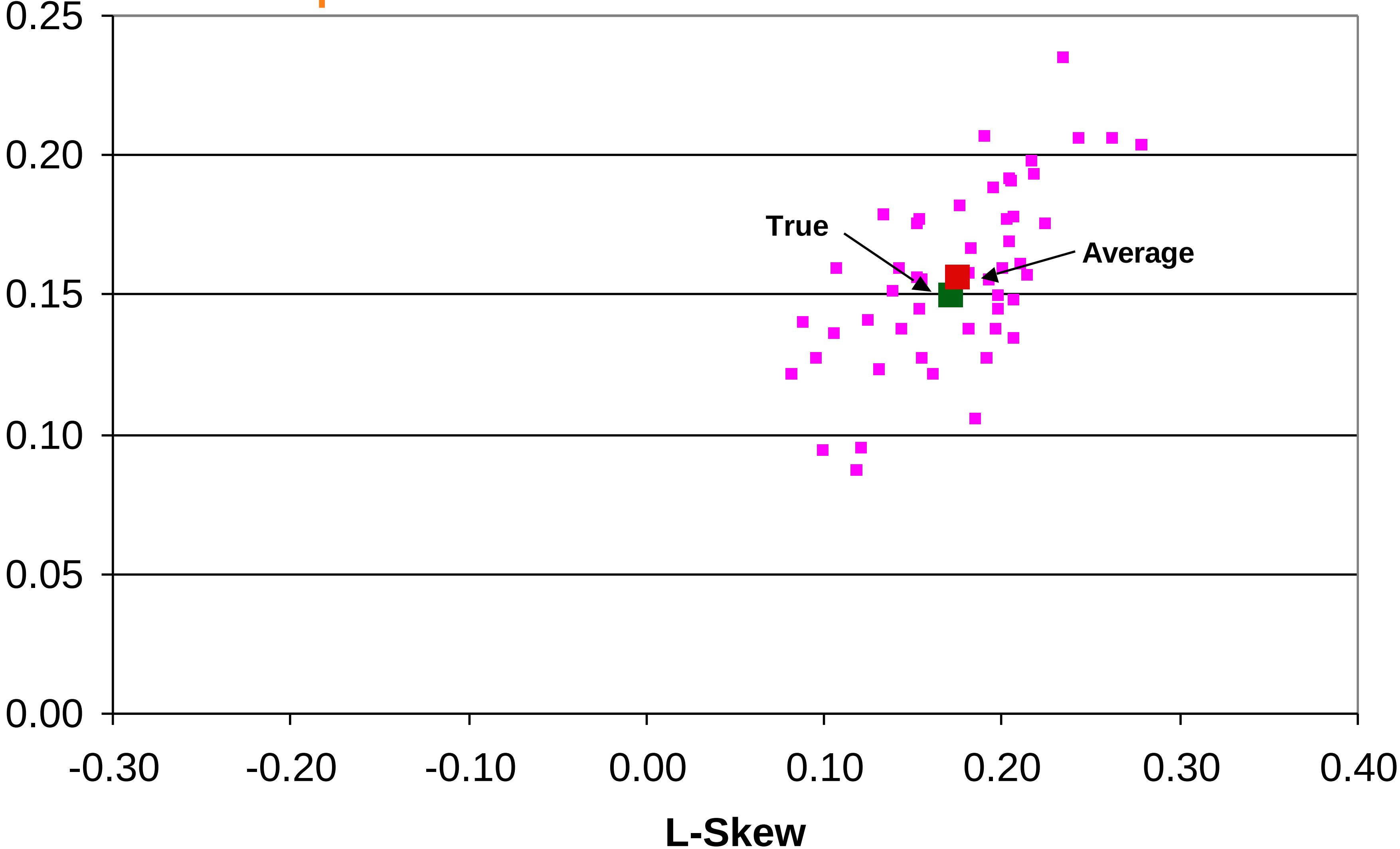
Product-Moment Skew-Kurtosis estimators: n=100

Samples drawn from a Gumbel distribution.



L-Moment Skew-Kurtosis estimators: n=100

Samples drawn from a Gumbel distribution.



Lesson: Skew–Kurtosis Diagrams

- For small n, product–moment estimators have bounds that prevents representation of true moments.
- For large n, skew–kurtosis estimators are highly variable and so highly correlated that they do not represent true moments.
- While also variable, L–skew and L–kurtosis estimators are approximately unbiased so regional averages can represent true values.

L-Moment Diagrams: Selecting a Distribution

- Goodness-of-fit statistics

  (such as probability plots) can show how well a member of each family fits a sample. This identifies most flexible family, not necessarily family from which samples were drawn.

- L-moment diagrams

  focus on character of sample statistics which describe the "parent" distribution for the phenomena of interest.