Coastal Structures ('07 Fall)
Final Exam (12/18 12:00 – 12/19 10:00)

You have to solve this exam by yourself without talking to other persons.

1. (30) We would like to show that the probability density function of wave height H is given by the Rayleigh probability density function for waves of narrow-band normal (or Gaussian) process. A narrow-band random process is defined as one whose spectral density function is sharply concentrated in the neighborhood of a certain frequency ω_0. This implies that the random process $x(t)$ has a constant frequency, and they may be written as

$$x(t) = A(t)\cos[\omega_0 t + \varepsilon(t)] \tag{1}$$

where $A(t)$ is the amplitude and $\varepsilon(t)$ is the phase; both are random variables.

On the other hand, assuming that the random process $x(t)$ is a normal random process with zero mean and variance σ^2, $x(t)$ may be written as

$$x(t) = \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t) \tag{2}$$

where

$$a_n = \frac{2}{T}\int_0^T x(t)\cos n\omega t dt \tag{3}$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin n\omega t dt \tag{4}$$

Here, the coefficients a_n and b_n are normally distributed with zero mean and variance σ^2.

(a) By writing $n\omega t$ as $(n\omega - \omega_0)t + \omega_0 t$, express $x(t)$ in the form

$$x(t) = x_c(t)\cos\omega_0 t - x_s(t)\sin\omega_0 t \tag{5}$$

and obtain the expressions of $x_c(t)$ and $x_s(t)$.

(b) Obtain the expressions of $x_c(t)$ and $x_s(t)$ in terms of $A(t)$ and $\varepsilon(t)$.

(c) We may write $x_c(t)$, $x_s(t)$, $A(t)$ and $\varepsilon(t)$ as the random variables x_c, x_s, A and ε for a given time t. Since x_c and x_s are the summations of normal random variables, they are also normally distributed. It can be proved that x_c and x_s are statistically independent normal random variables with zero mean and variance which is equal to twice the area under the spectral density function of $x(t)$.

That is,

$$E[x_c] = E[x_s] = 0 \tag{6}$$

$$E[x_c x_s] = 0 \tag{7}$$

$$E[x_c^2] = E[x_s^2] = \sigma^2 = \int_0^\infty S(\omega)d\omega \tag{8}$$

where $S(\omega)$ = spectral density function of $x(t)$. For proofs, refer to Davenport and Root (1958)[1].

When the random variables X and Y are the functions of other random variables x and y, and the joint probability density function of X and Y, $f(X,Y)$, is known, it can be transformed to the joint probability density function of x and y, $f(x,y)$, by the following relationship:

$$f(x,y) = [f(X,Y)]_{\substack{X=fn(x,y)\\ Y=fn(x,y)}} \begin{vmatrix} \dfrac{\partial X}{\partial x} & \dfrac{\partial X}{\partial y} \\ \dfrac{\partial Y}{\partial x} & \dfrac{\partial Y}{\partial y} \end{vmatrix} \tag{9}$$

We may write the joint probability density function x_c and x_s as

$$f(x_c, x_s) = \frac{1}{2\pi\sigma^2} e^{-A^2/2\sigma^2} \qquad -\infty < x_c < \infty \qquad -\infty < x_s < \infty \tag{10}$$

Obtain the joint probability density function $f(A,\varepsilon)$ using the above transformation.

(d) Obtain the marginal probability density function $f(A)$ by integrating $f(A,\varepsilon)$ from 0 to 2π with respect to ε.

(e) Express $f(A)$ in terms of H and H_{rms} and compare it with the Rayleigh probability density function $f(H)$ given by Eq. (7.18) in Dean & Dalrymple's book.

[1] Davenport, W. B. and Root, W. L. (1958). An introduction to the theory of random signals and noise. McGraw-Hill, New York.

2. (40) Two gages will be used to separate the incident and reflected waves in front of a structure as shown in Fig. 1 where $\Delta\ell$ = unknown distance between the two gages; h = water depth; $S_i(f)$ = incident wave spectrum with f = frequency; H_s = incident significant wave height; and T_s = incident significant wave period. The wavemaker is assumed to be ideal and generate $S_i(f)$ as specified as input to the wavemaker. The unknown gage spacing $\Delta\ell$ may be selected to maximize the incident wave energy resolved by the two gages.

$$\text{maximize} \quad E = \int_{f_{\min}}^{f_{\max}} S_i(f)df \quad \text{with respect to} \quad \Delta\ell \tag{1}$$

where $f_{\max}$ and $f_{\min}$ are the maximum and minimum frequencies that can be resolved by the two gages on the basis of the recommendation in Goda's book (p 359).

To simplify the following analysis, $S_i(f)$ is assumed to be given by the Bretschneider-Mitsuyasu spectrum (Eq. (2.10) in Goda's book). Furthermore, the frequencies $f_{\max}$ and $f_{\min}$ are assumed to correspond to deep-water and shallow-water waves, respectively.

$$\tanh(k_{\max}h) \cong 1; \quad \tanh(k_{\min}h) \cong k_{\min}h \tag{2}$$

where $k_{\max}$ and $k_{\min}$ are the wave numbers corresponding to $f_{\max}$ and $f_{\min}$, respectively.

(a) Derive an equation based on Eq. (1) for the optimal gage spacing $\Delta\ell$. Show that the value of $\Delta\ell^* = \Delta\ell/(T_s\sqrt{gh})$ is uniquely determined for given $L_s^* = T_s\sqrt{g/h}$.

(b) Find the value of $\Delta\ell$ for h = 0.4 m and T_s = 2 s and show that the assumptions given in Eq. (2) are satisfied approximately. You may need the Newton-Raphson iteration method.

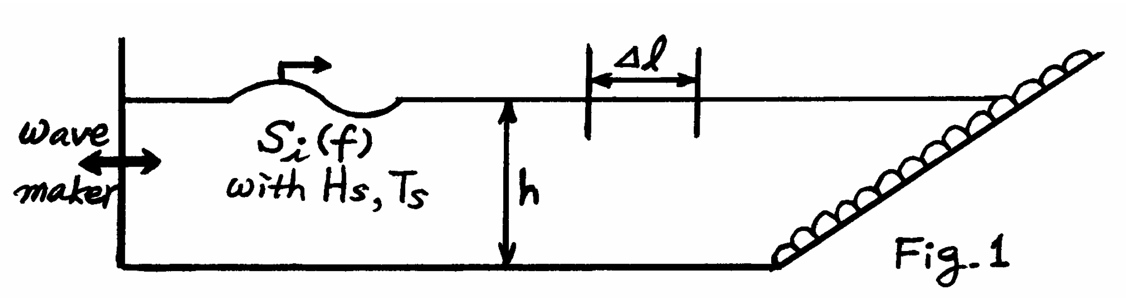

Fig. 1

3. (30) A structure is subjected to a characteristic load S that has a mean value μ_s = 80 kN and is normally distributed with a standard deviation σ_s = 18 kN. For the type of structure we are designing, the characteristic resistance R is known to have σ_r = 20 kN.

(a) We want to design the structure with the safety factor $\Gamma = \gamma_r \gamma_s = 1.0$, $Z_s = 0$ and $Z_r = -1.64$, so that the mean load can be resisted by the structure 95% of the time, where $Z_s = (S - \mu_s)/\sigma_s$, $Z_r = (R - \mu_r)/\sigma_r$, and γ_r and γ_s are the partial safety factors for resistance and load, respectively. Describe the design equation in the partial safety factor system. Calculate the mean resistance μ_r and the probability of failure P_f.

(b) We want to design the structure so that the probability of failure is the same as that calculated in part (a) but the mean load can be resisted by the structure 50% of the time. Calculate the safety factor Γ.

(c) In the case of $\Gamma = 1.3$, $Z_s = 0$ and $Z_r = -1.64$, calculate the mean resistance μ_r and the probability of failure P_f.