Eng Math. Mid Term (4/23/2008)

(Closed book and note: 120 min.)

1. Solve the differential equation [points]:

   a) $\frac{dy}{dx} - \frac{y}{x} = x^5, x > 0$

   Sol) $-\frac{y}{x} + \frac{1}{5}x^5 = C$

   b) $xdy - ydx - y^2dx = 0$

   Sol) By dividing with y^2, the equation above simply becomes $-d\left(\frac{x}{y}\right) - dx = 0$

   By integration, $y = \frac{x}{c - x}$

2. Solve the nonhomogeneous equation [points]:

   a) $y'' + y = \sum_{k=1}^{n} a_k e^{kx}$

   Sol)

$$y_k = Ae^{kx}$$

$$y'' + y = a_k e^{kx}, A = \frac{a_k}{1+k^2}$$

$$y_k = \frac{a_k e^{kx}}{1+k^2}$$

$$\therefore y = \sum_{k=1}^{n} y_k = \sum_{k=1}^{n} \frac{a_k e^{kx}}{1+k^2}$$

b) $x^3 y''' - 3x^2 y'' + 6xy' - 6y = x^4 \ln x$

Sol) p.119

3. Prove that $e^{\lambda x}, xe^{\lambda x}, x^2 e^{\lambda x} ..., x^{m-1} e^{\lambda x}$ will be the linearly independent solutions of a homogeneous linear ODE with constant coefficients (order n) if the characteristic equation of the high order ODE has a multiple real root of order m (note: n>m) [20 points]

Sol)
$L[y]=[D^n + a_{n-1}D^{n-1} + . . . + a_0]y$
For $y=e^{\lambda x}$, $L[e^{\lambda x}]=[\lambda^n + a_{n-1}\lambda^{n-1} + . . . + a_0]e^{\lambda x}$
Let λ_1 be an *m*th order of the polynomial, then
$L[e^{\lambda x}]=(\lambda - \lambda_1)^m h(\lambda)\, e^{\lambda x}$
Differentiate with respect to λ

$$\frac{\partial}{\partial \lambda} L[e^{\lambda x}] = m(\lambda-\lambda_1)^{m-1}\, h(\lambda)\, e^{\lambda x} + (\lambda-\lambda_1)^m \frac{\partial}{\partial \lambda}[h(\lambda)e^{\lambda x}] \quad \text{----------} \; 1)$$

$$\frac{\partial}{\partial \lambda} L[e^{\lambda x}] = L[\frac{\partial}{\partial \lambda} e^{\lambda x}] = L[xe^{\lambda x}]$$

When $\lambda=\lambda_1$, eqn 1) is equal to zero, and $xe^{\lambda_1 x}$ is a solution of $[D^n + a_{n-1}D^{n-1}+ . . . + a_0]y= 0$

we can repeat this step and produce $x^2e^{\lambda_1 x}$, . . . , $x^{m-1}e^{\lambda_1 x}$ by another m–2 such diffentiations!

4. Solve the systems of ODE by finding the eigenvalues and eigenvectors of the

coefficient matrix: $\underline{X}' = \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix} \underline{X} + \begin{bmatrix} 3t \\ e^{-t} \end{bmatrix}$ [points]

Sol)

$$\det(\underline{A} - \lambda \underline{I}) = \begin{vmatrix} -3-\lambda & 1 \\ 2 & -4-\lambda \end{vmatrix} = 0,\ \lambda = -2 \text{ or } -5$$

And the corresponding eigenvectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$, respectively.

So, the solution vectors of the system are then $\underline{X}_1 = \begin{bmatrix} e^{-2t} \\ e^{-2t} \end{bmatrix}$ and $\underline{X}_2 = \begin{bmatrix} e^{-5t} \\ -2e^{-5t} \end{bmatrix}$.

Let $\underline{X}^{(h)} = \underline{\Phi}(t) \cdot \underline{c}$, then $\underline{X}^{(h)} = \underline{\Phi}(t) \cdot \underline{c} = \begin{bmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

When $\underline{X}^{(p)} = \underline{\Phi}(t) \cdot \underline{u}$, then $\underline{u}' = \underline{\Phi}^{-1}(t) \cdot \begin{bmatrix} 3t \\ e^{-t} \end{bmatrix}$ by substituting into the problem.

So, $\underline{X}^{(p)} = \begin{bmatrix} \frac{6}{5}t - \frac{27}{50} + \frac{e^{-t}}{4} \\ \frac{3}{5}t - \frac{21}{50} + \frac{e^{-t}}{2} \end{bmatrix}$.

As a result, $\underline{X} = \begin{bmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} \frac{6}{5}t - \frac{27}{50} + \frac{e^{-t}}{4} \\ \frac{3}{5}t - \frac{21}{50} + \frac{e^{-t}}{2} \end{bmatrix}$

April 30, 2008

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ENGINEERING MATHEMATICS

Mid Term Exam

Solution key

5. (20 points)

$$m\frac{dv}{dt} = mg - kv^2 \quad w/ \quad v(0) = v_0$$

let $\quad p = \sqrt{\frac{mg}{k}}$

$$\therefore \frac{dv}{dt} = -\frac{g}{p^2}\left(v^2 - p^2\right)$$

by separation of variables (SOV):

$$\frac{dv}{v^2 - p^2} = -\frac{g}{p^2}\,dt$$

$$\frac{1}{2p}\left[\frac{dv}{(v-p)} - \frac{dv}{(v+p)}\right] = -\frac{g}{p^2}\,dt$$

by integration:

$$\frac{1}{2p}\left\{\ln(v-p) - \ln(v+p)\right\} = -\frac{g}{p^2}t$$

$$\ln\left(\frac{v-p}{v+p}\right) = -\frac{2g}{p}t + C_1 \text{ (integration constant)}$$

or $$\frac{v-p}{v+p} = Ce^{-\left(\frac{2g}{p}\right)t}$$

$$\therefore v(t) = \boxed{p\cdot\frac{1+Ce^{-\frac{2g}{p}t}}{1-Ce^{-\frac{2g}{p}t}}} \leftarrow \text{Ans.}$$

physical meaning:

as $t \to \infty$

$v \to p$

(terminal velocity)

From IC: $v(0) = v_0$

$$v_0 = p\cdot\frac{1+C}{1-C} \quad \therefore \boxed{C = \frac{v_0 - p}{v_0 + p}} \leftarrow \text{Ans.}$$

6. (20 points)

$$r^2 \frac{d^2V}{dr^2} + 2r\frac{dV}{dr} = 0 \qquad \frac{dV}{dr} = V' \quad \frac{d^2V}{dr^2} = V''$$

$$\boxed{r^2 V'' + 2r V' = 0}$$

$$V = r^m \qquad V' = m r^{m-1} \quad V'' = m(m-1) r^{m-2}$$

$m(m-1) + 2m = 0$: Auxiliary Eqn

$m(m+1) = 0 \quad \therefore m = 0 \quad m = -1$

$$\therefore V = c_1 + c_2\left(\frac{1}{r}\right)$$

From BC's : $V(a) = V_a$
$V(b) = V_b$

$$V_a = c_1 + c_2\frac{1}{a} \qquad c_1 = \frac{-\frac{1}{b}V_a + \frac{1}{a}V_b}{1/a - 1/b}$$

$$V_b = c_1 + c_2\frac{1}{b} \qquad c_2 = \frac{V_a - V_b}{\frac{1}{a} - \frac{1}{b}}$$

$$\therefore V = \frac{-\frac{1}{b}V_a + \frac{1}{a}V_b}{\frac{1}{a} - \frac{1}{b}} + \frac{\frac{1}{r}(V_a - V_b)}{\frac{1}{a} - \frac{1}{b}}$$

$$\therefore V\left(\frac{1}{a} - \frac{1}{b}\right) = -\frac{1}{b}(V_a - V_b) - \frac{1}{b}V_b + \frac{1}{a}V_b + \frac{1}{r}(V_a - V_b)$$

$$V\left(\frac{1}{a} - \frac{1}{b}\right) = (V_a - V_b)\left(\frac{1}{r} - \frac{1}{b}\right) + V_b\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$\therefore (V - V_b)\left(\frac{1}{a} - \frac{1}{b}\right) = (V_a - V_b)\left(\frac{1}{r} - \frac{1}{b}\right)$$

$$\therefore \boxed{\frac{V - V_b}{V_a - V_b} = \frac{\frac{1}{r} - \frac{1}{b}}{\frac{1}{a} - \frac{1}{b}}} \leftarrow \text{Ans.} \qquad \text{or } \boxed{\frac{V - V_a}{V_a - V_b} = \frac{\frac{1}{r} - \frac{1}{a}}{\frac{1}{a} - \frac{1}{b}}}$$

7. (30 points)

$$x^3 y''' + x y' - y = x^2 \qquad y(1) = 2$$

Nonhomogeneous, Linear $\qquad y'(1) = 8$

3rd order ODE $\qquad y''(1) = 3.$

$y = x^m$

$m(m-1)(m-2) + m - 1 = 0$ Auxiliary eq'n

$(m-1)\{m^2 - 2m + 1\} = 0$

$(m-1)^3 = 0$ triple root!!

$\therefore y_h = \boxed{\{c_1 + c_2 \ln x + c_3 (\ln x)^2\} x}$

$y_p = Ax^n$ by standard rule

$\{n(n-1)(n-2) + (n-1)\} A x^n = (n-1)^3 A x^n = x^2$

$\therefore n = 2, \; A = 1$

$\therefore y_p = \boxed{x^2}$

$\therefore y = y_h + y_p = \{c_1 + c_2 \ln x + c_3 (\ln x)^2\} x + x^2$

(by linearity)

$\therefore y'(x) = \left\{\frac{c_2}{x} + 2c_3 \frac{\ln x}{x}\right\} x + c_1 + c_2 \ln x + c_3 (\ln x)^2 + 2x$

$y''(x) = \frac{2c_3}{x} + \frac{c_2}{x} + 2c_3 \frac{\ln x}{x} + 2$

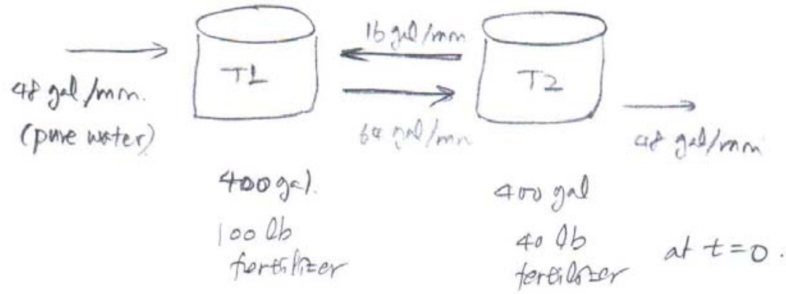
$y(1) = c_1 + 1 = 2 \quad \boxed{c_1 = 1}$

$y'(1) = c_2 + c_1 + 2 = 8 \quad \boxed{c_2 = 5}$

$y''(1) = 2c_3 + c_2 + 2 = 3 \quad \boxed{c_3 = -2}$

$\therefore y = \boxed{x + 5x \ln x - 2x (\ln x)^2 + x^2}$ ← Ans.

8. (30 points)



$$y_1' = \frac{16}{400} y_2 - \frac{64}{400} y_1$$

time change of fertilizer content in T1

$$y_2' = \frac{64}{400} y_1 - \frac{48}{400} y_2 - \frac{16}{400} y_2$$

$$\therefore \quad y_1' = \frac{4}{100} y_2 - \frac{16}{100} y_1 \qquad \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -0.16 & 0.04 \\ 0.16 & -0.16 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$y_2' = \frac{16}{100} y_1 - \frac{16}{100} y_2 \qquad \underline{y}' = \underline{\underline{A}}\, \underline{y}$$

try $\boxed{\underline{y} = \underline{x} e^{\lambda t}}$

$$\underline{y}' = \lambda \underline{x} e^{\lambda t} = \underline{\underline{A}}\, \underline{x} e^{\lambda t} \quad \therefore \quad \boxed{\underline{\underline{A}}\, \underline{x} = \lambda \underline{x}}$$: eigenvalue problem

where $\underline{\underline{A}} = \begin{bmatrix} -0.16 & 0.04 \\ 0.16 & -0.16 \end{bmatrix}$

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \begin{vmatrix} -0.16-\lambda & 0.04 \\ 0.16 & -0.16-\lambda \end{vmatrix} = (\lambda+0.16)^2 - 0.04 \cdot 0.16$$

$$= \lambda^2 + 0.32\lambda + 0.16 \cdot 0.12$$

$\boxed{\lambda_1 = -0.08 \quad \lambda_2 = -0.24}$; eigenvalues $\quad = (\lambda+0.08)(\lambda+0.24) = 0$

for $\lambda_1 = -0.08$

$$-0.08 y_1 + 0.04 y_2 = 0 \qquad y_1 = 1 \ \& \ y_2 = 2 \quad \therefore \underline{y}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

for $\lambda_2 = -0.24$ $\quad +0.08 y_1 + 0.04 y_2 = 0 \qquad y_1 = 1 \ \& \ y_2 = -2 \quad \underline{y}^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

eigenvectors

$$\therefore \boxed{\underline{y} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-0.08t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-0.24t}} \leftarrow \text{Ans}$$

ICs $y_1(0) = 100$ & $y_2(0) = 40$

$$\underline{y}(0) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ 2c_1 - 2c_2 \end{bmatrix} = \begin{bmatrix} 100 \\ 40 \end{bmatrix}$$

$$\begin{aligned} & c_1 + c_2 = 100 \\ +)\ & c_1 - c_2 = 20 \\ \hline & 2c_1 = 120 \end{aligned} \qquad c_1 = 60 \text{ \& } c_2 = 40$$

$$\therefore \underline{y} = \boxed{60 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-0.08t} + 40 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-0.24t}} \leftarrow \text{Ans.}$$