**<2009 Final Exam Solution>**

**Problem 1 (2pi each)**

(a) The purpose of Monte Carlo simulation is to obtain input probabilities from output distribution: __**T**__ (True or False)

(b) The pseudo-random numbers are repeated with a pattern: __**T**__ (True or False)

(c) The reliability index is the coefficient of variation of the safety margin: __**F**__ (True or False)

(d) The dependency between capacity and demand becomes more important when the capacity varies more than the demand when the other characteristics are same: __**F**__ (True or False) (Fig 9.1.10)

(e) The *k*-out-of-*m* system is always less reliable than the corresponding parallel system (i.e. *k*=1): __**T**__ (True or False) (Eq. 9.2.12)

**Problem 2 (5pt each)**

A charcoal filter can remove chlorinated hydrocarbons from drinking water. An environmental engineer collected 12 independent samples if water that passed through such a filter. The average of the 12 observed removal rates was $\overline{X}$= 99.27% with sample standard deviation $s$= 0.25%. Assuming that the measurements are normally distributed about the true removal rate $\mu$

(a) What is 99% confidence interval of $\mu$?

**(Solution)**

For $n$= 12, df= 11, $t_{0.005,11}$= 3.106

so a 99% CI for the unknown mean is: $\overline{X} \pm t_{0.005,11} \times s/\sqrt{n}$ → 99.05% to 99.49%

(b) Does the interval computed above indicate that precise information about the mean $\mu$ is available?

**(Solution)**

We are 99% confident that our computed interval contains the true mean, and the interval is narrow, therefore, our estimate of $\mu$ is relatively precise.

(c) The engineer plans to use this same sampling procedure on 12 other filters which her supervisor will specify. What is the probability that all 12 of the intervals that will be generated will contain the true means for the respective roads?

**(Solution)**

Probability that all 12 randomly constructed CI's contain respective true means is $(0.99)^{12}$= 0.886

**Problem 3 (20pt)**

In studying the effect of air quality on a lake, an experimenter takes observations on the pH of the water and the air quality as measured on an air quality index. The index goes from 0 to 100 with larger numbers representing high pollution. It is expected that the pH will decrease as the air quality index increases indicating more pollution. Here is a summary of the data obtained:

| | | | | | |
|---|---|---|---|---|---|
| Lake Water pH, $Y_i$: | 4.5 | 4.1 | 4.8 | ... | 6.1 |
| Air Quality Index, $X_i$: | 40 | 50 | 30 | ... | 15 |

$n = 10$ $\quad \overline{X} = 41$ $\quad s_X = 24.8$ $\quad \overline{Y} = 4.61$ $\quad s_Y = 0.96$ $\quad \sum(X_i - \overline{X})(Y_i - \overline{Y}) = -205.6$

(a) Compute an unbiased estimate of the variance of the $\varepsilon$ (4pt)

**(Solution)**

$s_\varepsilon^2 = [9 \times (0.96^2) - (-0.037) \times (-205.6)]/8 = 0.086 = (0.293)^2$

(b) What fraction of the observed variability in the water pH is explained by the model? (3pt)

**(Solution)**

$R^2 = 1 - (n-k)\, s_\varepsilon^2 \;/\; [(n-1)\, S_{yy}^2\,] = 1 - 8 \times (0.086) \;/\; [9 \times (0.96^2)] = 91.7\%$

$R^2$ never decreases when variables are added to the model. But we want a parsimonious model. The adjusted-$R^2$ corrects for the number of coefficients estimated and only increases if additional variables really add to the model's explanatory power.

(c) What is the standard deviation of estimator of $\beta$? (4pt)

**(Solution)**

Standard deviation of b= $\sqrt{}[\, s_\varepsilon^2 \;/\; \{(n-1) \times S_{XX}^2\,] = \sqrt{}[\, 0.086 \;/\; \{9 \times 242.8^2\,] = 0.004$

(d) The concern here is a relationship between air quality and water pH. What is the rejection region for a 5% test of whether or not $\beta$ is zero? Why have you selected either a one-sided or two-sided test? (5pt)

**(Solution)**

Reject $H_0$: $\beta = 0$ versus

$H_a$: $\beta < 0$ if $t_{test} \leq t_{0.05,\ 8} = 2.042$

Use a one-sided lower tail test because believe the water pH will decrease as the air quality index increase → inverse relationship.

$t_{test}$ = (-0.037 - 0)/0.004= -9.25, p-value= $P[T_8 \leq -9.25] = 0$

(e) What is a 98% prediction interval for the true value of the pH given when the air quality index is 23? (4pt)

**(Solution)**

98% PI for x*= 23: $\hat{\alpha} + \hat{\beta}x^* \pm t_{0.01,8}s_e\sqrt{[\ 1 + 1/n + (x^* - \bar{x})^2/\{(n-1)s_x^2\}\ ]}$

$\rightarrow 6.13 + (-0.037)(23) \pm 2.896(0.293)\sqrt{[\ 1 + 1/10 + (23 - 41)^2/\{(9)24.8^2\}\ ]}$

$\rightarrow$ 4.37 to 6.19

**Problem 4 (20pt)**

An engineer wants to determine the mean of an output with a given accuracy $1-\alpha$ and thus to generate a set of random numbers.

(a) White $1-\alpha$ confidence limits on the population mean $\mu$ using the normal approximation. (4pt)

**(Solution)**

$P[-z_{1-\alpha/2}\ \sigma(O)/\sqrt{m} < \bar{O} - \mu(O) < z_{1-\alpha/2}\ \sigma(O)/\sqrt{m}] = 1-\alpha$

where $\bar{O}$= sample mean of the output ; $\sigma(O)$= population variance

$z_{1-\alpha/2}$ = $1-\alpha/2$ quantile of the normal distribution with mean zero and variance one

$1-\alpha$= confidence level ; $m$= sample size

(b) If the sample mean of the output must be within $0.2\sigma$ of $\mu$ with 99%, at least how many samples should be generated? (5pt)

**(Solution)**

$z_{1-0.01/2}\ \sigma(O)/\sqrt{m} = 0.2\sigma(O)$ ; $2.576 / \sqrt{m} = 0.2 \rightarrow m = 166$

(c) Should he generate more or less samples if he wants a larger confidence interval when its accuracy remain same? Make your comment on why. (3pt)

**(Solution)**

In $P[-k_{\alpha/2}\ \sigma(O)/\sqrt{m} < \bar{O} - \mu(O) < k_{\alpha/2}\ \sigma(O)/\sqrt{m}] = 1-\alpha$

If he wants a larger confidence interval when its accuracy remain same, he generate less samples.
To large confidence interval (i.e. $-k_{\alpha/2}\ \sigma(O)/\sqrt{m} \rightarrow \downarrow$ ; $k_{\alpha/2}\ \sigma(O)/\sqrt{m} \rightarrow \uparrow$), the size of $m$ should be smaller than before.

(d) Should he generate more or less samples if he wants a lower accuracy when he wants to remain the same confidence interval? Make your comment on why. (3pt)

**(Solution)**

In $P[-k_{\alpha/2}\ \sigma(O)/\sqrt{m} < \bar{O} - \mu(O) < k_{\alpha/2}\ \sigma(O)/\sqrt{m}] = 1-\alpha$

If he wants a lower accuracy when he wants to remain the same confidence interval, he generate less samples.
To lower the accuracy (i.e. $1-\alpha \rightarrow \downarrow$ ; $\alpha \rightarrow \uparrow$; $k_{\alpha/2} \rightarrow \downarrow$; $k_{\alpha/2}\ \sigma(O)/\sqrt{m} \rightarrow \downarrow$), the size of $m$ should be smaller than before.

**<u>Problem 5 (10pt)</u>**

Consider a timber beam subject to flexure. The stress at the extreme fiber at a distance $X_2$ from the neutral axis acted upon by a bending moment $X_3$ is given by $X_2X_3/X_4$, where $X_4$ denotes the moment of inertia of the section. Assume that the factors are normal variates as follows:

| Factor | Expected value | Coefficient of variation |
|---|---|---|
| Bending moment, $X_3$ | 6 kN□cm | 0.25 |
| Moment of inertia, $X_4$ | 90 $cm^4$ | 0.10 |
| Distance from neutral axis, X2 | 20 cm | 0.05 |

Further assume that $X_2$, $X_3$, and $X_4$ are independent of each other. Find the reliability of the system if the capacity $X_1$ of the beam is a normal variate with a mean of 4 $kN/cm^2$ and coefficient of variation 30 percent.

**(Solution)**

$g(X1,\ X2,\ X3,\ X4) = X_1 - X_2X_3/X_4$

| $i$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\mu_i$ | 4 | 20 | 6 | 90 |
| $\sigma_i$ | 1.2 | 1 | 1.5 | 9 |

$\partial g/\partial X_i' = (\partial g/\partial X_i)\,\sigma_i$

$(\partial g/\partial X_1')_f = \sigma_1 = 1.2$; $(\partial g/\partial X_2')_f = \sigma_2(-X_3/X_4) = -0.0667$; $(\partial g/\partial X_3')_f = \sigma_3(-X_2/X_4) = -0.333$
$(\partial g/\partial X_4')_f = \sigma_4(-X_2X_3/X_4^2) = 0.133$

$\alpha_1 = 1.2/\sqrt{(1.2^2+0.0667^2+0.333^2+0.133^2)} = 0.957$;
$\alpha_2 = -0.0667/\sqrt{(1.2^2+0.0667^2+0.333^2+0.133^2)} = -0.0532$;
$\alpha_3 = -0.333/\sqrt{(1.2^2+0.0667^2+0.333^2+0.133^2)} = -0.266$;
$\alpha_4 = 0.133/\sqrt{(1.2^2+0.0667^2+0.333^2+0.133^2)} = 0.106$

$x_{1(new)} = \mu_1 - \sigma_1\alpha_1\beta = 4 - 1.15\beta$ $x_{2(new)} = \mu_2 - \sigma_2\alpha_2\beta = 20 + 0.0532\beta$ $x_{3(new)} = \mu_3 - \sigma_3\alpha_3\beta = 6 + 0.399\beta$
$x_{4(new)} = \mu_4 - \sigma_4\alpha_4\beta = 90 - 0.957\beta$

Limit State Equation: $(4 - 1.15\beta) - (20 + 0.0532\beta) \times (6 + 0.399\beta) / (90 - 0.957\beta) = 0$

**$\beta = 2.12$**

2nd iteration

$x_{1(new)} = \mu_1 - \sigma_1\alpha_1\beta = 4 - 1.15\times 2.12 = 1.56$
$x_{2(new)} = \mu_2 - \sigma_2\alpha_2\beta = 20 + 0.0532\times 2.12 = 20.1$
$x_{3(new)} = \mu_3 - \sigma_3\alpha_3\beta = 6 + 0.399\times 2.12 = 6.85$
$x_{4(new)} = \mu_4 - \sigma_4\alpha_4\beta = 90 - 0.957\times 2.12 = 88$

$(\partial g/\partial X_1')_f = \sigma_1 = 1.2$; $(\partial g/\partial X_2')_f = \sigma_2(-X_3/X_4) = -0.778$; $(\partial g/\partial X_3')_f = \sigma_3(-X_2/X_4) = -0.343$
$(\partial g/\partial X_4')_f = \sigma_4(-X_2X_3/X_4^2) = 0.160$

$\alpha_1 = 1.2/\sqrt{(1.2^2+0.0667^2+0.333^2+0.133^2)} = 0.952$;
$\alpha_2 = -0.778/\sqrt{(1.2^2+0.0667^2+0.333^2+0.133^2)} = -0.0617$;
$\alpha_3 = -0.343/\sqrt{(1.2^2+0.0667^2+0.333^2+0.133^2)} = -0.272$;
$\alpha_4 = 0.160/\sqrt{(1.2^2+0.0667^2+0.333^2+0.133^2)} = 0.127$

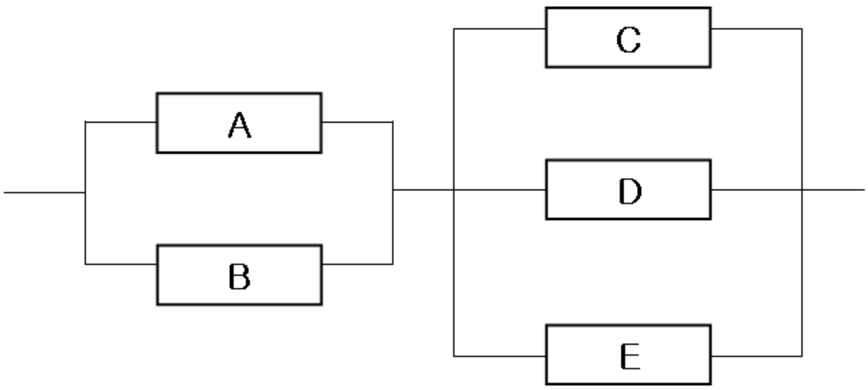
Limit State Equation: $(4 - 1.146\beta) - (20 + 0.0617\beta) \times (6 + 0.408\beta) / (90 - 1.14\beta) = 0$

**$\beta$= 2.12**

**$r$= $\Phi$(2.12)= 0.983**

**Problem 6 (25pt)**

Consider a pump system shown below. The risk of each pump of A, B, C, and D is 10%.




(a) What is the reliability of the entire pumping system? (7pt)

**(Solution)**

Reliability= $(1 - 0.1^2) \times (1 - 0.1^3) = 0.989$

(b) If we need to deliver more water and thus at least two of the right-sided parallel system should be running, how is the reliability of the entire system changed? (8pt)

**(Solution)**

Reliability= $(1 - 0.1^2) \times \{\sum_{x=2}^{3} \binom{3}{x} (1-p)^x p^{3-x}\}$= 0.962

(c) Now, you have to remove one pump from either of the left or the right parallel system. Which pump would you refer to remove in the system and why? (0pt)

Left remove: Reliability= $\{\sum_{x=1}^{2}\binom{2}{x}(1-p)^{x}p^{2-x}\}$ × (1 - $0.1^3$)= 0.899

Right remove: Reliability= (1 - $0.1^2$) × (1 - $0.1^2$)= 0.980

Therefore, you refer to remove one of right pumps in the system