# Midterm Exam with answers

October 25, 2010

1. Solve the problems using the tensor notation (n = 3).
   1) Expand $c_i(x_i + y_i)$. [2]

      $$c_i(x_i + y_i) = c_i x_i + c_i y_i = c_1x_1 + c_2x_2 + c_3x_3 + c_1y_1 + c_2y_2 + c_3y_3$$

   2) Show that $x \times y = -y \times x$. [3]

      $$x \times y = \varepsilon_{ijk}\ x_j y_k = \varepsilon_{i1k}x_1y_k + \varepsilon_{i2k}x_2y_k + \varepsilon_{i3k}x_3y_k$$
      $$= \varepsilon_{i12}x_1y_2 + \varepsilon_{i13}x_1y_3 + \varepsilon_{i21}x_2y_1 + \varepsilon_{i23}x_2y_3 + \varepsilon_{i31}x_3y_1 + \varepsilon_{i32}x_3y_2$$
      $$= \varepsilon_{312}x_1y_2 + \varepsilon_{213}x_1y_3 + \varepsilon_{321}x_2y_1 + \varepsilon_{123}x_2y_3 + \varepsilon_{231}x_3y_1 + \varepsilon_{132}x_3y_2$$
      $$= x_1y_2 - x_1y_3 - x_2y_1 + x_2y_3 + x_3y_1 - x_3y_2$$
      $$-y \times x = -\varepsilon_{ijk}\ y_j x_k = -\varepsilon_{i1k}y_1x_k - \varepsilon_{i2k}y_2x_k - \varepsilon_{i3k}y_3x_k$$
      $$= -\varepsilon_{i12}y_1x_2 - \varepsilon_{i13}y_1x_3 - \varepsilon_{i21}y_2x_1 - \varepsilon_{i23}y_2x_3 - \varepsilon_{i31}y_3x_1 - \varepsilon_{i32}y_3x_2$$
      $$= -\varepsilon_{312}y_1x_2 - \varepsilon_{213}y_1x_3 - \varepsilon_{321}y_2x_1 - \varepsilon_{123}y_2x_3 - \varepsilon_{231}y_3x_1 - \varepsilon_{132}y_3x_2$$
      $$= -y_1x_2 + y_1x_3 + y_2x_1 - y_2x_3 - y_3x_1 + y_3x_2$$

   3) Show that $\varepsilon_{ijk}\ a_j a_k = 0$. [3]

      $$\varepsilon_{ijk}\ a_j a_k = \varepsilon_{i1k}a_1a_k + \varepsilon_{i2k}a_2a_k + \varepsilon_{i3k}a_3a_k$$
      $$= \varepsilon_{i12}a_1a_2 + \varepsilon_{i13}a_1a_3 + \varepsilon_{i21}a_2a_1 + \varepsilon_{i23}a_2a_3 + \varepsilon_{i31}a_3a_1 + \varepsilon_{i32}a_3a_2$$
      $$= \varepsilon_{312}a_1a_2 + \varepsilon_{213}a_1a_3 + \varepsilon_{321}a_2a_1 + \varepsilon_{123}a_2a_3 + \varepsilon_{231}a_3a_1 + \varepsilon_{132}a_3a_2$$
      $$= a_1a_2 - a_1a_3 - a_2a_1 + a_2a_3 + a_3a_1 - a_3 = 0$$

   4) Show that $\varepsilon_{ijk}\ \delta_{jk} = 0$. [2]

      $$\varepsilon_{ijk}\ \delta_{jk} = \varepsilon_{i1k}\delta_{1k} + \varepsilon_{i2k}\delta_{2k} + \varepsilon_{i3k}\delta_{3k}$$
      $$= \varepsilon_{i12}\delta_{12} + \varepsilon_{i13}\delta_{13} + \varepsilon_{i21}\delta_{21} + \varepsilon_{i23}\delta_{23} + \varepsilon_{i31}\delta_{31} + \varepsilon_{i32}\delta_{32} = 0$$

2. Let the $x$, $y$, $z$ coordinate axes correspond to the principal axes of stress at a given point in a body, and let $\sigma_1$, $\sigma_2$, $\sigma_3$ be the principal stresses in the $x$, $y$, and $z$ directions, respectively. Assume that the given point is surrounded by an octahedron whose faces have direction cosines as follow.

   $\cos(n,x) = \pm\frac{1}{\sqrt{3}}$, $\cos(n,y) = \pm\frac{1}{\sqrt{3}}$, $\cos(n,z) = \pm\frac{1}{\sqrt{3}}$

   What are the magnitude of normal and shear stresses on each face? [10]

   $$\hat{n} = \pm\frac{1}{\sqrt{3}}(1,1,1) \quad \Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$
   $$t^{\hat{n}} = \pm\frac{1}{\sqrt{3}}(1,1,1)\cdot\Sigma = \pm\frac{1}{\sqrt{3}}(\sigma_1,\sigma_2,\sigma_3)$$
   $$\sigma_n = t^{\hat{n}}\cdot\hat{n} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$
   $$\overrightarrow{\tau_n} = t^{\hat{n}} - \sigma_n\hat{n} = \pm\frac{1}{3\sqrt{3}}(2\sigma_1 - \sigma_2 - \sigma_3, -\sigma_1 + 2\sigma_2 - \sigma_3, -\sigma_1 - \sigma_2 + 2\sigma_3)$$

$$\tau_n = \frac{1}{3\sqrt{3}}\sqrt{(2\sigma_1 - \sigma_2 - \sigma_3)^2 + (-\sigma_1 + 2\sigma_2 - \sigma_3)^2 + (-\sigma_1 - \sigma_2 + 2\sigma_3)^2}$$
$$= \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

3. Describe four stages of stress-strain response in uniaxial compression. [10]
   $0 \sim \sigma_a$: crack closure
   $\sigma_a \sim \sigma_{ci}$: elastic deformation
   $\sigma_{ci} \sim \sigma_{cd}$: stable crack propagation
   $\sigma_{cd} \sim \sigma_c$: unstable/irrecoverable deformation

4. Describe Coulomb's shear strength criterion with its merits and demerits. [10]
   $S = c + \sigma_n \tan\phi$ , 장점: simple하고 직관적이다. 단점: 전단파괴만을 가정한다. 파괴면이 이론과 항상 일치하지는 않는다. 실제 실험에 의한 결과에서는 non-linear 파괴포락선을 보인다.

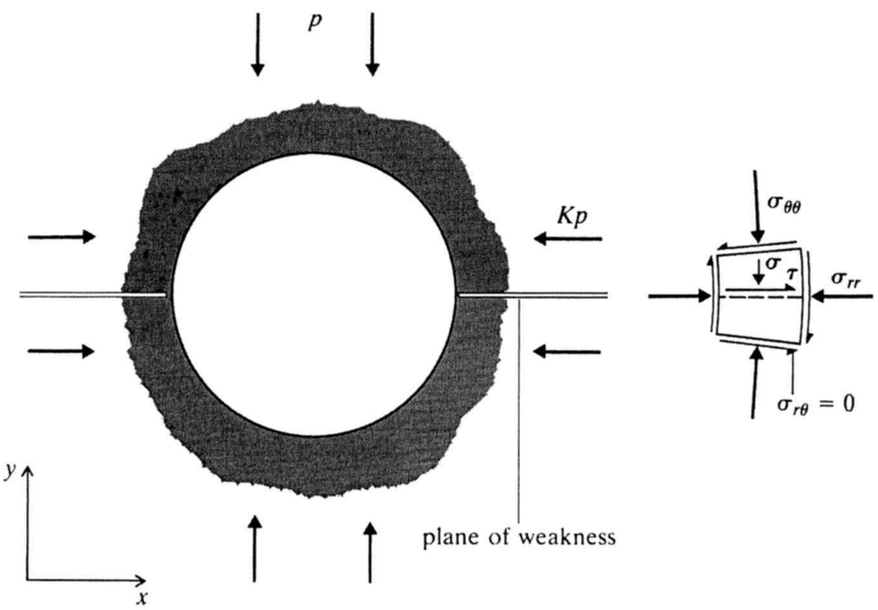
5. A triaxial compression test is to be carried out on a specimen of the granite with a joint plane inclined at 40° to the specimen axis. A confining pressure of $\sigma_3 = 2.0$ MPa and an axial stress of $\sigma_1 = 7.0$ MPa are to be applied. Then a joint water pressure will be introduced and gradually increased with $\sigma_1$ and $\sigma_3$ held constant. The rock joint is assumed to follow Barton's shear strength criterion: $\tau = \sigma_n \tan\left[\phi_r + JRC\ \log_{10}\frac{JCS}{\sigma_n}\right]$ where $\phi_r = 20°$, $JRC = 12$, and $JCS = \sigma_c = 100$ MPa. At what water pressure is slip on the joint expected to occur? [20]

$$\sigma_n = \frac{\sigma_1 + \sigma_3}{2} + \frac{\sigma_1 - \sigma_3}{2}\cos 2\theta = 4.5 + 2.5\cos 100^\circ = 4.07\ (\mathrm{MPa})$$
$$\tau = \frac{\sigma_1 - \sigma_3}{2}\sin 2\theta = 2.5\sin 100^\circ = 2.46\ (\mathrm{MPa})$$
$$\tau = (\sigma_n - u)\tan\left(\phi_r + JRC\log\frac{JCS}{\sigma_n - u}\right)$$
$$2.46 = (4.07 - u)\tan\left(20 + 12\log\frac{100}{4.07 - u}\right)$$

   By trial and error, $u = 0.92$ MPa

6. A circular horizontal opening whose radius is 2 m is to be excavated at depth of 200 m. A horizontal weakness plane whose friction angle is 25° passes through the expected opening center. The unit weight of the rock mass is 25 $kNm^{-3}$, and

the stress ratio, K (horizontal stress/vertical stress), of the site is 3.2. In-situ tests show that principal stresses are directed in vertical and horizontal. The rock mass is assumed to follow Coulomb's strength criterion, and its cohesion and friction angle are 20 MPa, and 35°, respectively. Find out the range of compressive (shear) failure at the opening boundary and width (or height) of a de-stressed zone in the rock mass around the opening. [20]




$\sigma_c = 76.84$ MPa 이고 공동주변 최대 접선응력은 43 MPa로 압축(전단)파괴는 발생하지 않는다.
이완대의 폭은 $\sigma_A = p(1-k+2q) = 5(-2.2+2W/H) = 0$ 에서 W/H = 1.1이므로 W = 4.4 m 가 된다. (ΔW = 0.2 m)

7. Two horizontal circular openings whose radius is 3 m and 2 m, respectively, are parallel to each other at the same depth. The distance between the centers is 10 m. The vertical principal (far-field) stress 20 MPa and the horizontal principal stress is 40 MPa. Let the failure criterion and mechanical properties of the rock mass are the same as those in Problem 6. The opening with 3 m of radius is assumed to have no plastic (failure) zone around it due to TBM excavation and installation of a support system. Describe the condition of failure at the boundary of opening whose radius is 2 m. [20]

2 m 공동중심에서의 응력:

$$\sigma_{rr} = \frac{20}{2}\left[3.0\left(1-\frac{9}{100}\right)+1.0\left(1-4\frac{9}{100}+3\frac{81}{10^4}\right)\right]=33.9MPa$$

$$\sigma_{\theta\theta} = \frac{20}{2}\left[3.0\left(1+\frac{9}{100}\right)-1.0\left(1+3\frac{81}{10^4}\right)\right]=22.4MPa$$

K = 33.9/22.4 = 1.51

$$\sigma_{\theta\theta} = 22.4\left[2.51-1.02\cos 2\theta\right]\geq 76.84$$

$$\cos 2\theta \leq -0.902, \quad 77.2^\circ \leq \theta \leq 102.8^\circ, \quad 257.2^\circ \leq \theta \leq 282.8^\circ$$ (압축파괴구간)

Kirsch equations:

$$\sigma_{rr} = \frac{p}{2}\left[(1+K)\left(1-\frac{a^2}{r^2}\right)-(1-K)\left(1-4\frac{a^2}{r^2}+3\frac{a^4}{r^4}\right)\cos 2\theta\right]$$

$$\sigma_{\theta\theta} = \frac{p}{2}\left[(1+K)\left(1+\frac{a^2}{r^2}\right)+(1-K)\left(1+3\frac{a^4}{r^4}\right)\cos 2\theta\right]$$

$$\sigma_{r\theta} = \frac{p}{2}(1-K)\left(1+2\frac{a^2}{r^2}-3\frac{a^4}{r^4}\right)\sin 2\theta$$

Boundary stresses of an elliptic opening:

$$\sigma_A = p(1-K+2q) = p\left(1-K+\sqrt{\frac{2W}{\rho_A}}\right)$$

$$\sigma_B = p\left(K-1+\frac{2K}{q}\right) = p\left(K-1+K\sqrt{\frac{2W}{\rho_B}}\right)$$

$q = W/H$