Engineering Mathematics

- Final Exam -

Friday, June 15th, 2012

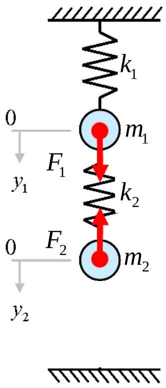
Time: 14:00-17:00 (3 hours)

Name	
SNU ID #	

Note: Budget your time wisely. Some parts of this exam could take you much longer time than others. Move on if you are stuck and return to the problem later.

Problem Number		1				2			3			4			5	
		(a)	(b)	(c)	(d)	(a)	(b)	(c)	(a)	(b)	(c)	(a)	(b)	(c)	(a)	Total
Grade r	max															100

1. (Two masses on springs) The mechanical system in Figure 1 consists of two bodies and two springs. The small masses of the springs are negligible, and damping is assumed to be practically zero.



m_1 : Mass of upper body
m_2 : Mass of lower body
k_1 : Spring coefficient of upper spring
k_2 : Spring coefficient of lower spring
F_1 : External force exerted on upper body
F_2 : External force exerted on lower body

Figure 1 : Mass-spring-damper system

Masses of the upper body and lower body are $1kg$, and spring coefficients of upper and lower springs are $4N/m$. The initial conditions are given by

$y_1(0) = 1,\ y_1{}'(0) = 1,\ y_2(0) = 1,\ y_2'(0) = -1.$

Answer the questions.

(a) Set up the model for the un-damped system in terms of

$$\mathbf{y}'' = \mathbf{A}\mathbf{y} + \mathbf{F}.$$

단, $\mathbf{y} = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$, $\mathbf{y}'' = \begin{bmatrix} y_1''(t) \\ y_2''(t) \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $y_1' = \dfrac{dy_1}{dt}$, $y_2' = \dfrac{dy_2}{dt}$, $\mathbf{F} = \begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix}$ 이다.

(b) Find the homogenous solution for the system of ODEs shown above. In this case, explain why we try an exponential function of t, $\mathbf{y} = \mathbf{x}e^{\lambda t}$

(c) In the above steps of solving for eigenvalue λ, explain why λ is determined from equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. The following words should be included in the explanation.

(Nontrivial solution, Rank, Linear independence)

(d) Find the particular solution. When $F_1(t)$ equals to $11 \sin t$, and $F_2(t)$ equals to $-11 \sin t$.

2. (Stretching of an elastic membrane) There is an elastic membrane on the xy-plane with quadrate boundary shown in Figure 1.

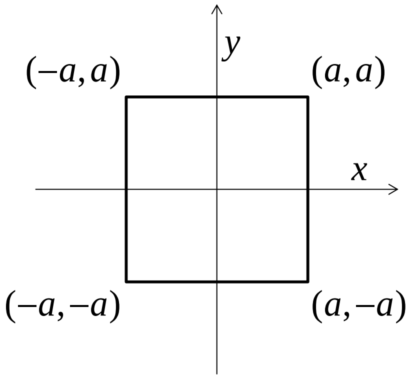


Figure 2 : An elastic membrane

The elastic membrane is stretched so that a point $\mathrm{P}(u_1, u_2)$ goes over into the point $\mathrm{Q}(z_1, z_2)$ given by

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \mathbf{A}\mathbf{u} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Answer the questions.

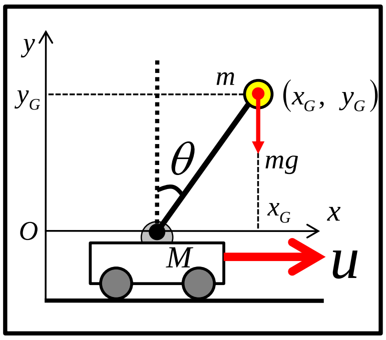
(a) Find transformed boundary of the quadrate boundary based on the concept of eigenvalues and eigenvectors.

(b) The higher power of $\mathbf{A}$, the closer its columns approach the steady state. It means the powers $\mathbf{A}^k$ of this matrix $\mathbf{A}$ approaches a limit as $k \to \infty$. Find $\mathbf{A}^{100}$.

(Hint : $\mathbf{A}^{100}$ should be found by using the eigenvalues of $\mathbf{A}$, not by multiplying 100 matrices.)

(c) This particular $\mathbf{A}$ is a Markov matrix. When its entries are positive and every column adds to 1. About 3×3 matrix, prove that those facts guarantee the largest eigenvalue λ is 1.

3. (*Jordan Form*) An inverted pendulum mounted on a motor-driven cart is shown in Figure 13. Here we consider only a two-dimensional problem in which the pendulum moves only in the plane of the page. The control force u is applied to the cart.



(x_G, y_G) : The center of gravity of the pendulum ball

x : The displacement of the cart in x-direction

l : The length of the rod

θ: The angle of the rod from vertical line

m: The mass of the pendulum ball

M: The mass of the cart

Figure 3. Inverted-pendulum system

The mathematical model for this system is given as follows:

$$Ml\ddot{\theta} = (M+m)g\theta - u\,, \quad M\ddot{x} = u - mg\theta\,.$$

Answer the questions.

(a) We defined state variables $x_1, x_2, x_3,$ and x_4 by $x_1 = \theta,\ x_2 = \dot{\theta},\ x_3 = x,\ x_4 = \dot{x}$. Then find a state-space representation of the system in terms of vector-matrix equations. Let $M = 1,\ m = 0.6,\ g = 10,\ l = 1$.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} u, \quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

(b) Evaluate controllability of this system using "**Jordan form**" of the matrix $\mathbf{A}$, and explain the advantage of using the "**Jordan form**".

4. (Laplace transform) Consider the spring-mass-damper system mounted on a ground as shown in Figure 4. In this system, $u(t)$ is the control force and is the input to the system. At $t = 0$, the cart is moving at a constant speed. The displacement $y(t)$ of the mass is the output. In this system, m denotes the mass, c denotes the friction coefficient, and k denotes the spring constant. We assume that the friction force of the dashpot is proportional to $\dot{y}$ and that the spring is a linear spring. $r(t)$ is the desired position of the mass.

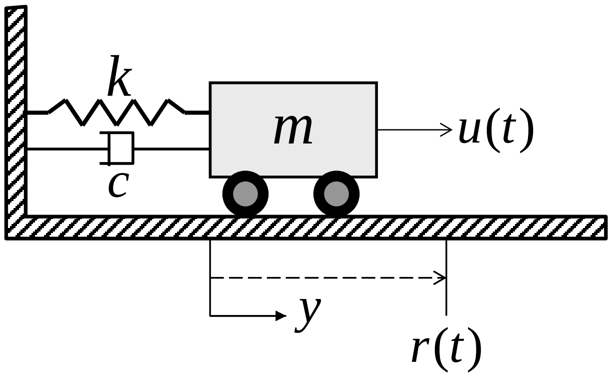


Figure 4. Spring-mass-damper system

Applying Newton's second law to the present system, we obtain

$$m\frac{d^2y}{dt^2} = -c\frac{dy}{dt} - ky + u(t) \quad \text{or} \quad m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = u(t).$$

Answer the questions.

(a) Take the Laplace transform of this equation, assuming zero initial condition. And take the ratio of $Y(s)$ to $U(s)$, then we can find the transfer function $G(s)$ of the system. $G(s) = \frac{Y(x)}{U(x)}$.

(b) Explain the reason why we assume zero initial condition: $y(0) = 0,\ y'(0) = 0.$

(c) In case that the proportional controller is used, the control force u equals to $K_p e(t)$, where $e(t) = r(t) - y(t)$ and $r(t)$ is regarded as constant. Solve the algebraic equation for $Y(s)$ by applying $\mathscr{L}^{-1}$ on the Laplace transformed equation.

5. (SVD) Consider the data $(0,\ 0),\ (1,\ 4),\ (2,\ 4)$ shown in Figure 5. If we try to match the data points with the function $y = ax + b$, then we wish to find a and b that satisfy the system of equations.

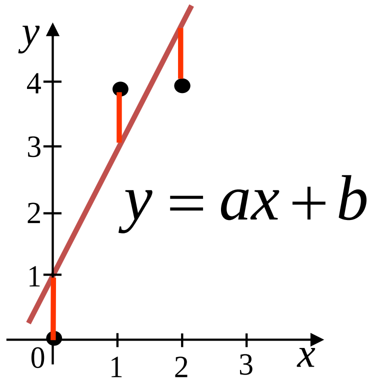


Figure 5. Line and data

But unfortunately it is an over-determined system and has no solution.

$$0 \cdot a + b = 0$$
$$1 \cdot a + b = 4$$
$$2 \cdot a + b = 4$$

These equations can be represented in terms of vector-matrix equations.

$$\mathbf{Y} = \mathbf{Ax}$$

, where $\mathbf{Y} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$, $\mathbf{X} = \begin{bmatrix} a \\ b \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$.

Thus we shall be content to find a vector $\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}$ so that the right side $\mathbf{Ax}$ is close to the left side $\mathbf{Y}$. We need a criterion that defines the concept of "best fit". Find the **singular value decomposition** of matrix $\mathbf{A}$ and determine the line of best fit.