Stress Analysis

1. Basic assumption:

- Rock is an isotropic, homogeneous, and linear elastic material.
- Reasonable for small piece (in-tact) of rock

2. Forces

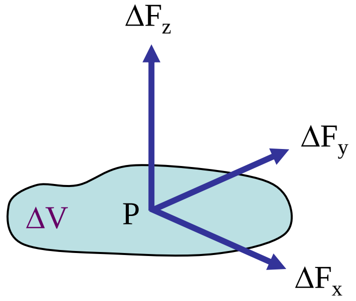
(1) Body forces: Without physical contact with other bodies.
i.e. gravitational, magnetic and inertial forces
(2) Surface forces: External forces resulting from physical contact with other bodies

3. Body force intensity (Force per unit volume)

$$X = \lim_{\Delta V \to 0} \frac{\Delta F_x}{\Delta V}$$

$$Y = \lim_{\Delta V \to 0} \frac{\Delta F_y}{\Delta V}$$

$$Z = \lim_{\Delta V \to 0} \frac{\Delta F_z}{\Delta V}$$

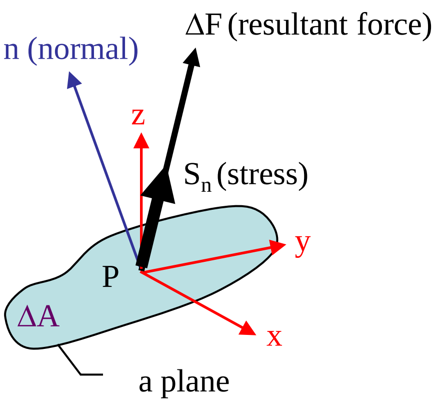


Stress Analysis

4. Stress (Surface force - a Tensor amount)

(1) Force per unit area (Force – a vector amount)

The stress acting on the plane whose normal is n



$$S_n = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}$$

ΔF resolved in Cartesian coordinates

$$S_{nx} = \lim_{\Delta A \to 0} \frac{\Delta F_x}{\Delta A}$$

$$S_{ny} = \lim_{\Delta A \to 0} \frac{\Delta F_y}{\Delta A}$$

$$S_{nz} = \lim_{\Delta A \to 0} \frac{\Delta F_z}{\Delta A}$$

(2) Magnitude of the stress

$$S_n^2 = S_{nx}^2 + S_{ny}^2 + S_{nz}^2 \qquad \cdots\cdots (1)$$

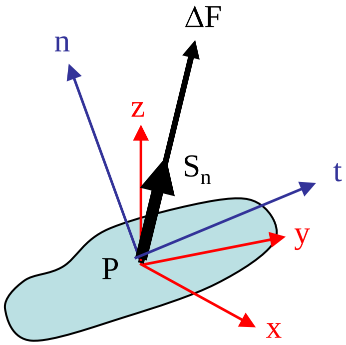
Stress Analysis

(3) Using direction cosines,

$$S_{nx} = S_n \cos(S_n, x) \quad ......(2)$$
$$S_{ny} = S_n \cos(S_n, y) \quad ......(3)$$
$$S_{nz} = S_n \cos(S_n, z) \quad ......(4)$$

(4) Resolving ΔF into two components

$$\sigma_{nn} = \lim_{\Delta A \to 0} \frac{\Delta F_n}{\Delta A} : \text{normal stress}$$
$$\tau_{nt} = \lim_{\Delta A \to 0} \frac{\Delta F_t}{\Delta A} : \text{tangential (shear) stress}$$



(5) Magnitude of the stress

$$S_n^{\ 2} = \sigma_{nn}^{\ 2} + \tau_{nt}^{\ 2} \quad ......(5)$$
$$\sigma_{nn} = S_n \cos(S_n, n) \quad ......(6)$$
$$\tau_{nt} = S_n \cos(S_n, t) \quad ......(7)$$

Stress Analysis

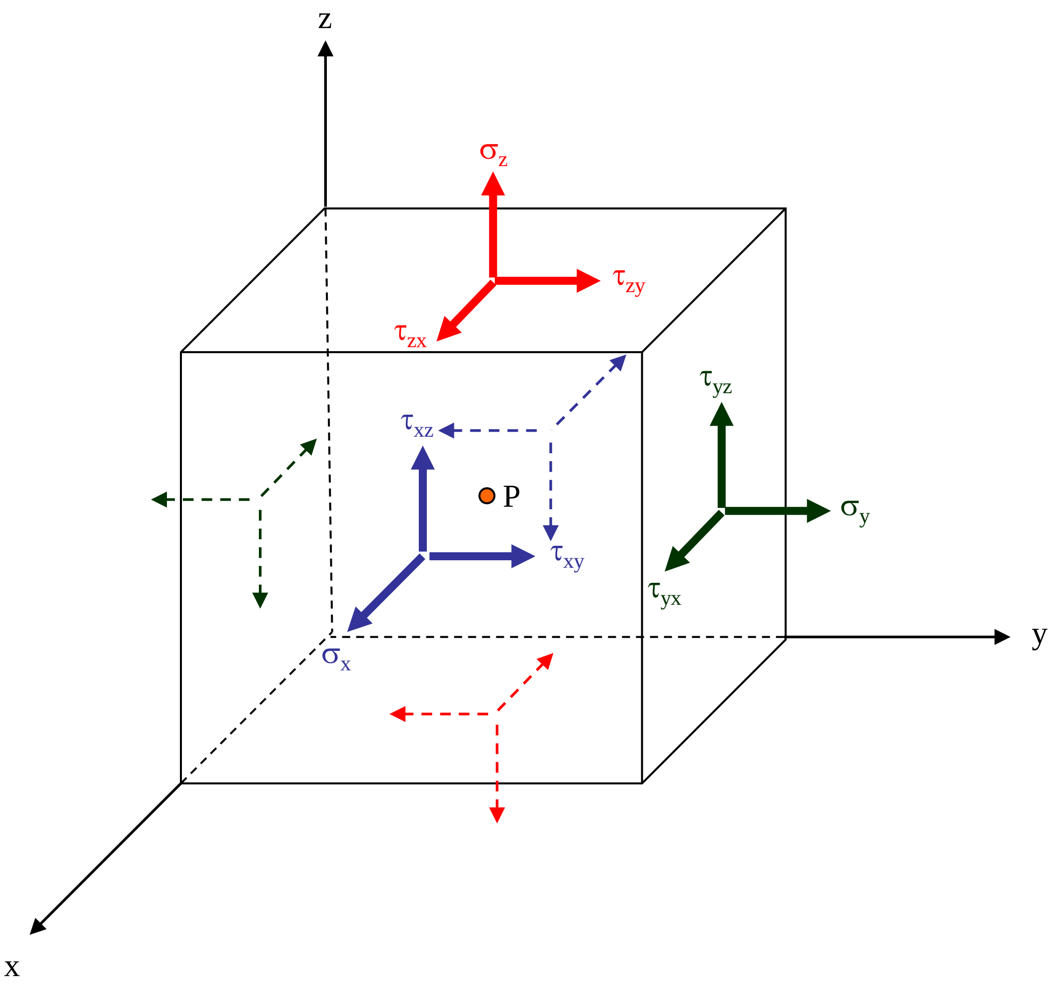
(6) Convention for designation

$\sigma_x = \sigma_{xx}$ = Normal stress acting on the plane normal to the x-axis

τ_{xy} = Shear stress acting in the y-direction and on a plane normal to the x-axis

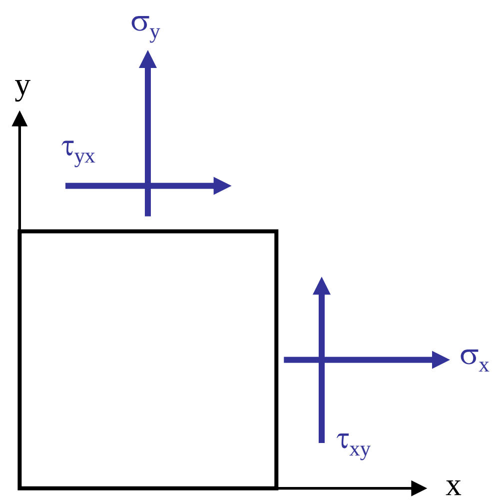
Normal stress ⊕ when directed outward (tensile)

Shear stress ⊕ when directed in the ⊕ direction on a plane whose outward normal points ⊕ direction



Stress Analysis

(7) In a 2-dimensional case



(8) By moment sum around P

$$\sum M_x = \tau_{yz} \cdot \frac{dy}{2} \cdot dx \cdot dz + \tau_{yz} \cdot \frac{dy}{2} \cdot dx \cdot dz - \tau_{zy} \cdot \frac{dz}{2} \cdot dx \cdot dy - \tau_{zy} \cdot \frac{dy}{2} \cdot dx \cdot dy = 0$$

$$\therefore \tau_{yz} = \tau_{zx}$$

Similarly,

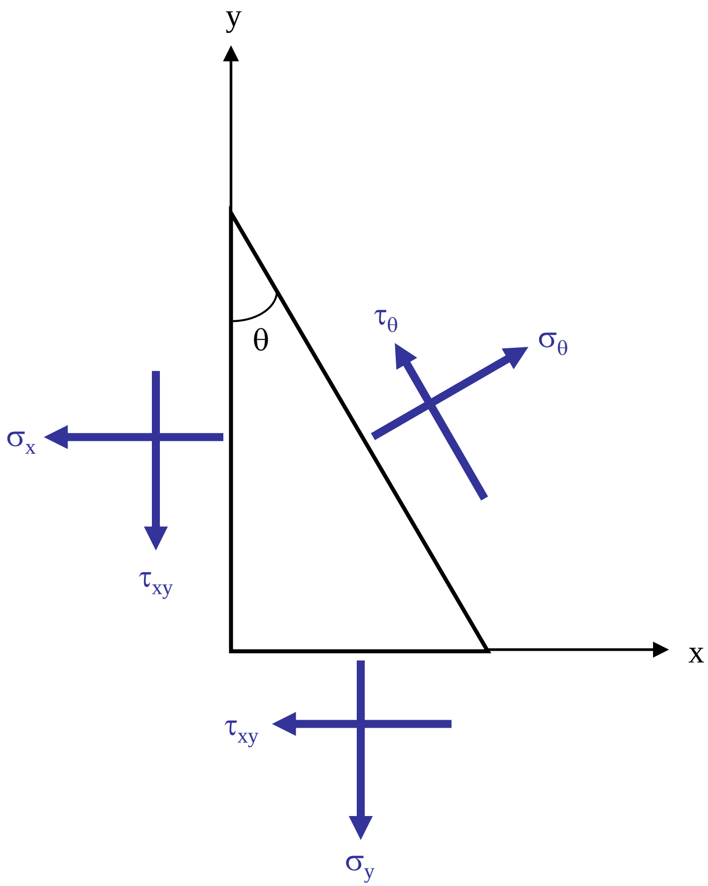
$$\tau_{xy} = \tau_{yx}, \; \tau_{zx} = \tau_{xz}$$

There are 6 independent stress components $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$

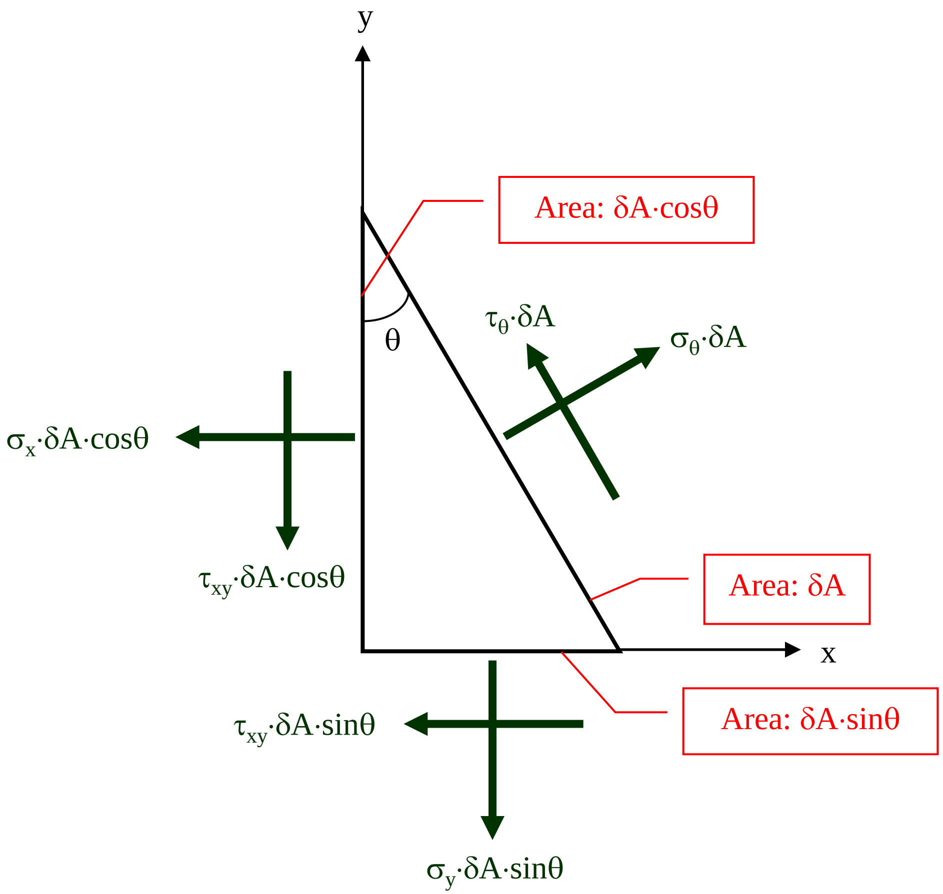
(9) Many problems of elasticity solved in 2-D due to simplicity and conservativeness

Stress Analysis

(10) Stresses and forces on an element of a body in the xy plane

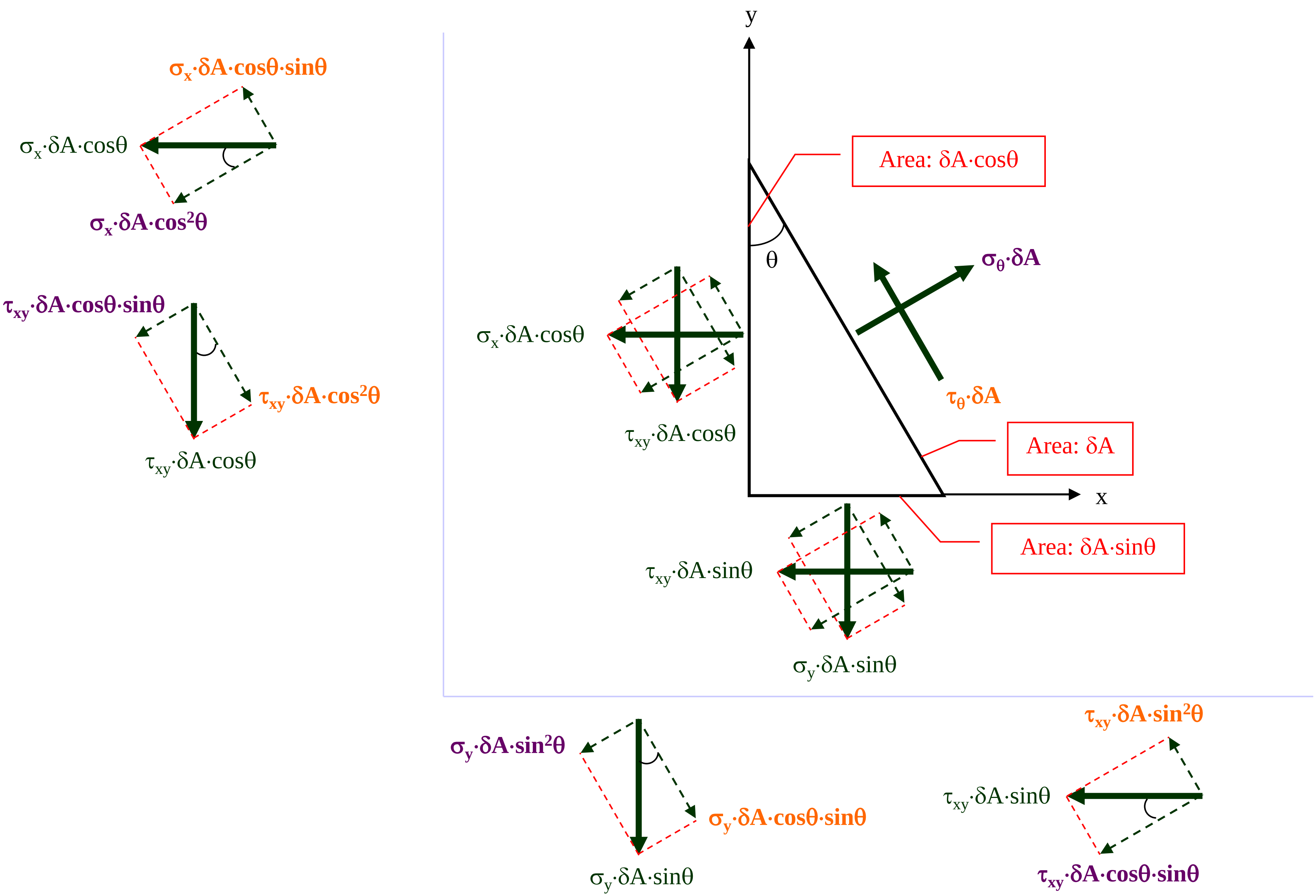


Stress diagram



Force diagram for unit thickness

Stress Analysis



Stress Analysis

(11) Equilibrium of the element

$$\sum F_{\sigma_\theta} = \sigma_\theta \delta A - \sigma_x \delta A \cos^2\theta - \sigma_y \delta A \sin^2\theta - 2\tau_{xy}\delta A \cos\theta \sin\theta = 0$$

$$\sum F_{\tau_\theta} = \tau_\theta \delta A + \tau_{xy} \delta A \sin^2\theta - \tau_{xy} \delta A \cos^2\theta + \sigma_x \delta A \cos\theta \sin\theta - \sigma_y \delta A \cos\theta \sin\theta = 0$$

$$\sigma_\theta = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \cos\theta \sin\theta$$

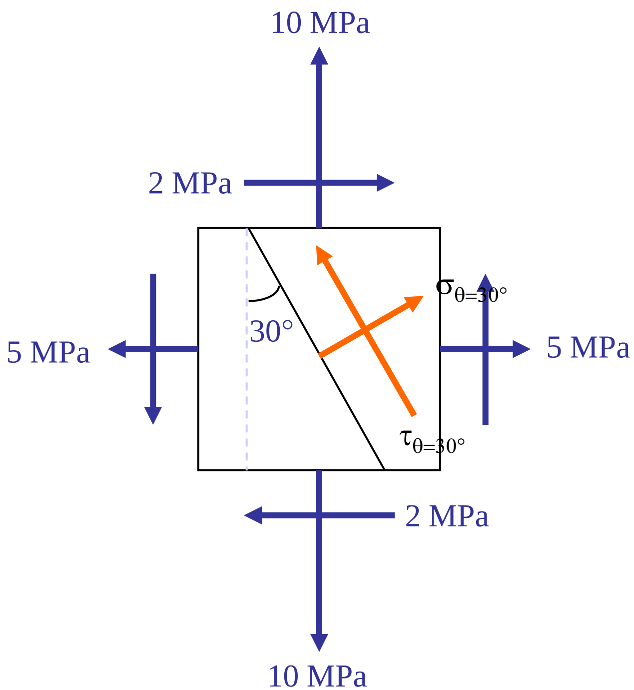
$$\tau_\theta = \tau_{xy}\left(\cos^2\theta - \sin^2\theta\right) - \left(\sigma_x - \sigma_y\right)\cos\theta \sin\theta$$

$$\sigma_\theta = \frac{1}{2}\left(\sigma_x + \sigma_y\right) + \frac{1}{2}\left(\sigma_x - \sigma_y\right)\cos 2\theta + \tau_{xy} \sin 2\theta \quad \ldots\ldots(8)$$

$$\tau_\theta = -\frac{1}{2}\left(\sigma_x - \sigma_y\right)\sin 2\theta + \tau_{xy} \cos 2\theta \quad \ldots\ldots(9)$$

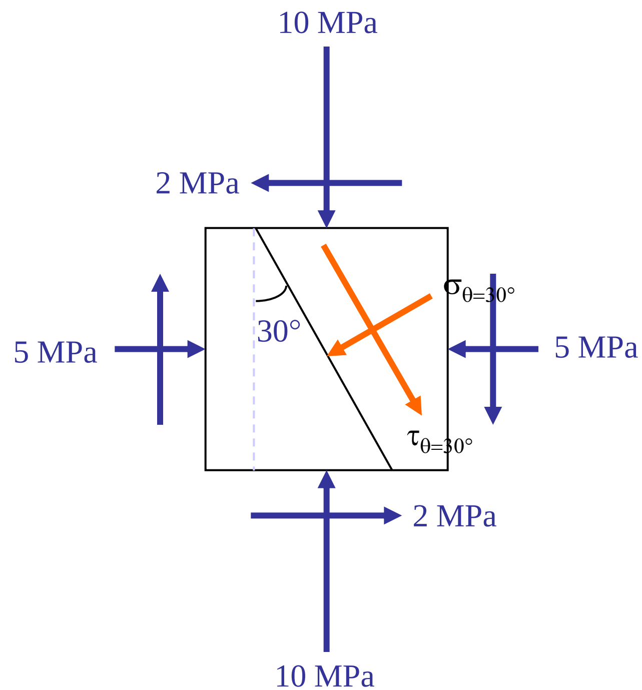
Stress Analysis

Example. Find the normal and shear stresses acting on the inclined plane in the figure.



$$\sigma = \frac{1}{2}(5+10)+\frac{1}{2}(5-10)\cos 60° + 2\cdot\sin 60°$$
$$\cong 7.98\ (\mathrm{MPa})$$

$$\tau = 2\cdot\cos 60° - \frac{1}{2}(5-10)\sin 60°$$
$$\cong 3.17\ (\mathrm{MPa})$$



$$\sigma = \frac{1}{2}(-5-10)+\frac{1}{2}(-5+10)\cos 60° + (-2)\cdot\sin 60°$$
$$\cong -7.98\ (\mathrm{MPa})$$

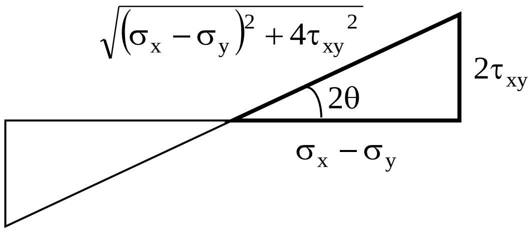
$$\tau = (-2)\cdot\cos 60° - \frac{1}{2}(-5+10)\sin 60°$$
$$\cong -3.17\ (\mathrm{MPa})$$

Stress Analysis

(12) Normal stress σ_θ and shear stress $\tau_\theta=f(\theta)$

Finding the maximum and minimum,

$$\frac{d\sigma_\theta}{d\theta} = -(\sigma_x - \sigma_y)\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0 \Rightarrow \therefore \theta = \frac{1}{2}\tan^{-1}\frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad \ldots\ldots(10)$$



$$\begin{cases} \sin 2\theta = \pm\dfrac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} \\ \cos 2\theta = \pm\dfrac{\sigma_x - \sigma_y}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} \end{cases}$$

Then, $$\sigma_\theta = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \quad \ldots\ldots(11)$$

Two principal stresses are

$$\sigma_1 = \sigma_{max} = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \quad \ldots\ldots(12)$$

$$\sigma_2 = \sigma_{min} = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \quad \ldots\ldots(13)$$

Stress Analysis

(13) At this θ, $\tau_\theta=0$. [Eq.(10)→Eq.(9)]

No shear stresses act on planes where the normal stresses are maximum and minimum.

(14) The maximum and minimum τ_θ [From Eq.(9)]

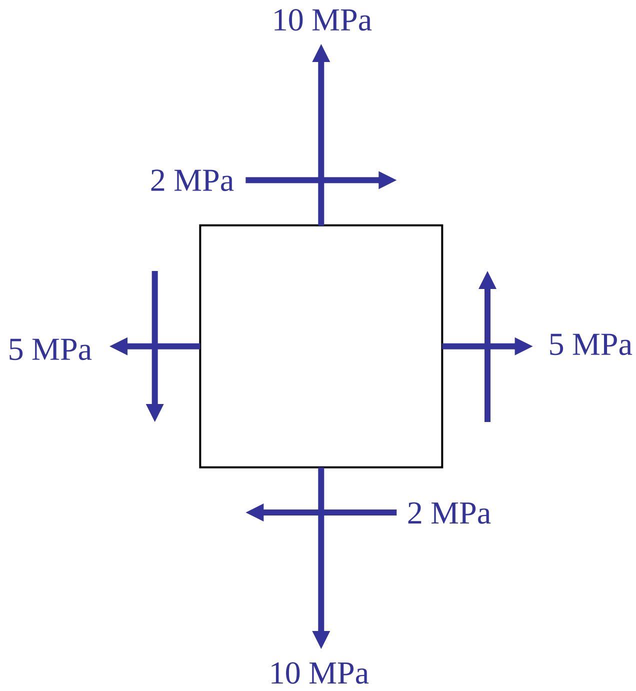
$$\frac{d\tau_\theta}{d\theta} = -2\tau_{xy}\sin 2\theta - (\sigma_x - \sigma_y)\cos 2\theta = 0 \quad \Rightarrow \quad \boxed{\theta = \frac{1}{2}\tan^{-1}\frac{\sigma_y - \sigma_x}{2\tau_{xy}}} \quad \ldots\ldots(14)$$

$$\tau_{max} = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}{}^2} \quad \ldots\ldots\ldots\ldots\ldots(15)$$

$$\tau_{min} = -\frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}{}^2} \quad \ldots\ldots\ldots\ldots\ldots(16)$$

Stress Analysis

Example. Find the principal stresses and the corresponding angles.



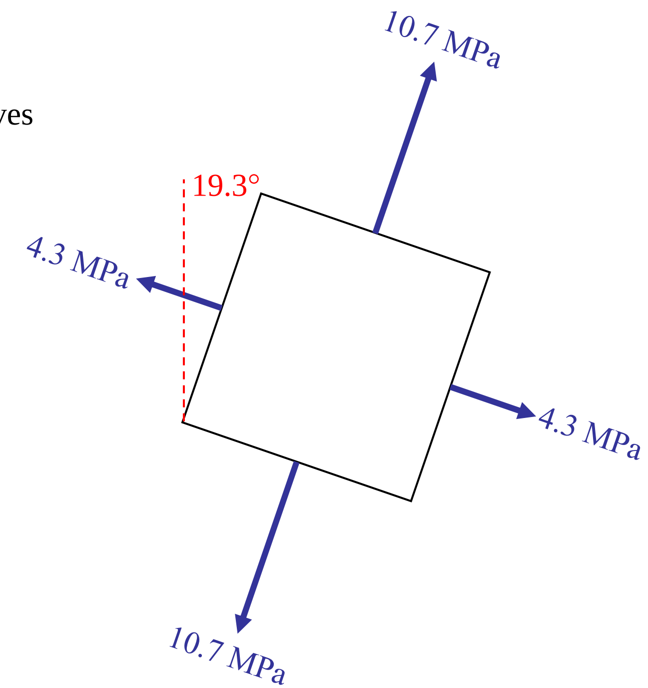
$$\sigma_1 = \frac{1}{2}(5+10) + \frac{1}{2}\sqrt{(5-10)^2 + 4\cdot 2^2} \cong 10.7 \ \ (\text{MPa})$$

$$\sigma_2 = \frac{1}{2}(5+10) - \frac{1}{2}\sqrt{(5-10)^2 + 4\cdot 2^2} \cong 4.3 \ \ (\text{MPa})$$

$$\theta = \frac{1}{2}\tan^{-1}\frac{2\cdot 2}{5-10} \cong -19.3°$$

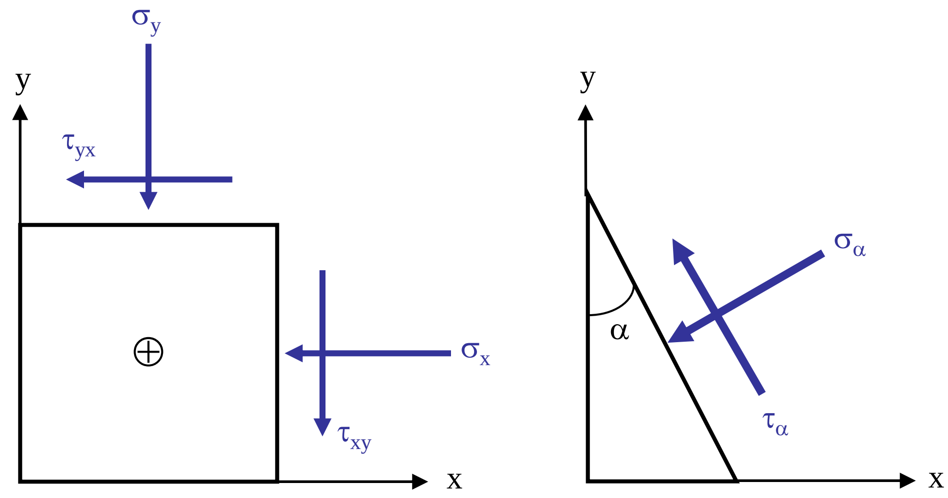
Maximum or minimum?
Plugging θ =-19.3° into Eq.(8) gives

$$\sigma_\theta = 4.3 \ \text{MPa} = \sigma_2$$



Stress Analysis

Stress Transformation (Caution: Different Sign Convention)



$$\sigma_\alpha = \sigma_x \cos^2\alpha + \sigma_y \sin^2\alpha + \tau_{xy}\cos 2\alpha$$

$$\tau_\alpha = (\sigma_y - \sigma_x)\frac{\sin 2\alpha}{2} + \tau_{xy}\cos 2\alpha$$

$$\sigma_1 \text{ and } \sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) \pm \sqrt{\tau_{xy}^2 + \frac{1}{4}(\sigma_x - \sigma_y)^2}$$

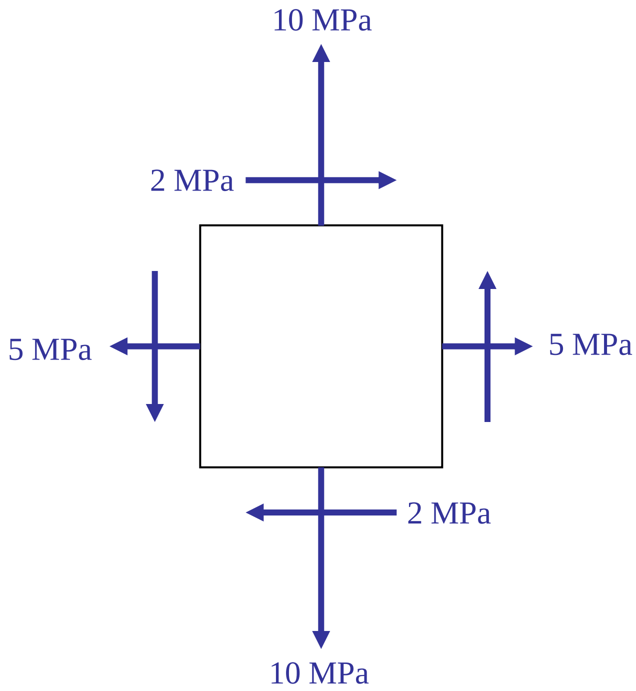
σ_1 points at α counterclockwise from x

$$\beta = \tan^{-1}\frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$2\alpha = \beta$ if $\sigma_x > \sigma_y$
$2\alpha = \beta + \pi$ if $\sigma_x < \sigma_y$ and $\tau_{xy} > 0$
$2\alpha = \beta - \pi$ if $\sigma_x < \sigma_y$ and $\tau_{xy} < 0$

Stress Analysis

Example

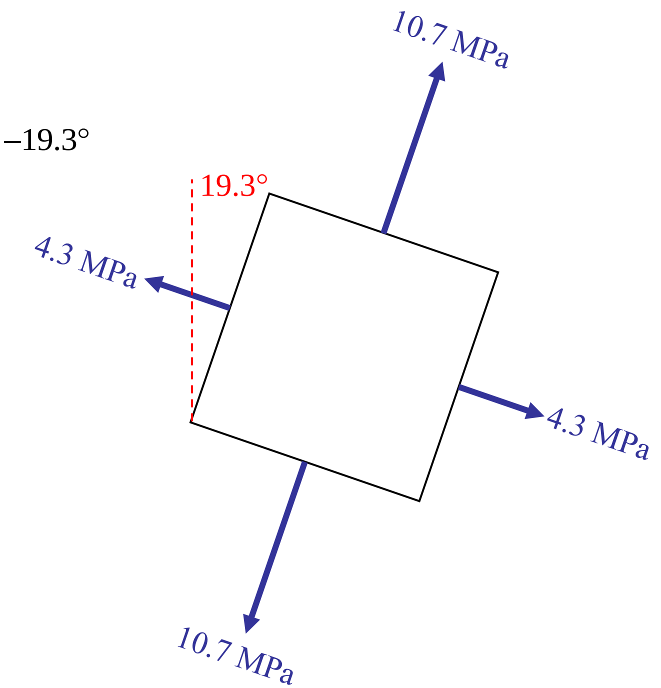


$$\sigma_x = -5\,\text{MPa}$$
$$\sigma_y = -10\,\text{MPa}$$
$$\tau_{xy} = -2\,\text{MPa}$$

$$\sigma_1 \text{ and } \sigma_2 = \frac{1}{2}(-5-10) \pm \sqrt{(-2)^2 + \frac{1}{4}(-5+10)^2}$$
$$= -4.3 \text{ or } -10.7\,(\text{MPa})$$

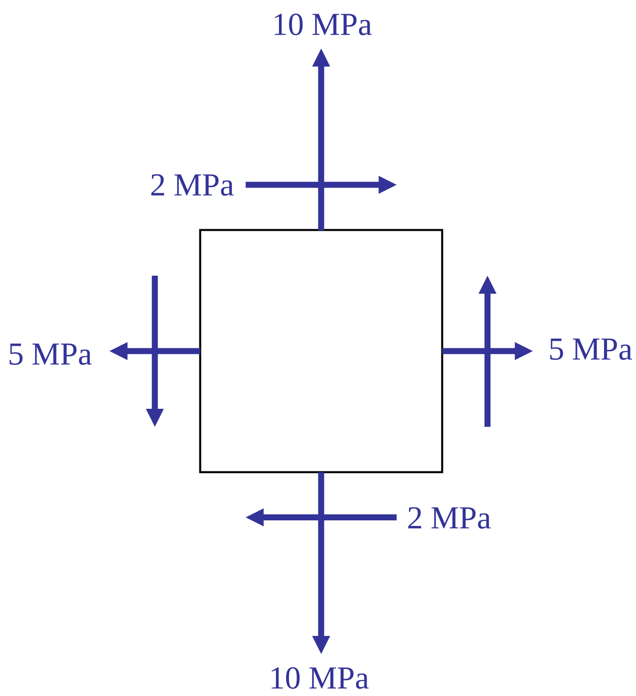
$$\beta = \tan^{-1}\frac{2\cdot(-2)}{-5+10} \cong -19.3°$$

$$\sigma_x > \sigma_y \;\Rightarrow\; 2\theta = \beta :\; \theta = \frac{\beta}{2} = -19.3°$$



Stress Analysis

Example. Find the maximum and minimum shear stresses and the corresponding angles.

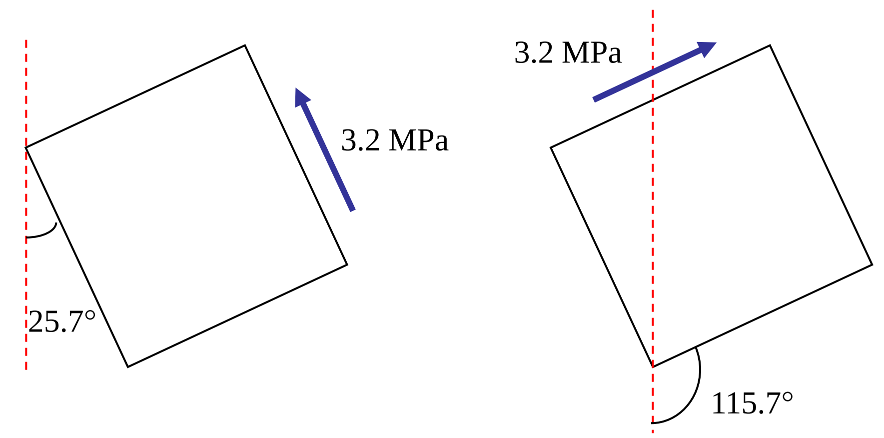


$$\tau_{max} \ \& \ \tau_{min} = \pm\frac{1}{2}\sqrt{(5-10)^2 + 4\cdot 2^2} = 3.2 \ \text{or} \ -3.2 \ (\text{MPa})$$

$$\theta = \frac{1}{2}\tan^{-1}\frac{10-5}{2\cdot 2} \cong 25.7°$$

Plugging θ = 25.7° into Eq.(9) gives τ = 3.2 MPa.

τ_{max} = 3.2 Mpa @ θ = 25.7°
τ_{min} = -3.2 Mpa @ θ = 115.7°



Stress Analysis

(15) Comparing Eq.(10) and Eq.(14), $2\theta = 90°$

$$\therefore \theta = \frac{1}{2}\tan^{-1}\frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad \ldots\ldots(10)$$

$$\theta = \frac{1}{2}\tan^{-1}\frac{\sigma_y - \sigma_x}{2\tau_{xy}} \quad \ldots\ldots(14)$$

The plane of the maximum normal stress
& the plane of the maximum shear stress
→ 45° difference

(16) When he shear stress is the maximum, the normal stress [Eq.(14) → Eq.(8)]

$$\sigma_\theta = \sigma_{\theta+90°} = \frac{\sigma_x + \sigma_y}{2} \quad \ldots\ldots\ldots\ldots\ldots(17)$$

(17) Eq.(12) - Eq.(13)

$$\sigma_{max} - \sigma_{min} = \sigma_1 - \sigma_2 = \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^{\ 2}}$$

Comparing this with Eq.(15)

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2}$$

τ_{max} = (Difference of principal stresses) / 2 @ 45° from the principal planes

Stress Analysis

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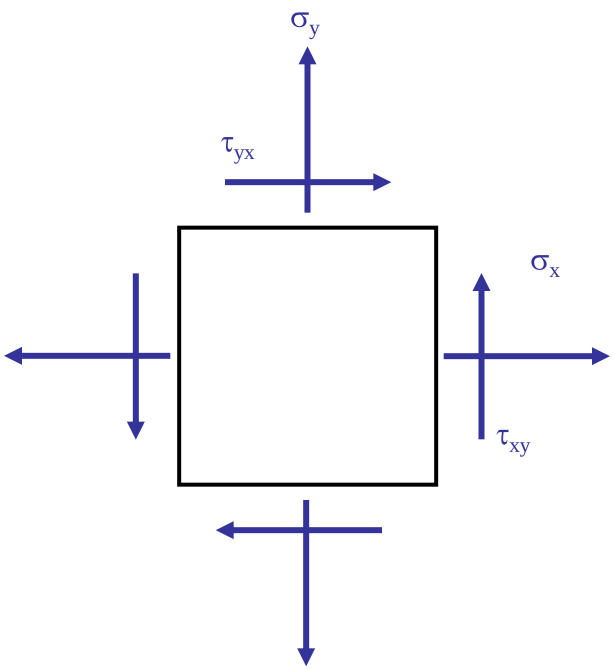
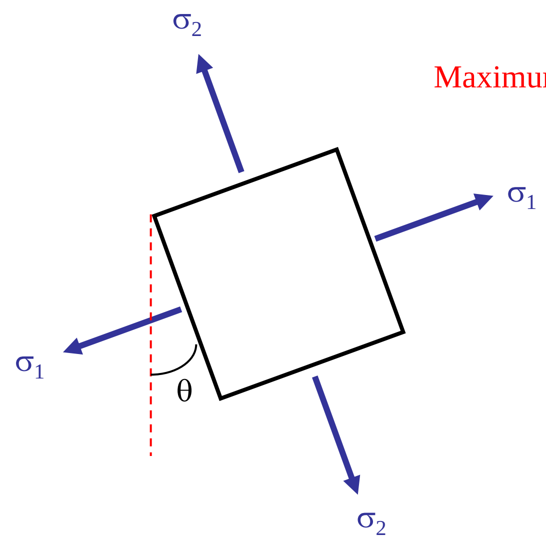
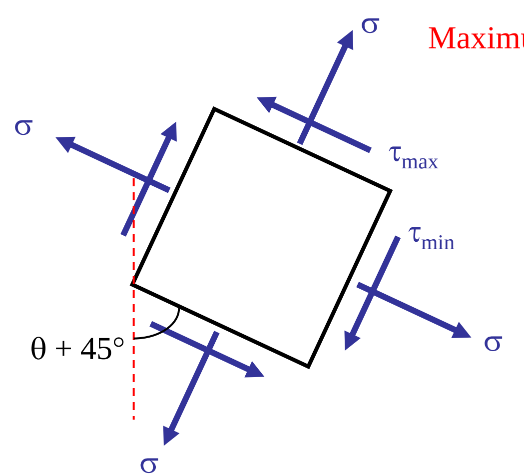




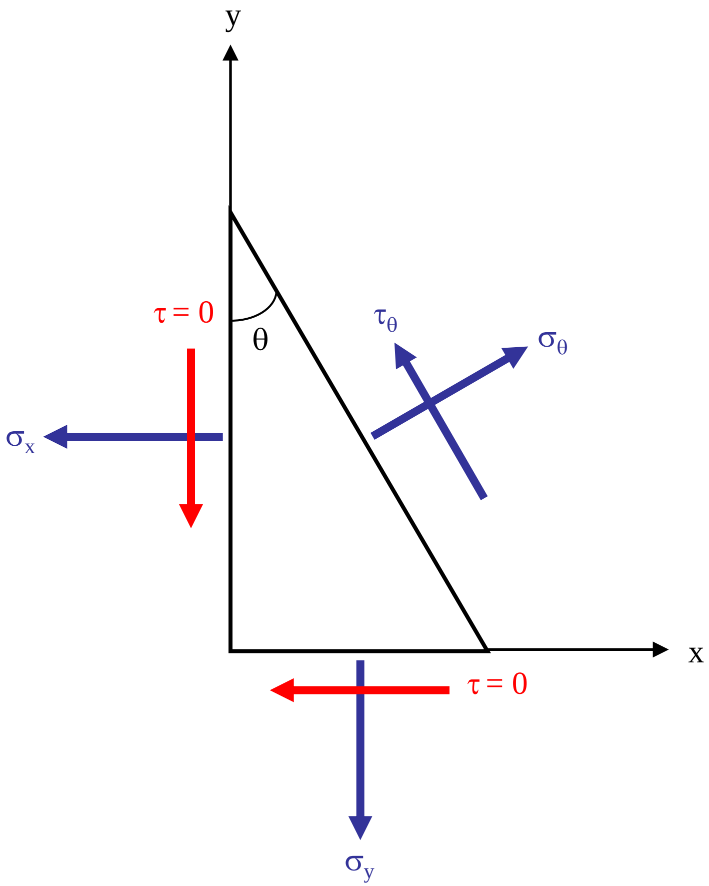
Maximum and minimum normal stresses
(Principal stresses)
$\tau = 0$



Maximum and minimum shear stresses
$\sigma \neq 0$, $\sigma = (\sigma_x + \sigma_y) / 2$
$\tau_{max} = (\sigma_1 - \sigma_2) / 2$

Stress Analysis

(18) Principal stress → stresses on an inclined plane



$$\sigma_\theta = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta \quad \text{……………..…(8)}$$

$$= \frac{1}{2}(\sigma_1 + \sigma_2) + \frac{1}{2}(\sigma_1 - \sigma_2)\cos 2\theta \quad \text{……………...…(18)}$$

$$\tau_\theta = \tau_{xy}\cos 2\theta - \frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta \quad \text{……………..…(9)}$$

$$= -\frac{1}{2}(\sigma_1 - \sigma_2)\sin 2\theta \quad \text{……………...…(19)}$$