

Chapter 11. Geometric Programming

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Chapter 11. Geometric Programming

11.1 Introduction

- Geometric Programming : Adaptable to problems where **the objective function** and **the constraints equations** are sum of polynomials of variables
- The first stage of the solution is to find the optimum value of the function.
- In this chapter, **degree of difficulty** (presented at 11.3) is **0** to be suited to geometric programming.

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11.2 Form of The objective Function and Constraint

- It can treat both of a constrained and unconstrained objective functions.
For example;

- Unconstrained

$$\text{Minimize } y = 5x + \frac{10}{\sqrt{x}}$$

- Constrained

$$\text{Minimize } y = 5x_1\sqrt{x_2} + 2x_1^2 + x_2^{3/2}$$

$$\text{Where } x_1x_2 = 50$$

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11.3 Degree of Difficulty(DOD)

- Definition;

$$\text{DoD} \equiv T - (N + 1)$$

Where T : Number of terms in objective function and constrains

N : Number of variables

- Example;

$$y = 6 - 3x + 10x^{0.5} \quad \rightarrow \quad \begin{array}{l} T = 2 (-3x, 10x^{0.5}) \\ N = 1 (x) \\ \therefore \text{DoD} = 0 \end{array}$$

$$\begin{array}{l} y = 5x_1x_2^{0.5} + 2x_1^2 + x_1^{0.5} \\ x_1x_2 = 50 \end{array} \quad \rightarrow \quad \begin{array}{l} T = 4 (5x_1x_2^{0.5}, 2x_1^2, x_1^{0.5}, x_1x_2) \\ N = 2 (x_1, x_2) \\ \therefore \text{DoD} = 1 \end{array}$$

In this chapter, however, DOD=0

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11.4 Mechanics of Solution for One Independent Variable, Unconstrained

- A basic mechanics of geometric programming is suggested.
- For the optimal value y^*

$$y = c_1 x^{a_1} + c_2 x^{a_2} = u_1 + u_2$$

- Using geometric programming, y^* can be represented in product form. Then,

$$y^* = g^* = \left(\frac{c_1 x^{a_1}}{w_1} \right)^{w_1} \left(\frac{c_2 x^{a_2}}{w_2} \right)^{w_2}$$

$$y^* = c_1 (x^*)^{a_1} + c_2 (x^*)^{a_2} = u_1^* + u_2^* \quad \text{Where } u_1^* = c_1 (x^*)^{a_1}, \quad u_2^* = c_2 (x^*)^{a_2}$$

- Provided that $w_1 + w_2 = 1$ (Optimized)
 $a_1 w_1 + a_2 w_2 = 0$ (Optimized)

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11.4 Mechanics of Solution for One Independent Variable, Unconstrained

- At the optimum value,

$$y^* = g^* = \left(\frac{c_1 x^{a_1}}{w_1} \right)^{w_1} \left(\frac{c_2 x^{a_2}}{w_2} \right)^{w_2}$$

- Provided that $w_1 + w_2 = 1$, and $a_1 w_1 + a_2 w_2 = 0$;

$$x^{a_1 w_1 + a_2 w_2} = 1$$

- So, from a given equation

$$\therefore w_1 = \frac{u_1^*}{y^*} = \frac{u_1^*}{u_1^* + u_2^*} \quad w_2 = \frac{u_2^*}{y^*} = \frac{u_2^*}{u_1^* + u_2^*}$$

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Example 11.1

- Determine **the optimum pipe diameter** which results in minimum first plus operating cost for 100 m of pipe.

(Given)

- The objective function, the cost y (\$), in terms of pipe diameter, D (mm) is then

$$y = 160D + \frac{32 \times 10^{12}}{D^5}$$

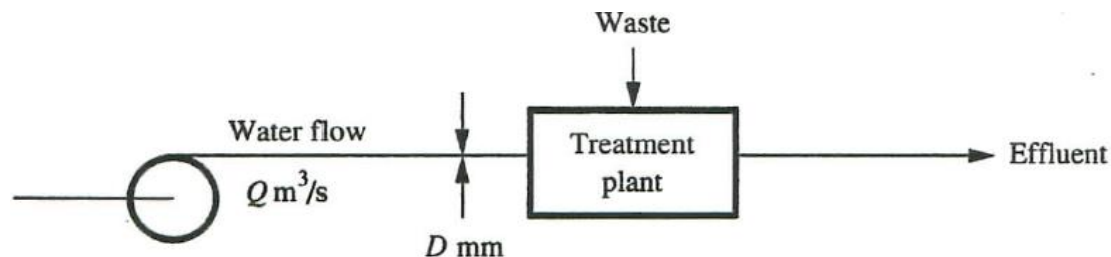


Fig. Waste-treatment system in Example 11.1 and 11.4

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Example 11.1

(Solution)

- For the optimum value y^*

$$y^* = g^* = 160D^* + \frac{32 \times 10^{12}}{(D^*)^5}$$

$$y^* = g^* = \left(\frac{160D}{w_1}\right)^{w_1} \left(\frac{32 \times 10^{12}}{D^5 w_2}\right)^{w_2}$$

- Provided that $w_1 + w_2 = 1$

$$a_1 w_1 + a_2 w_2 = w_1 - 5w_2 = 0$$

- Solving gives $w_1 = \frac{5}{6}$, $w_2 = \frac{1}{6}$

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Example 11.1

(Solution)

- By substituting w_1 and w_2 to the above optimizing equation,

$$y^* = g^* = \left(\frac{160}{5/6}\right)^{5/6} \left(\frac{32 \times 10^{12}}{1/6}\right)^{1/6} = \$ 19,200$$

$$w_1 = \frac{5}{6} = \frac{u_1^*}{y^*} = \frac{160D^*}{19,200} \quad \rightarrow \quad \therefore D^* = 100 \text{ mm}$$

(Answer)

- The optimum diameter : 100 mm (at minimum cost, \$19,200)

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Example 11.2

- Determine the maximum power of which this engine is capable and the rotative speed at which the maximum occurs.

(Given)

- The torque T (N·m) is represented by

$$T = 23.6\omega^{0.7} - 3.17\omega$$

Where ω : the rotative speed (rad/s)

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Example 11.2

(Solution)

- The power P (Watt) is the product of the torque and the rotative speed.

$$P = T\omega = 23.6\omega^{1.7} - 3.17\omega^2$$

$$y^* = g^* = \left(\frac{23.6}{w_1}\right)^{w_1} \left(\frac{-3.17}{w_2}\right)^{w_2}$$

- Provided that $w_1 + w_2 = 1$, and $a_1w_1 + a_2w_2 = 1.7w_1 + 2w_2 = 0$
- Solving gives $w_1 = 6.667$, $w_2 = -5.667$

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Example 11.2

(Solution)

$$y^* = g^* = \left(\frac{23.6}{6.667}\right)^{6.667} \left(\frac{-3.17}{-5.667}\right)^{-5.667} = 122,970 \text{ W}$$

$$w_1 = 6.667 = \frac{u_1^*}{y^*} = \frac{23.6(\omega^*)^{1.7}}{122,970} \quad \rightarrow \quad \omega^* = 469 \text{ rad/s}$$

- If we focus on w_2 instead of w_1 , then the result is also

$$w_2 = -5.667 = \frac{u_2^*}{y^*} = \frac{-3.17(\omega^*)^2}{122,970} \quad \rightarrow \quad \omega^* = 469 \text{ rad/s}$$

(Answer)

- The maximum power : 122,970 W (at the rotative speed, 469 rad/s)

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11.5 Why Geometric Programming Works; One Independent Variable

- Substantiation :

$$y = c_1 x^{a_1} + c_2 x^{a_2} = u_1 + u_2 \quad \text{Where } u_1 = c_1 x^{a_1}, u_2 = c_2 x^{a_2}$$

$$g = \left(\frac{u_1}{w_1}\right)^{w_1} \left(\frac{u_2}{w_2}\right)^{w_2} = \left(\frac{c_1 x^{a_1}}{w_1}\right)^{w_1} \left(\frac{c_2 x^{a_2}}{w_2}\right)^{w_2} \quad \text{Where } w_1 + w_2 = 1$$

- Using logarithm, it can be represented as

$$\ln g = w_1(\ln u_1 - \ln w_1) + w_2(\ln u_2 - \ln w_2)$$

$$\text{Subject to } \emptyset = w_1 + w_2 - 1 = 0$$

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11.5 Why Geometric Programming Works; One Independent Variable

- To find a combination w_1, w_2 at which function g has the maximum value, Lagrange multiplier is used

$$\nabla(\ln g) - \lambda \nabla \phi = 0$$

$$\phi = 0$$

- which provide the three equations:

$$w_1: \quad \ln u_1 - 1 - \ln w_1 - \lambda = 0$$

$$w_2: \quad \ln u_2 - 1 - \ln w_2 - \lambda = 0$$

$$w_1 + w_2 - 1 = 0$$

- The unknowns are w_1, w_2, λ , and the solutions for w_1 and w_2 are

$$w_1 = \frac{u_1}{u_1 + u_2} \quad w_2 = \frac{u_2}{u_1 + u_2}$$

$$\therefore g = \left(\frac{u_1}{w_1}\right)^{w_1} \left(\frac{u_2}{w_2}\right)^{w_2} = \left(\frac{u_1}{u_1/(u_1 + u_2)}\right)^{\frac{u_1}{(u_1 + u_2)}} \left(\frac{u_2}{u_2/(u_1 + u_2)}\right)^{\frac{u_2}{(u_1 + u_2)}} = u_1 + u_2 = y$$

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11.5 Why Geometric Programming Works; One Independent Variable

- To get optimum value x^* , y^* , g^* , the derivative of y is used.

$$y' = a_1 c_1 x^{(a_1-1)} + a_2 c_2 x^{(a_2-1)} = 0$$

- Multiplying by x and so,

$$xy' = a_1 c_1 x^{a_1} + a_2 c_2 x^{a_2} = 0$$

$$a_1 u_1^* + a_2 u_2^* = 0$$

- Where u_1^* and u_2^* are the values of u_1 and u_2 at the optimum value of y

$$u_1^* = -\frac{a_2 u_2^*}{a_1}$$

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11.5 Why Geometric Programming Works; One Independent Variable

- When substituting u_1^* and u_2^* into w_1 and w_2 ,

$$w_1 = \frac{u_1^*}{u_1^* + u_2^*} = \frac{-\frac{a_2 u_2^*}{a_1}}{-\frac{a_2 u_2^*}{a_1} + u_2^*} = \frac{-a_2}{a_1 + a_2} \quad w_2 = \frac{u_2^*}{u_1^* + u_2^*} = \frac{u_2^*}{-\frac{a_2 u_2^*}{a_1} + u_2^*} = \frac{a_1}{a_1 + a_2}$$

- When these values of w_1 and w_2 are substituted into the solution for g ,

$$g^* = \left(\frac{c_1 x^{a_1}}{w_1} \right)^{\frac{-a_2}{a_1+a_2}} \left(\frac{c_2 x^{a_2}}{w_2} \right)^{\frac{a_1}{a_1+a_2}} = \left(\frac{c_1}{w_1} \right)^{\frac{-a_2}{a_1+a_2}} \left(\frac{c_2}{w_2} \right)^{\frac{a_1}{a_1+a_2}} \quad \because x^{\frac{-a_2 a_1 + a_2 a_1}{a_1+a_2}} = 1$$

$$\therefore g^* = \left(\frac{c_1}{w_1} \right)^{w_1} \left(\frac{c_2}{w_2} \right)^{w_2} = y^*$$

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11.7 Unconstrained, Multivariable Optimization

- The solving procedure for one variable extend to multivariable optimizations.
- If **DoD=0**, a two variable problem have an objective function containing 3 terms.
- An example 11.4 represents a detailed solution for two variable problem.

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Example 11.4

- Use geometric programming to optimize the total system.

(Given)

- The objective function, the cost y (\$), in terms of pipe diameter, D (mm) and flow rate, Q (m³/s) is then

$$y = 160D + \frac{32 \times 10^{15} Q^2}{D^5} + \frac{150}{Q}$$

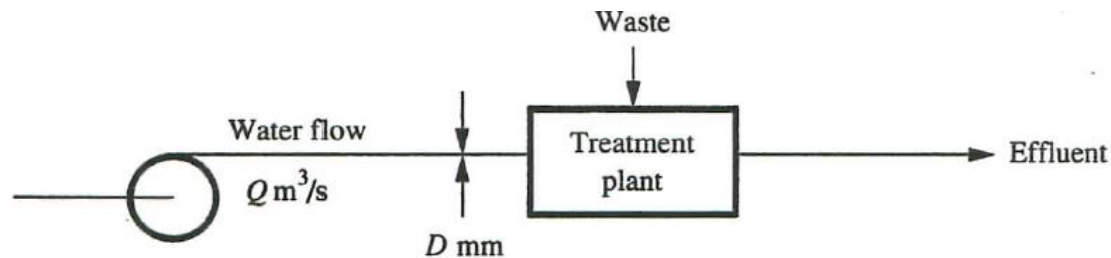


Fig. Waste-treatment system in Example 11.1 and 11.4

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Example 11.4

(Solution)

$$y = 160D + \frac{32 \times 10^{15} Q^2}{D^5} + \frac{150}{Q}$$

$$y^* = g^* = \left(\frac{160}{w_1}\right)^{w_1} \left(\frac{220 \times 10^{15}}{w_2}\right)^{w_2} \left(\frac{150}{w_3}\right)^{w_3}$$

- Provided that $w_1 + w_2 + w_3 = 1$

$$\text{D:} \quad w_1 - 5w_2 = 0$$

$$\text{Q:} \quad 2w_2 - w_3 = 0$$

- Solving gives $w_1 = \frac{5}{8}$, $w_2 = \frac{5}{8}$, $w_3 = \frac{2}{8}$

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Example 11.4

(Solution)

$$\left(\frac{160}{5/8}\right)^{5/8} \left(\frac{220 \times 10^{15}}{1/8}\right)^{1/8} \left(\frac{150}{1/4}\right)^{1/4} = \$30,224$$

$$u_1^* = 160D = \frac{5}{8}(30,224) \rightarrow D^* = 118 \text{ mm}$$

$$u_3^* = \frac{150}{Q^*} = \frac{2}{8}(30,224) \rightarrow Q^* = 0.0198 \text{ m}^3/\text{s}$$

(Answer)

- The optimum diameter : 118 mm
- The optimum flow rate : 0.0198 m³/s
- The minimum cost : \$30,224

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11.8 Constrained Optimization with Zero Degree of Difficulty

- Suppose that the objective function to be minimized and its constraint is

$$y = u_1 + u_2 + u_3$$

$$u_4 + u_5 = 1$$

Where $u_i = f_i(x_1, x_2, x_3, x_4)$

- The objective function can be rewritten

$$y = g = \left(\frac{u_1}{w_1}\right)^{w_1} \left(\frac{u_2}{w_2}\right)^{w_2} \left(\frac{u_3}{w_3}\right)^{w_3}$$

- Provided that

$$w_1 + w_2 + w_3 = 0$$

$$w_i = \frac{u_i}{u_1 + u_2 + u_3}$$

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11.8 Constrained Optimization with Zero Degree of Difficulty

- The constraint equation can also be rewritten as

$$u_4 + u_5 = 1 = \left(\frac{u_4}{w_4}\right)^{w_4} \left(\frac{u_5}{w_5}\right)^{w_5}$$

- Provided that

$$w_4 + w_5 = 1$$

$$w_4 = \frac{u_4}{u_4 + u_5} = u_4 \quad w_5 = \frac{u_5}{u_4 + u_5} = u_5 \quad \because u_4 + u_5 = 1$$

- The above equation can be raised to the M th power

$$1 = \left(\frac{u_4}{w_4}\right)^{Mw_4} \left(\frac{u_5}{w_5}\right)^{Mw_5} \quad M \text{ is an arbitrary constant}$$

- When multiplying the transformed objective function by the transformed constraint equation

$$y = g = \left(\frac{u_1}{w_1}\right)^{w_1} \left(\frac{u_2}{w_2}\right)^{w_2} \left(\frac{u_3}{w_3}\right)^{w_3} \left(\frac{u_4}{w_4}\right)^{Mw_4} \left(\frac{u_5}{w_5}\right)^{Mw_5}$$

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11.8 Constrained Optimization with Zero Degree of Difficulty

- Return to the original equation and solve by Lagrange multipliers

$$\nabla(u_1 + u_2 + u_3) - \lambda[\nabla(u_4 + u_5)] = 0$$

$$u_4 + u_5 = 1$$

- For u_i 's variable x_1, x_2, x_3, x_4 , the number of possible equations is 4

$$\left\{ \begin{array}{l} a_{11}u_1^* + a_{21}u_2^* + a_{31}u_3^* - \lambda a_{41}u_4^* - \lambda a_{51}u_5^* = 0 \\ a_{12}u_1^* + a_{22}u_2^* + a_{32}u_3^* - \lambda a_{42}u_4^* - \lambda a_{52}u_5^* = 0 \\ a_{13}u_1^* + a_{23}u_2^* + a_{33}u_3^* - \lambda a_{43}u_4^* - \lambda a_{53}u_5^* = 0 \\ a_{14}u_1^* + a_{24}u_2^* + a_{34}u_3^* - \lambda a_{44}u_4^* - \lambda a_{54}u_5^* = 0 \end{array} \right.$$

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11.8 Constrained Optimization with Zero Degree of Difficulty

- Dividing the combinations of 4 equations by y^* , $\frac{u_i^*}{y^*} \rightarrow w_i$
- Since M was arbitrary, let it equal; $M = \frac{-\lambda}{y^*}$

$$\begin{cases} a_{11}w_1 + a_{21}w_2 + a_{31}w_3 - Ma_{41}w_4 - Ma_{51}w_5 = 0 \\ a_{12}w_1 + a_{22}w_2 + a_{32}w_3 - Ma_{42}w_4 - Ma_{52}w_5 = 0 \\ a_{13}w_1 + a_{23}w_2 + a_{33}w_3 - Ma_{43}w_4 - Ma_{53}w_5 = 0 \\ a_{14}w_1 + a_{24}w_2 + a_{34}w_3 - Ma_{44}w_4 - Ma_{54}w_5 = 0 \end{cases}$$

$$w_1 + w_2 + w_3 = 1$$

$$Mw_4 + Mw_5 = M$$

- These 6 equations can be solved for the 6 unknowns $w_1, w_2, w_3, Mw_4, Mw_5, M$

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11.8 Constrained Optimization with Zero Degree of Difficulty

- Because u_i is a polynomial of x_i , all of the x terms in u can be cancelled
- So the y equation can be simplified as

$$\therefore y = g = \left(\frac{c_1}{w_1}\right)^{w_1} \left(\frac{c_2}{w_2}\right)^{w_2} \left(\frac{c_3}{w_3}\right)^{w_3} \left(\frac{c_4}{w_4}\right)^{Mw_4} \left(\frac{c_5}{w_5}\right)^{Mw_5}$$

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Example 11.5

- A water pipeline extends 30 km. Select the number of pumps and the pipe diameter that results in the minimum total cost for the system.

(Given)

Cost of each pump = $2500 + 0.00032\Delta p^{1.2}$ (\$)

Δp : pressure drop in each pipe, Pa

Cost of 30 km of pipe = $2,560,000D^{1.5}$ (\$)

D : diameter of pipe, m

- A friction factor : 0.02
- A flow rate : $0.16 \text{ m}^3/\text{s}$

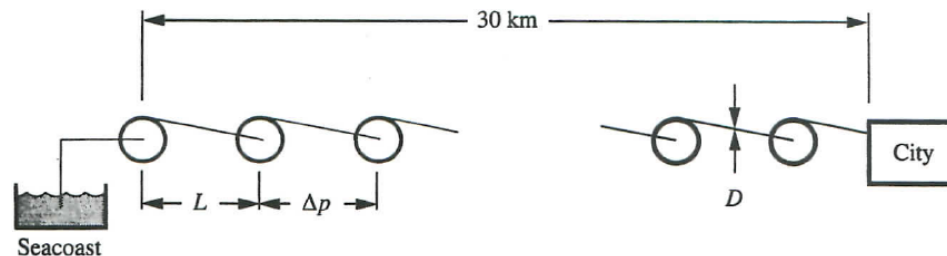


Fig. Water pipeline in Example 11.5

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Example 11.5

(Solution)

$$y = n(2500 + 0.00032\Delta p^{1.2}) + 2,560,000D^{1.5} \quad n = \frac{30,000 \text{ m}}{L}$$

- Where L is the length of each pipe section (m)

$$\Delta p = f \frac{L V^2}{D} \rho = (0.02) \frac{L}{D} \left(\frac{0.16}{\frac{\pi D^2}{4}} \right)^2 \frac{1}{2} (1000 \text{ kg/m}^3) \quad \rightarrow \quad \frac{\Delta p D^5}{L} = 0.4150$$

- The statement of the problem is

$$y = \frac{75,000,000}{L} + \frac{9.6\Delta p^{1.2}}{L} + 2,560,000D^{1.5}$$

$$\frac{2.410\Delta p D^5}{L} = 1$$

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Example 11.5

(Solution)

$$y^* = \left(\frac{75,000,000}{w_1} \right)^{w_1} \left(\frac{9.6}{w_2} \right)^{w_2} \left(\frac{2,560,000}{w_3} \right)^{w_3} \left(\frac{2.41}{w_4} \right)^{w_4}$$

- Provided that

$$L: \quad -w_1 - w_2 \quad - Mw_4 = 0$$

$$\Delta p: \quad 1.2w_2 \quad + Mw_4 = 0$$

$$D: \quad 1.5w_3 + 5Mw_4 = 0$$

$$w_1 + w_2 + w_3 = 1$$

$$Mw_4 = M$$

$$\therefore w_1 = 0.0385, w_2 = 0.1923, w_3 = 0.7692, M = -0.2308, Mw_4 = -0.2308$$

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Example 11.5

(Answer)

$$y^* = \left(\frac{75,000,000}{0.0385}\right)^{0.0385} \left(\frac{9.6}{0.1923}\right)^{0.1923} \left(\frac{2,560,000}{0.7692}\right)^{0.7692} \left(\frac{2.41}{1}\right)^{-0.2308} = \$410,150$$

$$u_1^* = (410,150)(0.0385) = \frac{75,000,000}{L^*} \rightarrow L^* = 4750 \text{ m}$$

$$u_3^* = (410,150)(0.769) = 2,560,000 D^{1.5} \rightarrow D^* = 0.246 \text{ m}$$

$$\Delta p^* = \frac{L^*}{2.410 D^{*5}} = 2,188,000 \text{ Pa} = 2188 \text{ kPa}$$