

Lecture Notes 414.341

선박해양유체역학

MARINE HYDRODYNAMICS

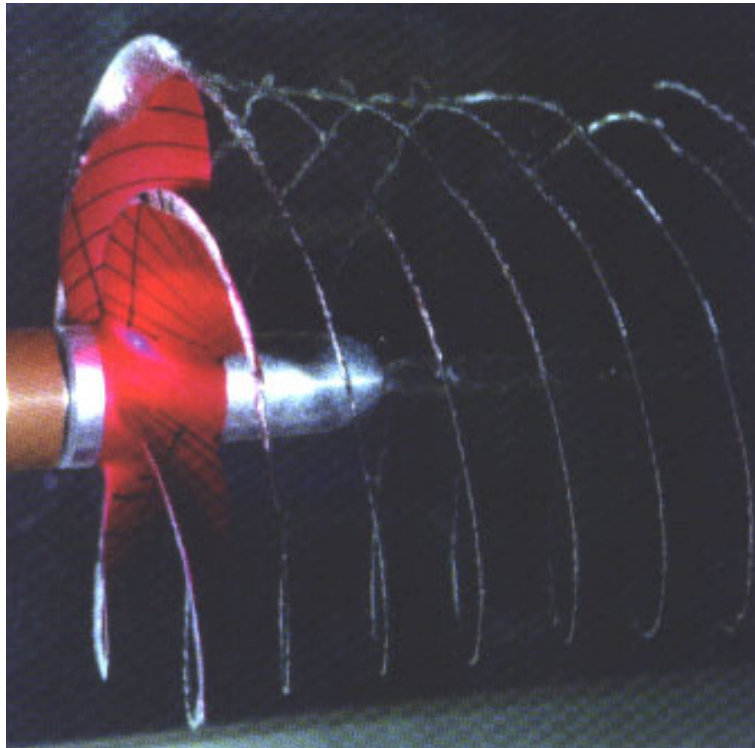
2019년 6월 28일

Suh, Jung-Chun
서정천

Seoul National Univ., Dept. NAOE
서울대학교 공과대학 조선해양공학과

선박해양유체역학

MARINE HYDRODYNAMICS



Suh, Jung-Chun
서정천

Contents

0. OUTLINE OF COURSE	1
0.1 Course Description	1
0.2 References	2
0.3 Course Contents	2
1. INTRODUCTION	5
1.1 Continuum Mechanics	6
1.1.1 Definition of Fluid	6
1.1.2 Assumptions and Axioms	7
1.1.3 Basic Equations	10
1.2 Characteristics of Hydrodynamics	12
1.2.1 Types of Fluid Flow	12
1.2.2 Various Characteristic Effects	14
1.2.3 Characteristics of Ship/Marine Hydrodynamics	14
1.2.4 Ocean Environment	16
1.3 Mathematical Prerequisites: Vector Analysis	24
1.3.1 Fundamental Function Analysis	24
1.3.2 Vector Calculus	25
1.3.3 Expansion Formulas	30
1.3.4 Divergence Theorem (Gauss Theorem)	31

1.3.5 Stokes' Theorem	32
1.3.6 Dyadic Products	33
1.3.7 Reynolds Transport Theorem	34
1.3.7.1 Example of the Reynolds transport theorem in 1-D	35
1.3.8 Moving Coordinate Systems	37
2. MODEL TESTING	39
2.1 Introduction	41
2.1.1 Dimensional Analysis	41
2.1.2 Flow Similarity and Model Studies	42
2.1.3 Nature of Dimensional Analysis: Example	43
2.1.4 Significant Dimensionless Numbers	44
2.1.5 Error Estimates in Uncertainty Analysis	45
2.1.6 Flow Visualization	45
2.2 Drag Force on a Sphere	46
2.2.1 Dimensional Analysis	46
2.2.2 Pressure Drag Variation with Reynolds Numbers	48
2.3 Viscous Drag on a Flat Plate	50
2.3.1 Dimensional Analysis for Frictional Drag	50
2.3.2 Transition Range of Reynolds Numbers	52
2.4 Viscous Drag on General Bodies	54
2.4.1 Infeasible Tests of Geosims	54
2.4.2 Frictional Drag and Pressure Drag	54
2.5 Hydrofoil Lift and Drag	58
2.5.1 Lifting Surfaces	58
2.5.2 Lift and Drag on Hydrofoil	61
2.5.3 Remarks: Induced Drag for 3-D Lifting Surfaces	63

2.6 Screw Propeller	63
2.7 Drag on a Ship Hull	71
2.8 Propeller-Hull Interactions.	77
2.8.1 Propeller and Ship Powering	80
2.9 Unsteady Force on an Accelerating Body	85
2.9.1 Added Mass	87
2.10 Vortex Shedding.	87
2.11 Water Waves	93
2.11.1 Dispersion Relation of a Progressive Wave in Infinite Depth	93
2.11.2 Secondary Effects: Viscosity, Air, Surface tension, Nonlinear Effects.	96
2.11.3 Solutions for Finite Depth	100
2.11.4 Shallow Water Limit	101
2.11.5 Superposition of Waves: Group Velocity	106
2.11.6 Wave Force on a Stationary Body.	108
2.11.7 Body Motions in Waves	110
2.11.8 Ship Motions in Waves	111
 3. KINEMATICS	 115
3.1 Description of Fluid Motion	116
3.1.1 Definition of Fluid Particle	116
3.1.2 Lagrangian Description: Path Lines	116
3.1.3 Eulerian Description	117
3.1.3.1 Local derivative	117
3.1.3.2 Material derivative	118
3.1.4 Particle Tracing Lines	119
3.1.4.1 Streamlines.	119
3.1.4.2 Streaklines	120

3.1.5 Example of Particle Tracing Lines	120
3.1.5.1 Velocity field.	120
3.1.5.2 Pathlines	121
3.1.5.3 Streamlines.	122
3.1.5.4 Streaklines	123
3.2 Conservation of Mass: Continuity Equation.	124
3.2.1 Material Volume Approach	124
3.2.2 Control Volume Approach.	125
3.2.3 Special Cases: Steady Motion and Incompressible Flow.	128
3.3 Vorticity and Circulation	129
3.3.1 Definition of Vorticity	129
3.3.2 Vortex Line and Vortex Tube	130
3.3.3 Circulation and Vorticity Flux	131
3.3.4 Vortex Strength	132
4. DYNAMICS	135
4.1 Forces	136
4.1.1 Body Forces	136
4.1.2 Surface Forces.	137
4.1.3 Stress and Stress Tensor	138
4.1.4 Surface Tension	139
4.2 Equations of Motion: Navier-Stokes Equations	141
4.2.1 Newton's Second Law for Material Volume	141
4.2.2 Surface Forces for Differential Control Volume	142
4.2.3 Stress and Strain Rate in a Newtonian Fluid	143
4.2.4 Navier-Stokes Equations for Incompressible Newtonian Fluids	148
4.2.4.1 Alternate forms of the convective and diffusion terms.	149

4.2.5 Navier-Stokes Equations in a Moving Frame	150
4.2.5.1 Kinematic description	150
4.2.5.2 Representation of velocity field	151
4.2.5.3 Governing equations	152
4.2.6 Boundary Conditions	153
4.3 Example: Low Reynolds Number Flows	153
4.3.1 Velocity Field: Solution of the Simplified Navier-Stokes Equations . .	154
4.3.2 Stress Tensor and Drag	157
4.4 Simple Problems for Viscous Flows	158
4.4.1 Flow between Two Parallel Walls (Plane Couette Flow)	158
4.4.2 Flow through a Pipe(Poiseuille Flow)	160
4.4.3 External Flow Past One Flat Plate: Unsteady Motion of a Flat Plate . .	160
4.4.3.1 Stokes 1st problem	161
4.4.3.2 Stokes 2nd problem	162
4.5 Boundary Layer Theory	164
4.5.1 Introduction	164
4.5.2 Laminar Boundary Layer: Flat Plate	165
4.5.3 Laminar Boundary Layer: Steady 2-D Flow	171
4.5.3.1 Von Karman integral relation	174
4.5.3.2 Laminar boundary layers: Closing remarks	175
4.6 Turbulent Flows: General Aspects	176
4.6.1 Introduction	176
4.6.2 Analysis of Turbulent Flows	178
4.6.3 Turbulent Boundary Layer on a Flat Plate	181
4.6.4 Turbulent Boundary Layers: Closing Remarks	187

5. POTENTIAL FLOWS	195
5.1 Euler Equations for Inviscid Flows	196
5.1.1 General Aspects	196
5.1.2 Conservation of Circulation: Kelvin's Theorem	199
5.1.2.1 Introduction	199
5.1.2.2 Time rate of circulation for barotropic fluids	199
5.1.2.3 Case of inviscid flows	201
5.1.2.4 Another derivation of Kelvin's theorem	202
5.1.2.5 Circulation generation: Kutta condition	203
5.1.3 Irrotational Flow and Velocity Potential	204
5.2 Energy Equation: Bernoulli Equation	205
5.2.1 Integration of Euler Equations for Inviscid Fluids	205
5.2.2 Case 1: Inviscid Steady (possibly Rotational) Incompressible Flow along Streamline	206
5.2.2.1 Steady barotropic flows	207
5.2.3 Case 2: Inviscid Unsteady Irrotational Incompressible Flow	209
5.2.3.1 Irrotational barotropic flows	209
5.3 Boundary Conditions	210
5.3.1 Kinematic Boundary Condition	210
5.3.1.1 Alternate form	211
5.3.2 Dynamic Boundary Condition: Free Surface Condition	212
5.4 Simple Potential Flows	213
5.4.1 Irrotational Inviscid Flows: Potential Flows	213
5.4.1.1 Uniqueness of Laplace equation	215
5.4.2 Techniques for Solving the Laplace Equation	215
5.4.3 Source/Sink/Dipole/Vortex in Free Stream (Uniform Flow)	216
5.4.3.1 Flow past a sphere	220

5.4.4	Flow around a Circular Cylinder	221
5.4.4.1	Circular cylinder with circulation.	222
5.5	Stream Function.	223
5.5.1	Various Singularity and Simple Potential Flows	225
5.6	Separation of Variables	227
5.6.1	Fixed Bodies and Moving Bodies	230
5.7	Method of Images.	231
5.8	Green’s Theorem and Distributions of Singularities	236
5.8.1	Scalar Identity	236
5.9	Hydrodynamics Pressure Forces: Forces and Moments Acting on a Body.	239
5.10	Force on a Moving Body in an Unbounded Fluid	242
5.10.1	Far-field Behavior of Velocity Potential	242
5.10.2	Decomposition of Velocity Potential: 6 Degrees of Freedom	243
5.11	Added Mass	245
5.11.1	General Properties of Added-Mass Coefficients	246
5.11.2	Added Mass of Simple Forms	249
5.12	Body-Mass Force: Equations of Motion	251
A.	VECTOR ANALYSIS	255
A.1	Introduction	257
A.1.1	Vector and Tensor Notation	257
A.1.2	Fundamental Function Analysis	258
A.2	Vector Calculus	260
A.2.1	Definition of Vector Quantity.	260
A.2.2	Basic Unit Tensors	263
A.2.2.1	Kronecker delta tensor.	263
A.2.2.2	Permutation tensor	263
A.2.2.3	Multiplication of basic tensors	264
A.2.2.4	Example of permutation tensor	265

A.2.3	Multiplication of Vectors	266
A.2.4	Vector Derivatives	267
A.2.4.1	Gradient	267
A.2.4.2	Divergence	268
A.2.4.3	Curl.	269
A.2.4.4	Laplacian	269
A.2.4.5	Differential operators	270
A.2.4.6	Directed derivative	270
A.2.5	Expansion Formulas	271
A.3	Integral Theorems	272
A.3.1	Divergence Theorem.	272
A.3.2	Stokes' Theorem	274
A.3.3	Volume Integrals of a Vector	276
A.4	Curvilinear Orthogonal Coordinates	278
A.4.1	Line element	278
A.4.2	Gradient	280
A.4.3	Divergence	281
A.4.4	Curl.	281
A.4.5	Laplacian	282
A.4.6	Convection term	283
A.5	Tensors of Second Order	284
A.5.1	Dyadic Products	284
A.5.2	Gradient of a Vector	285
A.6	Reynolds Transport Theorem	287
A.6.1	Mathematical Derivation of Transport Theorem.	287
A.6.2	Alternative Derivation of Transport Theorem	290

A.7 Moving Coordinate Systems	291
A.7.1 Velocity due to Rigid Body Rotation	291
A.7.2 Transformations of Moving Coordinates	293
A.8 Mathematical Identities	295
A.8.1 Green's Scalar Identity	295
A.8.2 Uniqueness of Scalar Identity	298
A.8.3 Type of Boundary Conditions	299
A.8.4 Vector Identity	301
A.8.5 Integral Expression of Helmholtz Decomposition	305
A.8.6 Uniqueness of Vector Identity	307
A.8.7 Improper Integrals	308
A.8.8 Green Functions	310

List of Figures

1.1 Behavior of a solid and a fluid, under the action of constant shear force.	6
1.2 Definition of density.	8
1.3 Measured data to define density with variation of length scale.	9
1.4 Possible classification of fluid mechanics.	13
1.5 Schematic diagram of various types of fluid flow.	13
1.6 Variation of density with salinity and temperature at atmospheric pressure. . .	17
1.7 Typical variation of depth versus density for different global latitudes.	19
1.8 Variation of surface temperature, salinity and density with latitude–average for all oceans.	20
1.9 Typical viscous velocity gradient.	21
1.10 Molecular explanation of surface tension.	22
1.11 Sequential change of the interval of the integral (1.106).	35
1.12 Moving coordinate system.	38
2.1 The drag coefficient of a sphere.	44
2.2 Definition of bias and precision error.	45
2.3 Probability error estimates.	46
2.4 Typical PIV system for cavitation tunnel.	47
2.5 The drag coefficient of a sphere.	48
2.6 The drag coefficient of a sphere for moderate Reynolds number.	49
2.7 Wake of a sphere for moderate Reynolds number.	49

2.8	The drag coefficient of a sphere at critical Reynolds number.	50
2.9	Wake of a sphere at critical Reynolds number.	51
2.10	Wake of a sphere at critical Reynolds number.	51
2.11	Schoenherr's flat-plate frictional drag coefficient.	52
2.12	Transition from laminar to turbulent flow for a flat plate drag coefficient.	53
2.13	Flow over a flat plate normal to inflow.	55
2.14	Drag coefficient for two-dimensional cylinders.	56
2.15	Drag on a strut: Contributions of frictional drag and pressure drag to total drag as a function of thickness-length ratio.	56
2.16	Drag coefficient for a circular cylinder.	57
2.17	Drag coefficient for a sphere.	57
2.18	Three-dimensional lifting surface.	58
2.19	Geometry of a hydrofoil section.	59
2.20	Flow past a hydrofoil section.	60
2.21	Flow past a foil without circulation.	60
2.22	Assumed flow past a foil with circulation.	60
2.23	Lift coefficient for a two-dimensional hydrofoil.	61
2.24	Lift characteristics of hydrofoils.	62
2.25	Drag coefficient for a two-dimensional hydrofoil.	64
2.26	Trailing vortex system of a wing.	65
2.27	Downwash distribution for trailing vortex system on a wing.	65
2.28	Derivation of induced drag on a wing.	66
2.29	Propeller coordinate system.	67
2.30	Propeller blade geometry in 3 coordinate systems.	68
2.31	Perspective view of a propeller and its shaft.	69
2.32	2-D view of a propeller blade section.	70
2.33	Experimental setup for propeller open water test in towing tank.	70

2.34	Components of a typical marine propulsion system.	71
2.35	Decomposition of ship resistance.. . . .	72
2.36	Components of ship resistance.	72
2.37	Froude’s hypothesis for prediction of resistance of a full-scale ship.	73
2.38	Calculation of model and full scale frictional drag coefficients.	74
2.39	Total drag coefficients of geosim models.	75
2.40	Total drag coefficients of geosim models: Validation check on Froude hypothesis.	76
2.41	Total drag coefficients of geosim models: Viscous form drag.	77
2.42	Frictional drag coefficients.	78
2.43	Total drag coefficients of geosim models: Larger models effect.. . . .	79
2.44	Extrapolation using form factor k	79
2.45	Schematic diagram of propeller-hull interaction.	80
2.46	Thrust identity condition between open-water propeller test and self-propulsion test.	81
2.47	Components of a typical marine propulsion system.	81
2.48	Schematic diagram of propeller and ship powering.	82
2.49	Transmission efficiency of mechanical drives and electric drives.	83
2.50	Development of unsteady flows about an impulsively started circular cylinder.. . . .	86
2.51	Force coefficient for an impulsively started circular cylinder.. . . .	88
2.52	Karman vortex street.	88
2.53	Karman vortex street in nature.	89
2.54	Lateral lift force and shedding frequency.	89
2.55	Strouhal number for a circular cylinder at low Reynolds numbers.	90
2.56	Drag coefficient and Strouhal number for a circular cylinder at moderate and high Reynolds numbers.. . . .	91
2.57	Lock-in phenomena between the lateral lift force and vortex-shedding frequency.. . . .	91

2.58	Example of VIV(Vortex Induced Vibration) phenomena: Collapse of the Tacoma bridge..	92
2.59	Sketch of a periodic progressive wave in a fluid of mean depth h	93
2.60	Velocity field of a plane progressive wave in deep water.	96
2.61	Surface tension for two stationary fluids in 2-dimensions.	98
2.62	Surface tension for a stationary spherical bubble..	98
2.63	The “steepest wave” profile of nonlinear waves.	99
2.64	Velocity field of a plane progressive wave in finite depth.	101
2.65	Particle trajectories of a plane progressive wave in finite depth.	102
2.66	Phase velocity, wave length, and depth ratios of a plane progressive wave in finite depth.	103
2.67	Propagation of plane progressive waves into shallow water.	104
2.68	Propagation of the Indian Ocean Tsunami of 26 December 2004.	105
2.69	Wave group resulting from the superposition of two nearly equal plane waves..	105
2.70	Phase velocity and group velocity ratios for a plane progressive wave as a function of the water depth.	106
2.71	Sequence of photographs showing the phase velocity and the group velocity for a plane progressive wave system advancing into calm water..	107
2.72	Heave response of slender spar buoy in regular waves.	112
2.73	Ship motions in 6 degrees of freedom.	113
2.74	Roll and pitch response of a 319 m ship..	114
3.1	Example of various flow lines.	121
3.2	Material surface of interface between two fluids..	124
3.3	Differential control volume in Cartesian coordinates.	126
3.4	Differential control volume in cylindrical coordinates.	129
3.5	Viscous shearing and vorticity generation by no-slip boundary condition at wall boundary.	130

3.6	Line and surface integrals for Stokes' theorem.	131
3.7	Circulation about a vortex tube.	132
3.8	Vortex filament approximation of a vortex tube.	133
4.1	Stress tensor in Cartesian coordinates.	138
4.2	Stress vector at surface.	138
4.3	Force diagram for a spherical bubble with surface tension.	140
4.4	Stresses in the x -direction on a differential control volume in Cartesian coordinates.	142
4.5	Motion of a fluid particle.	143
4.6	Deformation of fluid element in 2-D flows.	144
4.7	Rotational motion of a fluid particle.	144
4.8	Squeeze motion of a fluid particle.	145
4.9	Pressure diagram of a fluid.	146
4.10	Stress and strain for simple shear flow of a Newtonian fluid.	148
4.11	Moving coordinate system.	150
4.12	Notation for low Reynolds number flow about a spherical body in uniform flow.	154
4.13	Viscous flow around a foil.	158
4.14	Plane Couette flow between two parallel walls.	159
4.15	Stokes 1st problem: Sinusoidal motion of flat plate.	162
4.16	Velocity profile near an impulsively started flat plate (Stokes 2nd problem).	163
4.17	Boundary layer thickness in Stokes 2nd problem for an impulsively started flat plate.	164
4.18	Variation of boundary layer thickness with Reynolds numbers.	165
4.19	Boundary layer on a flat plate.	166
4.20	Velocity profile in a Blasius laminar boundary layer.	168
4.21	Boundary layer thickness definition.	169

4.22	Development of laminar boundary layer along a flat plate.	169
4.23	Flat-plate frictional resistance coefficient: Blasius laminar boundary layer solution.	170
4.24	Laminar boundary layer: Steady 2-D flow.	171
4.25	Boundary layer flow with pressure gradient.	172
4.26	Flow past a body with separation.	172
4.27	Boundary layer equations with pressure gradient.	173
4.28	Velocity profiles and growth of the boundary layer near the leading edge of a foil.	176
4.29	Attached and separated flows about a foil.	176
4.30	Flow visualization of laminar, transition and turbulent flows in pipe.	177
4.31	Decomposition of turbulent quantities into an average and a fluctuating part.	179
4.32	Comparison of laminar and mean turbulent velocity profiles.	181
4.33	Mean velocity profile of turbulent flow close to a smooth wall.	184
4.34	Logarithmic form of the velocity defect law for a turbulent boundary layer of a smooth wall.	185
4.35	Boundary layers of turbulent flows with wall regions.	185
4.36	Schoenherr's flat plate frictional resistance coefficient (ATTC line).	186
4.37	Eddy Structure of turbulent boundary layers near a flat plate.	188
4.38	Turbulent viscous flows generated on body surface.	189
4.39	Comparison of laminar and turbulent separation on a body surface.	189
4.40	Flow over sphere: Effect of Reynolds number.	190
4.41	Modification of boundary layer flows.	190
4.42	Control of flow separation on a sphere by turbulence stimulator.	191
4.43	Examples of simulation by CFD.	192
4.44	Examples of flow over bodies.	192
4.45	Examples of coupled solid-fluid interaction.	193
4.46	Pattern of the turbulent kinetic energy spectrum.	193

4.47	Measurements of turbulent flows	194
5.1	Schematic view of the Hele-Shaw flow for visualization of inviscid potential flow.	198
5.2	Initial development of the circulatory flow past a hydrofoil by Kutta condition.. . . .	203
5.3	Open and closed line integrals in region of an irrotational flow: Path independent integrals of velocity potential.	204
5.4	Semi-infinite half-body generated by a point source at the origin in free stream.	217
5.5	Rankine ovoid generated by a point source and sink combination in free stream.	218
5.6	Schematic concept of dipole singularity by approach of source and sink.	219
5.7	Spherical coordinate system for showing the flow about a sphere..	219
5.8	Pressure distribution over the surface of a sphere.	221
5.9	Measure of flux across a contour: Stream function.	224
5.10	Streamlines about a Rankine ovoid..	226
5.11	Streamlines about a ship forebody.	228
5.12	Method of images: Source above a wall.	232
5.13	Method of images: Circular cylinder above a wall.	233
5.14	Method of images: Vortex above a wall..	234
5.15	Method of images: Circular cylinder and vortex inside an L-corner.	234
5.16	Method of images: Circular cylinder in two parallel walls..	235
5.17	Surfaces of integration for Green's theorem.	237
5.18	Fixed control surface S_C surrounding the moving body surface S_B	239
5.19	Position vectors in inertial coordinate system fixed in space and moving system fixed in a body.	245
5.20	Approximation of added mass for a square by using the inscribed circle and the circumscribed circle..	251
5.21	Added mass for a spheroid or an ellipsoid of revolution.	252

A.1 Two Cartesian coordinate systems rotated with respect to one another.	261
A.2 Cylindrical and spherical coordinate systems.	279
A.3 Change of material volume in transport of physical quantity.	290
A.4 Rotation of a rigid body.	292
A.5 Moving coordinate system.	293
A.6 Two-dimensional drawing of a simply connected region for deriving the scalar identity.	296
A.7 Small sphere region containing a singular point.	297

List of Tables

1.1 Density variations with temperature (salinity 3.5%).	18
1.2 Density variations with temperature (fresh water).	18
1.3 Viscosity of sea water with temperature (salinity 3.5%).	22
1.4 Viscosity of fresh water with temperature.	22
1.5 Saturation vapour pressure p_v for fresh and sea water.	22
1.6 Typical values of surface tension for sea and fresh water with temperature.	23
2.1 Dimensions of fluid properties	42
2.2 Dispersive relation of deep water ocean waves.	95
2.3 Relative effect of nonlinear waves	100
4.1 Boundary layers of turbulent flows with wall regions	183
4.2 Comparison between laminar and turbulent boundary layers	188
5.1 Velocity potential versus stream function	225
5.2 Various singularity and simple potential flows	227
5.3 Comparison of flows for fixed bodies and moving bodies: circle and sphere of radius r_0	231
5.4 Far-field behavior of singularities	242
5.5 Added mass of various simple 2-D bodies.	250

1

INTRODUCTION

1.1 Continuum Mechanics	6
1.1.1 Definition of Fluid	6
1.1.2 Assumptions and Axioms	7
1.1.3 Basic Equations.	10
1.2 Characteristics of Hydrodynamics.	12
1.2.1 Types of Fluid Flow	12
1.2.2 Various Characteristic Effects	14
1.2.3 Characteristics of Ship/Marine Hydrodynamics.	14
1.2.4 Ocean Environment.	16
1.3 Mathematical Prerequisites: Vector Analysis	24
1.3.1 Fundamental Function Analysis	24
1.3.2 Vector Calculus	25
1.3.3 Expansion Formulas	30

1.3.4 Divergence Theorem (Gauss Theorem)	31
1.3.5 Stokes' Theorem	32
1.3.6 Dyadic Products	33
1.3.7 Reynolds Transport Theorem.	34
1.3.8 Moving Coordinate Systems	37

1.1 Continuum Mechanics

1.1.1 Definition of Fluid

- Fluid can not withstand shearing forces when a shear stress is applied: Fluids continuously deform and Solids deform or bend.
- While solid can be in stable equilibrium under shear stress oblique to the surface separating any two parts, fluid cannot be in stationary equilibrium.

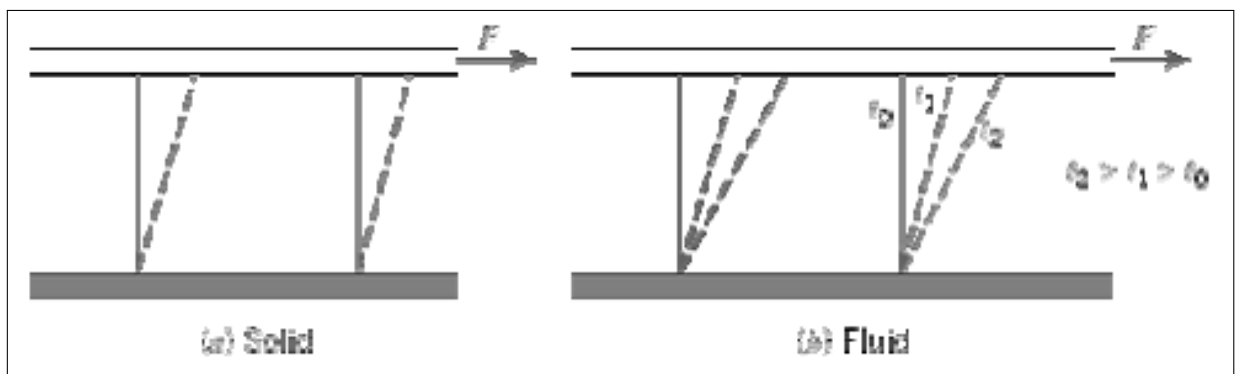


Figure 1.1 Behavior of a solid and a fluid, under the action of constant shear force. Solid (*left*); Fluid (*right*). (From Fox, McDonald & Pritchard 2004)

- Resistance to rate of shear deformation from viscosity gives rise to drag for bodies. We can easily recognize that such shear stresses do exist in fluids: e.g., consider how the fluid in a rotating circular vessel takes on the rotating motion of the vessel eventually.

- Other observed properties of fluids are:
 - resistance to volumetric compression and tension in general,
 - no shape or preferred orientation (Solids: definite shape; Fluids: no preferred shape),
 - homogeneous matter in general, and
 - has mass.
- There are two kinds of fluids depending on bulk elasticity (compressibility): ¹
 - Liquid forms a free surface(density $\rho \approx 0$ above free surface).
 - Gas expands to fill container.

1.1.2 Assumptions and Axioms

- Continuum Viewpoint: Fluid as a continuum in macroscopic scale compared with molecules.

We assume that the fluid is continuous and homogeneous in structure.

- Actually this is not so since matter is ultimately made up of molecules and atoms, but in many applications the dimensions we are concerned with are large compared to the molecular structure, and the smallest sample of fluid that concerns us contains a very great number of molecules (i.e., number of about $2.687 \times 10^{19}/cm^3$). ²
- In such cases, the properties of any sample are the average values over many molecules, and the approximation of a continuum is found to be acceptable and useful.

* Measurable smallest scale: length $l \sim O(10^{-5}) m$, volume $V \sim O(10^{-15}) m^3$.

¹On the mechanism of formation of liquid and vapor, see Brennen, C. (1995), *Cavitation and Bubble Dynamics*, Oxford University Press, pp. 1–6.

²Avogadro no./1 mol = $6.02214 \times 10^{23}/22.414$ liter. This number is about 10 times as many as the total no. of stars in the known universe, and about 100 times as many as grains of sand on all beaches and deserts. (See Garrison, T. (2007), *Oceanography: An Invitation to Marine Science*, 6th ed., Thomson Brooks/Cole.) Suppose a supercomputer can count at the rate of one billion molecules per second. For a cubic centimeter of a fluid we would take a very long time for counting the occupied molecules: 2.687×10^{10} sec \approx 850 yr.

- Nano scale devices: $l \sim O(10^{-8}) m$
- Ocean current diameter: $l \sim O(10^6) m$
- * No. of air molecules in the volume at standard pressure: $\sim 3 \times 10^{10}$.
- * No. of water molecules in the volume at standard pressure: $\sim 10^{13}$.

Example: definition of density at a point (see Figures 1.2 and 1.3)

$$\rho \equiv \lim_{\delta V \rightarrow \delta V'} \frac{\delta m}{\delta V} = \rho(x, y, z, t) \quad (1.1)$$

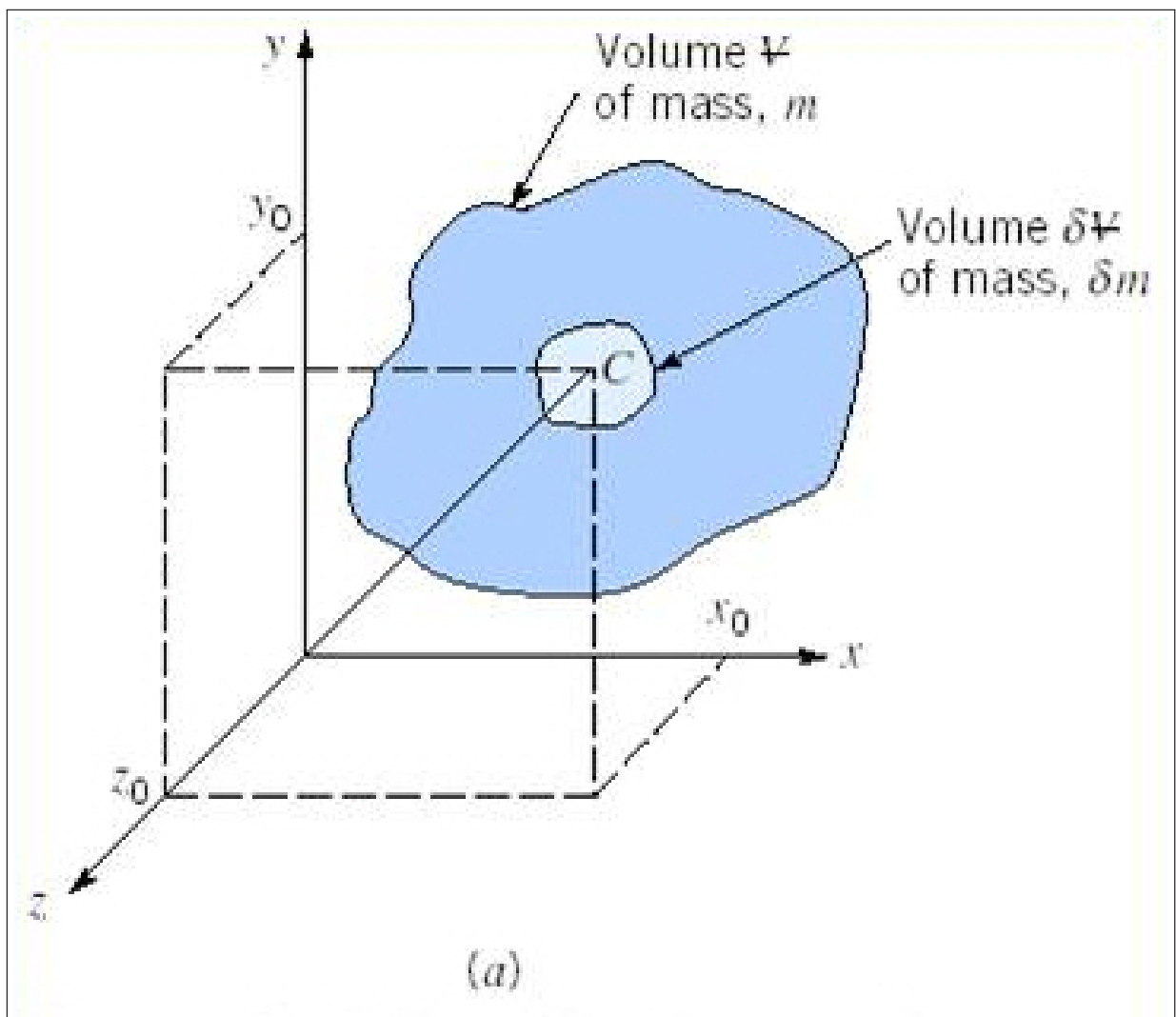


Figure 1.2 Definition of density. (From Fox, McDonald & Pritchard 2004)

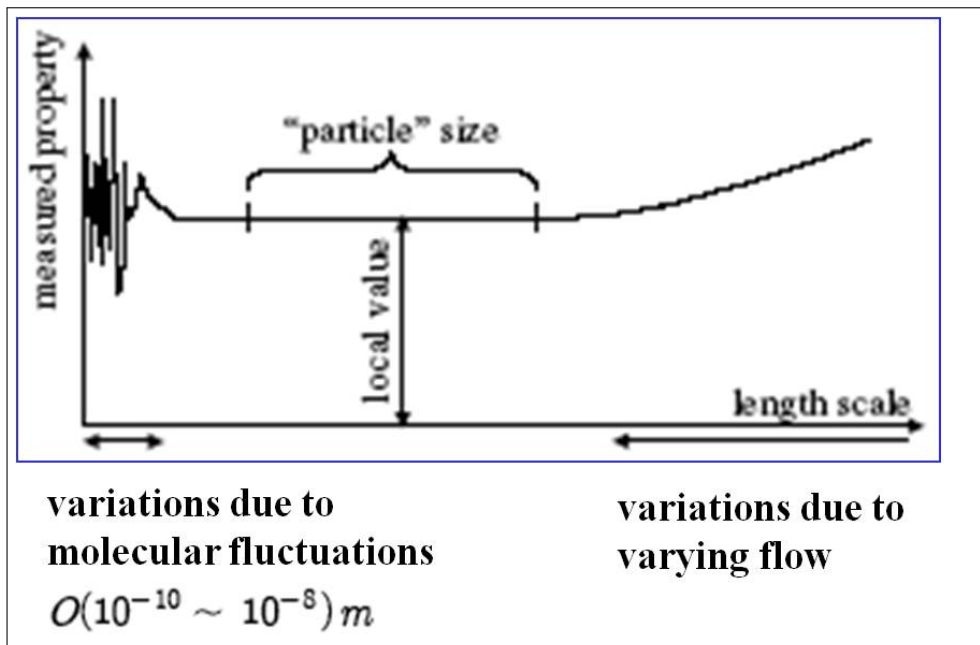


Figure 1.3 Measured data to define density with variation of length scale. (From Pantton 1984)

- Nevertheless, results obtained on the assumption of a continuum may be erroneous whenever the molecular structure dimensions are relatively large.
 - For example, at very high altitudes (low pressures), the molecular spacing is so great that air is not even approximately a continuum in its contact with a body.
 - Failures of the continuum assumption occur probably in the cases of that body size compares with molecular dimensions (e.g., a very small body in a fluid) or with distances between molecules (e.g., a body in a rarefied gas).
- Other acceptable and useful assumptions are those as follows:
 - (1) that physical laws are independent of the coordinate system used to express them (frame indifference),
 - (2) that natural laws are independent of the dimensions of physical quantities that occur in the expressions (dimensional homogeneity),
 - (3) that derivations of physical quantities with respect to space and time exist to the required order (smoothness of quantities), and

- (4) that the present motion is a function of its history and not the future (memory of history).

1.1.3 Basic Equations

- Fundamental Laws of Continuum Mechanics:
 - Conservation of mass: Continuity equation
 - Conservation of momentum (Newton's law of motion)
 - * Principle of momentum and angular momentum
 - * Navier-Stokes equations for viscous flow
 - * Euler's equation for inviscid flow
 - * Bernoulli equation as an energy equation from integration of Euler's equation
 - Conservation of energy (First law of thermodynamics)
 - * The First Law of Thermodynamics: Energy conservation for heat and work interactions
 - * The Second Law of Thermodynamics: Heat flow in direction of entropy increase

We postulate that mass, momentum and energy are conserved: Conservation of mass, Conservation of momentum, Conservation of energy. Since these notes tend to deal mostly with incompressible flows, we do not examine the conservation of energy.

- Newton's equations of motion are derived from rigid body mechanics:

$$\underline{F} = m \frac{d^2 \underline{x}}{dt^2} \quad (1.2)$$

where m = mass of particle or body, \underline{F} = sum of external forces, \underline{x} = position vector, and t = time.

- Three differential equations

- Applicable to mass particle or system of mass particles, differential element or whole part of continuum
- Concept of dynamic and static equilibrium for given external forces
- General equations of motion applicable to arbitrary elements of bodies in concern:

$$\underline{F} = \frac{d\underline{M}}{dt}, \quad \underline{L} = \frac{d\underline{H}}{dt} \quad (1.3)$$

where \underline{M} = linear momentum, \underline{L} = sum of moment in action, \underline{H} = angular momentum.

- Our use of these laws are based on continuum hypothesis.

Equations of motion for continuum:

$$\rho \frac{d^2 \underline{x}}{dt^2} = \rho \underline{b} + \nabla \cdot \underline{T} \quad (1.4)$$

where \underline{T} = stress tensors (internal or surface forces), \underline{b} = body force per unit mass.

- Constitutive Laws

- Solids: Hooke's law, stress = f (strain)
force/area \sim relative displacement/length
- Fluids: stress = f (rate of strain)
force/area \sim velocity gradient

We need the constitutive equations which are a sort of relationships between the stress tensor and the strain tensor, under some assumptions such as homogeneous, isotropic, continuous, elastic (Newtonian) continuum.

Relation to the strain in solid and the strain rate in fluid:

$$\underline{F} \propto \Delta \ell, \quad \underline{F} \propto \Delta u \quad (1.5)$$

where the proportional factors are the elasticity and the viscosity coefficients, respectively.

- Isotropic stress tensors having linear relationship with strain tensor and

strain rate tensor, respectively: ³

$$\underline{\underline{T}} = \lambda' (\nabla \cdot \underline{d}) \underline{\underline{I}} + \lambda [\nabla \underline{d} + (\nabla \underline{d})^T] \quad (1.6)$$

$$\underline{\underline{T}} = [-p + \mu' (\nabla \cdot \underline{u})] \underline{\underline{I}} + \mu [\nabla \underline{u} + (\nabla \underline{u})^T] \quad (1.7)$$

where \underline{d} and \underline{u} are, respectively, the displacement and the velocity, λ' , λ , μ' and μ are the proportional factors.

The differential equations for displacement and velocity, respectively:

$$\rho \frac{\partial^2 \underline{d}}{\partial t^2} = \rho \underline{f}_{body} + (\lambda' + \lambda) \nabla (\nabla \cdot \underline{d}) + \lambda \nabla^2 \underline{d} \quad (1.8)$$

$$\rho \frac{\partial^2 \underline{u}}{\partial t^2} = \rho \underline{f}_{body} - \nabla p + (\mu' + \mu) \nabla (\nabla \cdot \underline{u}) + \mu \nabla^2 \underline{u} \quad (1.9)$$

- Liquids and gases depending on compressibility (bulk elasticity).
 - Liquids: ‘Hydrodynamics’ (Hydrodynamic flows are treated as incompressible.)
 - Gases: ‘Aerodynamics’ (Aerodynamic flows are treated as compressible.)
- Classification of fluid mechanics

1.2 Characteristics of Hydrodynamics

1.2.1 Types of Fluid Flow

With the principal types of fluid flow and their associated phenomena, it is possible to make up practically any flow combination in nature, even the complex system around a moving ship:

- Potential Flow

³For detailed information on difference and similarity among various fields in continuum mechanics and their historical background, see the article: 이승준 (1992), “재료역학과 고체역학: 유체역학자의 관점에서”, 대한조선학회지, 제29권, 제3호.

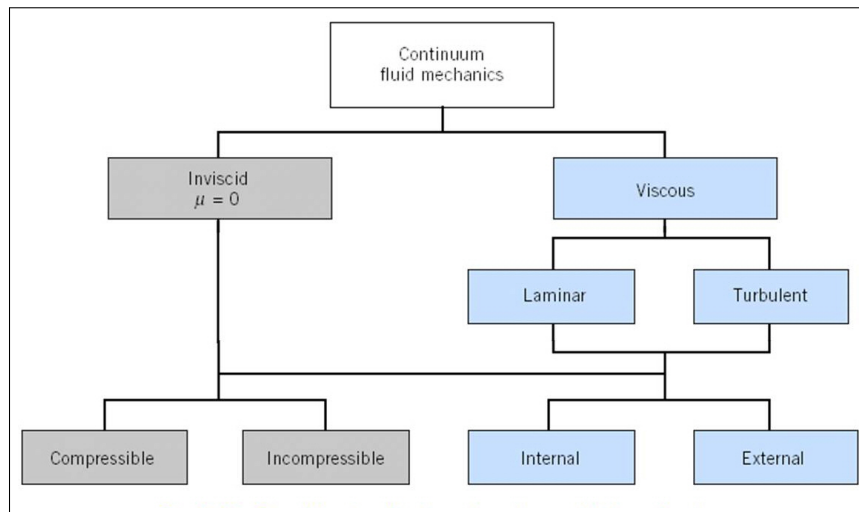


Figure 1.4 Possible classification of fluid mechanics. (From Fox, McDonald & Pritchard 2004)

- Viscous Flow
- Turbulent Flow
- Separation of Flow from a Surface
- Cavitation
- Wavemaking
- Circulatory or Vortical Flow
- Elastic or Compressible Flow

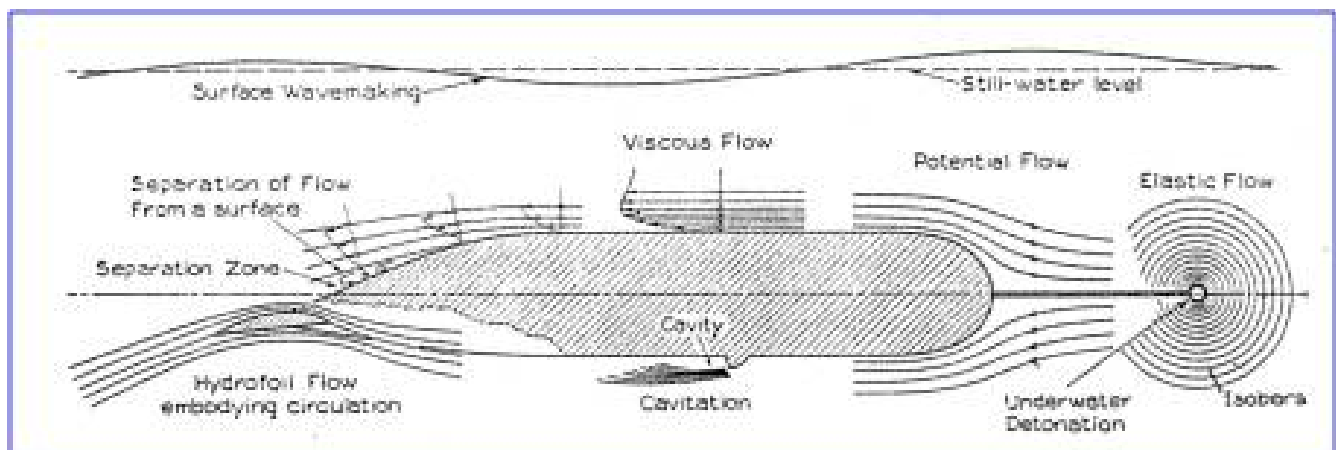


Figure 1.5 Schematic diagram of various types of fluid flow. (From Saunders 1957)

1.2.2 Various Characteristic Effects

Such flow phenomena can be characterized by several principal effects which constitute the basis for important relationships in the form of non-dimensional numbers:

- Velocity effects have the phenomena which are function of the rate at which a body moves in a fluid (e.g. detonation wave around a torpedo exploded).
- Acceleration effects is associated with the acceleration imparted to fluid particles by differences in pressure and other causes (e.g. flow around a propeller blade section).
- Force effects are the application of forces of special nature which is closely related to accelerative effects (e.g. dynamic lift developed by a planning form).
- Inertia effects involve the mass density of the fluid, the velocity of the moving fluid particles, and the necessity for changing their direction (e.g. dynamic pressure on the blunt face).
- Gravity effects result in the changes in potential energy under the influence of the gravity, which occur at an interface with gas (e.g. wavemaking).
- Viscous effects are due to the internal resistance of the fluid to deformation in the shear forces (e.g. flow past a solid surface).
- Elastic effects are due to the compressibility of the fluid (e.g. acoustic wave traveling through water).
- Surface tension effects appear as the attraction between the surface molecules of the real fluid (e.g. air bubbles in water).

1.2.3 Characteristics of Ship/Marine Hydrodynamics

- Ship/Marine Hydrodynamic Aspects: Applicable to naval architecture and ocean engineering

- Many Separate Topics: Propulsion/steering, behavior in waves of a moored buoy or oil-drilling platform
- Related Applications of the General Field of Hydrodynamics
 - Lifting Surfaces: Propellers, Rudders, Anti-rolling fins, Yacht keels, Sails
 - Equations of Motion: Unsteady ship, Buoy, or Platform motions in waves, Maneuvering of ships or submarines in non-straight paths.
- Broad Level of Sophistication: From empirical design methodology to theoretical research activities (as well as intuition and experiment)
 - Diverse Fields of Technology: Fluid mechanics, solid mechanics, control theory, statistics, random process, data acquisition
- Necessary Background
 - Intelligent evaluation and application of empirical procedure
 - Introduction to specialized study on the advanced research
 - Continuum Mechanics: Force and motion in smooth and continuous manner
- Complicated Force Mechanisms
 - 3 Principal Types: Inertial, Gravitational, Viscous
 - Secondary Effects: Surface tension, Elastic, Cavitation
- Physical Parameters: Length, Velocity, Density, Gravity, Viscosity, Pressure

$$\text{– Inertial forces} \sim \text{mass} \times \text{acceleration} \sim (\rho l^3) \left(\frac{U^2}{l} \right) = \rho U^2 l^2 \quad (1.10)$$

$$\text{– Viscous forces} \sim \text{shear stress} \left(\mu \frac{\partial u}{\partial y} \right) \times \text{area} \sim \left(\mu \frac{U}{l} \right) (l^2) = \mu U l \quad (1.11)$$

$$\text{– Gravitational forces} \sim \text{mass} \times \text{gravity} \sim (\rho l^3) g \quad (1.12)$$

$$\text{– Pressure forces} \sim \text{pressure} \times \text{area} \sim (p - p_0) l^2 \quad (1.13)$$

- Dynamical Similarity

$$\frac{\text{Inertial Force}}{\text{Gravitational Force}} = \frac{\rho U^2 l^2}{\rho g l^3} = \frac{U^2}{g l} = (\text{Froude No.})^2 \quad (1.14)$$

$$\frac{\text{Inertial Force}}{\text{Viscous Force}} = \frac{\rho U^2 l^2}{\mu U l} = \frac{\rho U l}{\mu} = \text{Reynolds No.} \quad (1.15)$$

$$\frac{\text{Gravitational Force}}{\text{Viscous Force}} = \frac{\rho g l^3}{\mu U l} = \frac{\rho g l^2}{\mu U} \quad (1.16)$$

- Simultaneous scaling is not possible. Scaling dilemma!
- Cavitation: Change of physical state below vapor pressure at very high speeds

$$\text{Cavitation No.: } \sigma = \frac{p_0 - p_v}{\frac{1}{2} \rho U^2} \quad (1.17)$$

- Difficulty of Navier-Stokes Equations: System of coupled nonlinear P.D.E.
 - Inviscid Assumptions: Mathematical solutions (with free surface effects).
 - Froude's Hypothesis: Total resistance = frictional + residual resistance.
 - Boundary Layer: Viscous effect within thin viscous layer at large Reynolds number.
 - CFD: Numerical simulation by using discretization and approximation of governing equations with physical modelling.

1.2.4 Ocean Environment

- Density of Water
 - Dependence on temperature and salinity.
 - * Greater influence of temperature at a given salinity in a higher temperature regions.
 - * Greater effect of salinity in a lower temperature regions.

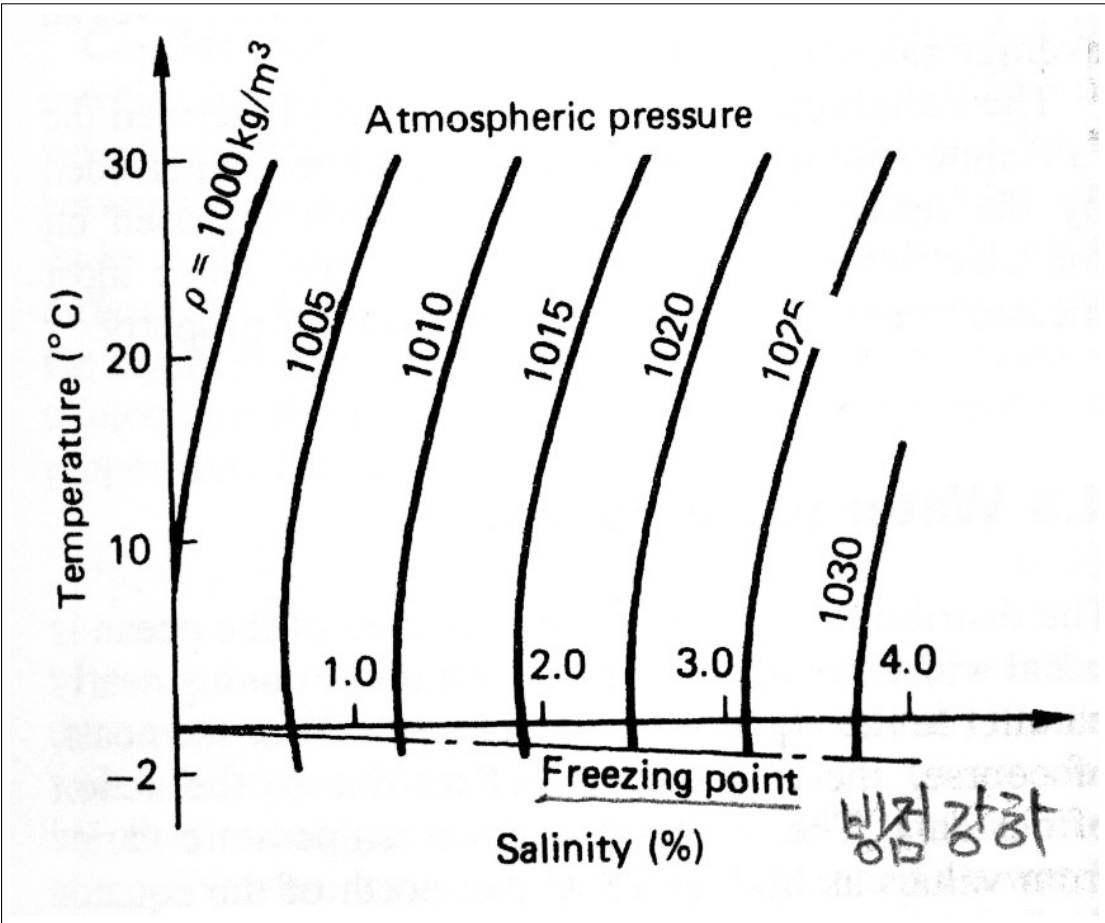


Figure 1.6 Variation of density with salinity and temperature at atmospheric pressure. (From Carlton 1994)

Table 1.1 Density variations with temperature (salinity 3.5%). (Adapted from Carlton 1994)

Temperature ($^{\circ}C$)	0	5	10	15	20	25	30
Density (kg/m^3)	1028.1	1027.7	1026.8	1025.9	1024.7	1023.3	1021.7

Table 1.2 Density variations with temperature (fresh water). (Adapted from Carlton 1994)

Temperature ($^{\circ}C$)	0	4	5	10	15	20	25	30
Density (kg/m^3)	999.8	1000	999.9	999.6	999.0	998.1	996.9	995.6

– Increase with depth.

Sea water: $\rho = 1,026 \text{ kg/m}^3$ at $15^{\circ}C$.

- Salinity = 3.47% at sea surface:

$$\text{Salinity} = 1.80655 \times \text{Chlorinity} \quad \text{in \%} \quad (1.18)$$

- Water Temperature

– From $T = 28^{\circ}C$ at equator to $T = -2^{\circ}C$ near ice in high latitudes.

– Three thermal layers:

* Upper layer between $D = 50 \text{ m}$ and $D = 200 \text{ m}$ below surface:
 $T = 20^{\circ}C$ at surface

* Transition layer to $D = 1,000 \text{ m}$: $T = 8^{\circ}C$ at $D = 500 \text{ m}$ and
 $T = 5^{\circ}C$ at $D = 1,000 \text{ m}$

* Deep ocean region: $T = 2^{\circ}C$ at $D = 4,000 \text{ m}$.

- Viscosity

$$\tau_{yx} = \mu \frac{\partial u}{\partial y} \quad (1.19)$$

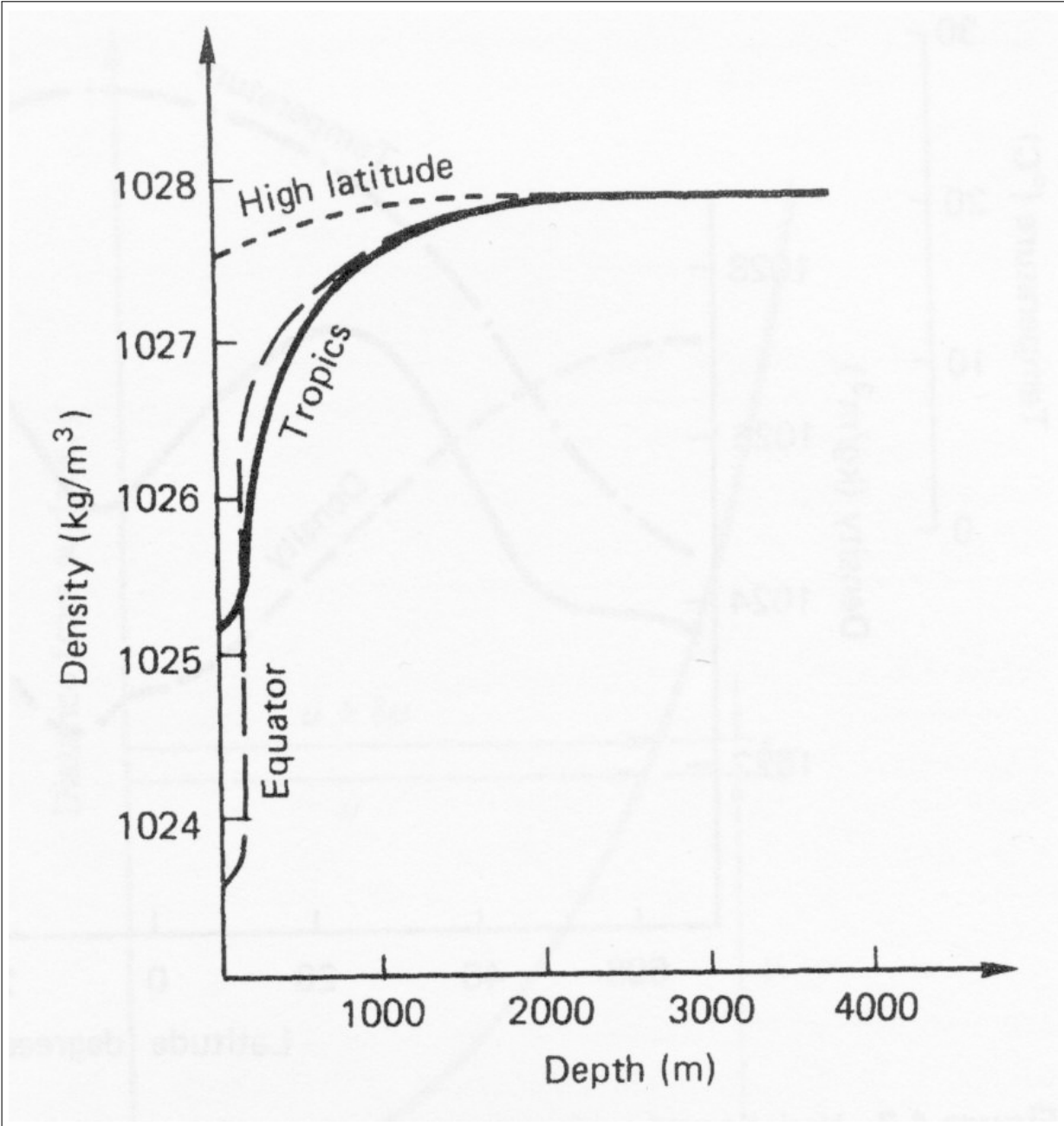


Figure 1.7 Typical variation of depth versus density for different global latitudes. (From Carlton 1994)

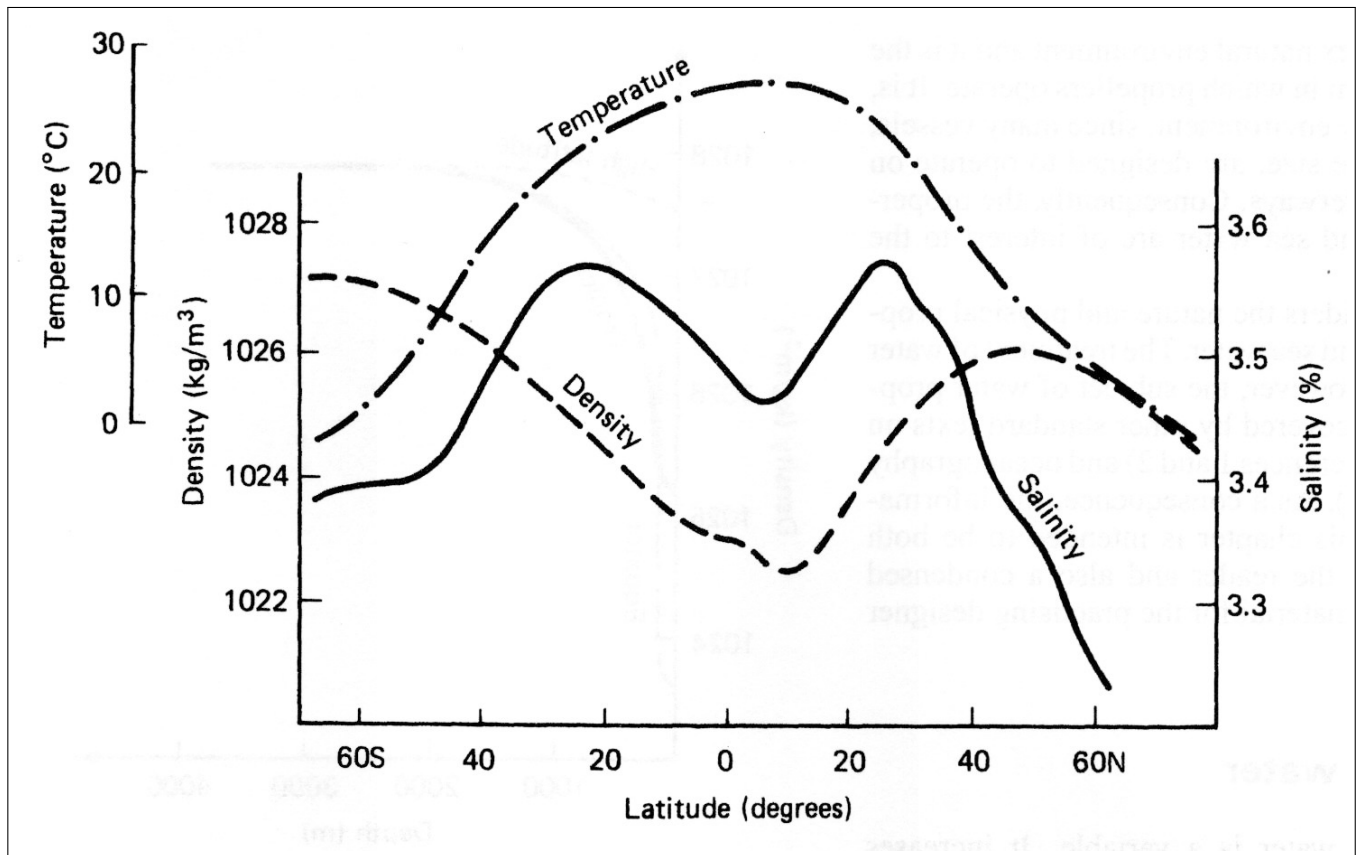


Figure 1.8 Variation of surface temperature, salinity and density with latitude—average for all oceans. (From Carlton 1994)

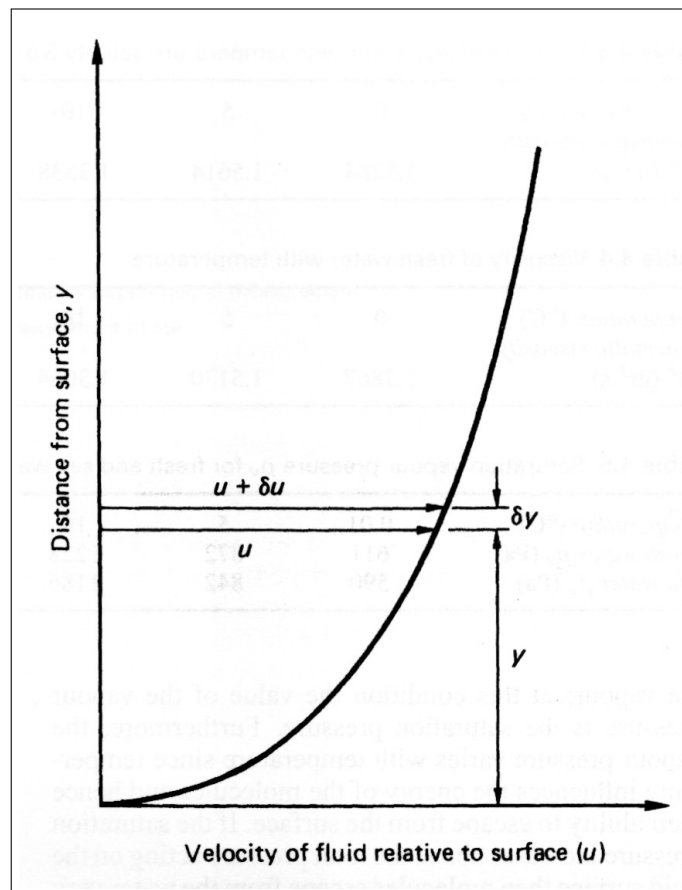


Figure 1.9 Typical viscous velocity gradient. (From Carlton 1994)

Table 1.3 Viscosity of sea water with temperature (salinity 3.5%). (Adapted from Carlton 1994)

Temperature ($^{\circ}C$)	0	5	10	15	20	25	30
Kinematic Viscosity $\times 10^6 (m^2/s)$	1.8284	1.5614	1.3538	1.1883	1.0537	0.9425	0.8493

Table 1.4 Viscosity of fresh water with temperature. (Adapted from Carlton 1994)

Temperature ($^{\circ}C$)	0	5	10	15	20	25	30
Kinematic Viscosity $\times 10^6 (m^2/s)$	1.7867	1.5170	1.3064	1.1390	1.0037	0.8929	0.8009

- Vapour Pressure

Table 1.5 Saturation vapour pressure p_v for fresh and sea water. (Adapted from Carlton 1994)

Temperature ($^{\circ}C$)	0.01	5	10	15	20	25	30
Sea Water $p_v(Pa)$	590	842	1186	1646	2296	3058	4097
Fresh Water $p_v(Pa)$	611	872	1228	1704	2377	3166	4241

- Surface Tension

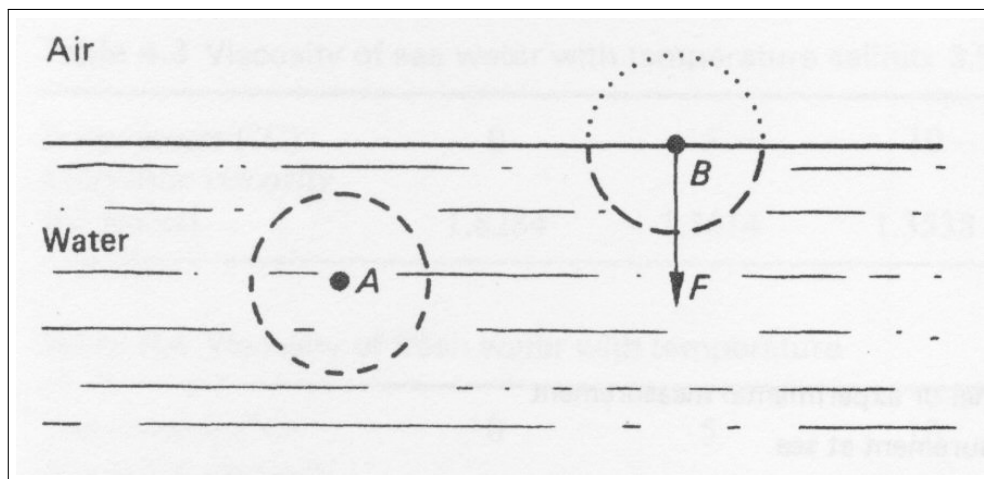


Figure 1.10 Molecular explanation of surface tension. (From Carlton 1994)

Table 1.6 Typical values of surface tension for sea and fresh water with temperature. (Adapted from Carlton 1994) Note: 1 *dyne* = 10^{-5} *Newton*.

Temperature ($^{\circ}C$)	0	5	10	15	20	25	30
Sea Water (<i>dynes/cm</i>)	76.41	75.69	74.97	74.25	73.55	72.81	72.09
Fresh Water (<i>dynes/cm</i>)	75.64	74.92	74.20	73.48	72.76	72.04	71.32

- Incompressibility of Water ⁴

- The elastic force in a fluid is

$$\text{Elastic force} \sim p l^3 \sim \rho C^2 l^2 \quad (1.20)$$

since the speed of sound C in a fluid is related to pressure and density:

$$C = \frac{p}{\rho}. \text{ Then} \quad (1.21)$$

- Mach number: ratio of characteristic fluid velocity in a flow to speed of sound in the medium:

$$\text{Mach number } M = \frac{\text{inertia force}}{\text{elastic force}} = \sqrt{\frac{\rho U^2 l^2}{\rho C^2 l^2}} = \frac{U}{C} \quad (1.22)$$

where U is the characteristic fluid velocity.

- The average speed of sound in air and water is: $C_{air} \sim 350m/s$, $C_{water} \sim 1,500m/s$ Therefore the average ratio of the speed of sound in water to air is $C_{water}/C_{air} \sim 4$.
- Because the average water to air density ratio is 1,000, it is ‘harder’ to move in water and therefore, typically, it is $U_{water} \ll U_{air}$ giving thus typical values of Mach numbers in the order of:

$$M_{air} \sim O(1) \Rightarrow \text{Compressible flow} \quad (1.23)$$

$$M_{water} \ll 1 \Rightarrow \text{Incompressible flow} \quad (1.24)$$

⁴ Movie: Shock Flows around Airfoil
./mmfm_movies/545.mov

- Only 0.4 % change under 100 atmospheric pressure (at 1 km depth in sea water).
- Note: An incompressible flow does not mean constant density.

1.3 Mathematical Prerequisites: Vector Analysis

1.3.1 Fundamental Function Analysis

- If $\lim_{x \rightarrow c} \phi(x) = \phi(c)$, the function $\phi(x)$ is said to be continuous at the point $x = c$.
- The base of natural logarithm e ,

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = 2.7182818285 \dots \quad (1.25)$$

Euler formula: $e^{i\theta} = \cos \theta + i \sin \theta$

Hyperbolic sine and cosine functions:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2} \quad (1.26)$$

- A definite integral in the sense of Riemann sum:

$$\int_a^b f(x) dx = \lim_{N \rightarrow \infty} \sum_{i=1}^N f\left(a + i \frac{b-a}{N}\right) \frac{b-a}{N} \quad (1.27)$$

- The rule for change of variable :

$$\int_{x_1}^{x_2} f(x) dx = \int_{u_1}^{u_2} f(x(u)) \frac{dx}{du} du \quad (1.28)$$

The integral for functions of two variables

$$\iint_{\Omega_{xy}} f(x, y) dx dy = \iint_{\Omega_{uv}} f(x(u, v), y(u, v)) |J| du dv, \quad (1.29)$$

where Jacobian $J \equiv \frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$.

- Leibnitz's rule in 1-D :

$$\frac{d}{dt} \int_{a(t)}^{b(t)} f(x, t) dx = f[b(t), t] b'(t) - f[a(t), t] a'(t) + \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t} dx \quad (1.30)$$

The Reynolds' Transport Theorem for 2-D and 3-D region.

- Tensor notation in 3D space

- Range convention:

$$x_i \quad (i = 1, 2, 3) \rightarrow x_1, x_2, x_3 \quad (1.31)$$

- Summation convention:

$$a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 \quad (i = 1, 2, 3) \quad (1.32)$$

- For example, $a_i = x_{ij} n_j$ denotes three equations, one for each $i = 1, 2, 3$ and j is the dummy index.

- Tensors.

- A scalar is called a zero-order tensor.
- A vector is a first-order tensor.
- Dyads are second-order tensors: a 3×3 matrix form. (e.g. stress tensor)
- The alternating tensor ϵ_{ijk} is a special third-order tensor.

1.3.2 Vector Calculus

- The simplest vector: *line vectors*.

A line vector is transformed from one coordinate system to another.

- Consider two Cartesian coordinate systems rotated with respect to one another.
 - a_{11}, a_{21}, a_{31} : the direction cosines of the x'_1 axis, with respect to the x_1, x_2, x_3 axes, respectively.
 - Transform between 2 coordinates in a summation notation:

$$x'_i = \sum_{j=1}^3 a_{ji} x_j \quad i = 1, 2, 3 \quad (1.33)$$

$$x_i = \sum_{j=1}^3 a_{ij} x'_j \quad i = 1, 2, 3 \quad (1.34)$$

- A *vector* is defined as :

$$u'_i = \sum_{j=1}^3 a_{ji} u_j \quad i = 1, 2, 3 \quad (1.35)$$

- Example (a) Velocity of a point, dx_i/dt .

$$\frac{dx'_i}{dt} = \frac{d}{dt} \sum_{j=1}^3 a_{ji} x_j = \sum_{j=1}^3 a_{ji} \frac{dx_j}{dt}. \quad (1.36)$$

- Example (b) Gradient of a scalar function, $\partial u / \partial x_i$.

Let $w_i \equiv \partial u / \partial x_i$; and $w'_i \equiv \partial u / \partial x'_i$.

$$w'_i = \sum_{j=1}^3 \frac{\partial u}{\partial x_j} \frac{\partial x_j}{\partial x'_i} = \sum_{j=1}^3 \frac{\partial u}{\partial x_j} a_{ji} = \sum_{j=1}^3 a_{ji} w_j \quad (1.37)$$

- *Unit base vectors*: \underline{i} , \underline{j} , and \underline{k} .
For curvilinear coordinate systems, \underline{e}_1 , \underline{e}_2 , and \underline{e}_3 .
- Any vector \underline{a} as the sum of its components :

$$\underline{a} = a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}. \quad (1.38)$$

The position vector \underline{x} :

$$\underline{x} = x \underline{i} + y \underline{j} + z \underline{k}. \quad (1.39)$$

The distance of \underline{x} from the origin as $r \equiv |\underline{x}| = \sqrt{x^2 + y^2 + z^2}$.

- Tensor notation:

Example) Kronecker delta:

$$\delta_{ij} = 1 \text{ if } i = j; \quad \delta_{ij} = 0 \text{ if } i \neq j \quad (1.40)$$

Example) Alternating tensor of permutation symbol:

$$\left. \begin{aligned} \epsilon_{ijk} &= 0 && \text{if any } i, j, k \text{ equal} \\ \epsilon_{ijk} &= 1 && \text{if } (ijk) = (123), (231), (312) \\ \epsilon_{ijk} &= -1 && \text{if } (ijk) = (132), (213), (321) \end{aligned} \right\} \quad (1.41)$$

The basic formulas :

$$\delta_{ii} = 3, \quad \delta_{ij} u_{klmi} = u_{klmj}, \quad \delta_{ij} \epsilon_{ijk} = 0, \quad (1.42)$$

$$\epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}, \quad \epsilon_{ijk} \epsilon_{ijk} = 6, \quad (1.43)$$

$$\epsilon_{ijk} \epsilon_{ljk} = 2 \delta_{il}. \quad (1.44)$$

- *Scalar product:*

$$\underline{a} \cdot \underline{b} = a b \cos(\underline{a} \cdot \underline{b}) \quad (1.45)$$

or,

$$\underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \quad (1.46)$$

or,

$$\underline{a} \cdot \underline{b} = \delta_{ij} a_i b_j = a_i b_i, \quad (\text{summation convention}) \quad (1.47)$$

- *Vector product:*

$$\underline{c} = \underline{a} \times \underline{b}; \quad w = a b \sin(\underline{a}, \underline{b}). \quad (1.48)$$

In a form of tensor-notation, $\underline{a} \times \underline{b} = \epsilon_{ijk} a_j b_k$.

- *Scalar triple product:*

$$\underline{a} \cdot (\underline{b} \times \underline{c}) = a_i \epsilon_{ijk} b_j c_k \quad (1.49)$$

$$\underline{a} \cdot \underline{b} \times \underline{c} = \underline{a} \times \underline{b} \cdot \underline{c} = \underline{b} \cdot \underline{c} \times \underline{a} \text{ etc.} \quad (1.50)$$

- *Vector triple product:*

$$\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c}) \underline{b} - (\underline{a} \cdot \underline{b}) \underline{c} \quad (1.51)$$

i.e.,

$$\begin{aligned} \underline{a} \times (\underline{b} \times \underline{c}) &= \epsilon_{mli} a_l \epsilon_{ijk} b_j c_k \\ &= (\delta_{mj} \delta_{lk} - \delta_{mk} \delta_{lj}) a_l b_j c_k \\ &= a_k b_j c_k - a_j b_k c_k. \end{aligned} \quad (1.52)$$

The basic formula :

$$\underline{a} \times (\underline{b} \times \underline{c}) + \underline{b} \times (\underline{c} \times \underline{a}) + \underline{c} \times (\underline{a} \times \underline{b}) = 0 \quad (1.53)$$

- *Gradient:*

$$\nabla u \equiv \lim_{V \rightarrow 0} \frac{1}{V} \oint_S u \underline{n} dS \quad (1.54)$$

where S is the area enclosing the volume V , dS is the element of area, and \underline{n} is the unit vector normal to the surface.

In limit,

$$\nabla u = \underline{i} \frac{\partial u}{\partial x} + \underline{j} \frac{\partial u}{\partial y} + \underline{k} \frac{\partial u}{\partial z} \quad (1.55)$$

Symbol: ∇u , grad u , or $\frac{\partial u}{\partial x_i}$.

- *Divergence:*

$$\nabla \cdot \underline{v} = \lim_{V \rightarrow 0} \frac{1}{V} \oint_S \underline{n} \cdot \underline{v} dS \quad (1.56)$$

$$\nabla \cdot \underline{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \quad (1.57)$$

Symbol: $\nabla \cdot \underline{v}$, $\text{div } \underline{v}$, or $\frac{\partial v_i}{\partial x_i}$.

• *Curl:*

$$\nabla \times \underline{v} \equiv \lim_{V \rightarrow 0} \frac{1}{V} \oint_S \underline{n} \times \underline{v} \, dS \quad (1.58)$$

$$\nabla \times \underline{v} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) \underline{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \right) \underline{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) \underline{k} \quad (1.59)$$

or,

$$\nabla \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} \quad (1.60)$$

Symbol: $\text{curl } \underline{v}$, $\nabla \times \underline{v}$, or $\epsilon_{ijk} \frac{\partial v_k}{\partial x_j}$.

• *Laplacian :*

$$\nabla^2 u \equiv \nabla \cdot (\nabla u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \quad (1.61)$$

The Laplacian of a vector function

$$\nabla^2 \underline{v} = \underline{i} \nabla^2 v_1 + \underline{j} \nabla^2 v_2 + \underline{k} \nabla^2 v_3 \quad (1.62)$$

• *Differential operator: $d\underline{r} \cdot \nabla$*

$$du = d\underline{r} \cdot \nabla u \quad (1.63)$$

where \underline{r} is the position vector and $d\underline{r}$ is any directed line element.

- This means that du is the increment of u corresponding to a position increment $d\underline{r}$.
- In rectangular Cartesian coordinates,

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = (d\underline{r} \cdot \nabla)u \quad (1.64)$$

– Similarly, for a vector function $\underline{v}(x, y, z)$,

$$\begin{aligned} d\underline{v} &\equiv \underline{i} dv_1 + \underline{j} dv_2 + \underline{k} dv_3 \\ &= \left(dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} + dz \frac{\partial}{\partial z} \right) (\underline{i} v_1 + \underline{j} v_2 + \underline{k} v_3) \\ &= (d\underline{r} \cdot \nabla) \underline{v} \end{aligned} \quad (1.65)$$

– The symbol ∇ is a *vector operator*:

$$\nabla = \underline{i} \frac{\partial}{\partial x} + \underline{j} \frac{\partial}{\partial y} + \underline{k} \frac{\partial}{\partial z}. \quad (1.66)$$

1.3.3 Expansion Formulas

- ϕ : any differentiable scalar function of x, y, z .
 $\underline{u}, \underline{v}, \underline{w}$: any such vector functions.

$$\nabla \cdot (\phi \underline{u}) = \underline{u} \cdot \nabla \phi + \phi \nabla \cdot \underline{u} \quad (1.67)$$

$$\nabla \times (\phi \underline{u}) = (\nabla \phi) \times \underline{u} + \phi \nabla \times \underline{u} \quad (1.68)$$

$$\nabla \cdot (\underline{v} \times \underline{w}) = \underline{w} \cdot \nabla \times \underline{v} - \underline{v} \cdot \nabla \times \underline{w} \quad (1.69)$$

$$\nabla \times (\underline{v} \times \underline{w}) = \underline{w} \cdot \nabla \underline{v} + \underline{v} \nabla \cdot \underline{w} - \underline{w} \nabla \cdot \underline{v} - \underline{v} \cdot \nabla \underline{w} \quad (1.70)$$

$$\nabla(\underline{v} \cdot \underline{w}) = \underline{v} \cdot \nabla \underline{w} + \underline{w} \cdot \nabla \underline{v} + \underline{v} \times (\nabla \times \underline{w}) + \underline{w} \times (\nabla \times \underline{v}) \quad (1.71)$$

$$\nabla \cdot (\nabla \times \underline{v}) = 0 \quad (1.72)$$

$$\nabla \times (\nabla \phi) = 0 \quad (1.73)$$

$$\nabla \times (\nabla \times \underline{v}) = \nabla(\nabla \cdot \underline{v}) - \nabla^2 \underline{v} \quad (1.74)$$

- The position vector: $\underline{x} = x_1 \underline{i} + x_2 \underline{j} + x_3 \underline{k}$.
Magnitude of the position vector $r = |\underline{x}| = \sqrt{\underline{x} \cdot \underline{x}}$ and a constant vector \underline{a} :

$$\nabla r = \underline{x}/r \quad (1.75)$$

$$\nabla \cdot \underline{x} = 3 \quad (1.76)$$

$$\nabla \times \underline{x} = 0 \quad (1.77)$$

$$\nabla r^n = n r^{n-2} \underline{x} \quad (1.78)$$

$$\nabla \cdot (r^n \underline{x}) = (n + 3) r^n \quad (1.79)$$

$$\nabla \times (r^n \underline{x}) = 0 \quad (1.80)$$

$$\nabla^2(r^n) = n(n + 1) r^{n-2} \quad (1.81)$$

$$\nabla \cdot (\underline{a} \times \underline{x}) = 0 \quad (1.82)$$

$$\nabla(\underline{a} \cdot \underline{x}) = \underline{a} \quad (1.83)$$

$$\nabla \times (\underline{a} \times \underline{x}) = 2 \underline{a} \quad (1.84)$$

$$\nabla \cdot (\underline{a} \times \nabla r) = 0 \quad (1.85)$$

$$\nabla \cdot (r \underline{a}) = (\underline{x} \cdot \underline{a})/r \quad (1.86)$$

$$\nabla \times (r \underline{a}) = (\underline{x} \times \underline{a})/r \quad (1.87)$$

1.3.4 Divergence Theorem (Gauss Theorem)

- Consider the surface integral $\oint_S u \underline{n} dS$.

If the volume V is subdivided into small volume V_i ,

$$\oint_S u \underline{n} dS = \sum \oint_{S_i} u \underline{n} dS \quad (1.88)$$

In the limit,

$$\oint_{S_i} u \underline{n} dS = \int_{V_i} \nabla u dV.$$

Definition of the gradient, eq. (1.54),

$$\oint_S u \underline{n} dS = \int_V \nabla u dV \quad (1.89)$$

- Use the definitions of the divergence and curl,

$$\oint_S \underline{n} \cdot \underline{v} \, dS = \int_V \nabla \cdot \underline{v} \, dV \quad (1.90)$$

$$\oint_S \underline{n} \times \underline{v} \, dS = \int_V \nabla \times \underline{v} \, dV \quad (1.91)$$

General form:

$$\boxed{\oint_S (\underline{n} * f) \, dS = \int_V (\nabla * f) \, dV} \quad (1.92)$$

- An example,

$$\begin{aligned} \int_V \nabla^2 u \, dV &= \int_V \nabla \cdot (\nabla u) \, dV \\ &= \oint_S \underline{n} \cdot \nabla u \, dS = \oint_S \frac{\partial u}{\partial n} \, dS \end{aligned} \quad (1.93)$$

where $\partial u / \partial n$ is the directed derivative in the outward direction.

1.3.5 Stokes' Theorem

- From our definitions for ∇u and $\nabla \times \underline{v}$

$$\underline{n} \times \nabla u \approx \frac{1}{S} \oint_C u \, d\underline{r} \quad (1.94)$$

$$\underline{n} \cdot \nabla \times \underline{v} \approx \frac{1}{S} \oint_C \underline{v} \cdot d\underline{r} \quad (1.95)$$

where S denotes a very small surface element in the fluid, C is the small contour that forms the boundary of S , and \underline{n} is a unit normal to S .

- The transformation theorems relate certain surface integrals to contour in-

tegrals:

$$\int_S \underline{n} \times \nabla u \, dS = \oint_C u \, d\underline{r} \quad (1.96)$$

$$\begin{aligned} \int_S \underline{n} \cdot \nabla \times \underline{v} \, dS &= - \int_S (\underline{n} \times \nabla) \cdot \underline{v} \, dS \\ &= \oint_C \underline{v} \cdot d\underline{r} \end{aligned} \quad (1.97)$$

The unified form of Stokes' theorem :

$$\boxed{\int_S (\underline{n} \times \nabla) * f \, dS = \oint_C d\underline{r} * f} \quad (1.98)$$

1.3.6 Dyadic Products

- The *dyadic product*: a special form of *second-order tensor* $\underline{u} \underline{v}$:

$$\begin{aligned} (\underline{u} \underline{v}) \cdot \underline{w} &\equiv \underline{u}(\underline{v} \cdot \underline{w}) \\ \underline{w} \cdot (\underline{u} \underline{v}) &\equiv (\underline{w} \cdot \underline{u})\underline{v} \end{aligned} \quad (1.99)$$

- The gradient of a vector, $\nabla \underline{v}$, in the Navier-Stokes equations:

If the vector \underline{v} is a velocity resolved into a symmetric and antisymmetric form:

$$\begin{aligned} \nabla \underline{v} &= \frac{1}{2} [(\nabla \underline{v} + \nabla \underline{v}^T) + (\nabla \underline{v} - \nabla \underline{v}^T)] \\ &= \frac{1}{2} \text{def}(\underline{v}) + \frac{1}{2} \text{rot}(\underline{v}) \end{aligned} \quad (1.100)$$

- A second-order tensor to be a 3×3 matrix.
- The superscript T stand for transpose of the matrix.
- The first term causes stress: (i) normal strain rate and (ii) shear strain rate.
- The second term: rigid body rotation of a fluid element.

1.3.7 Reynolds Transport Theorem

- The rate of change of an integral taken over a volume moving through a field $F(\underline{x}, t)$

$$\frac{d}{dt} \iiint_{V(t)} F(\underline{x}, t) dV \quad (1.101)$$

The path of points in $V(t)$:

$$\underline{x} = \underline{x}(\underline{\xi}, t) \quad (1.102)$$

where $\underline{\xi}$ is the initial point of \underline{x} .

- The Reynolds transport theorem

$$\frac{d}{dt} \iiint_{V(t)} F dV = \iiint_V \left[\frac{\partial F}{\partial t} + \nabla \cdot (\underline{v} F) \right] dV \quad (1.103)$$

or

$$\boxed{\frac{d}{dt} \iiint_{V(t)} F dV = \iiint_{V(t)} \frac{\partial F}{\partial t} dV + \iint_{S(t)} \underline{n} \cdot (\underline{v} F) dS} \quad (1.104)$$

- Here \underline{v} is the velocity of the point \underline{x} , and \underline{n} is the outward unit vector normal to the boundary $S(t)$.
- The first integral is the rate of change in volume, and the second integral is the rate of change associated with motion of surface bounding volume.
- Similar to Leibnitz's rule for 1-dimensional region:

$$\begin{aligned} \frac{d}{dt} \int_{a(t)}^{b(t)} f(x, t) dx = \\ \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t} dx + f[b(t), t] b'(t) - f[a(t), t] a'(t) \end{aligned} \quad (1.105)$$

1.3.7.1 Example of the Reynolds transport theorem in 1-D

- Consider an integral with $a(t) = t + 1$, $b(t) = 2t + 2$, $F(x, t) = x t$, and $x(\xi, t) = \xi t + \xi$:

$$I(t) = \int_{a(t)}^{b(t)} F(x, t) dx, \quad \frac{dI(t)}{dt} = \frac{d}{dt} \int_{a(t)}^{b(t)} F(x, t) dx \quad (1.106)$$

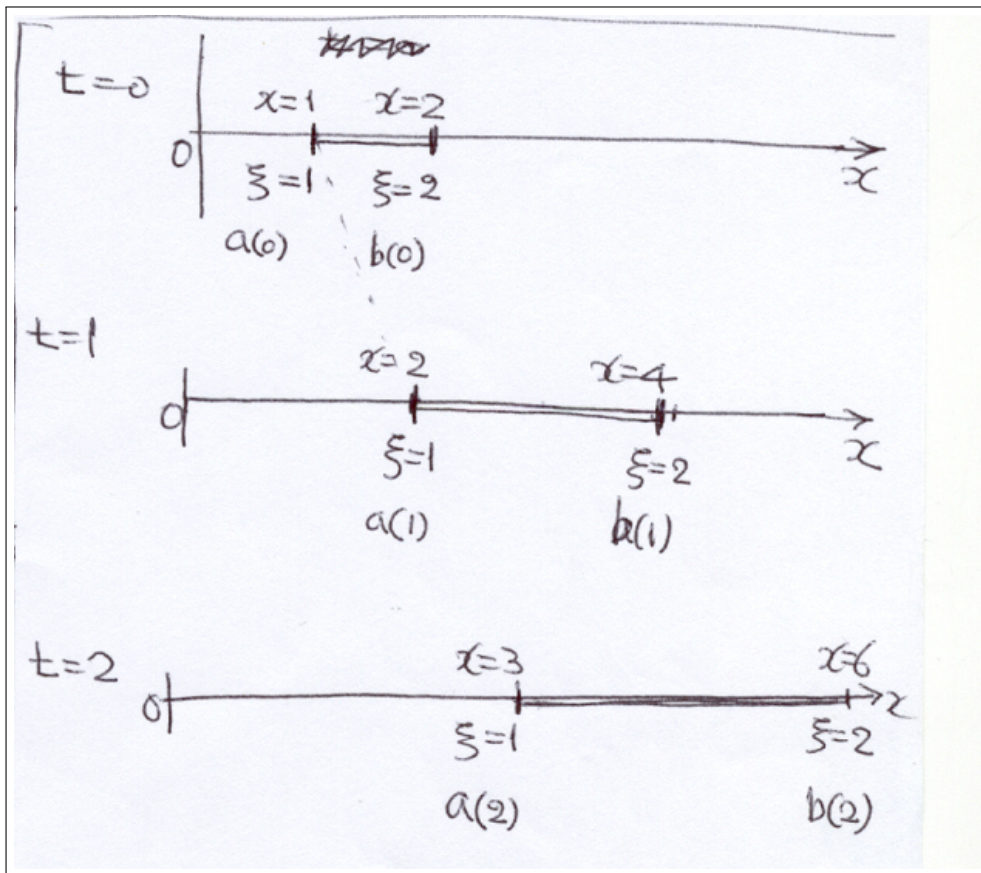


Figure 1.11 Sequential change of the interval of the integral (1.106).

- Lagrangian variable $\xi = \frac{x}{t+1}$ from the given relation $x(\xi, t) = \xi t + \xi$, then the integrand

$$F^*(\xi, t) = (\xi t + \xi)t \quad (1.107)$$

- Velocity and divergence of velocity:

$$\underline{v}^*(\xi, t) = \left. \frac{\partial x}{\partial t} \right|_{\xi=\text{const.}} = \xi \underline{i} \equiv v^* \underline{i} \quad \text{in Lagrangian manner} \quad (1.108)$$

$$\underline{v}(x, t) = \frac{x}{t+1} \underline{i} \equiv v \underline{i} \quad \text{in Eulerian manner} \quad (1.109)$$

$$\nabla \cdot \underline{v}(x, t) = \frac{1}{t+1} \equiv \frac{\partial v}{\partial x} \quad \text{in 1-dimensions} \quad (1.110)$$

- Chain rule (i.e., Jacobian)

$$\frac{\partial x}{\partial \xi} = t+1 \equiv J \text{ (Jacobian)} \quad (1.111)$$

- The time derivative of the integral

$$\begin{aligned} \frac{dI(t)}{dt} &= \frac{d}{dt} \int_{a(t)}^{b(t)} F(x, t) dx \\ &= \frac{d}{dt} \int_{a(0)}^{b(0)} F^*(\xi, t) \frac{\partial x}{\partial \xi} d\xi \quad \text{(by change of variable)} \\ &= \frac{d}{dt} \int_{a(0)}^{b(0)} \{(\xi t + \xi)t\} J d\xi \quad \text{(with Jacobian)} \\ &= \int_{a(0)}^{b(0)} \left[\left. \frac{\partial \{(\xi t + \xi)t\}}{\partial t} \right|_{\xi=\text{const.}} J + \{(\xi t + \xi)t\} \underbrace{\frac{\partial J}{\partial t}}_{\boxed{A}} \right] d\xi \end{aligned} \quad (1.112)$$

where \boxed{A} is equivalent to

$$\begin{aligned} \frac{\partial J}{\partial t} &= \frac{\partial}{\partial t} \left(\frac{\partial x}{\partial \xi} \right) = \frac{\partial}{\partial \xi} \left(\frac{\partial x}{\partial t} \right) \quad \text{(interchanging the differential order)} \\ &= \frac{\partial v^*}{\partial \xi} = \frac{\partial v}{\partial x} \frac{\partial x}{\partial \xi} \quad \text{(by chain rule)} \\ &= (\nabla \cdot \underline{v})J \implies \left(\frac{1}{t+1} \right) (t+1) \end{aligned} \quad (1.113)$$

- Rearranging the terms of the integrand and then converting the integral with respect to the variable ξ into one with respect to the original variable x , we have

$$\begin{aligned}
\frac{dI(t)}{dt} &= \int_{a(t)}^{b(t)} \left[\underbrace{\frac{\partial}{\partial t}(xt)}_{\frac{\partial F}{\partial t}} + \underbrace{\frac{\partial x}{\partial t} \Big|_{\xi=\text{const.}} \frac{\partial}{\partial x}(xt)}_{\underline{v} \cdot \nabla F} + \underbrace{(xt) \left(\frac{1}{t+1} \right)}_{F \nabla \cdot \underline{v}} \right] \underbrace{(t+1)}_J d\xi \\
&= \int_{a(t)}^{b(t)} \left[\frac{\partial F}{\partial t} + \nabla \cdot (\underline{v} F) \right] dx \\
&= \int_{a(t)}^{b(t)} \left[\frac{\partial F}{\partial t} + \frac{\partial}{\partial x} (v F) \right] dx \\
&= \int_{a(t)}^{b(t)} \frac{\partial F}{\partial t} dx + [v F(x, t)] \Big|_{x=b(t)} - [v F(x, t)] \Big|_{x=a(t)} \\
&= \int_{a(t)}^{b(t)} \frac{\partial F}{\partial t} dx + F[b(t), t] b'(t) - F[a(t), t] a'(t) \tag{1.114}
\end{aligned}$$

1.3.8 Moving Coordinate Systems

- Two coordinate systems:

The position vector \underline{x}' in the space-fixed system is related to the position vector \underline{x} in the moving system:

$$\underline{x}' = \underline{x} + \underline{R} \tag{1.115}$$

where \underline{R} is the distance vector between the origins of two coordinate systems.

- The derivative (d'/dt) observed in the space-fixed system and the derivative (d/dt) observed in the moving system:

$$\frac{d'}{dt} = \frac{d}{dt} + \underline{\Omega} \times \tag{1.116}$$

where $\underline{\Omega}$ is the angular velocity of the moving system.

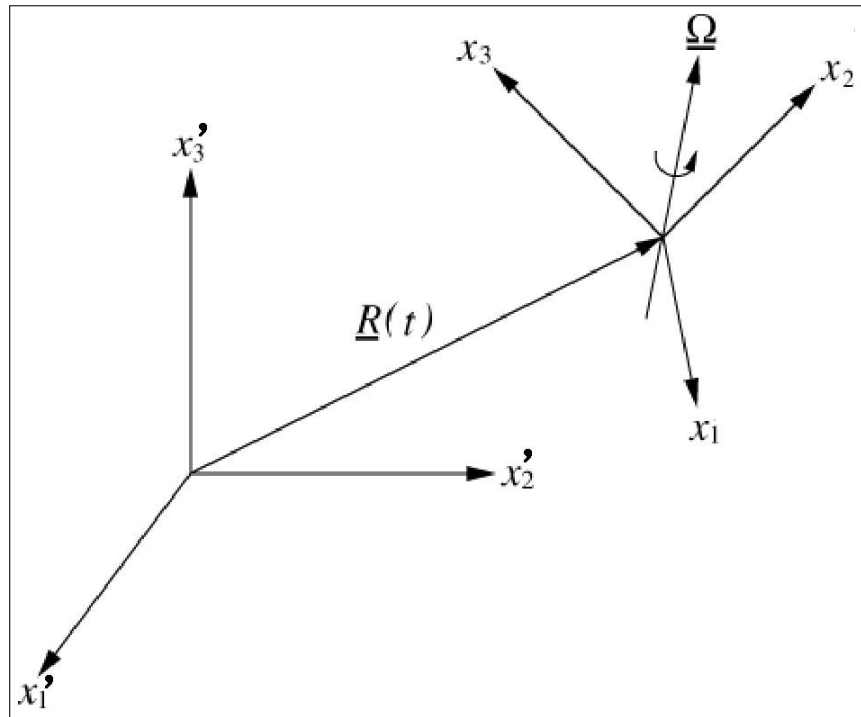


Figure 1.12 Moving coordinate system.

- The velocity vectors in the two coordinate systems:

$$\underline{q}' = \underline{q} + \underline{\Omega} \times \underline{x} + \underline{\dot{R}} \quad (1.117)$$

where $\underline{\dot{R}}$ represents the translation velocity of the moving frame.

- Acceleration vectors \underline{a} in the space-fixed system :

$$\underline{a}' \equiv \frac{d^2 \underline{x}'}{dt^2} = \underline{a} + 2\underline{\Omega} \times \underline{q} + \frac{d\underline{\Omega}}{dt} \times \underline{x} + \underline{\Omega} \times (\underline{\Omega} \times \underline{x}) + \underline{\ddot{R}} \quad (1.118)$$

- The first term : the acceleration viewed in the moving system.
- The second term: the *Coriolis acceleration*.
- The fourth term: the generalized centripetal acceleration, since

$$|\underline{\Omega} \times (\underline{\Omega} \times \underline{x})| = \Omega^2 \underline{x} \sin(\underline{\Omega}, \underline{x}) \quad (1.119)$$

- For the self-rotation of earth with constant angular speed, its effect (a form of gradient of a scalar function) is already included in gravitational acceleration values.

