Lecture Notes 414.341

선 박 해 양 유 체 역 학

MARINE HYDRODYNAMICS

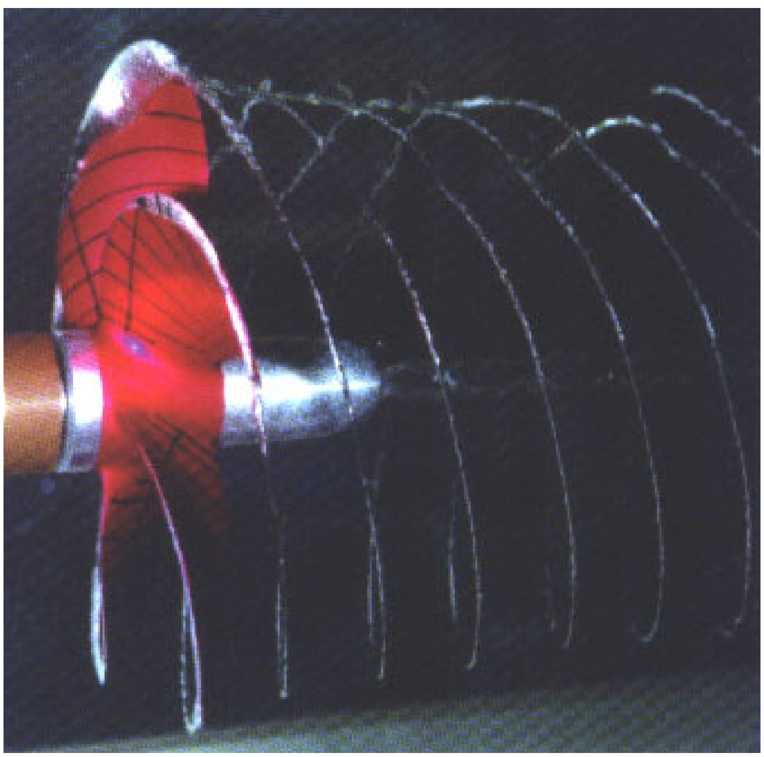
2019년 6월 28일

Suh, Jung-Chun
서 정 천

Seoul National Univ., Dept. NAOE
서울대학교 공과대학 조선해양공학과

선 박 해 양 유 체 역 학

MARINE HYDRODYNAMICS



Suh, Jung-Chun
서 정 천

Contents

List of Figures

List of Tables

1

INTRODUCTION

1.1 Continuum Mechanics

1.1.1 Definition of Fluid

- Fluid can not withstand shearing forces when a shear stress is applied: Fluids continuously deform and Solids deform or bend.

- While solid can be in stable equilibrium under shear stress oblique to the surface separating any two parts, fluid cannot be in stationary equilibrium.

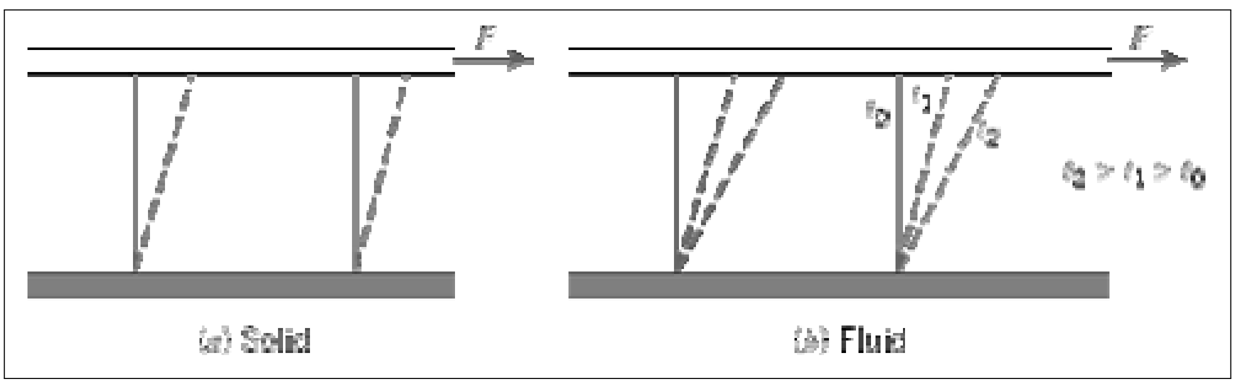


Figure 1.1 Behavior of a solid and a fluid, under the action of constant shear force. Solid (*left*); Fluid (*right*). (From Fox, McDonald & Pritchard 2004)

- Resistance to rate of shear deformation from viscosity gives rise to drag for bodies. We can easily recognize that such shear stresses do exist in fluids: e.g., consider how the fluid in a rotating circular vessel takes on the rotating motion of the vessel eventually.

- Other observed properties of fluids are:
  - resistance to volumetric compression and tension in general,
  - no shape or preferred orientation (Solids: definite shape; Fluids: no preferred shape),
  - homogeneous matter in general, and
  - has mass.
- There are two kinds of fluids depending on bulk elasticity (compressibility): [1]
  - Liquid forms a free surface(density $\rho \approx 0$ above free surface).
  - Gas expands to fill container.

1.1.2 Assumptions and Axioms

- Continuum Viewpoint: Fluid as a continuum in macroscopic scale compared with molecules.
  We assume that the fluid is continuous and homogeneous in structure.
  - Actually this is not so since matter is ultimately made up of molecules and atoms, but in many applications the dimensions we are concerned with are large compared to the molecular structure, and the smallest sample of fluid that concerns us contains a very great number of molecules (i.e., number of about $2.687 \times 10^{19}/cm^3$). [2]
  - In such cases, the properties of any sample are the average values over many molecules, and the approximation of a continuum is found to be acceptable and useful.
    * Measurable smallest scale: length $l \sim O(10^{-5})\, m$, volume $V \sim O(10^{-15})\, m^3$.

[1] On the mechanism of formation of liquid and vapor, see Brennen, C. (1995), *Cavitation and Bubble Dynamics*, Oxford University Press, pp. 1–6.

[2] Avogadro no./1 mol = $6.02214 \times 10^{23}/22.414$ liter. This number is about 10 times as many as the total no. of stars in the known universe, and about 100 times as many as grains of sand on all beaches and deserts. (See Garrison, T. (2007), *Oceanography: An Invitation to Marine Science*, 6th ed., Thomson Brooks/Cole.) Suppose a supercomputer can count at the rate of one billion molecules per second. For a cubic centimeter of a fluid we would take a very long time for counting the occupied molecules: 2.687×10^{10} sec ≈ 850 yr.

- $\cdot$ Nano scale devices: $l \sim O(10^{-8})\,m$
- $\cdot$ Ocean current diameter: $l \sim O(10^{6})\,m$

$*$ No. of air molecules in the volume at standard pressure: $\sim 3 \times 10^{10}$.

$*$ No. of water molecules in the volume at standard pressure: $\sim 10^{13}$.

Example: definition of density at a point (see Figures **1.2** and **1.3**)

$$\rho \equiv \lim_{\delta V \to \delta V'} \frac{\delta m}{\delta V} = \rho(x, y, z, t) \tag{1.1}$$

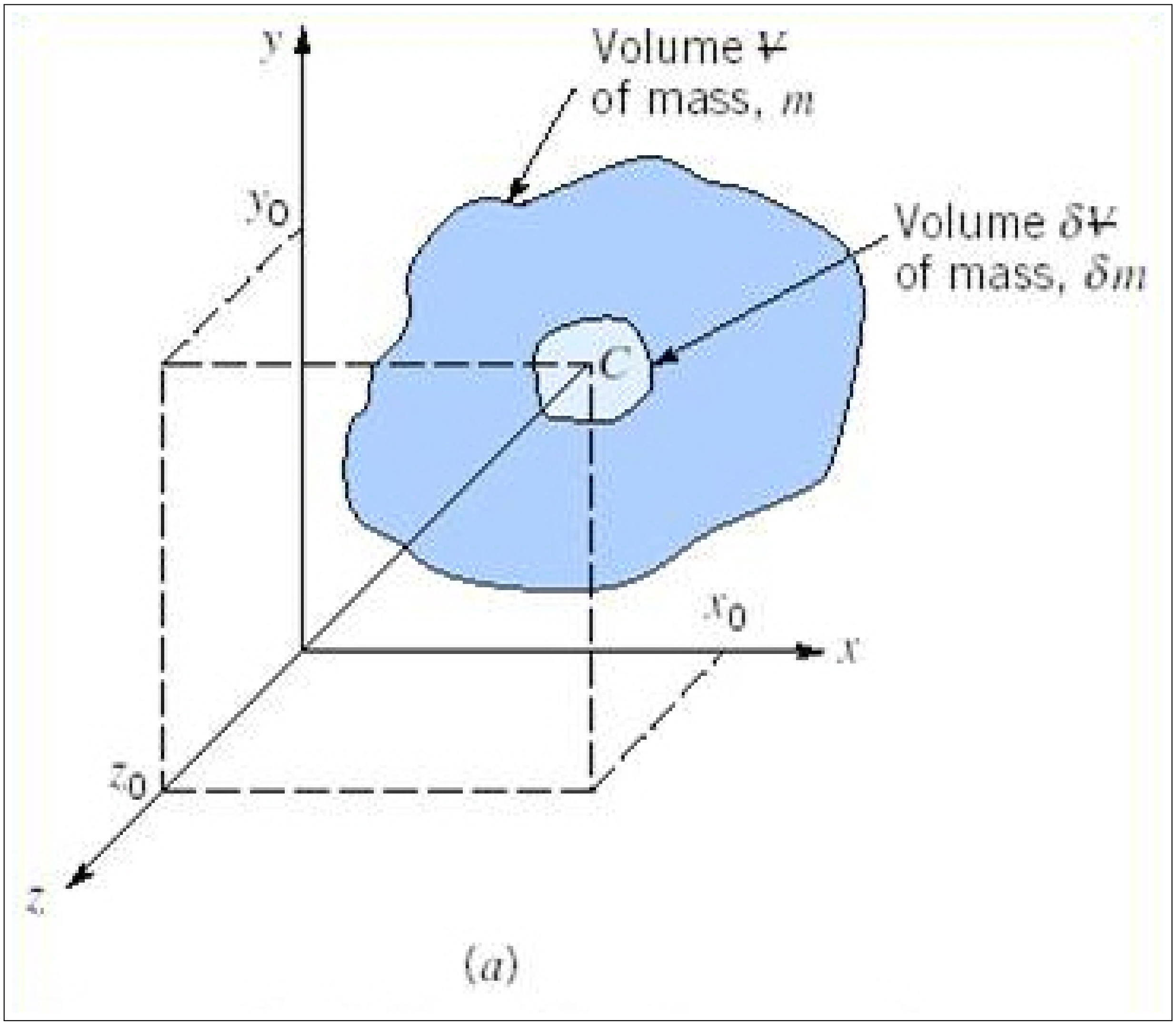


Figure 1.2 Definition of density. (From Fox, McDonald & Pritchard 2004)

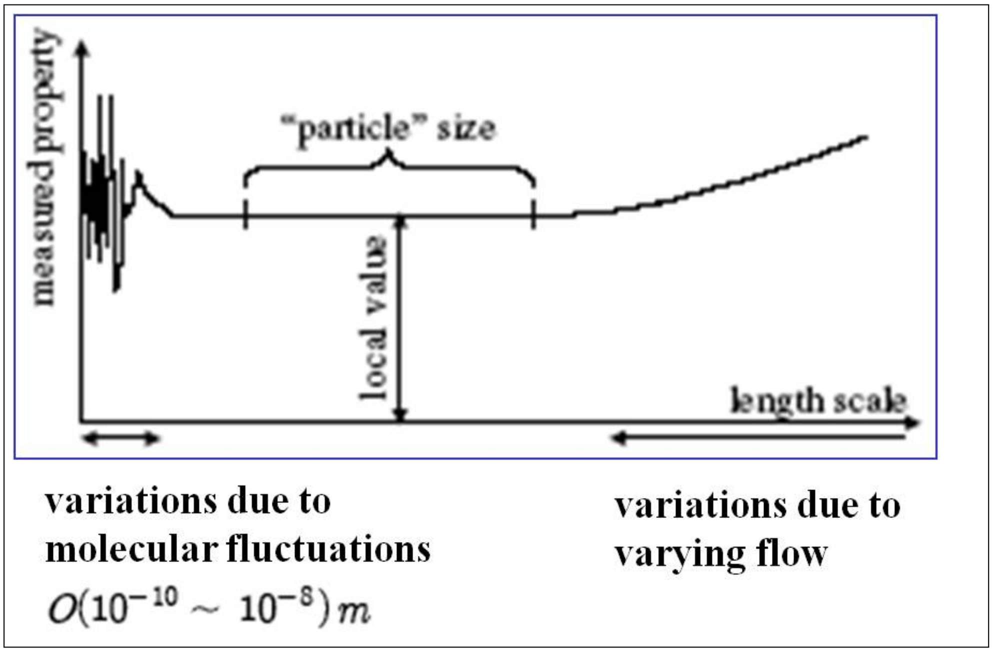


Figure 1.3 Measured data to define density with variation of length scale. (From Panton 1984)

- Nevertheless, results obtained on the assumption of a continuum may be erroneous whenever the molecular structure dimensions are relatively large.
  - For example, at very high altitudes (low pressures), the molecular spacing is so great that air is not even approximately a continuum in its contact with a body.
  - Failures of the continuum assumption occur probably in the cases of that body size compares with molecular dimensions (e.g., a very small body in a fluid) or with distances between molecules (e.g., a body in a rarefied gas).
- Other acceptable and useful assumptions are those as follows:

  (1) that physical laws are independent of the coordinate system used to express them (frame indifference),

  (2) that natural laws are independent of the dimensions of physical quantities that occur in the expressions (dimensional homogeneity),

  (3) that derivations of physical quantities with respect to space and time exist to the required order (smoothness of quantities), and

(4) that the present motion is a function of its history and not the future (memory of history).

1.1.3 Basic Equations

- Fundamental Laws of Continuum Mechanics:
  - Conservation of mass: Continuity equation
  - Conservation of momentum (Newton's law of motion)
    - Principle of momentum and angular momentum
    - Navier-Stokes equations for viscous flow
    - Euler's equation for inviscid flow
    - Bernoulli equation as an energy equation from integration of Euler's equation
  - Conservation of energy (First law of thermodynamics)
    - The First Law of Thermodynamics: Energy conservation for heat and work interactions
    - The Second Law of Thermodynamics: Heat flow in direction of entropy increase

  We postulate that mass, momentum and energy are conserved: Conservation of mass, Conservation of momentum, Conservation of energy. Since these notes tend to deal mostly with incompressible flows, we do not examine the conservation of energy.

- Newton's equations of motion are derived from rigid body mechanics:

$$\underline{F} = m\frac{d^2\underline{x}}{dt^2} \tag{1.2}$$

  where m = mass of particle or body, $\underline{F}$ = sum of external forces, $\underline{x}$ = position vector, and t = time.

  - Three differential equations

- Applicable to mass particle or system of mass particles, differential element or whole part of continuum
- Concept of dynamic and static equilibrium for given external forces
- General equations of motion applicable to arbitrary elements of bodies in concern:
$$\underline{F} = \frac{d\underline{M}}{dt}, \quad \underline{L} = \frac{d\underline{H}}{dt} \tag{1.3}$$
where $\underline{M}$ = linear momentum, $\underline{L}$ = sum of moment in action, $\underline{H}$ = angular momentum.

- Our use of these laws are based on continuum hypothesis.
Equations of motion for continuum:

$$\rho \frac{d^2\underline{x}}{dt^2} = \rho \underline{b} + \nabla \cdot \underline{\underline{T}} \tag{1.4}$$

where $\underline{\underline{T}}$ = stress tensors (internal or surface forces), $\underline{b}$ = body force per unit mass.

- Constitutive Laws
  - Solids: Hooke's law, stress = f (strain)
  force/area $\sim$ relative displacement/length
  - Fluids: stress = f (rate of strain)
  force/area $\sim$ velocity gradient

We need the constitutive equations which are a sort of relationships between the stress tensor and the strain tensor, under some assumptions such as homogeneous, isotropic, continuous, elastic (Newtonian) continuum.
Relation to the strain in solid and the strain rate in fluid:

$$\underline{F} \propto \triangle \ell, \quad \underline{F} \propto \triangle u \tag{1.5}$$

where the proportional factors are the elasticity and the viscosity coefficients, respectively.

- Isotropic stress tensors having linear relationship with strain tensor and strain rate tensor, respectively: [3]

$$\underline{\underline{T}} = \lambda' (\nabla \cdot \underline{d}) \underline{\underline{I}} + \lambda \left[\nabla \underline{d} + (\nabla \underline{d})^T \right] \tag{1.6}$$

$$\underline{\underline{T}} = \left[-p + \mu' (\nabla \cdot \underline{u}) \right] \underline{\underline{I}} + \mu \left[\nabla \underline{u} + (\nabla \underline{u})^T \right] \tag{1.7}$$

where $\underline{d}$ and $\underline{u}$ are, respectively, the displacement and the velocity, λ', λ, μ' and μ are the proportional factors.
The differential equations for displacement and velocity, respectively:

$$\rho \frac{\partial^2 \underline{d}}{\partial t^2} = \rho \underline{f}_{body} + (\lambda' + \lambda) \nabla (\nabla \cdot \underline{d}) + \lambda \nabla^2 \underline{d} \tag{1.8}$$

$$\rho \frac{\partial^2 \underline{u}}{\partial t^2} = \rho \underline{f}_{body} - \nabla p + (\mu' + \mu) \nabla (\nabla \cdot \underline{u}) + \mu \nabla^2 \underline{u} \tag{1.9}$$

- Liquids and gases depending on compressibility (bulk elasticity).
  - Liquids: ‘Hydrodynamics’ (Hydrodynamic flows are treated as incompressible.)
  - Gases: ‘Aerodynamics’ (Aerodynamic flows are treated as compressible.)
- Classification of fluid mechanics

1.2 Characteristics of Hydrodynamics

1.2.1 Types of Fluid Flow

With the principal types of fluid flow and their associated phenomena, it is possible to make up practically any flow combination in nature, even the complex system around a moving ship:

- Potential Flow

[3]For detailed information on difference and similarity among various fields in continuum mechanics and their historical background, see the article: 이승준 (1992), “재료역학과 고체역학: 유체역학자의 관점에서”, *대한조선학회지*, 제29권, 제3호.

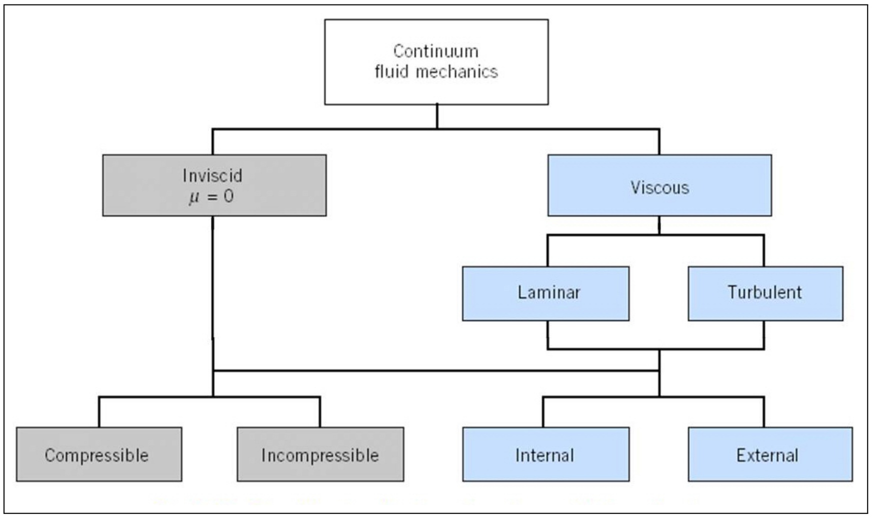


Figure 1.4 Possible classification of fluid mechanics. (From Fox, McDonald & Pritchard 2004)

- Viscous Flow
- Turbulent Flow
- Separation of Flow from a Surface
- Cavitation
- Wavemaking
- Circulatory or Vortical Flow
- Elastic or Compressible Flow

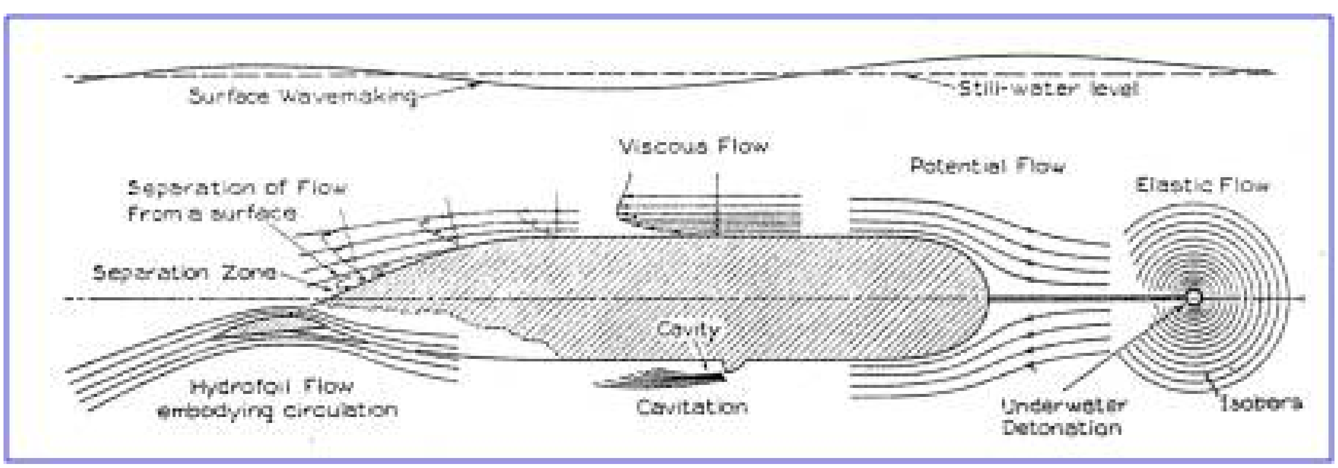


Figure 1.5 Schematic diagram of various types of fluid flow. (From Saunders 1957)

1.2.2 Various Characteristic Effects

Such flow phenomena can be characterized by several principal effects which constitute the basis for important relationships in the form of non-dimensional numbers:

- Velocity effects have the phenomena which are function of the rate at which a body moves in a fluid (e.g. denotation wave around a torpedo exploded).
- Acceleration effects is associated with the acceleration imparted to fluid particles by differences in pressure and other causes (e.g. flow around a propeller blade section).
- Force effects are the application of forces of special nature which is closely related to accelerative effects (e.g. dynamic lift developed by a planning form).
- Inertia effects involve the mass density of the fluid, the velocity of the moving fluid particles, and the necessity for changing their direction (e.g. dynamic pressure on the blunt face).
- Gravity effects result in the changes in potential energy under the influence of the gravity, which occur at an interface with gas (e.g. wavemaking).
- Viscous effects are due to the internal resistance of the fluid to deformation in the shear forces (e.g. flow past a solid surface).
- Elastic effects are due to the compressibility of the fluid (e.g. acoustic wave traveling through water).
- Surface tension effects appear as the attraction between the surface molecules of the real fluid (e.g. air bubbles in water).

1.2.3 Characteristics of Ship/Marine Hydrodynamics

- Ship/Marine Hydrodynamic Aspects: Applicable to naval architecture and ocean engineering

  - Many Separate Topics: Propulsion/steering, behavior in waves of a moored buoy or oil-drilling platform

- Related Applications of the General Field of Hydrodynamics
  - Lifting Surfaces: Propellers, Rudders, Anti-rolling fins, Yacht keels, Sails
  - Equations of Motion: Unsteady ship, Buoy, or Platform motions in waves, Maneuvering of ships or submarines in non-straight paths.

- Broad Level of Sophistication: From empirical design methodology to theoretical research activities (as well as intuition and experiment)
  - Diverse Fields of Technology: Fluid mechanics, solid mechanics, control theory, statistics, random process, data acquisition

- Necessary Background
  - Intelligent evaluation and application of empirical procedure
  - Introduction to specialized study on the advanced research
  - Continuum Mechanics: Force and motion in smooth and continuous manner

- Complicated Force Mechanisms
  - 3 Principal Types: Inertial, Gravitational, Viscous
  - Secondary Effects: Surface tension, Elastic, Cavitation

- Physical Parameters: Length, Velocity, Density, Gravity, Viscosity, Pressure
  - Inertial forces $\sim$ mass $\times$ acceleration $\sim \left(\rho l^3\right)\left(\frac{U^2}{l}\right) = \rho U^2 l^2$ (1.10)
  - Viscous forces $\sim$ shear stress $\left(\mu \frac{\partial u}{\partial y}\right) \times$ area $\sim \left(\mu \frac{U}{l}\right)(l^2) = \mu U l$ (1.11)
  - Gravitational forces $\sim$ mass $\times$ gravity $\sim (\rho l^3) g$ (1.12)
  - Pressure forces $\sim$ pressure $\times$ area $\sim (p - p_0) l^2$ (1.13)

- Dynamical Similarity

$$\frac{\text{Inertial Force}}{\text{Gravitational Force}} = \frac{\rho U^2 l^2}{\rho g l^3} = \frac{U^2}{gl} = (\text{Froude No.})^2 \quad (1.14)$$

$$\frac{\text{Inertial Force}}{\text{Viscous Force}} = \frac{\rho U^2 l^2}{\mu U l} = \frac{\rho U l}{\mu} = \text{Reynolds No.} \quad (1.15)$$

$$\frac{\text{Gravitational Force}}{\text{Viscous Force}} = \frac{\rho g l^3}{\mu U l} = \frac{\rho g l^2}{\mu U} \quad (1.16)$$

- Simultaneous scaling is not possible. Scaling dilemma!
- Cavitation: Change of physical state below vapor pressure at very high speeds

$$\text{Cavitation No.:} \quad \boxed{\sigma = \frac{p_0 - p_v}{\frac{1}{2}\rho U^2}} \quad (1.17)$$

- Difficulty of Navier-Stokes Equations: System of coupled nonlinear P.D.E.
  - Inviscid Assumptions: Mathematical solutions (with free surface effects).
  - Froude's Hypothesis: Total resistance = frictional + residual resistance.
  - Boundary Layer: Viscous effect within thin viscous layer at large Reynolds number.
  - CFD: Numerical simulation by using discretization and approxiamtion of governing equations with physical modelling.

1.2.4 Ocean Environment

- Density of Water
  - Dependence on temperature and salinity.
    - Greater influence of temperature at a given salinity in a higher temperature regions.
    - Greater effect of salinity in a lower temperature regions.

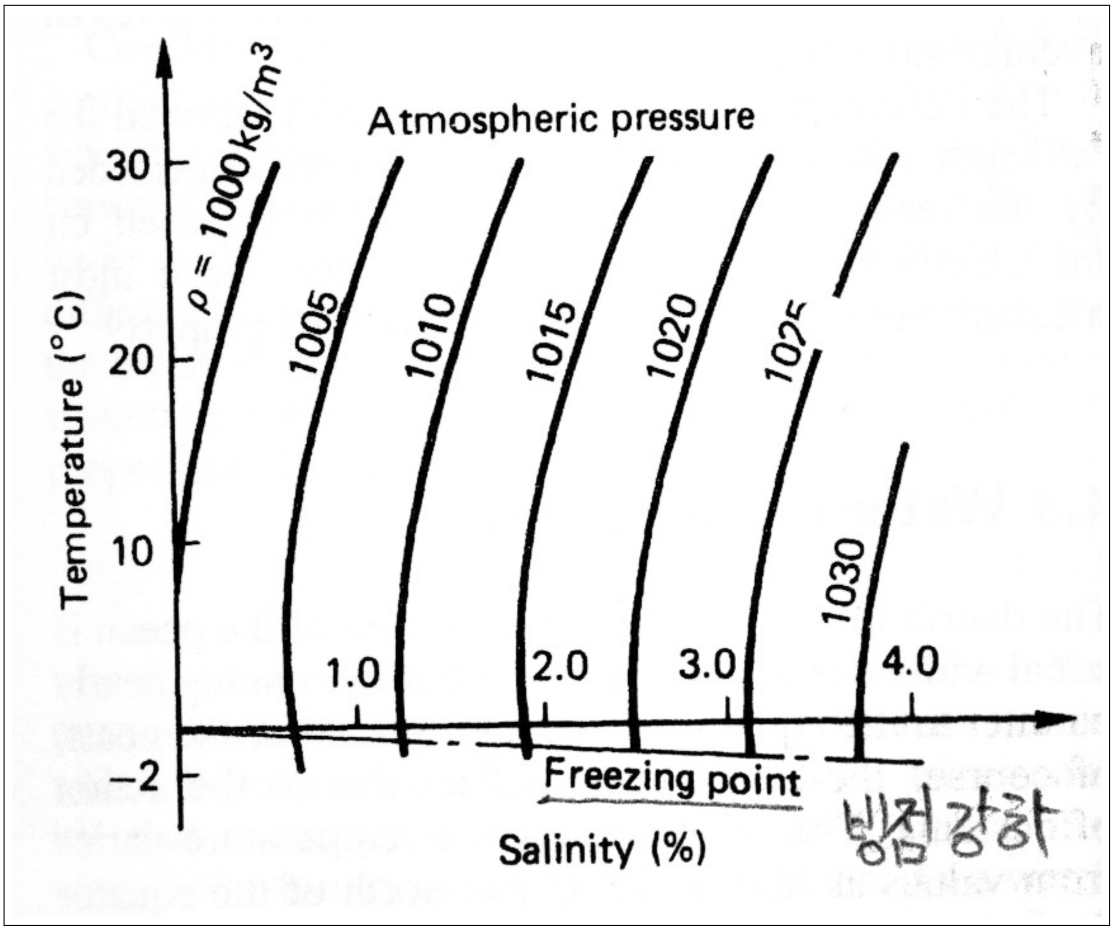


Figure 1.6 Variation of density with salinity and temperature at atmospheric pressure. (From Carlton 1994)

Table 1.1 Density variations with temperature (salinity 3.5%). (Adapted from Carlton 1994)

Temperature (^0C)	0	5	10	15	20	25	30
Density (kg/m^3)	1028.1	1027.7	1026.8	1025.9	1024.7	1023.3	1021.7

Table 1.2 Density variations with temperature (fresh water). (Adapted from Carlton 1994)

Temperature (^0C)	0	4	5	10	15	20	25	30
Density (kg/m^3)	999.8	1000	999.9	999.6	999.0	998.1	996.9	995.6

  - Increase with depth.
    Sea water: $\rho = 1,026\,kg/m^3$ at 15^oC.

- Salinity = 3.47% at sea surface:

$$\text{Salinity} \;=\; 1.80655 \times \text{ Chlorinity} \quad \text{in } \% \tag{1.18}$$

- Water Temperature

  - From $T = 28^o\,C$ at equator to $T = -2^o\,C$ near ice in high latitudes.
  - Three thermal layers:
    * Upper layer between $D = 50\,m$ and $D = 200\,m$ below surface: $T = 20^o\,C$ at surface
    * Transition layer to $D = 1,000\,m$: $T = 8^o\,C$ at $D = 500\,m$ and $T = 5^o\,C$ at $D = 1,000\,m$
    * Deep ocean region: $T = 2^o\,C$ at $D = 4,000\,m$.

- Viscosity

$$\tau_{yx} = \mu\frac{\partial u}{\partial y} \tag{1.19}$$

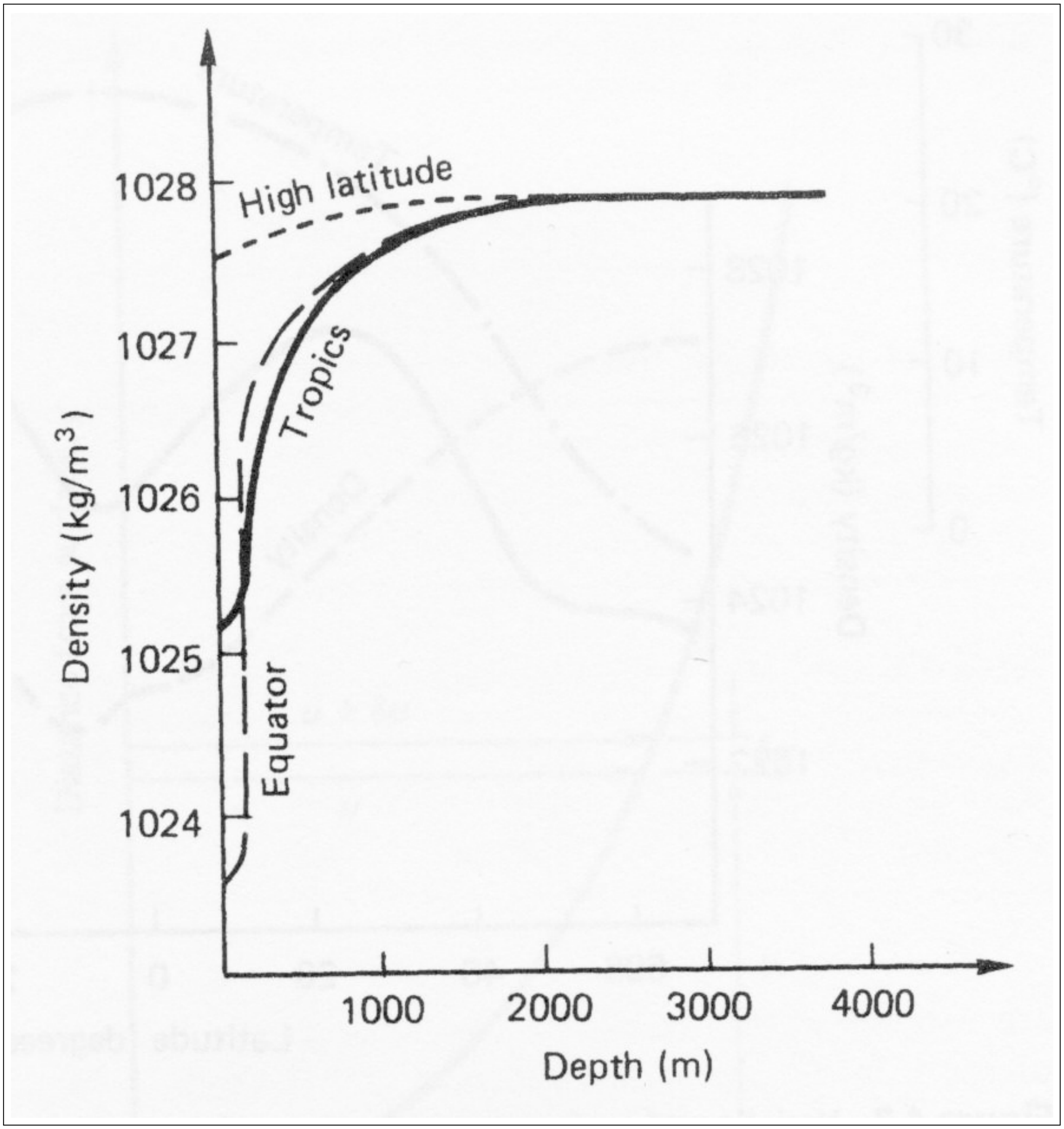


Figure 1.7 Typical variation od depth versus density for different global latitudes. (From Carlton 1994)

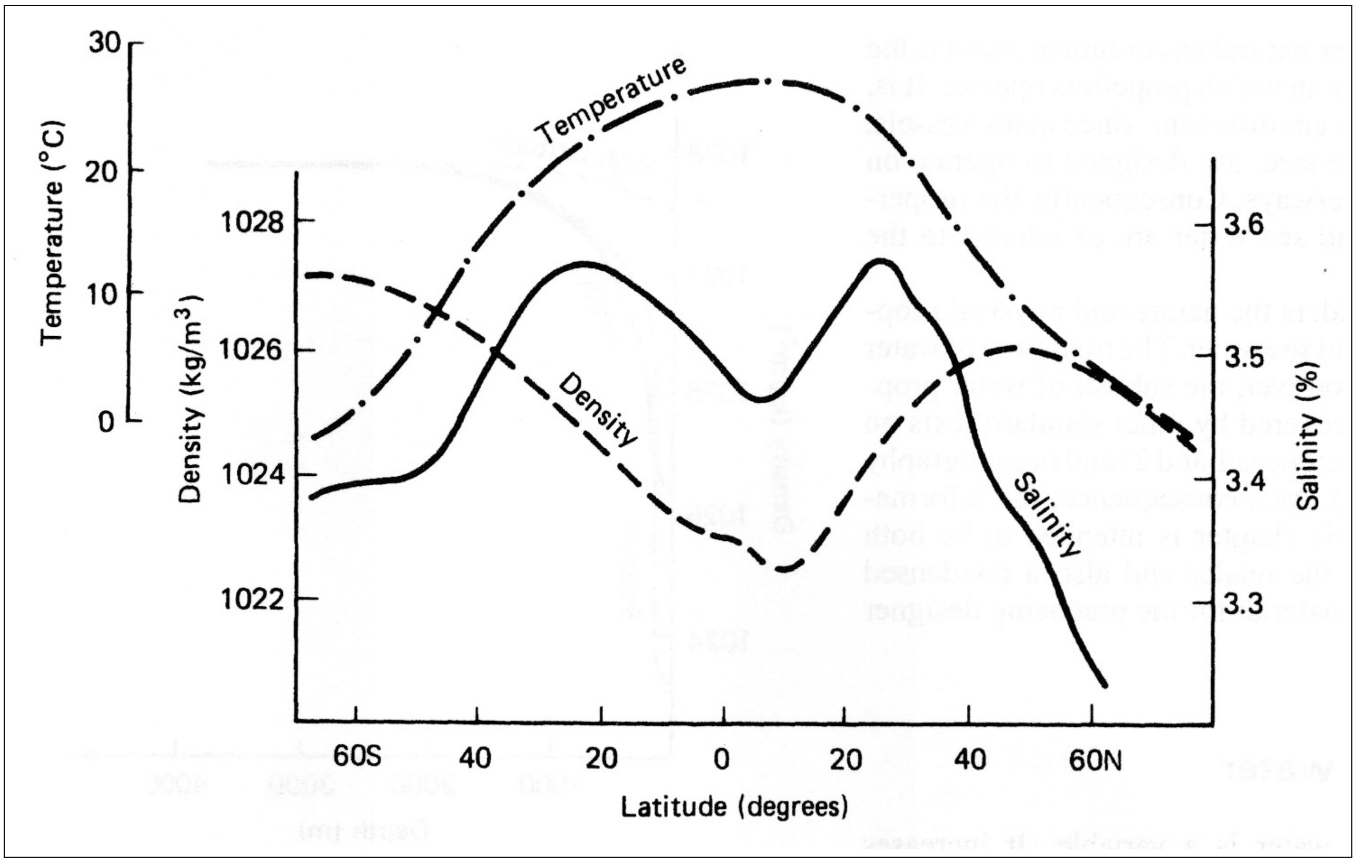


Figure 1.8 Variation of surface temperature, salinity and density with latitude–average for all oceans. (From Carlton 1994)

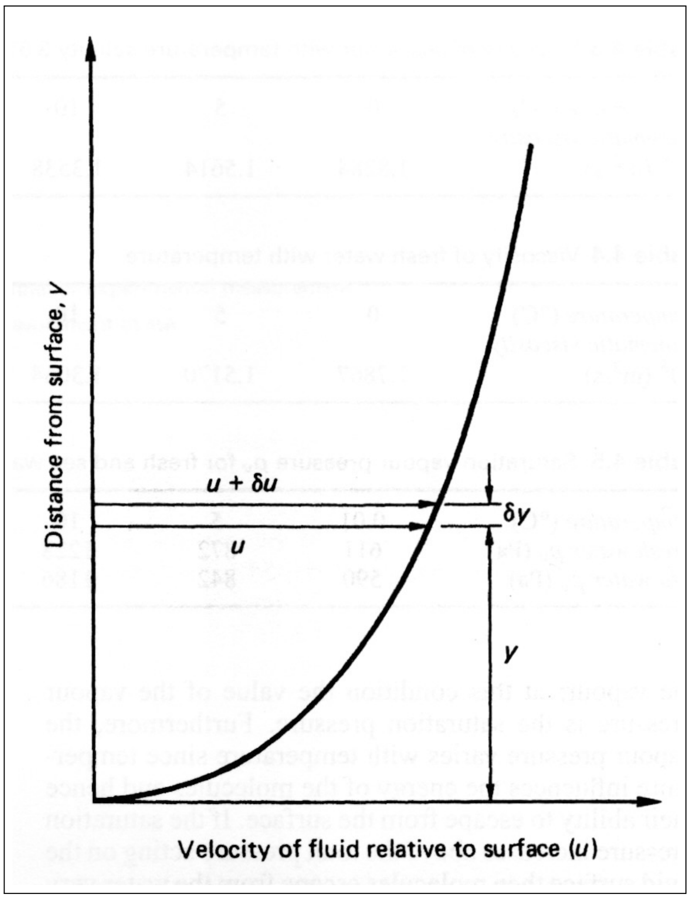


Figure 1.9 Typical viscous velocity gradient. (From Carlton 1994)

Table 1.3 Viscosity of sea water with temperature (salinity 3.5%). (Adapted from Carlton 1994)

Temperature (^0C)	0	5	10	15	20	25	30
Kinematic Viscosity $\times 10^6$ (m^2/s)	1.8284	1.5614	1.3538	1.1883	1.0537	0.9425	0.8493

Table 1.4 Viscosity of fresh water with temperature. (Adapted from Carlton 1994)

Temperature (^0C)	0	5	10	15	20	25	30
Kinematic Viscosity $\times 10^6$ (m^2/s)	1.7867	1.5170	1.3064	1.1390	1.0037	0.8929	0.8009

- Vapour Pressure

Table 1.5 Saturation vapour pressure p_v for fresh and sea water. (Adapted from Carlton 1994)

Temperature (^0C)	0.01	5	10	15	20	25	30
Sea Water $p_v(Pa)$	590	842	1186	1646	2296	3058	4097
Fresh Water $p_v(Pa)$	611	872	1228	1704	2377	3166	4241

- Surface Tension

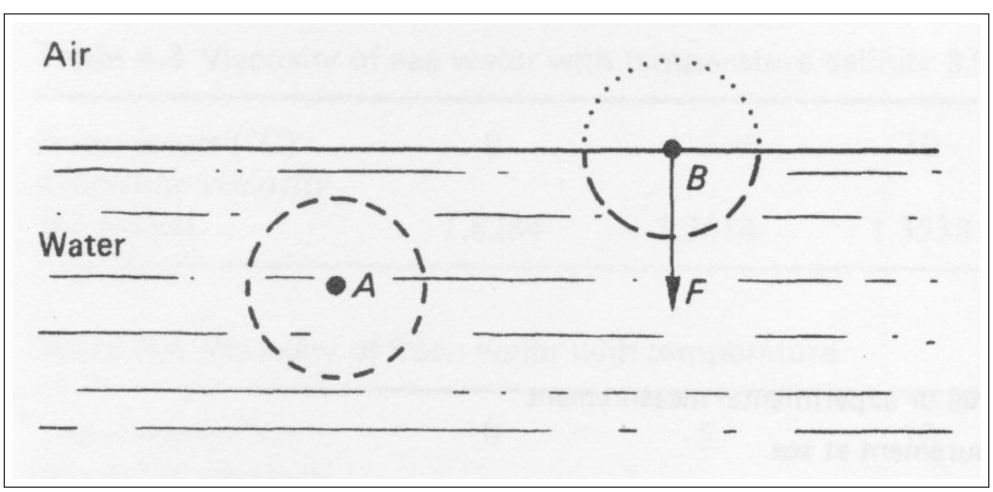


Figure 1.10 Molecular explanation of surface tension. (From Carlton 1994)

Table 1.6 Typical values of surface tension for sea and fresh water with temperature. (Adapted from Carlton 1994) Note: $1\,dyne = 10^{-5}\,Newton$.

Temperature (^0C)	0	5	10	15	20	25	30
Sea Water ($dynes/cm$)	76.41	75.69	74.97	74.25	73.55	72.81	72.09
Fresh Water ($dynes/cm$)	75.64	74.92	74.20	73.48	72.76	72.04	71.32

- Incompressibility of Water [4]
  - The elastic force in a fluid is

$$\text{Elastic force} \sim p\,l^3 \sim \rho\,C^2\,l^2 \tag{1.20}$$

since the speed of sound C in a fluid is related to pressure and density: $C = \frac{p}{\rho}$. Then

$$\tag{1.21}$$

  - Mach number: ratio of characteristic fluid velocity in a flow to speed of sound in the medium:

$$\text{Mach number} \quad M = \frac{\text{inertia force}}{\text{elastic force}} = \sqrt{\frac{\rho\,U^2\,l^2}{\rho\,C^2\,l^2}} = \frac{U}{C} \tag{1.22}$$

where U is the characteristic fluid velocity.

  - The average speed of sound in air and water is: $C_{air} \sim 350m/s$, $C_{water} \sim 1,500m/s$ Therefore the average ratio of the speed of sound in water to air is $C_{water}/C_{air} \sim 4$.
  - Because the average water to air density ratio is 1,000, it is 'harder' to move in water and therefore, typically, it is $U_{water} \ll U_{air}$ giving thus typical values of Mach numbers in the order of:

$$M_{air} \sim O(1) \Rightarrow \text{Compressible flow} \tag{1.23}$$

$$M_{water} \ll 1 \Rightarrow \text{Incompressible flow} \tag{1.24}$$

[4] Movie: Shock Flows around Airfoil
./mmfm_movies/545.mov

– Only 0.4 % change under 100 atmospheric pressure (at 1 km depth in sea water).

– Note: An incompressible flow does not mean constant density.

1.3 Mathematical Prerequisites: Vector Analysis

1.3.1 Fundamental Function Analysis

- If $\lim_{x \to c} \phi(x) = \phi(c)$, the function $\phi(x)$ is said to be continuous at the point $x = c$.

- The base of natural logarithm e,

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = 2.7182818285 \cdots . \tag{1.25}$$

  Euler formula: $e^{i\theta} = \cos\theta + i \sin\theta$
  Hyperbolic sine and cosine functions:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2} \tag{1.26}$$

- A definite integral in the sense of Riemann sum:

$$\int_a^b f(x)\, dx = \lim_{N \to \infty} \sum_{i=1}^{N} f\left(a + i\frac{b-a}{N}\right) \frac{b-a}{N} \tag{1.27}$$

- The rule for change of variable :

$$\int_{x_1}^{x_2} f(x)\, dx = \int_{u_1}^{u_2} f(x(u)) \frac{dx}{du}\, du \tag{1.28}$$

  The integral for functions of two variables

$$\iint_{\Omega_{xy}} f(x, y)\, dxdy = \iint_{\Omega_{uv}} f(x(u, v), y(u, v))\, |J|\, dudv, \tag{1.29}$$

where Jacobian $J \equiv \dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial x}{\partial v}\dfrac{\partial y}{\partial u}$.

- Leibnitz's rule in 1-D :

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\,dx =$$
$$f\,[b(t),t]\,b'(t) - f\,[a(t),t]\,a'(t) + \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}\,dx \qquad (1.30)$$

  The Reynolds' Transport Theorem for 2-D and 3-D region.

- Tensor notation in 3D space
  - Range convention:

$$x_i\ (i=1,2,3) \rightarrow x_1, x_2, x_3 \qquad (1.31)$$

  - Summation convention:

$$a_i\,b_i = a_1\,b_1 + a_2\,b_2 + a_3\,b_3\ \ (i=1,2,3) \qquad (1.32)$$

  - For example, $a_i = x_{ij}\,n_j$ denotes three equations, one for each $i = 1,2,3$ and j is the dummy index.
- Tensors.
  - A scalar is called a zero-order tensor.
  - A vector is a first-order tensor.
  - Dyads are second-order tensors: a 3×3 matrix form. (e.g. stress tensor)
  - The alternating tensor ϵ_{ijk} is a special third-order tensor.

1.3.2 Vector Calculus

- The simplest vector: *line vectors*.
  A line vector is transformed from one coordinate system to another.

- Consider two Cartesian coordinate systems rotated with respect to one another.
  - a_{11}, a_{21}, a_{31}: the direction cosines of the x'_1 axis, with respect to the x_1, x_2, x_3 axes, respectively.
  - Transform between 2 coordinates in a summation notation:

$$x'_i = \sum_{j=1}^{3} a_{ji} x_j \quad i = 1, 2, 3 \tag{1.33}$$

$$x_i = \sum_{j=1}^{3} a_{ij} x'_j \quad i = 1, 2, 3 \tag{1.34}$$

- A *vector* is defined as :

$$u'_i = \sum_{j=1}^{3} a_{ji} u_j \quad i = 1, 2, 3 \tag{1.35}$$

  - Example (a) Velocity of a point, dx_i/dt.

$$\frac{dx'_i}{dt} = \frac{d}{dt} \sum_{j=1}^{3} a_{ji} x_j = \sum_{j=1}^{3} a_{ji} \frac{dx_j}{dt}. \tag{1.36}$$

  - Example (b) Gradient of a scalar function, $\partial u/\partial x_i$.
    Let $w_i \equiv \partial u/\partial x_i$; and $w'_i \equiv \partial u/\partial x'_i$.

$$w'_i = \sum_{j=1}^{3} \frac{\partial u}{\partial x_j} \frac{\partial x_j}{\partial x'_i} = \sum_{j=1}^{3} \frac{\partial u}{\partial x_j} a_{ji} = \sum_{j=1}^{3} a_{ji} w_j \tag{1.37}$$

- *Unit base vectors*: $\underline{i}$, $\underline{j}$, and $\underline{k}$.
  For curvilinear coordinate systems, $\underline{e}_1$, $\underline{e}_2$, and $\underline{e}_3$.

- Any vector $\underline{a}$ as the sum of its components :

$$\underline{a} = a_1 \, \underline{i} + a_2 \, \underline{j} + a_3 \, \underline{k}. \tag{1.38}$$

The position vector $\underline{x}$:

$$\underline{x} = x\,\underline{i} + y\,\underline{j} + z\,\underline{k}. \tag{1.39}$$

The distance of $\underline{x}$ from the origin as $r \equiv |\underline{x}| = \sqrt{x^2 + y^2 + z^2}$.

- Tensor notation:
  Example) Kronecker delta:

$$\delta_{ij} = 1 \text{ if } i = j; \quad \delta_{ij} = 0 \text{ if } i \neq j \tag{1.40}$$

Example) Alternating tensor of permutation symbol:

$$\left.\begin{aligned} \epsilon_{ijk} &= 0 && \text{if any } i, j, k \text{ equal} \\ \epsilon_{ijk} &= 1 && \text{if } (ijk) = (123), (231), (312) \\ \epsilon_{ijk} &= -1 && \text{if } (ijk) = (132), (213), (321) \end{aligned}\right\} \tag{1.41}$$

The basic formulas :

$$\delta_{ii} = 3, \quad \delta_{ij}\,u_{klmi} = u_{klmj}, \quad \delta_{ij}\,\epsilon_{ijk} = 0, \tag{1.42}$$

$$\epsilon_{ijk}\,\epsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}, \quad \epsilon_{ijk}\,\epsilon_{ijk} = 6, \tag{1.43}$$

$$\epsilon_{ijk}\,\epsilon_{ljk} = 2\,\delta_{il}. \tag{1.44}$$

- *Scalar product*:

$$\underline{a} \cdot \underline{b} = a\,b\,\cos(\underline{a} \cdot \underline{b}) \tag{1.45}$$

or,

$$\underline{a} \cdot \underline{b} = a_1\,b_1 + a_2\,b_2 + a_3\,b_3 \tag{1.46}$$

or,

$$\underline{a} \cdot \underline{b} = \delta_{ij}\,a_i\,b_j = a_i\,b_i, \quad \text{(summation convention)} \tag{1.47}$$

- *Vector product*:

$$\underline{c} = \underline{a} \times \underline{b}; \quad w = a\,b\,\sin(\underline{a}, \underline{b}). \tag{1.48}$$

In a form of tensor-notation, $\underline{a} \times \underline{b} = \epsilon_{ijk}\,a_j\,b_k$.

- *Scalar triple product*:

$$\underline{a} \cdot (\underline{b} \times \underline{c}) = a_i\, \epsilon_{ijk}\, b_j\, c_k \tag{1.49}$$

$$\underline{a} \cdot \underline{b} \times \underline{c} = \underline{a} \times \underline{b} \cdot \underline{c} = \underline{b} \cdot \underline{c} \times \underline{a} \;\; \text{etc.} \tag{1.50}$$

- *Vector triple product*:

$$\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c})\, \underline{b} - (\underline{a} \cdot \underline{b})\, \underline{c} \tag{1.51}$$

i.e.,

$$\begin{aligned} \underline{a} \times (\underline{b} \times \underline{c}) &= \epsilon_{mli}\, a_l\, \epsilon_{ijk}\, b_j\, c_k \\ &= (\delta_{mj}\, \delta_{lk} - \delta_{mk}\, \delta_{lj})\, a_l\, b_j\, c_k \\ &= a_k\, b_j\, c_k - a_j\, b_j\, c_k. \end{aligned} \tag{1.52}$$

The basic formula :

$$\underline{a} \times (\underline{b} \times \underline{c}) + \underline{b} \times (\underline{c} \times \underline{a}) + \underline{c} \times (\underline{a} \times \underline{b}) = 0 \tag{1.53}$$

- *Gradient*:

$$\nabla u \equiv \lim_{V \to 0} \frac{1}{V} \oint_S u\, \underline{n}\, dS \tag{1.54}$$

where S is the area enclosing the volume V, dS is the element of area, and $\underline{n}$ is the unit vector normal to the surface.
In limit,

$$\nabla u = \underline{i}\, \frac{\partial u}{\partial x} + \underline{j}\, \frac{\partial u}{\partial y} + \underline{k}\, \frac{\partial u}{\partial z} \tag{1.55}$$

Symbol: ∇u, grad u, or $\dfrac{\partial u}{\partial x_i}$.

- *Divergence*:

$$\nabla \cdot \underline{v} = \lim_{V \to 0} \frac{1}{V} \oint_S \underline{n} \cdot \underline{v}\, dS \tag{1.56}$$

$$\nabla \cdot \underline{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \tag{1.57}$$

Symbol: $\nabla \cdot \underline{v}$, div $\underline{v}$, or $\dfrac{\partial v_i}{\partial x_i}$.

- *Curl*:

$$\nabla \times \underline{v} \equiv \lim_{V \to 0} \frac{1}{V} \oint_S \underline{n} \times \underline{v}\, dS \tag{1.58}$$

$$\nabla \times \underline{v} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) \underline{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \right) \underline{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) \underline{k} \tag{1.59}$$

or,

$$\nabla \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} \tag{1.60}$$

Symbol: curl $\underline{v}$, $\nabla \times \underline{v}$, or $\epsilon_{ijk} \dfrac{\partial v_k}{\partial x_j}$.

- *Laplacian* :

$$\nabla^2 u \equiv \nabla \cdot (\nabla u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \tag{1.61}$$

The Laplacian of a vector function

$$\nabla^2 \underline{v} = \underline{i}\, \nabla^2 v_1 + \underline{j}\, \nabla^2 v_2 + \underline{k}\, \nabla^2 v_3 \tag{1.62}$$

- *Differential operator*: $d\underline{r} \cdot \nabla$

$$du = d\underline{r} \cdot \nabla u \tag{1.63}$$

where $\underline{r}$ is the position vector and $d\underline{r}$ is any directed line element.

  - This means that du is the increment of u corresponding to a position increment $d\underline{r}$.
  - In rectangular Cartesian coordinates,

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = (d\underline{r} \cdot \nabla) u \tag{1.64}$$

– Similarly, for a vector function $\underline{v}(x, y, z)$,

$$
\begin{aligned}
d\underline{v} &\equiv \underline{i}\, dv_1 + \underline{j}\, dv_2 + \underline{k}\, dv_3 \\
&= \left(dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} + dz \frac{\partial}{\partial z} \right) (\underline{i}\, v_1 + \underline{j}\, v_2 + \underline{k}\, v_3) \\
&= (d\underline{r} \cdot \nabla)\underline{v} \qquad (1.65)
\end{aligned}
$$

– The symbol ∇ is a *vector operator*:

$$\nabla = \underline{i}\, \frac{\partial}{\partial x} + \underline{j}\, \frac{\partial}{\partial y} + \underline{k}\, \frac{\partial}{\partial z}. \qquad (1.66)$$

1.3.3 Expansion Formulas

- ϕ: any differentiable scalar function of x, y, z.
  $\underline{u}, \underline{v}, \underline{w}$: any such vector functions.

$$\nabla \cdot (\phi\, \underline{u}) = \underline{u} \cdot \nabla\phi + \phi\, \nabla \cdot \underline{u} \qquad (1.67)$$

$$\nabla \times (\phi\, \underline{u}) = (\nabla\phi) \times \underline{u} + \phi\, \nabla \times \underline{u} \qquad (1.68)$$

$$\nabla \cdot (\underline{v} \times \underline{w}) = \underline{w} \cdot \nabla \times \underline{v} - \underline{v} \cdot \nabla \times \underline{w} \qquad (1.69)$$

$$\nabla \times (\underline{v} \times \underline{w}) = \underline{w} \cdot \nabla\underline{v} + \underline{v}\, \nabla \cdot \underline{w} - \underline{w}\, \nabla \cdot \underline{v} - \underline{v} \cdot \nabla\underline{w} \qquad (1.70)$$

$$\nabla(\underline{v} \cdot \underline{w}) = \underline{v} \cdot \nabla\underline{w} + \underline{w} \cdot \nabla\underline{v} + \underline{v} \times (\nabla \times \underline{w}) + \underline{w} \times (\nabla \times \underline{v}) \qquad (1.71)$$

$$\nabla \cdot (\nabla \times \underline{v}) = 0 \qquad (1.72)$$

$$\nabla \times (\nabla\phi) = 0 \qquad (1.73)$$

$$\nabla \times (\nabla \times \underline{v}) = \nabla(\nabla \cdot \underline{v}) - \nabla^2\underline{v} \qquad (1.74)$$

- The position vector: $\underline{x} = x_1\, \underline{i} + x_2\, \underline{j} + x_3\, \underline{k}$.
  Magnitude of the position vector $r = |\underline{x}| = \sqrt{\underline{x} \cdot \underline{x}}$ and a constant vector $\underline{a}$:

$$\nabla r = \underline{x}/r \qquad (1.75)$$

$$\nabla \cdot \underline{x} = 3 \qquad (1.76)$$

$$\nabla \times \underline{x} = 0 \tag{1.77}$$

$$\nabla r^n = n\, r^{n-2}\, \underline{x} \tag{1.78}$$

$$\nabla \cdot (r^n\, \underline{x}) = (n+3)\, r^n \tag{1.79}$$

$$\nabla \times (r^n\, \underline{x}) = 0 \tag{1.80}$$

$$\nabla^2 (r^n) = n(n+1)\, r^{n-2} \tag{1.81}$$

$$\nabla \cdot (\underline{a} \times \underline{x}) = 0 \tag{1.82}$$

$$\nabla (\underline{a} \cdot \underline{x}) = \underline{a} \tag{1.83}$$

$$\nabla \times (\underline{a} \times \underline{x}) = 2\, \underline{a} \tag{1.84}$$

$$\nabla \cdot (\underline{a} \times \nabla r) = 0 \tag{1.85}$$

$$\nabla \cdot (r\, \underline{a}) = (\underline{x} \cdot \underline{a})/r \tag{1.86}$$

$$\nabla \times (r\, \underline{a}) = (\underline{x} \times \underline{a})/r \tag{1.87}$$

1.3.4 Divergence Theorem (Gauss Theorem)

- Consider the surface integral $\oint_S u\, \underline{n}\, dS$.
  If the volume V is subdivided into small volume V_i,

$$\oint_S u\, \underline{n}\, dS = \sum \oint_{S_i} u\, \underline{n}\, dS \tag{1.88}$$

In the limit,

$$\oint_{S_i} u\, \underline{n}\, dS = \int_{V_i} \nabla u\, dV.$$

Definition of the gradient, eq. (1.54),

$$\oint_S u\, \underline{n}\, dS = \int_V \nabla u\, dV \tag{1.89}$$

- Use the definitions of the divergence and curl,

$$\oint_S \underline{n} \cdot \underline{v} \, dS = \int_V \nabla \cdot \underline{v} \, dV \tag{1.90}$$

$$\oint_S \underline{n} \times \underline{v} \, dS = \int_V \nabla \times \underline{v} \, dV \tag{1.91}$$

General form:

$$\boxed{\oint_S (\underline{n} * f) \, dS = \int_V (\nabla * f) \, dV} \tag{1.92}$$

- An example,

$$\begin{aligned} \int_V \nabla^2 u \, dV &= \int_V \nabla \cdot (\nabla u) \, dV \\ = \oint_S \underline{n} \cdot \nabla u \, dS &= \oint_S \frac{\partial u}{\partial n} \, dS \end{aligned} \tag{1.93}$$

where $\partial u / \partial n$ is the directed derivative in the outward direction.

1.3.5 Stokes' Theorem

- From our definitions for ∇u and $\nabla \times \underline{v}$

$$\underline{n} \times \nabla u \approx \frac{1}{S} \oint_C u \, d\underline{r} \tag{1.94}$$

$$\underline{n} \cdot \nabla \times \underline{v} \approx \frac{1}{S} \oint_C \underline{v} \cdot d\underline{r} \tag{1.95}$$

where S denotes a very small surface element in the fluid, C is the small contour that forms the boundary of S, and $\underline{n}$ is a unit normal to S.

- The transformation theorems relate certain surface integrals to contour integrals:

$$\int_S \underline{n} \times \nabla u \, dS = \oint_C u \, d\underline{r} \tag{1.96}$$

$$\begin{aligned} \int_S \underline{n} \cdot \nabla \times \underline{v} \, dS &= -\int_S (\underline{n} \times \nabla) \cdot \underline{v} \, dS \\ &= \oint_C \underline{v} \cdot d\underline{r} \end{aligned} \tag{1.97}$$

The unified form of Stokes' theorem :

$$\boxed{\int_S (\underline{n} \times \nabla) * f \, dS = \oint_C \, d\underline{r} * f} \tag{1.98}$$

1.3.6 Dyadic Products

- The *dyadic product*: a special form of *second-order tensor* $\underline{u}\ \underline{v}$:

$$\begin{aligned} (\underline{u}\ \underline{v}) \cdot \underline{w} \equiv \underline{u}(\underline{v} \cdot \underline{w}) \\ \underline{w} \cdot (\underline{u}\ \underline{v}) \equiv (\underline{w} \cdot \underline{u})\underline{v} \end{aligned} \tag{1.99}$$

- The gradient of a vector, $\nabla \underline{v}$, in the Navier-Stokes equations:
  If the vector $\underline{v}$ is a velocity resolved into a symmetric and antisymmetric form:

$$\begin{aligned} \nabla \underline{v} &= \frac{1}{2}\left[\left(\nabla \underline{v} + \nabla \underline{v}^T\right) + \left(\nabla \underline{v} - \nabla \underline{v}^T\right)\right] \\ &= \frac{1}{2} def(\underline{v}) + \frac{1}{2} rot(\underline{v}) \end{aligned} \tag{1.100}$$

  - A second-order tensor to be a 3×3 matrix.
  - The superscript T stand for transpose of the matrix.
  - The first term causes stress: (i) normal strain rate and (ii) shear strain rate.
  - The second term: rigid body rotation of a fluid element.

1.3.7 Reynolds Transport Theorem

- The rate of change of an integral taken over a volume moving through a field $F(\underline{x}, t)$

$$\frac{d}{dt} \iiint\limits_{V(t)} F(\underline{x}, t)\; dV \tag{1.101}$$

The path of points in V(t):

$$\underline{x} = \underline{x}(\underline{\xi}, t) \tag{1.102}$$

where $\underline{\xi}$ is the initial point of $\underline{x}$.

- The Reynolds transport theorem

$$\frac{d}{dt} \iiint\limits_{V(t)} F\; dV = \iiint\limits_{V} \left[\frac{\partial F}{\partial t} + \nabla \cdot (\underline{v}\, F)\right] dV \tag{1.103}$$

or

$$\boxed{\frac{d}{dt} \iiint\limits_{V(t)} F\; dV = \iiint\limits_{V(t)} \frac{\partial F}{\partial t}\; dV + \iint\limits_{S(t)} \underline{n} \cdot (\underline{v}\, F)\; dS} \tag{1.104}$$

  - Here $\underline{v}$ is the velocity of the point $\underline{x}$, and $\underline{n}$ is the outward unit vector normal to the boundary $S(t)$.
  - The first integral is the rate of change in volume, and the second integral is the rate of change associated with motion of surface bounding volume.
  - Similar to Leibnitz's rule for 1-dimensional region:

$$\frac{d}{dt} \int_{a(t)}^{b(t)} f(x, t)\; dx =$$
$$\int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}\; dx + f[b(t), t]\, b'(t) - f[a(t), t]\, a'(t) \tag{1.105}$$

1.3.7.1 Example of the Reynolds transport theorem in 1-D

- Consider an integral with $a(t) = t + 1$, $b(t) = 2t + 2$, $F(x,t) = x\,t$, and $x(\xi,t) = \xi\,t + \xi$:

$$I(t) = \int_{a(t)}^{b(t)} F(x,t)\, dx, \quad \frac{dI(t)}{dt} = \frac{d}{dt}\int_{a(t)}^{b(t)} F(x,t)\, dx \tag{1.106}$$

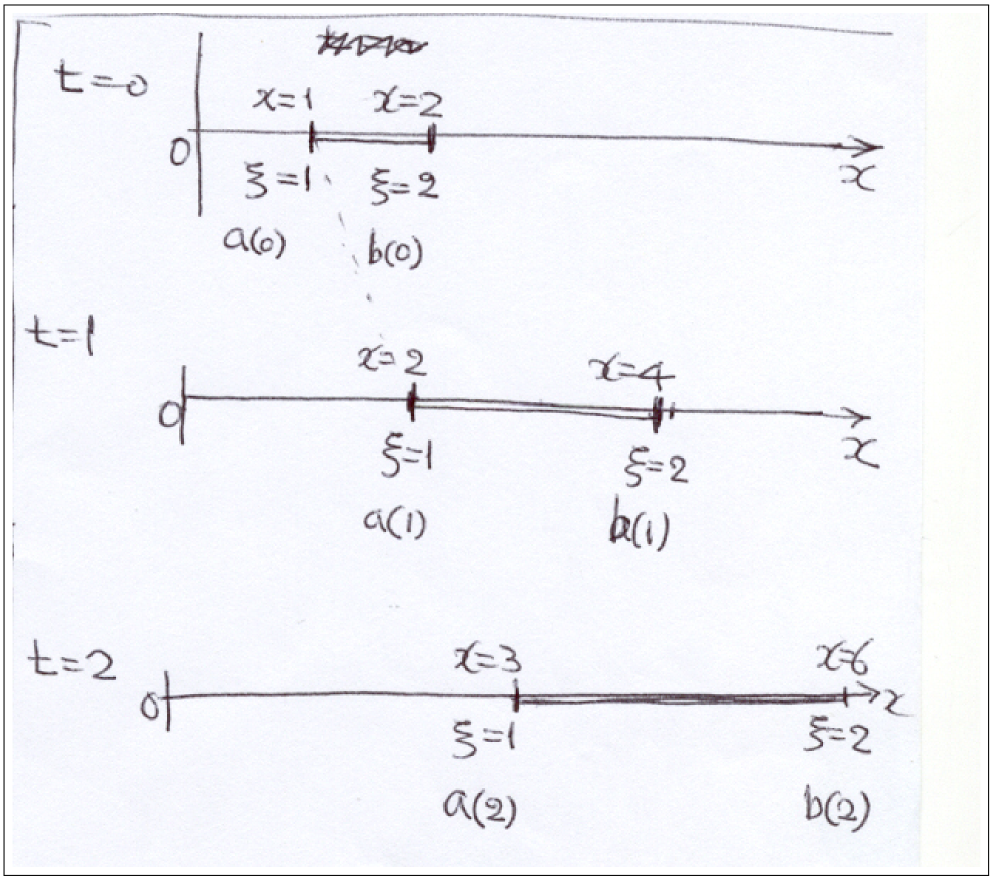


Figure 1.11 Sequential change of the interval of the integral (1.106).

- Lagrangian variable $\xi = \dfrac{x}{t+1}$ from the given relation $x(\xi,t) = \xi\,t + \xi$, then the integrand

$$F^*(\xi,t) = (\xi t + \xi)t \tag{1.107}$$

- Velocity and divergence of velocity:

$$\underline{v}^*(\xi,t) = \left.\frac{\partial x}{\partial t}\right|_{\xi=\text{const.}} = \xi\,\underline{i} \equiv v^*\underline{i} \qquad \text{in Lagrangian manner} \tag{1.108}$$

$$\underline{v}(x,t) = \frac{x}{t+1}\,\underline{i} \equiv v\,\underline{i} \qquad \text{in Eulerian manner} \tag{1.109}$$

$$\nabla\cdot\underline{v}(x,t) = \frac{1}{t+1} \equiv \frac{\partial v}{\partial x} \qquad \text{in 1-dimensions} \tag{1.110}$$

- Chain rule (i.e., Jacobian)

$$\frac{\partial x}{\partial \xi} = t+1 \equiv J \;\; (\text{Jacobian}) \tag{1.111}$$

- The time derivative of the integral

$$\begin{aligned}
\frac{dI(t)}{dt} &= \frac{d}{dt}\int_{a(t)}^{b(t)} F(x,t)\,dx \\
&= \frac{d}{dt}\int_{a(0)}^{b(0)} F^*(\xi,t)\,\frac{\partial x}{\partial \xi}\,d\xi \qquad \text{(by change of variable)} \\
&= \frac{d}{dt}\int_{a(0)}^{b(0)} \{(\xi\,t+\xi)t\}\;J\,d\xi \qquad \text{(with Jacobian)} \\
&= \int_{a(0)}^{b(0)} \left[\left.\frac{\partial\{(\xi\,t+\xi)t\}}{\partial t}\right|_{\xi=\text{const.}} J + \;\{(\xi\,t+\xi)t\}\,\underbrace{\frac{\partial J}{\partial t}}_{\boxed{A}}\right] d\xi
\end{aligned} \tag{1.112}$$

where $\boxed{A}$ is equivalent to

$$\begin{aligned}
\frac{\partial J}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{\partial x}{\partial \xi}\right) = \frac{\partial}{\partial \xi}\left(\frac{\partial x}{\partial t}\right) \qquad \text{(interchanging the differential order)} \\
&= \frac{\partial v^*}{\partial \xi} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \xi} \qquad \text{(by chain rule)} \\
&= (\nabla\cdot\underline{v})J \quad \Longrightarrow \quad \left(\frac{1}{t+1}\right)(t+1)
\end{aligned} \tag{1.113}$$

- Rearranging the terms of the integrand and then converting the integral with respect to the variable ξ into one with respecto to the original variable x, we have

$$
\begin{aligned}
\frac{dI(t)}{dt} &= \int_{a(0)}^{b(0)} \Bigg[\underbrace{\frac{\partial}{\partial t}(x\,t)}_{\dfrac{\partial F}{\partial t}} + \underbrace{\left.\frac{\partial x}{\partial t}\right|_{\xi=\text{const.}} \frac{\partial}{\partial x}(x\,t)}_{\underline{v}\cdot\nabla F} + \underbrace{(x\,t)\left(\frac{1}{t+1}\right)}_{F\,\nabla\cdot\underline{v}} \Bigg] \underbrace{(t+1)}_{J}\, d\xi \\
&= \int_{a(t)}^{b(t)} \left[\frac{\partial F}{\partial t} + \nabla\cdot(\underline{v}\,F) \right] dx \\
&= \int_{a(t)}^{b(t)} \left[\frac{\partial F}{\partial t} + \frac{\partial}{\partial x}(v\,F) \right] dx \\
&= \int_{a(t)}^{b(t)} \frac{\partial F}{\partial t}\, dx + [v\,F(x,t)]\Big|_{x=b(t)} - [v\,F(x,t)]\Big|_{x=a(t)} \\
&= \int_{a(t)}^{b(t)} \frac{\partial F}{\partial t}\, dx + F[b(t),t]\,b'(t) - F[a(t),t]\,a'(t)
\end{aligned}
\tag{1.114}
$$

1.3.8 Moving Coordinate Systems

- Two coordinate systems:
  The position vector $\underline{x}'$ in the space-fixed system is related to the position vector $\underline{x}$ in the moving system:

$$\underline{x}' = \underline{x} + \underline{R} \tag{1.115}$$

  where $\underline{R}$ is the distance vector between the origins of two coordinate systems.

- The derivative(d'/dt) observed in the space-fixed system and the derivative(d/dt) observed in the moving system:

$$\frac{d'}{dt} = \frac{d}{dt} + \underline{\Omega}\times \tag{1.116}$$

  where $\underline{\Omega}$ is the angular velocity of the moving system.

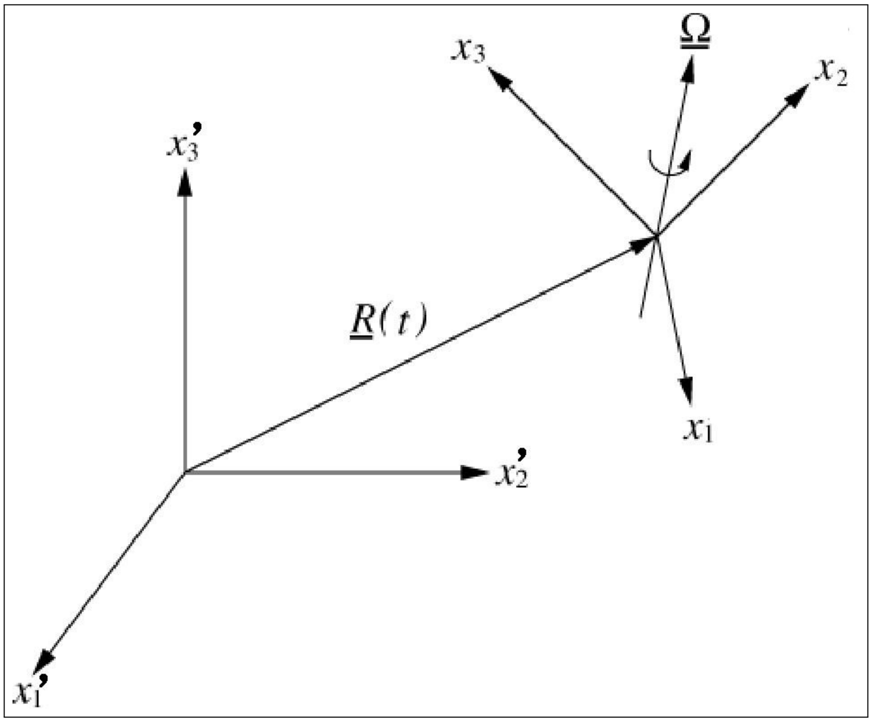


Figure 1.12 Moving coordinate system.

- The velocity vectors in the two coordinate systems:

$$\underline{q}' = \underline{q} + \underline{\Omega} \times \underline{x} + \dot{\underline{R}} \tag{1.117}$$

where $\dot{\underline{R}}$ represents the translation velocity of the moving frame.

- Acceleration vectors $\underline{a}$ in the space-fixed system :

$$\underline{a}' \equiv \frac{d^2\underline{x}'}{dt^2} = \underline{a} + 2\underline{\Omega} \times \underline{q} + \frac{d\underline{\Omega}}{dt} \times \underline{x} + \underline{\Omega} \times (\underline{\Omega} \times \underline{x}) + \ddot{\underline{R}} \tag{1.118}$$

  - The first term : the acceleration viewed in the moving system.
  - The second term: the *Coriolis acceleration*.
  - The fourth term: the generalized centripetal acceleration, since

$$|\underline{\Omega} \times (\underline{\Omega} \times \underline{x})| = \Omega^2 \, \underline{x} \, \sin(\underline{\Omega}, \underline{x}) \tag{1.119}$$

  - For the self-rotation of earth with constant angular speed, its effect (a form of gradient of a scalar function) is already included in gravitational acceleration values.