Lecture Notes 414.341

선 박 해 양 유 체 역 학

MARINE HYDRODYNAMICS

2019년 6월 28일

Suh, Jung-Chun
서 정 천

Seoul National Univ., Dept. NAOE
서울대학교 공과대학 조선해양공학과

3

KINEMATICS

3.1 Description of Fluid Motion

3.1.1 Definition of Fluid Particle

- Although one of our assumption on a fluid is that it is a continuum and does not consist of discrete particles, we introduce the term "fluid particles," such as, "velocity of a particle," etc, to identify simply an infinitesimal portion or sample of the fluid by mathematically tagging it. [1]
- There are two common ways of representing equations to describe a fluid flow. [2]

3.1.2 Lagrangian Description: Path Lines

- We may take the tag to be the initial position, denoted by $\underline{\xi}(a, b, c)$. Let a, b, c denote the coordinates of any fluid particle at the time $t = 0$.
- Let x, y, z denote the coordinates of the same particle at time t. Then the flow geometry is completely specified if we know $x = x(a, b, c, t)$, $y = y(a, b, c, t)$, $z = z(a, b, c, t)$. These give the trajectories of various particles.
- The pathline of a particle is the curve $\underline{x} = \underline{x}(\underline{\xi}, t)$, where $\underline{x}$ is the position vector. [3] The velocity is $\underline{q}(a, b, c, t) = \partial\underline{x}/\partial t$ and the acceleration is $\partial\underline{q}/\partial t = \partial^2\underline{x}/\partial t^2$.

[1] Movie: Fluid particles: Velocity vector, Block & cylinder
./mmfm_movies/Vectors2.mov ./mmfm_movies/1_06re1000_block.mov

[2] Movie: Flat plate(Lagrangian frame & Eulerian frame)
./mmfm_movies/impflws_cntrstd.mov

[3] Movie: Pathline (Smoke over roof)
./mmfm_movies/722.mov ./mmfm_movies/721.mov

- Any other physical quantities would be given by a function, say, $f = f(a,b,c,t)$. This description is called Lagrangian, material, or convective description of motion.

3.1.3 Eulerian Description

- Instead of following individual particles as above, in Eulerian description we fix our attention on a point in space, x, y, z. Consider any property of the fluid, for example, the density ρ, and calculate its differential:

$$\begin{aligned} \rho &= \rho(x,y,z,t) && (3.1) \\ d\rho &= \frac{\partial\rho}{\partial x}\,dx + \frac{\partial\rho}{\partial y}\,dy + \frac{\partial\rho}{\partial z}\,dz + \frac{\partial\rho}{\partial t}\,dt \\ &= d\underline{x}\cdot\nabla\rho + \frac{\partial\rho}{\partial t}\,dt && (3.2) \end{aligned}$$

- For any given particle as it moves along, dx, dy, dz are not independent; in fact, $dx = u\ dt, dy = v\ dt$, and $dz = w\ dt$, i.e., $d\underline{x} = \underline{q}\ dt$, where $\underline{q}(x,y,z,t)$ is the velocity.
  Thus, the rate of change of the density of a particle is

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial z} = \frac{\partial\rho}{\partial t} + \underline{q}\cdot\nabla\rho \qquad (3.3)$$

3.1.3.1 Local derivative

- The time rate of change of a flow quantity at a fixed point $\underline{x}$ is given by

$$\left.\frac{\partial}{\partial t}\right|_{\underline{x}=\text{const}} \qquad (3.4)$$

The flow is then called steady if the first term vanishes, that is, it does not vary with time. [4]

[4] Movie: Block(Unsteady pressure)
./mmfm_movies/thoung_02.mov

3.1.3.2 Material derivative

- We use the symbol $\dfrac{D}{Dt}$ for this type of derivative, sometimes called the "convective or material derivative":

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \underline{q} \cdot \nabla \tag{3.5}$$

- The time rate of change of a flow quantity following a particle is given by

$$\left.\frac{\partial}{\partial t}\right|_{\underline{\xi}=\mathrm{const}} \equiv \frac{D}{Dt} \tag{3.6}$$

- The velocity of a particle is the material derivative of the position vector of the particle:

$$\underline{q}^*(\underline{\xi}, t) = \left.\frac{\partial \underline{x}}{\partial t}\right|_{\underline{\xi}} = \frac{D\underline{x}}{Dt} = \underline{q}(\underline{x}, t) \tag{3.7}$$

- This can be applied to any fluid property including vector properties. The acceleration of a particle, for example, is

$$\frac{D\underline{q}}{Dt} \equiv \frac{\partial \underline{q}}{\partial t} + \underline{q} \cdot \nabla \underline{q} \tag{3.8}$$

- A similar description for the evolution of the material line element [5] is

$$\frac{D(d\underline{\xi})}{Dt} = d\underline{q} = dx_j \frac{\partial \underline{q}}{\partial x_j} = d\underline{x} \cdot \nabla \underline{q}. \tag{3.9}$$

- If $F(\underline{x}, t)$ is some property of the flow field, then

$$\left.\frac{\partial F}{\partial t}\right|_{\underline{\xi}} = \left.\frac{\partial F}{\partial t}\right|_{\underline{x}} + \underline{q} \cdot \nabla F \tag{3.10}$$

[5] A material line is a line composed of the same fluid particles in a moving fluid. Similarly a material surface and a material volume are, respectively, a surface and a volume composed of the same particles. A material surface may be a bounding surface and every impenetrable bounding surface must be a material surface.

3.1.4 Particle Tracing Lines

In the previous section, the material and spatial descriptions of the flow were described. Below we list some additional prerequisites about particle tracing lines. [6]

3.1.4.1 Streamlines [7]

- A streamline is defined as a line everywhere parallel to velocity $\underline{q}$. Namely, the tangent of the streamline at each point is parallel to the fluid velocity at that point.

- We can produce a streamline by taking a short time exposure picture of a flow for which numerous particles have been tagged. We try to trace out curves on the photograph such that each curve is tangent to the velocity vector at a point.

- Let the fluid velocity be denoted by the vector $\underline{q}$; then $\underline{q} = \underline{q}(x, y, z, t) = (u, v, w)$. Differential equations for streamlines are

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}. \tag{3.11}$$

- If $\underline{x}(\sigma)$ (where σ is the parameter) describes the position vector of a streamline, then $\dfrac{d\underline{x}}{d\sigma}$ is tangent to a streamline and parallel to the velocity at $\underline{x}(\sigma)$. Hence we can express the differential equation for streamlines in terms of the parameter σ:

$$\frac{d\underline{x}}{d\sigma} \times \underline{q}(\underline{x}(\sigma), t) = 0, \quad \text{or} \quad \frac{d\underline{x}}{d\sigma} = \underline{q}(\underline{x}(\sigma), t) \tag{3.12}$$

[6] Movie: Particle tracing lines (Smoke over roof), Timeline (Tunnel flow)
./mmfm_movies/roof_ypth_yovr.mov ./mmfm_movies/timelines.mov

[7] Movie: Streamline, (Smoke over roof)
./mmfm_movies/724.mov ./mmfm_movies/723.mov

3.1.4.2 Streaklines

- At time t, a streakline through a fixed point $\underline{y}$ is the curve traced out by particles each of which have gone through $\underline{y}$ since time $t_0 < t$. (Typically $t_0 = 0$).
- Physically we construct a streakline by making (or tagging) all particles that pass a point, e.g., by continuously emitting dye at that point. The dye trail marks the streakline.
- A particle is on the streakline at time of observation t if it had been at $\underline{y}$ at time s where s lies in the inteval $t_0 \leq s \leq t$. The material coordinates fo the particle that went through $\underline{y}$ at s are $\underline{\xi} = \underline{\xi}(\underline{y}, s)$. At time t, the particle is at the spatial position

$$\underline{x} = \underline{x}\left(\underline{\xi}(\underline{y}, s), t\right) \tag{3.13}$$

  where $\underline{y}$ and t are to be assigned and s varies from t_0 to t to trace out the streakline.
- For steady flows, a pathline, a streamline and a streakline coincide.

3.1.5 Example of Particle Tracing Lines

3.1.5.1 Velocity field

- The concepts of various flow lines may be illustrated by the 2-D case for which the particle velocity is considered to be

$$\underline{q}^*(\underline{\xi}, t) = \xi_1\, \underline{i} + \xi_2\, e^t\, \underline{j} \tag{3.14}$$

- This means that at the initial time $t_0 = 0$ the particle velocity is equal to the position vector: $\underline{q}^*(\underline{\xi}, 0) = \underline{\xi}$, and as time proceeds from $t = 0$, the horizontal component of the velocity remains unchanged but the vertical velocity component grows exponentially with time.

- Typical particle tracing lines of the flow patterns are illustrated in Figure 3.1 .

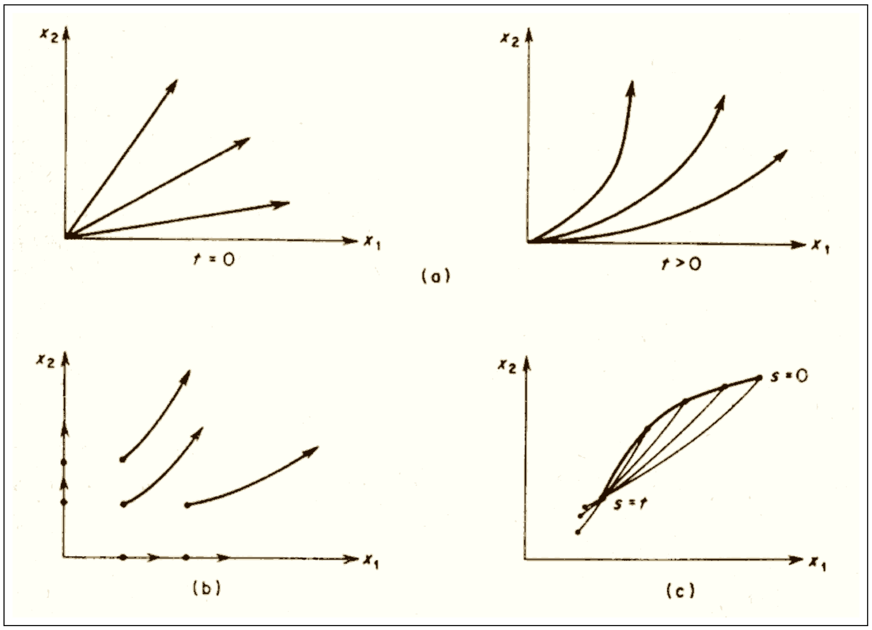

Figure 3.1 Example of various flow lines. (a) streamlines at $t = 0$ and $t > 0$; (b) path lines; (c) streaklines. (From Aris 1962, p. 82)

3.1.5.2 Pathlines

- The pathline of the particle that was initially at $\underline{\xi}$ is the curve

$$\underline{x} = \underline{\xi} + \int_0^t \underline{q}^*(\underline{\xi}, t)\, dt = \xi_1(1+t)\,\underline{i} + \xi_2\, e^t\, \underline{j} \qquad (3.15)$$

- Spatial coordinates and material coordinates can be related:

$$x_1 = \xi_1(1+t), \qquad x_2 = \xi_2\, e^t \qquad (3.16)$$

This is the parametric representation of the pathline.

- Eliminate the parameter t from the equation to find the pathline in the (x_1, x_2) plane:

$$x_2 = \xi_2\, e^{(x_1/\xi_1 - 1)} \qquad (3.17)$$

- The inverse of the pathline is the relation obtained by solving for $\underline{\xi}(\underline{x}, t)$

$$\underline{\xi} = \frac{x_1}{(1+t)}\,\underline{i} + \frac{x_2}{e^t}\,\underline{j} \tag{3.18}$$

- With the inverse of the pathlines known, the spatial description of the velocity vector can be constructed:

$$\begin{aligned} \underline{q}(\underline{x}, t) &= \underline{q}^*(\underline{\xi}(x,t), t) \\ &= \frac{x_1}{(1+t)}\,\underline{i} + \frac{x_2}{e^t}e^t\,\underline{j} \\ &= \frac{x_1}{(1+t)}\,\underline{i} + x_2\,\underline{j} \end{aligned} \tag{3.19}$$

- If the spatial description of the velocity vector were given, the differential equation of the particle pathline would be

$$\frac{\partial \underline{x}\left(\underline{\xi}, t\right)}{\partial t} = \underline{q}\left(\underline{x}(\underline{\xi}, t), t\right) \tag{3.20}$$

and, if solved, would give the same expressions as above.

3.1.5.3 Streamlines

- We can also use the spatial description of the velocity field to the find the position vector of a streamline, $\underline{x}(\sigma, t)$:

$$\begin{aligned} \left.\frac{\partial \underline{x}}{\partial \sigma}\right|_t &= \underline{q}\left(\underline{x}(\sigma), t)\right) \\ &= \frac{x_1(\sigma)}{(1+t)}\,\underline{i} + x_2(\sigma)\,\underline{j} \end{aligned} \tag{3.21}$$

From which we obtain

$$x_1(\sigma) = c_1\,e^{\sigma/(1+t)} \tag{3.22}$$

$$x_2 = c_2\,e^{\sigma} \tag{3.23}$$

- If we eliminate the parameter σ from these two equations, then in the (x_1, x_2) plane the streamlines are the curves:

$$x_2 = c_2\,(x_1/c_1)^{(1+t)} \tag{3.24}$$

Note that $x_2 = k\,x_1$ at $t = 0$.

3.1.5.4 Streaklines

- The streaklines are determined by finding the material coordinates of a particle that was a spatial position $\underline{y}$ at some time s. We use the inverse relations for the pathline to define the relationship:

$$\underline{\xi} = \frac{y_1}{(1+s)}\,\underline{i} + \frac{y_2}{e^s}\,\underline{j} \tag{3.25}$$

- Hence the streakline is

$$\underline{x}(s) = \frac{y_1}{(1+s)}\,(1+t)\,\underline{i} + \frac{y_2}{e^s}\,e^t\,\underline{j} \tag{3.26}$$

- At $s = t$, these relations give $\underline{x} = \underline{y}$, so that is the location of the particle just passing through the spatial point $\underline{y}$.

- At $s = 0$, the particle that was previously at $\underline{y}$ for $t = 0$ is to be found. To find the streakline definition for any time, we solve the i component for the relationship between s and the other variables:

$$s = \left(\frac{y_1}{x_1}\right)(1+t) - 1 \tag{3.27}$$

and from the second equation:

$$x_2 = y_2\,e^{t-(1+t)(y_1/x_1)+1} \tag{3.28}$$

- Thus for any particular time t this equation gives the equation of the streakline through the point $\underline{y}$.

3.2 Conservation of Mass: Continuity Equation

3.2.1 Material Volume Approach

- Consider an arbitrary volume $V(t)$ enclosed in a material surface $S(t)$. A material surface is always composed of the same fluid particles. As the volume moves through space it experiences deformation although the mass within the volume remains constant.

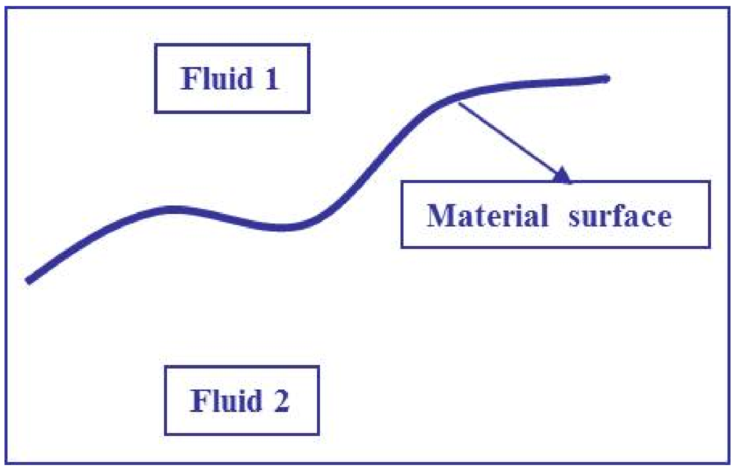

Figure 3.2 Material surface of interface between two fluids.

- The mass enclosed within $V(t)$ is given by, in an integral form for density ρ,

$$\int_{V(t)} \rho \, dV \tag{3.29}$$

where the integration is over the region of space occupied by V at time t.

- Since the mass of the material volume is constant, the time derivation of this expression is zero:

$$\frac{d}{dt} \int_{V(t)} \rho \, dV = 0 \tag{3.30}$$

- Using the (Reynolds) transport theorem, one obtains

$$\int_{V(t)} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \, \underline{q}) \right] \, dV = 0. \tag{3.31}$$

- This is the integral form of the continuity equation. Since the volume taken is arbitrary, the integrand must be zero at all points within V:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{q}) = 0. \tag{3.32}$$

- This is the spatial or Eulerian description of the continuity equation. The above derivation of the continuity equation was from the system analysis point of view for which the mass within a deformable bounding surface is constant.

3.2.2 Control Volume Approach

- Meanwhile, it is common to also use control volume analysis, for which one consider an arbitrary fixed volume V enclosed in a surface S. Let $\underline{n}$ be the outward unit normal vector.

- The mass of fluid in V is $\int_V \rho \, dV = m$, say. If m increases it means that fluid has entered through S:

$$\frac{dm}{dt} = -\int_S \rho \, \underline{n} \cdot \underline{q} \, dS \tag{3.33}$$

  and by the "divergence theorem", this surface integral is equal to

$$-\int_V \nabla \cdot (\rho \underline{q}) \, dV, \tag{3.34}$$

- V being a fixed volume, we can write

$$\frac{dm}{dt} = \int_V \frac{\partial \rho}{\partial t} \, dV \tag{3.35}$$

  Hence, for arbitrary choice of V, we have

$$\int_V \frac{\partial \rho}{\partial t} \, dV = -\int_V \nabla \cdot (\rho \underline{q}) \, dV. \tag{3.36}$$

- The control volume (CV) approach implies that

$$\begin{aligned} & \text{[Rate of change of mass inside the control volume]} \\ + & \text{[Net rate of mass flux out through the control surface]} \\ = & \ 0, \end{aligned} \tag{3.37}$$

$$\frac{\partial}{\partial t}\int_{CV} \rho\, dV + \oint_{CS} \rho\, \underline{q} \cdot \underline{n}\, dS = 0 \tag{3.38}$$

On the right side of the control volume $x + \dfrac{dx}{2}$, the density, the x- com-

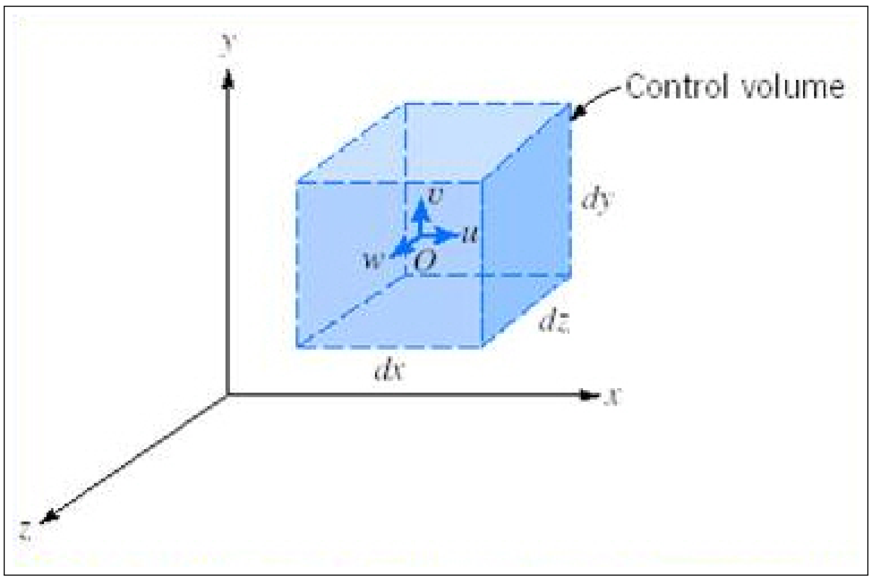

Figure 3.3 Differential control volume in Cartesian coordinates. (From Fox, McDonald & Pritchard 2004)

ponent of the velocity, and the mass flux are, respectively,

$$\rho_{(x+dx/2)} = \rho + \left(\frac{\partial \rho}{\partial x}\right)\frac{dx}{2} \tag{3.39}$$

$$u_{(x+dx/2)} = u + \left(\frac{\partial u}{\partial x}\right)\frac{dx}{2} \tag{3.40}$$

$$\begin{aligned} m_{(x+dx/2)} &= \left[\rho_{(x+dx/2)}\right]\left[u_{(x+dx/2)}\right] dy\, dz \\ &= \left\{\rho\, u + \left[\rho\left(\frac{\partial u}{\partial x}\right) + u\left(\frac{\partial \rho}{\partial x}\right)\right]\frac{dx}{2} + O\left[(dx)^2\right]\right\} dy\, dz \\ &= \left\{\rho\, u + \left[\frac{\partial(\rho u)}{\partial x}\right]\frac{dx}{2} + O\left[(dx)^2\right]\right\} dy\, dz \end{aligned} \tag{3.41}$$

On the left side of the control volume $x - \frac{dx}{2}$, the density, the x- component of the velocity, and the mass flux are, respectively,

$$\rho_{(x-dx/2)} = \rho - \left(\frac{\partial \rho}{\partial x}\right)\frac{dx}{2} \tag{3.42}$$

$$u_{(x-dx/2)} = u - \left(\frac{\partial u}{\partial x}\right)\frac{dx}{2} \tag{3.43}$$

$$\begin{aligned} m_{(x-dx/2)} &= -\left[\rho_{(x-dx/2)}\right]\left[u_{(x-dx/2)}\right] dy\, dz \\ &= -\left\{\rho\, u - \left[\rho\left(\frac{\partial u}{\partial x}\right) - u\left(\frac{\partial \rho}{\partial x}\right)\right]\frac{dx}{2} + O\left[(dx)^2\right]\right\} dy\, dz \\ &= -\left\{\rho\, u - \left[\frac{\partial(\rho\, u)}{\partial x}\right]\frac{dx}{2} + O\left[(dx)^2\right]\right\} dy\, dz \end{aligned} \tag{3.44}$$

Add the mass fluxes out through the right and the left side, and neglect the higher order terms to have

$$m_{(x+dx/2)} + m_{(x-dx/2)} = \left[\frac{\partial(\rho\, u)}{\partial x}\right] dx\, dy\, dz \tag{3.45}$$

Similarly, on the top and the bottom sides and on the front and the back sides, respectively,

$$m_{(y+dy/2)} + m_{(y-dy/2)} = \left[\frac{\partial(\rho\, v)}{\partial y}\right] dy\, dz\, dx \tag{3.46}$$

$$m_{(z+dz/2)} + m_{(z-dz/2)} = \left[\frac{\partial(\rho\, w)}{\partial z}\right] dz\, dx\, dy \tag{3.47}$$

The net rate of mass flux out through the control surface (6 sides of the differential control volume) is

$$\oint_{CS} \rho\, \underline{q} \cdot \underline{n}\, dS = \left[\frac{\partial(\rho\, u)}{\partial x} + \frac{\partial(\rho\, v)}{\partial y} + \frac{\partial(\rho\, w)}{\partial z}\right] dx\, dy\, dz \tag{3.48}$$

The rate of change of mass inside the differential control volume

$$\frac{\partial}{\partial t}\int_{CV} \rho\, dV = \frac{\partial \rho}{\partial t}\, dx\, dy\, dz \tag{3.49}$$

- The only way that these integrals can be equal for any and every choice of V is that their integrands be equal; thus we obtain the *General Equation of Continuity*:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \, \underline{q}) = 0 \tag{3.50}$$

3.2.3 Special Cases: Steady Motion and Incompressible Flow

- Noting that $\nabla \cdot (\rho \, \underline{q}) = \underline{q} \cdot \nabla \rho + \rho \nabla \cdot \underline{q}$, this equation can be expressed, in an alternative form, as

$$\frac{D\rho}{Dt} + \rho \, \nabla \cdot \underline{q} = 0 \tag{3.51}$$

- There are two important special cases:

  (1) Steady motion
  Since, for steady motion, all partial derivatives $\partial(\quad)/\partial t$ vanish, Eq. (3.50) becomes

$$\nabla \cdot (\rho \, \underline{q}) = 0 \tag{3.52}$$

  (2) Incompressible flow
  If the density of every particle is constant, $D\rho/Dt = 0$, and Eq. (3.51) gives us

$$\nabla \cdot \underline{q} = 0 \quad \text{or} \quad \frac{\partial q_i}{\partial x_i} = 0 \tag{3.53}$$

  Vector fields with this property are called solenoidal. Most of our work will deal with incompressible fluid.
  It is to be noted that this is correct whether the fluid is steady or not, and moreover it applies to the case of an inhomogeneous fluid, such as a stratified liquid, in which ρ varies throughout the fluid, provided each particle is incompressible.

- Continuity Equation in Cylindrical Coordinate System:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial \, (\rho \, r \, q_r)}{\partial r} + \frac{1}{r}\frac{\partial (\rho \, q_\theta)}{\partial \theta} + \frac{\partial (\rho \, q_z)}{\partial z} = 0 \tag{3.54}$$

For incompressible fluids, it reduces to

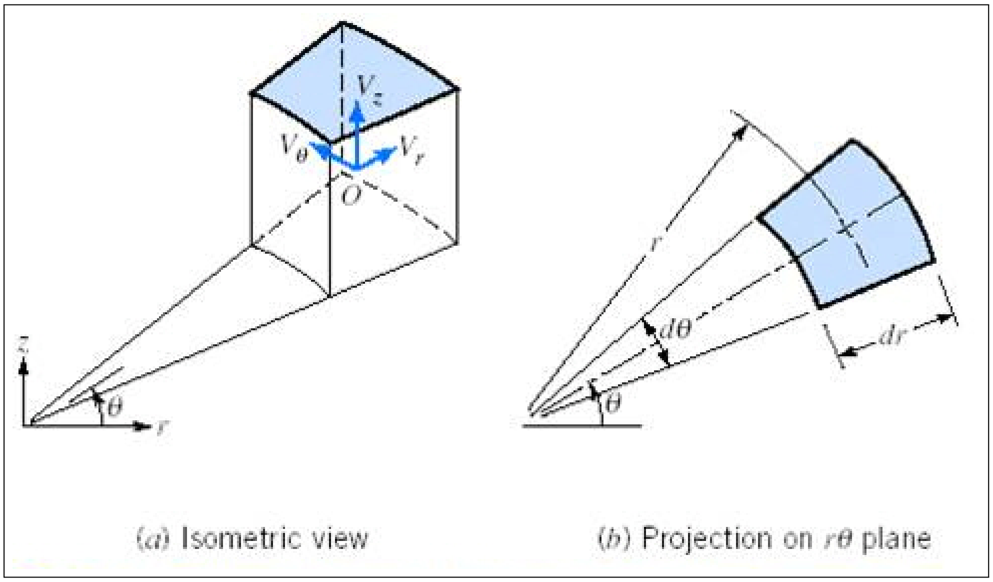

Figure 3.4 Differential control volume in cylindrical coordinates. (From Fox, McDonald & Pritchard 2004)

$$\nabla \cdot \underline{q} = \frac{1}{r}\frac{\partial\,(r\,q_r)}{\partial r} + \frac{1}{r}\frac{\partial q_\theta}{\partial \theta} + \frac{\partial q_z}{\partial z} = 0 \tag{3.55}$$

and, for steady flow,

$$\nabla \cdot (\rho\,\underline{q}) = \frac{1}{r}\frac{\partial\,(\rho\,r\,q_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho\,q_\theta)}{\partial \theta} + \frac{\partial(\rho\,q_z)}{\partial z} = 0 \tag{3.56}$$

3.3 Vorticity and Circulation

3.3.1 Definition of Vorticity

- The vector function $\nabla \times \underline{q}$, where $\underline{q}(x, y, z, t)$ is the velocity of the fluid, is called the vorticity. Its components are occasionally represented by the symbols ξ, η, ζ; namely, in rectangular Cartesian coordinates

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \quad \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \quad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{3.57}$$

- To give a physical feature of the meaning of vorticity, it is often said that $\nabla \times \underline{q}$ is twice the angular-velocity vector of the fluid particle. Since the particle is being deformed continually, perhaps we should say the average angular velocity at a point.

- Viscous Shearing and Vorticity at Boundary [8]

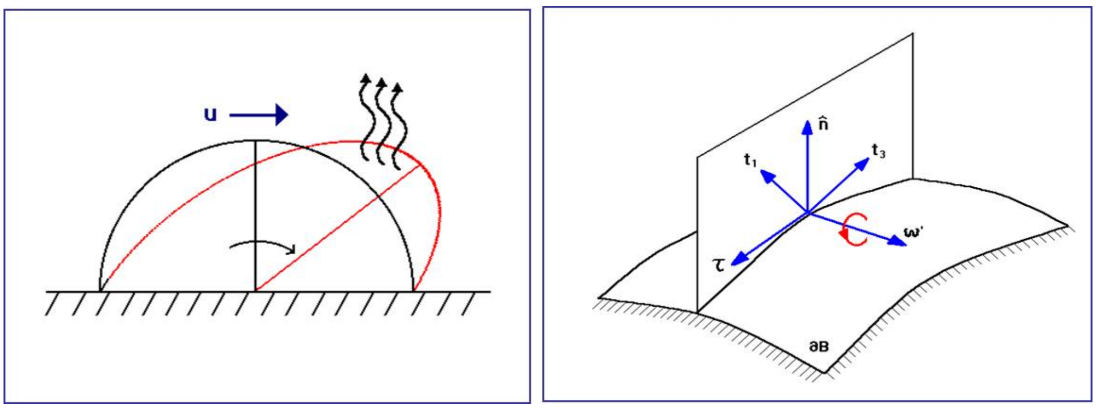

Figure 3.5 Viscous shearing and vorticity generation by no-slip boundary condition at wall boundary. (From Wu & Wu 1993)

3.3.2 Vortex Line and Vortex Tube

- A vortex line is a curve which is tangent at each point to the vorticity at the point. It is analogous to the stream line. Its differential equation is $dx/\xi = dy/\eta = dz/\zeta$ where the Cartesian component of $\underline{\omega}$ are ξ, η, ζ.

- Since the divergence of any curl of a vector must be zero, a continuity equation $\nabla \cdot \underline{\omega}$ for $\underline{\omega}$ must be invoked especially in the case that the vorticity field is itself to be sought with independence of the velocity field.

- The condition $\nabla \cdot \underline{\omega} = 0$ can be thought of as meaning that vortex lines do not begin nor end in the fluid. We call a tube whose walls are made up of vortex lines a vortex tube. (The analogous tube made up of streamlines would be called a stream tube.)

[8] Movie: Vortical flows
./mmfm_movies/laser_vortex.mov ./mmfm_movies/smoke_ring.mov

3.3.3 Circulation and Vorticity Flux

- We classify flows as irrotational and rotational, depending on whether $\nabla \times \underline{q}$ is or is not everywhere zero.

- The line integral

$$\Gamma = \oint_C \underline{q} \cdot d\underline{\ell} \qquad (3.58)$$

where $\underline{q}$ is the fluid velocity, taken about any closed curve C in space, is called the circulation about the contour C.

- By Stokes' theorem, it is clear that the circulation and vorticity are related, for

$$\Gamma = \oint_C \underline{q} \cdot d\underline{\ell} = \int_S \underline{n} \cdot \nabla \times \underline{q}\, dS = \int_S \underline{n} \cdot \underline{\omega}\, dS \qquad (3.59)$$

The transformation is only permissible, of course, when $\underline{q}$ is finite and has

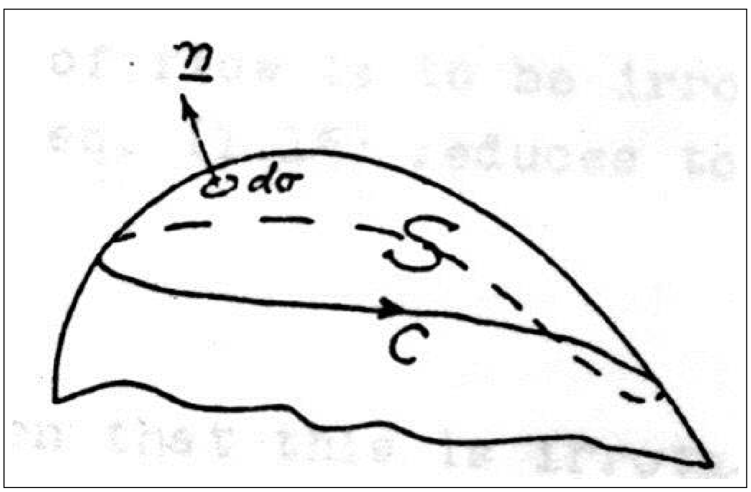

Figure 3.6 Line and surface integrals for Stokes' theorem. (From Sears 1970)

continuous partial derivatives at each point of S; we may encounter some cases where certain singularities have to be excluded from such processes.

- Obviously, if the flow is wholly irrotational, Γ will be zero for every contour. In any case, Γ is zero if C encloses only irrotational portions of the flow.

3.3.4 Vortex Strength

- Consider the application of Stokes' theorem to a cross-section of the vortex tube:

$$\int_{\Sigma} \underline{n} \cdot \underline{\omega}\, dS = \Gamma = \text{constant along tube} \tag{3.60}$$

Thus the average vorticity in the cross-section varies inversely as the

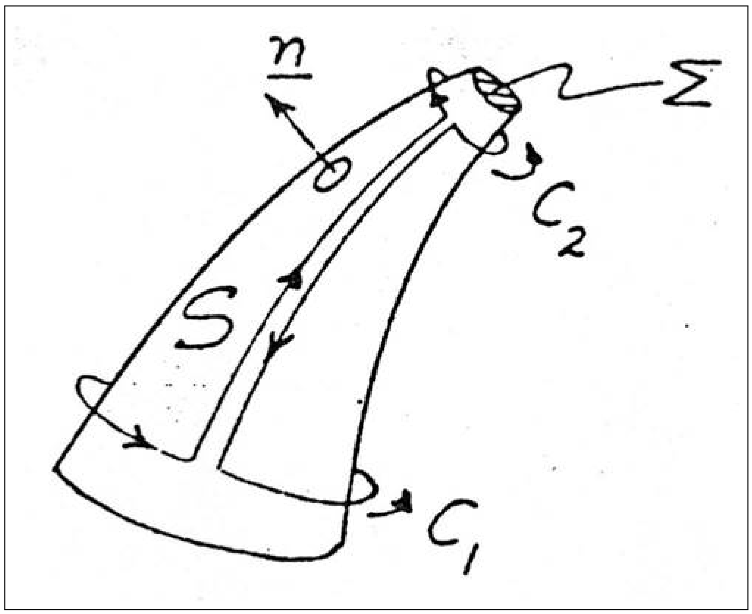

Figure 3.7 Circulation about a vortex tube. (From Sears 1970)

cross- sectional area. The vorticity becomes very small if the tube spreads out. This is the result of viscosity, for example; the vorticity is dissipated over a wide region.

- Suppose, on the other hand, that the tube is necked down; this makes the vorticity large. In the extreme case, we imagine that the tube is contracted to a line. Then the vorticity at this line becomes infinite, but the circulation is still the same, Γ. This is called a vortex filament, or briefly a "vortex", and Γ is its strength.

- It is a kind of mathematical approximation to the case where all the vorticity is confined to a tube of relatively small cross-section, as often occurs in nature – for example in a tornado. Outside the core of a tornado, the air is

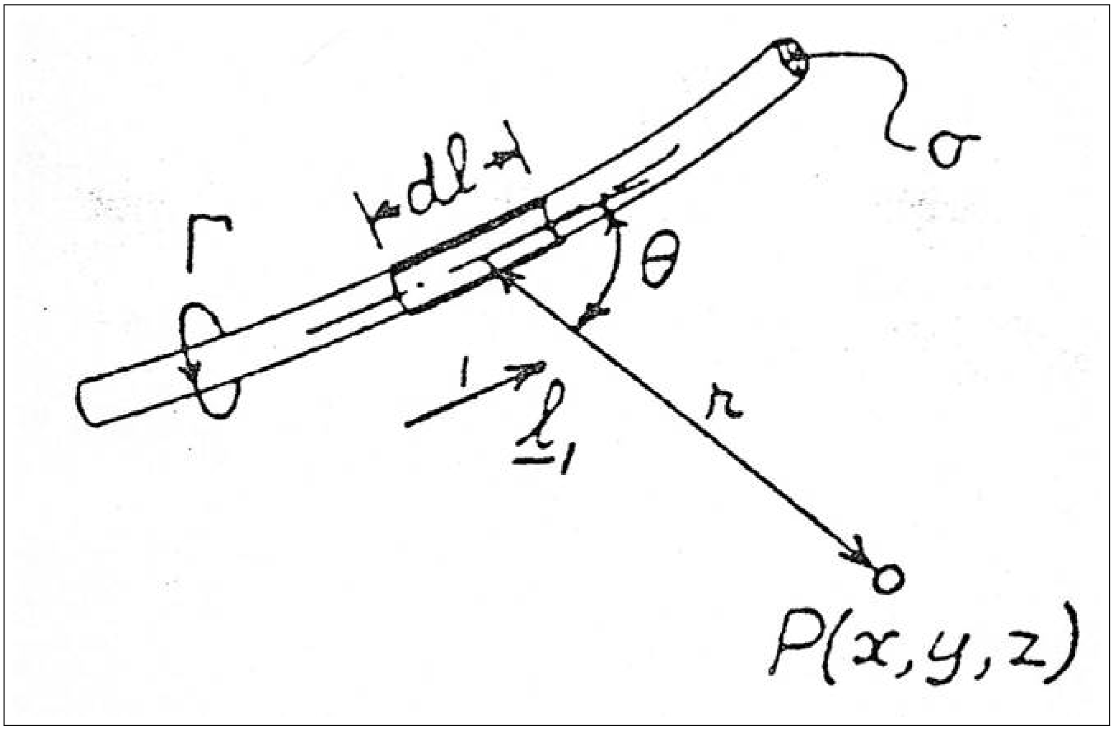

Figure 3.8 Vortex filament approximation of a vortex tube. (From Sears 1970)

in practically irrotational motion. [9]

- The irrotational concentric flow represents the case of a long, straight vortex filament; the singularity at the center is the filament, and there the vorticity is infinite, as predicted. Clearly, a vortex tube or filament, consisting of vortex lines, cannot begin nor end in the fluid. It can double back on itself in a ring or terminate at a boundary of the fluid.

[9] Movie: Tornado
./mmfm_movies/Tornado.mov