Lecture Notes 414.341

선 박 해 양 유 체 역 학

MARINE HYDRODYNAMICS

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A

VECTOR ANALYSIS

A.1 Introduction

A.1.1 Vector and Tensor Notation

(1) Concerned with both scalar and vector functions:

- Parametric description of curves in space.
- Results of algebra and calculus for scalars.
- Vector analysis.
- Coordinate systems.

(2) Tensor notation

- Range convention: Whenever a subscript appears only once in a term, the subscript takes all possible values. For example in 3D space:

$$x_i(i = 1, 2, 3) \rightarrow x_1, x_2, x_3 \tag{A.1}$$

- Summation convention: Whenever a subscript appears twice in the same term the repeated index is summed over the index parameter space. For example in 3D space:

$$a_i\, b_i = a_1 b_1 + a_2 b_2 + a_3 b_3\, (i = 1, 2, 3) \tag{A.2}$$

- Non repeated subscripts remain fixed during the summation. For example in 3D space, $a_i = x_{ij} n_j$ denotes three equations, one for each $i = 1, 2, 3$ and j is the dummy index.
- Note 1: To avoid confusion between fixed and repeated indices or different repeated indices, etc, no index can be repeated more than twice.
- Note 2: Number of free indices shows how many quantities are represented by a single term.

(3) Tensors.

- A scalar is called a zero-order tensor.

- A vector is a first-order tensor.
- Dyads are second-order tensors: a 3×3 matrix form. (e.g. stress tensor)
- The alternating tensor ϵ_{ijk} is a special third-order tensor.

A.1.2 Fundamental Function Analysis

A scalar field f is defined in a region D of two- or three-dimensional space with the property that the value of f varies from point to point in D. Some concepts and analysis for scalar functions are listed below.

(1) If $\lim_{x \to c} f(x) = f(c)$, the function $f(x)$ is said to be continuous at the point $x = c$.

(2) The base of natural logarithm is denoted by e, where $e = \lim_{n \to \pm\infty} \left(1 + \frac{1}{n}\right)^n = 2.7182818285 \cdots$. One often writes $\ln(x)$ for $\log_e x$.

(3) By using the Euler formula $e^{i\theta} = \cos\theta + i \sin\theta$, the real sine and cosine function can be combined into a single function.

(4) A definite integral of a function $f(x)$ which exists on the interval $a \le x \le b$, can be defined by the limiting process in the sense of Riemann sum: namely,

$$\int_a^b f(x)\, dx = \lim_{N \to \infty} \sum_{i=1}^{N} f\left(a + i\,\frac{b-a}{N}\right) \frac{b-a}{N} \tag{A.3}$$

(5) For function of one variable, the rule for change of variable in a definite integral is

$$\int_{x_1}^{x_2} f(x)\, dx = \int_{u_1}^{u_2} f(x(u))\, \frac{dx}{du}\, du \tag{A.4}$$

where we assume $f(x)$ and $f(x(u))$ are continuous in the range of integration and $x = x(u)$ is continuous and its derivative is continuous for $u_1 \le u \le u_2$.

(6) For functions of two variables, the integral becomes

$$\int_{S_{xy}} f(x,y)\,dxdy = \int_{S_{uv}} f(x(u,v),\,y(u,v))\,|J|\,dudv, \tag{A.5}$$

where Jacobian $J \equiv \dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial x}{\partial v}\dfrac{\partial y}{\partial u}$.

(7) For $f(x,t)$ and $\dfrac{\partial f}{\partial t}$ in a region $S_{xt}, \quad a(t) \le x \le b(t), t_1 \le t \le t_2$,

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\,dx = f\,[b(t),t]\;b'(t) - f\,[a(t),t]\;a'(t) + \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}\,dx \tag{A.6}$$

This relationship is called Leibnitz's rule. The corresponding expression for the integral over a two or three dimensional region is called Reynolds transport theorem, which will be derived later.

(8) Dirac delta functions
Dirac delta function is defined as the sense of generalized functions:

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1 \tag{A.7}$$

Also, the derivative of the unit-step function:

$$\frac{dU(t)}{dt} = \delta(t) \tag{A.8}$$

The definite integral of Dirac delta function:

$$\int_a^b \delta(t)\,dt = \begin{cases} 1, & \text{if } a < 0 < b \\ 0, & \text{otherwise} \end{cases} \tag{A.9}$$

Dirac delta function is combined with a regular function:

$$\int_a^b g(t)\,\delta(t)\,dt = g(0)\int_a^b \delta(t)\,dt \tag{A.10}$$

(9) Fourier transforms
For $f(x)$ periodic with period $2L$, then $f(x)$ can be expressed in a Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] \tag{A.11}$$

where

$$a_n = \frac{1}{L}\int_0^{2L} f(x) \cos\left(\frac{n\pi x}{L}\right)\, dx,\ b_n = \frac{1}{L}\int_0^{2L} f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx \tag{A.12}$$

The Fourier transform of a function and its inverse transform:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt \tag{A.13}$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega \tag{A.14}$$

(10) The Laplace transform:

$$F(s) = \int_0^{\infty} f(t)\, e^{-st}\, dt \tag{A.15}$$

$$f(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(s)\, e^{st}\, ds \tag{A.16}$$

A.2 Vector Calculus

A.2.1 Definition of Vector Quantity

(1) The simplest vector: *line vectors*.
A line vector is transformed from one coordinate system to another.

(2) Consider two Cartesian coordinate systems rotated with respect to one another.

a_{11}, a_{21}, a_{31}: the direction cosines of the x'_1 axis, with respect to the x_1, x_2, x_3 axes, respectively.

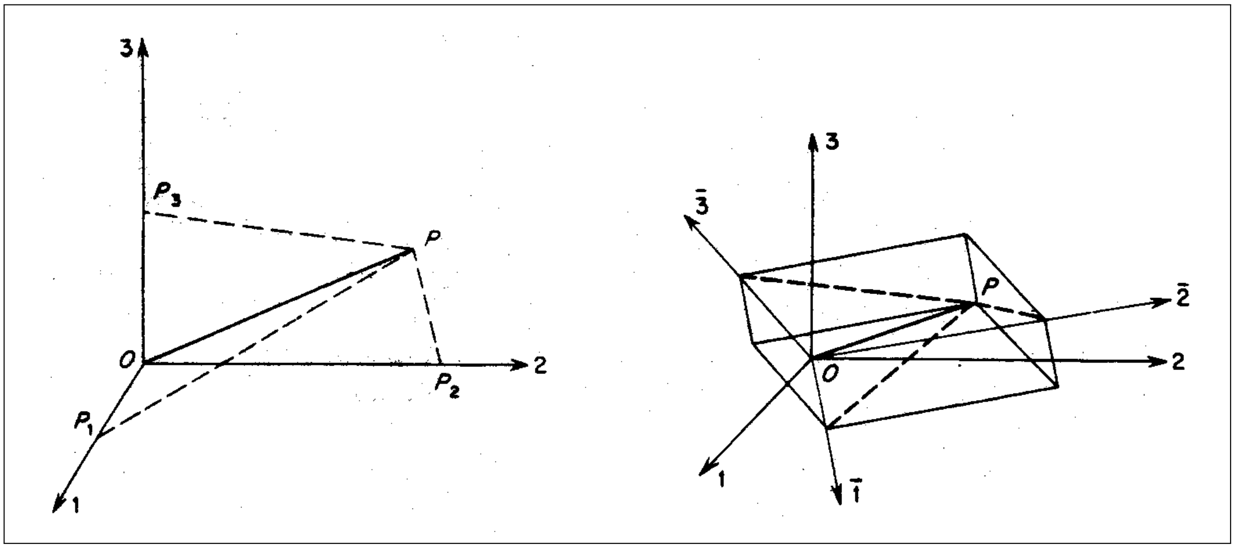

Figure A.1 Two Cartesian coordinate systems rotated with respect to one another. (From Aris 1962, p. 9)

(3) The new coordinates:

$$
\begin{aligned}
x'_1 &= a_{11}\, x_1 + a_{21}\, x_2 + a_{31}\, x_3 \\
x'_2 &= a_{12}\, x_1 + a_{22}\, x_2 + a_{32}\, x_3 \\
x'_3 &= a_{13}\, x_1 + a_{23}\, x_2 + a_{33}\, x_3
\end{aligned}
\tag{A.17}
$$

Also, transform from x'_1, x'_2, x'_3 to x_1, x_2, x_3:

$$
\begin{aligned}
x_1 &= a_{11}\, x'_1 + a_{12}\, x'_2 + a_{13}\, x'_3 \\
x_2 &= a_{21}\, x'_1 + a_{22}\, x'_2 + a_{23}\, x'_3 \\
x_3 &= a_{31}\, x'_1 + a_{32}\, x'_2 + a_{33}\, x'_3
\end{aligned}
\tag{A.18}
$$

A summation notation:

$$x'_i = \sum_{j=1}^{3} a_{ji} x_j \quad i = 1, 2, 3 \tag{A.19}$$

$$x_i = \sum_{j=1}^{3} a_{ij} x'_j \quad i = 1, 2, 3 \tag{A.20}$$

(4) A *vector* is defined as :

$$u'_i = \sum_{j=1}^{3} a_{ji} u_j \quad i = 1, 2, 3 \tag{A.21}$$

Let us consider two simple examples.

(a) Consider velocity of a point $P(x_1, x_2, x_3)$. The components of this quantity along the three axes are dx_1/dt, dx_2/dt, and dx_3/dt. Calculating the velocity in the primed system, we find

$$\frac{dx'_i}{dt} = \frac{d}{dt} \sum_{j=1}^{3} a_{ji}\, x_j = \sum_{j=1}^{3} a_{ji}\, \frac{dx_j}{dt}. \tag{A.22}$$

This has exactly the form required by Eq. (A.21). Hence the velocity of a point is a vector quantity.

(b) Consider the set of numbers $\partial u / \partial x_i$ where u is a scalar function $u(x_1, x_2, x_3)$. We see how $\partial u / \partial x'_i$ is expressed in terms of $\partial u / \partial x_i$:

$$\frac{\partial u}{\partial x'_i} = \sum_{j=1}^{3} \frac{\partial u}{\partial x_j} \frac{\partial x_j}{\partial x'_i} = \sum_{j=1}^{3} \frac{\partial u}{\partial x_j}\, a_{ji} \quad \text{from Eq. (A.20).} \tag{A.23}$$

Hence $\partial u / \partial x_i$ is a vector. It is actually a gradient of the scalar function.

A.2.2 Basic Unit Tensors

In general, in a 3-dimensional space a tensor of order (rank) m has 3^m components,

$$\tau_{ij\cdots k}\, \underline{e}_i\, \underline{e}_j \cdots \underline{e}_k \quad \text{for} \quad i, j, \cdots k = 1, 2, 3 \tag{A.24}$$

A.2.2.1 Kronecker delta tensor

(1) The most useful tensor of order 2 is the unit tensor, denoting by doubly-underlined upper-cased bold face:

$$\underline{\underline{\mathbf{I}}} = \delta_{ij}\, \underline{e}_i\, \underline{e}_j \tag{A.25}$$

with Kronecker delta δ_{ij} being defined by

$$\delta_{ij} = 1 \;\text{ if }\; i = j; \quad \delta_{ij} = 0 \;\text{ if }\; i \neq j \tag{A.26}$$

(2) The contraction (inner product) of 2 unit tensors gives

$$\underline{\underline{\mathbf{I}}} \cdot \underline{\underline{\mathbf{I}}} = \delta_{ij}\, \delta_{jk} = \delta_{ik} = \underline{\underline{\mathbf{I}}} \tag{A.27}$$

(3) The double contraction of 2 unit tensors (denoted by a colon) gives

$$\underline{\underline{\mathbf{I}}} : \underline{\underline{\mathbf{I}}} = \delta_{ij}\, \delta_{ji} = \delta_{ii} = d \tag{A.28}$$

where d is the dimension of the space that we dealt with; e.g., $d = 3$ in 3-dimensions.

A.2.2.2 Permutation tensor

As another example, the important tensor of order 3 is the permutation (alternating) tensor:

$$\underline{\underline{\underline{\mathbf{E}}}} = \epsilon_{ijk}\, \underline{e}_i\, \underline{e}_j\, \underline{e}_k \tag{A.29}$$

where ϵ_{ijk} are the Cartesian components of permutation symbol:

$$\left.\begin{array}{lll} \epsilon_{ijk} & = 0 & \text{if any } i,j,k \text{ equal} \\ \epsilon_{ijk} & = 1 & \text{if } (ijk) = (123),(231),(312) \\ \epsilon_{ijk} & = -1 & \text{if } (ijk) = (132),(213),(321). \end{array}\right\} \tag{A.30}$$

A.2.2.3 Multiplication of basic tensors

(1) We can easily see that the following formulas for δ_{ij} and ϵ_{ijk} holds from their definitions:

$$\delta_{ii} = 3, \tag{A.31}$$

$$\delta_{ij}\, u_{klmi} = u_{klmj}, \tag{A.32}$$

$$\delta_{ij}\, \epsilon_{ijk} = 0, \tag{A.33}$$

(2) The permutation tensor is used for cross (vector) product of vectors. If we need more than one cross products, the multiplication of two permutation tensors is involved. Let us start with the rule of vector product: [1]

$$\epsilon_{ijk} = \underline{e}_i \cdot (\underline{e}_j \times \underline{e}_k) = \begin{vmatrix} \underline{e}_i \cdot \underline{e}_1 & \underline{e}_i \cdot \underline{e}_2 & \underline{e}_i \cdot \underline{e}_3 \\ \underline{e}_j \cdot \underline{e}_1 & \underline{e}_j \cdot \underline{e}_2 & \underline{e}_j \cdot \underline{e}_3 \\ \underline{e}_k \cdot \underline{e}_1 & \underline{e}_k \cdot \underline{e}_2 & \underline{e}_k \cdot \underline{e}_3 \end{vmatrix} = \begin{vmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{vmatrix} \tag{A.34}$$

(3) From Eq. (A.34), the product of two permutation tensors is written as

$$\epsilon_{ijk}\epsilon_{mnl} = \left| \begin{bmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{bmatrix} \begin{bmatrix} \delta_{m1} & \delta_{m2} & \delta_{m3} \\ \delta_{n1} & \delta_{n2} & \delta_{n3} \\ \delta_{l1} & \delta_{l2} & \delta_{l3} \end{bmatrix} \right| = \begin{vmatrix} \delta_{im} & \delta_{in} & \delta_{il} \\ \delta_{jm} & \delta_{jn} & \delta_{jl} \\ \delta_{km} & \delta_{kn} & \delta_{kl} \end{vmatrix} \tag{A.35}$$

(4) Contraction with respect to k, l (i.e., $k = l$) yields

$$\epsilon_{ijk}\, \epsilon_{mnk} = \delta_{im}\, \delta_{jn} - \delta_{in}\, \delta_{jm} \tag{A.36}$$

[1] We will follow the procedure in the text, Wu, J.-Z, Ma, H.-Y. and Zhou, M.-D. (2006), *Vorticity and Vortex Dynamics*, Springer, pp. 697–698.

(5) Making the contraction with respect to j, n and continuing again give

$$\epsilon_{ijk}\,\epsilon_{mjk} = \delta_{im}\,\delta_{jj} - \delta_{ij}\,\delta_{jm} = 3\,\delta_{im} - \delta_{im} = 2\delta_{im} \tag{A.37}$$

$$\epsilon_{ijk}\,\epsilon_{ijk} = 2\,\delta_{ii} = 6 \tag{A.38}$$

(6) The corresponding formulas in a 2-dimensional space are given by

$$\epsilon_{ij3}\,\epsilon_{mn3} = \begin{vmatrix} \delta_{im} & \delta_{in} \\ \delta_{jm} & \delta_{jn} \end{vmatrix} = \delta_{im}\,\delta_{jn} - \delta_{in}\,\delta_{jm} \tag{A.39}$$

$$\epsilon_{ij3}\,\epsilon_{mj3} = \delta_{im}\,\delta_{jj} - \delta_{ij}\,\delta_{jm} = 2\,\delta_{im} - \delta_{im} = \delta_{im} \tag{A.40}$$

$$\epsilon_{ij3}\,\epsilon_{ij3} = 2 \tag{A.41}$$

A.2.2.4 Example of permutation tensor

(1) A special example of the permutation tensor can be observed in definition of vorticity: [2]

$$\underline{\omega} = \omega_i = \nabla \times \underline{q} = \epsilon_{ijk}\,\frac{\partial q_k}{\partial x_j} = \epsilon_{ijk}\,\frac{1}{2}\left(\frac{\partial q_k}{\partial x_j} - \frac{\partial q_j}{\partial x_k}\right) = \frac{1}{2}\epsilon_{ijk}\,\Omega_{jk} \tag{A.42}$$

where $\Omega_{jk} \equiv \left(\frac{\partial q_k}{\partial x_j} - \frac{\partial q_j}{\partial x_k}\right)$ is a spin(rotational) tensor.

(2) Also it is easily seen that, by multiplying the above equation by ϵ_{lmi} and using Eq. (A.34),

$$\begin{aligned} \epsilon_{lmi}\,\omega_i &= \epsilon_{lmi}\,\frac{1}{2}\epsilon_{ijk}\,\Omega_{jk} = \frac{1}{2}(\delta_{lj}\,\delta_{mk} - \delta_{lk}\,\delta_{jm})\,\Omega_{jk} \\ &= \frac{1}{2}(\Omega_{lm} - \Omega_{ml}) = \Omega_{lm} \end{aligned} \tag{A.43}$$

from which we have

$$\Omega_{ij} = \epsilon_{ijk}\,\omega_k. \tag{A.44}$$

[2] See Aris, R. (1962), *Vectors, Tensors and the Basic Equations of Fluid Mechanics*, Prentice Hall, p. 25 and Wu, J.-Z, Ma, H.-Y. and Zhou, M.-D. (2006), *Vorticity and Vortex Dynamics*, Springer, p. 698.

(3) The inner product of a vector $\underline{a}$ and an antisymmetric tensor $\underline{\underline{\Omega}}$ becomes

$$\underline{a} \cdot \underline{\underline{\Omega}} = a_i \, \epsilon_{ijk} \, \omega_k = \underline{\omega} \times \underline{a}, \quad \underline{\underline{\Omega}} \cdot \underline{a} = \epsilon_{ijk} \, \omega_k \, a_j = \underline{a} \times \underline{\omega}. \tag{A.45}$$

(4) If the relative velocity $\underline{v}$ of any two points is $\underline{\underline{\Omega}} \cdot \underline{x}$ where $\underline{x}$ is the relative position vector of the two points, then the motion is due to a rigid body rotation. Here $\underline{\underline{\Omega}}$ relates to the angular velocity.

(5) Similarly, we also have

$$\nabla \cdot \underline{\underline{\Omega}} = \frac{\partial}{\partial x_i} (\epsilon_{ijk} \, \omega_k) = -\nabla \times \underline{\omega}. \tag{A.46}$$

Such relations between vorticity $\underline{\omega}$ and the spin tensor $\underline{\underline{\Omega}}$ are useful to deduce the physical interpretation in vortex dynamics.

A.2.3 Multiplication of Vectors

(1) *Scalar product*:

$$\underline{a} \cdot \underline{b} = ab \cos(\underline{a} \cdot \underline{b}) \tag{A.47}$$

or

$$\underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{A.48}$$

or

$$\underline{a} \cdot \underline{b} = \delta_{ij} a_i b_j = a_i b_i, \quad \text{(summation convention)} \tag{A.49}$$

(2) *Vector product*:

$$\underline{c} = \underline{a} \times \underline{b}; \qquad w = ab \sin(\underline{a}, \underline{b}). \tag{A.50}$$

In a form of tensor-notation, $\underline{a} \times \underline{b} = \epsilon_{ijk} a_j b_k$.

(3) *Scalar triple product*:

$$\underline{a} \cdot (\underline{b} \times \underline{c}) = a_i \epsilon_{ijk} b_j c_k \tag{A.51}$$

$$\underline{a} \cdot \underline{b} \times \underline{c} = \underline{a} \times \underline{b} \cdot \underline{c} = \underline{b} \cdot \underline{c} \times \underline{a} \qquad \text{etc.} \tag{A.52}$$

(4) *Vector triple product*:

$$\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c})\, \underline{b} - (\underline{a} \cdot \underline{b})\, \underline{c} \tag{A.53}$$

i.e.,

$$\begin{aligned} \underline{a} \times (\underline{b} \times \underline{c}) &= \epsilon_{mli}\, a_l\, (\epsilon_{ijk}\, b_j\, c_k) \\ &= (\delta_{mj}\, \delta_{lk} - \delta_{mk}\, \delta_{lj})\, a_l\, b_j\, c_k \\ &= a_k\, b_j\, c_k - a_j\, b_j\, c_k. \end{aligned} \tag{A.54}$$

The basic formula :

$$\underline{a} \times (\underline{b} \times \underline{c}) + \underline{b} \times (\underline{c} \times \underline{a}) + \underline{c} \times (\underline{a} \times \underline{b}) = 0 \tag{A.55}$$

A.2.4 Vector Derivatives

A.2.4.1 Gradient: ∇u

(1) Consider a scalar function $u = u(x, y, z)$ that is differentiable and has continuous derivatives. Let us define the gradient of u at x, y, z as the limiting value of a certain surface integral over a surface surrounding the point x, y, z, as follows

$$\nabla u \equiv \lim_{V \to 0} \frac{1}{V} \oint_S u\, \underline{n}\, dS \tag{A.56}$$

where S is the area enclosing the volume V, dS is the element of area, and $\underline{n}$ is the unit vector normal to the surface at each point of the surface integration. [3]

(2) Now we can take V very small, in the form of a cube, say, with sides $\triangle x, \triangle y, \triangle z$. Then, neglecting second-order quantities, $V = \triangle x \triangle y \triangle z$,

[3] $\int_S \cdots dS$ and $\oint_S \cdots dS$ are the symbolism to indicate that the integration is over, respectively, an open surface and a closed surface.

and

$$\begin{aligned}\int_S u\,\underline{n}\,dS \;&\approx\; -u\,\underline{i}\,\triangle y\,\triangle z - u\,\underline{j}\,\triangle x\,\triangle z - u\,\underline{k}\,\triangle x\,\triangle y\\ &\quad + \left(u + \frac{\partial u}{\partial x}\triangle x\right)\underline{i}\,\triangle y\,\triangle z + \left(u + \frac{\partial u}{\partial y}\triangle y\right)\underline{j}\,\triangle x\,\triangle z\\ &\quad + \left(u + \frac{\partial u}{\partial z}\triangle z\right)\underline{k}\,\triangle x\,\triangle y\\ &\approx\; \left\{\frac{\partial u}{\partial x}\underline{i} + \frac{\partial u}{\partial y}\underline{j} + \frac{\partial u}{\partial z}\underline{k}\right\}V \end{aligned} \tag{A.57}$$

(3) Hence, in limit,

$$\nabla u = \underline{i}\,\frac{\partial u}{\partial x} + \underline{j}\,\frac{\partial u}{\partial y} + \underline{k}\,\frac{\partial u}{\partial z} \tag{A.58}$$

We recognize this as the vector. Another symbol often used for ∇u is grad u. In a form of tensor notation, it is $\dfrac{\partial u}{\partial x_i}$.

A.2.4.2 Divergence: $\nabla \cdot \underline{v}$

(1) Consider now a *vector function*, $\underline{v} = \underline{v}(x, y, z) \equiv v_1\ \underline{i} + v_2\ \underline{j} + v_3\ \underline{k}$, where v_1, v_2, and v_3 are all scalar functions of x, y, z, having continuous derivatives. We define

$$\nabla \cdot \underline{v} = \lim_{V\to 0} \frac{1}{V} \oint_S \underline{n} \cdot \underline{v}\,dS \tag{A.59}$$

(2) Now, by calculating for a small cubical volume, you can easily confirm the following equality:

$$\nabla \cdot \underline{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \tag{A.60}$$

(3) Another symbol used for $\nabla \cdot \underline{v}$ is div $\underline{v}$. In a form of tensor notation, it is $\dfrac{\partial v_i}{\partial x_i}$.

A.2.4.3 Curl: $\nabla \times \underline{v}$

(1) We define the curl of a vector

$$\nabla \times \underline{v} \equiv \lim_{V \to 0} \frac{1}{V} \oint_S \underline{n} \times \underline{v} \, dS \tag{A.61}$$

and find, by considering a small cube, that

$$\nabla \times \underline{v} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) \underline{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \right) \underline{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) \underline{k} \tag{A.62}$$

(2) Symbolically we write

$$\nabla \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} \tag{A.63}$$

(3) Another symbol used for curl $\underline{v}$ is $\nabla \times \underline{v}$. In a form of tensor notation, it is $\epsilon_{ijk} \dfrac{\partial v_k}{\partial x_j}$.

A.2.4.4 Laplacian: $\nabla^2 u$

(1) The Laplacian of a scalar function $u(x, y, z)$ is defined as

$$\nabla^2 u \equiv \nabla \cdot (\nabla u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \tag{A.64}$$

(2) By analogy, the Laplacian of a vector function is the vector whose rectangular Cartesian components are the Laplacian of the vector's correspond-

ing components [4]

$$\nabla^2 \underline{v} = \underline{i}\ \nabla^2 v_1 + \underline{j}\ \nabla^2 v_2 + \underline{k}\ \nabla^2 v_3 \tag{A.65}$$

A.2.4.5 Differential operator: ∇

(1) From the original definition of grad u, we can deduce that the differential du is given by the formula, in rectangular Cartesian coordinates,

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = d\underline{\ell} \cdot \nabla u \tag{A.66}$$

where $d\underline{\ell}$ is any directed line (vector) element. This means that du is the increment of u corresponding to a position increment $d\underline{\ell}$.

(2) Similarly, for a vector function $\underline{v}(x, y, z)$,

$$\begin{aligned} d\underline{v} &\equiv \underline{i}\ dv_1 + \underline{j}\ dv_2 + \underline{k}\ dv_3 \\ &= \left(dx\ \frac{\partial}{\partial x} + dy\ \frac{\partial}{\partial y} + dz\ \frac{\partial}{\partial z} \right) \left(\underline{i}\ v_1 + \underline{j}\ v_2 + \underline{k}\ v_3 \right) \\ &= d\underline{\ell} \cdot \nabla \underline{v} \end{aligned} \tag{A.67}$$

(3) In all of the formulas above, we consider the symbol ∇ as representing a *vector operator* $\underline{i}\ \dfrac{\partial}{\partial x} + \underline{j}\ \dfrac{\partial}{\partial y} + \underline{k}\ \dfrac{\partial}{\partial z}$.
If you treat this operator as a vector, with the appropriate vector-multiplication signs, you get the right result. Equations (A.66) and (A.67) are independent of the choice of coordinate system.

A.2.4.6 Directed derivative

(1) Equations (A.66) and (A.67) lead immediately to the formulas for the *directed derivative* in the direction of a given vector $\underline{s} \equiv s_1\, \underline{i} + s_2\, \underline{j} + s_3\, \underline{k}$ in

[4] We must do more work to find its expression in a non-Cartesian system.

rectangular Cartesian coordinate:

$$\frac{\partial u}{\partial s} = \underline{e}_s \cdot \nabla u \tag{A.68}$$

$$\frac{\partial \underline{v}}{\partial s} = \underline{e}_s \cdot \nabla \underline{v} \tag{A.69}$$

(2) Again we have defined a new vector operator:

$$\underline{e}_s \cdot \nabla = \frac{s_1}{s}\frac{\partial}{\partial x} + \frac{s_2}{s}\frac{\partial}{\partial y} + \frac{s_3}{s}\frac{\partial}{\partial z} \tag{A.70}$$

where s is the magnitude of $\underline{s}$.

A.2.5 Expansion Formulas

(1) The following formulas are of general utility. Let ϕ denote any differentiable scalar function of x, y, z, and $\underline{u}, \underline{v}$ and $\underline{w}$ any such vector functions.

$$\nabla \cdot (\phi\,\underline{u}) = \underline{u} \cdot \nabla\phi + \phi\,\nabla \cdot \underline{u} \tag{A.71}$$

$$\nabla \times (\phi\,\underline{u}) = (\nabla\phi) \times \underline{u} + \phi\,\nabla \times \underline{u} \tag{A.72}$$

$$\nabla \cdot (\underline{v} \times \underline{w}) = \underline{w} \cdot \nabla \times \underline{v} - \underline{v} \cdot \nabla \times \underline{w} \tag{A.73}$$

$$\nabla \times (\underline{v} \times \underline{w}) = \underline{w} \cdot \nabla\underline{v} + \underline{v}\nabla \cdot \underline{w} - \underline{w}\nabla \cdot \underline{v} - \underline{v} \cdot \nabla\underline{w} \tag{A.74}$$

$$\nabla(\underline{v} \cdot \underline{w}) = \underline{v} \cdot \nabla\underline{w} + \underline{w} \cdot \nabla\underline{v} + \underline{v} \times (\nabla \times \underline{w}) + \underline{w} \times (\nabla \times \underline{v}) \tag{A.75}$$

$$\nabla \cdot (\nabla \times \underline{v}) = 0 \tag{A.76}$$

$$\nabla \times (\nabla\,\phi) = 0 \tag{A.77}$$

$$\nabla \times (\nabla \times \underline{v}) = \nabla(\nabla \cdot \underline{v}) - \nabla^2\underline{v} \tag{A.78}$$

(2) Operation on the position vector $\underline{x} = x_1\,\underline{i} + x_2\,\underline{j} + x_3\,\underline{k}$ whose magnitude is denoted by $r = |\underline{x}| = \sqrt{\underline{x} \cdot \underline{x}}$, with a constant vector $\underline{a}$, is illustrated as follows:

$$\nabla r = \underline{x}/r \tag{A.79}$$

$$\nabla \cdot \underline{x} = 3 \tag{A.80}$$

$$\nabla \times \underline{x} = 0 \tag{A.81}$$

$$\nabla r^n = n\, r^{n-2}\, \underline{x} \tag{A.82}$$

$$\nabla \cdot (r^n\, \underline{x}) = (n+3)\, r^n \tag{A.83}$$

$$\nabla \times (r^n\, \underline{x}) = 0 \tag{A.84}$$

$$\nabla^2 (r^n) = n(n+1)\, r^{n-2} \tag{A.85}$$

$$\nabla \cdot (\underline{a} \times \underline{x}) = 0 \tag{A.86}$$

$$\nabla (\underline{a} \cdot \underline{x}) = \underline{a} \tag{A.87}$$

$$\nabla \times (\underline{a} \times \underline{x}) = 2\, \underline{a} \tag{A.88}$$

$$\nabla \cdot (\underline{a} \times \nabla r) = 0 \tag{A.89}$$

$$\nabla \cdot (r\, \underline{a}) = (\underline{x} \cdot \underline{a})/r \tag{A.90}$$

$$\nabla \times (r\, \underline{a}) = (\underline{x} \times \underline{a})/r \tag{A.91}$$

A.3 Integral Theorems

A.3.1 Divergence Theorem

(1) Let u and $\underline{v}$ denote arbitrary scalar and vector functions of x, y, z as before. These are assumed to be defined, continuous, and single-valued in a certain region of space, and, moreover, that their first derivatives with respect to x, y, and z satisfy the same requirements.

(2) Now consider the surface integral $\oint_S u\ \underline{n}\ dS$, carried over any closed surface S within the region, enclosing a volume V, $\underline{n}$ being the unit normal vector directed outward.

(3) It is clear that, if the volume V is subdivided into small volume V_i, this integral equals the sum of all the integrals $\oint_{S_i} u\ \underline{n}\ dS$ taken over the small surfaces S_i.

(4) Since integration over neighboring elements will cancel one another, and only the integration over the outside will remain:

$$\oint_S u \, \underline{n} \, dS = \sum \oint_{S_i} u \, \underline{n} \, dS \tag{A.92}$$

(5) But, in the limit, the surface integral over the small surface become $\nabla u \ dV$, according to our definition of the gradient, Eq. (A.56), and the summation becomes a volume integration:

$$\oint_S u \, \underline{n} \, dS = \int_V \nabla u \, dV \tag{A.93}$$

(6) In particular, if $u = const.$, Eq. (A.93) becomes

$$\oint_S \underline{n} \, dS = 0. \tag{A.94}$$

It means that the integral of vectorial surface element over a closed surface must vanish.

(7) If u is taken as a negative of static pressure acting on a body submerged fully into a fluid (i.e., $u = -p = \rho g z$, where z is vertically upward coordinate), the force acting on the body is

$$\underline{F} = \oint_S (-p) \, \underline{n} \, dS = \int_V \nabla(\rho g z) \, dV = \int_V (\rho \, g \, \underline{k}) \, dV = \rho \, g \, V \, \underline{k} \tag{A.95}$$

This relation is well known as the Archimedes principle for buoyancy force of a submerged body.

(8) By entirely analogous reasoning, using the definitions of the divergence and curl, you will verify that

$$\oint_S \underline{n} \cdot \underline{v} \, dS = \int_V \nabla \cdot \underline{v} \, dV \tag{A.96}$$

and

$$\oint_S \underline{n} \times \underline{v} \, dS = \int_V \nabla \times \underline{v} \, dV \tag{A.97}$$

Equation (A.96) is known as the divergence theorem, or Gauss theorem.

(9) If we take $\underline{v}$ as fluid velocity, Eqs. (A.96) and (A.97) become, respectively,

$$\oint_S \underline{n} \cdot \underline{v} \, dS = \int_V \theta \, dV \tag{A.98}$$

and

$$\oint_S \underline{n} \times \underline{v} \, dS = \int_V \underline{\omega} \, dV \tag{A.99}$$

These equations show that the velocity components over boundary are directly related with the field distribution of expansion (or compressing process) and vorticity in fluid region.

(10) The three types of the theorem above can be unified by a general form:

$$\oint_S (\underline{n} * f) \, dS = \int_V (\nabla * f) \, dV \tag{A.100}$$

where $*$ denotes one of differential operator, scalar product and vector product, and f is a scalar or vector function depending on the choice.

(11) As an example, take $\underline{f} = \nabla u$ to yield

$$\int_V \nabla^2 u \, dV = \int_V \nabla \cdot (\nabla u) \, dV = \oint_S \underline{n} \cdot \nabla u \, dS = \oint_S \frac{\partial u}{\partial n} \, dS \tag{A.101}$$

where $\partial u / \partial n$ is the directed derivative in the outward direction as defined in Eq (A.68).

A.3.2 Stokes' Theorem

(1) Let us apply Eq. (A.56) for definition of ∇u to a very small volume element of a thin disk with uniform height $\triangle h$ and base area $\triangle S$. Its volume then becomes $\triangle S \, \triangle h$.

(2) Consider the product of ∇u with the outward unit normal vector to the

upper surface $\underline{n}_u$. Then it is not difficult to prove that,

$$\underline{n}_u \times \nabla u \approx \underline{n}_u \times \frac{1}{\triangle V}\oint_S u\,\underline{n}\,dS \approx \frac{1}{\triangle S}\oint_C u\,d\underline{\ell} \tag{A.102}$$

where C is the small contour that forms the boundary of $\triangle S$. The line integral in Eq. (A.102) is taken in the direction that would advance a right-hand screw in the $\underline{n}$ direction.

(3) Now consider a volume element with the uniform thin height and an arbitrary base surface S. If this volume is subdivided into very small volume V_i with the same height, the above product in an integral sense can be expressed as the sum of all the integrals taken over the small line integrals:

$$\int_S \underline{n} \times \nabla u\,dS = \lim_{V_i \to 0}\sum \oint_{C_i} u\,d\underline{\ell} \tag{A.103}$$

(4) Since the line integration over neighboring contour elements will cancel one another, and only the integration over the outside contour will remain:

$$\int_S \underline{n} \times \nabla u\,dS = \oint_C u\,d\underline{\ell} \tag{A.104}$$

(5) With this knowledge, two more important transformation theorems follow:

$$\int_S \underline{n} \cdot \nabla \times \underline{v}\,dS \;=\; \oint_C \underline{v} \cdot d\underline{\ell} \tag{A.105}$$

$$\int_S (\underline{n} \times \nabla) \times \underline{v}\,dS \;=\; \oint_C d\underline{\ell} \times \underline{v} \tag{A.106}$$

The first of these is known as Stokes' theorem.

(6) If u is constant, Eq. (A.104) becomes

$$0 = \oint_C u\,d\underline{\ell} \tag{A.107}$$

and if $\underline{v} = \underline{x}$, Eq. (A.106) becomes, since $(\underline{n} \times \nabla) \times \underline{x} = -2\,\underline{n}$,

$$\int_S \underline{n}\; dS = \frac{1}{2}\oint_C \underline{x} \times d\underline{\ell}. \tag{A.108}$$

(7) If we consider $\underline{v}$ as fluid velocity, we have the well-known relation between vorticity flux through an open surface and circulation along the boundary of the surface:

$$\int_S \underline{n} \cdot \underline{\omega}\; dS = \oint_C \underline{v} \cdot d\underline{\ell} \tag{A.109}$$

(8) By analogous reasoning, we have used the relationship,

$$\underline{n} \cdot \nabla \times \underline{v} \approx \frac{1}{S}\oint_C \underline{v} \cdot d\underline{\ell} \tag{A.110}$$

(9) The conditions on u and $\underline{v}$ are analogous to those imposed above; that is, the functions and their first derivations must be finite, continuous, and single-valued in the region. The surface S enclosed by the contour C need not be flat; $\underline{n}$ is normal to S at every point, and the direction of C is chosen as described above. [5]

(10) The unified form of Stokes' theorem may be written by,

$$\int_S (\underline{n} \times \nabla) * f\; dS = \oint_C \; d\underline{\ell} * f \tag{A.111}$$

A.3.3 Volume Integrals of a Vector

(1) Using integration by parts, we can express the integration of $f(x)$ by the moment of $f'(x)$:

$$\int_a^b f(x)\; dx = b\, f(b) - a\, f(a) - \int_a^b x\, f'(x)\; dx \tag{A.112}$$

(2) In a similar fashion to this one-dimensional formula, a surface or volume integral can be cast to the integrals of the first moment of the derivative of

[5] For rigorous proof, see Arfken, G. (1970), *Mathematical Methods for Physicists*, 2nd ed., Academic Press, pp. 51–53.

f plus boundary integrals.

(3) With $d = 2, 3$ being the space dimension and $\underline{x}$ the position vector, we find the vector expansion formulas:

$$\begin{aligned} \nabla \cdot (\underline{f}\, \underline{x}) &= \underline{f} + \underline{x}\, (\nabla \cdot \underline{f}) && \text{(A.113)} \\ \nabla \cdot (\underline{x}\, \underline{f}) &= d\, \underline{f} + \underline{x} \cdot \nabla \underline{f} && \text{(A.114)} \\ \nabla (\underline{x} \cdot \underline{f}) &= \underline{f} + \underline{x} \cdot \nabla \underline{f} + \underline{x} \times (\nabla \times \underline{f}) && \text{(A.115)} \\ \underline{x} \times (\underline{n} \times \underline{f}) &= \underline{n}\, (\underline{f} \cdot \underline{x}) - (\underline{n} \cdot \underline{x})\, \underline{f}, && \text{(A.116)} \end{aligned}$$

(4) From the volume integral for Eq. (A.113), we apply the divergence theorem to find an identity:

$$\int_V \underline{f}\, dV = \oint_S (\underline{n} \cdot \underline{f})\, \underline{x}\, dS - \int_V \underline{x}\, (\nabla \cdot \underline{f})\, dV \qquad \text{(A.117)}$$

(5) Another form of Eq. (A.117) can be provided as follows:
First, subtracting Eq. (A.115) from Eq. (A.114) yields

$$\nabla \cdot (\underline{x}\, \underline{f}) - \nabla (\underline{x} \cdot \underline{f}) = (d-1)\, \underline{f} - \underline{x} \times (\nabla \times \underline{f}) \qquad \text{(A.118)}$$

Now we take volume integrals of this equation and apply the divergence theorem to find another identity, using Eq. (A.116):

$$\begin{aligned} \int_V \underline{f}\, dV &= \frac{1}{d-1} \left[\int_V \underline{x} \times (\nabla \times \underline{f})\, dV + \int_V \left\{ \nabla \cdot (\underline{x}\, \underline{f}) - \nabla (\underline{f} \cdot \underline{x}) \right\} dV \right] \\ &= \frac{1}{d-1} \left[\int_V \underline{x} \times (\nabla \times \underline{f})\, dV + \oint_S \left\{ (\underline{n} \cdot \underline{x})\, \underline{f} - \underline{n}\, (\underline{f} \cdot \underline{x}) \right\} dS \right] \\ &= \frac{1}{d-1} \left[\int_V \underline{x} \times (\nabla \times \underline{f})\, dV - \oint_S \underline{x} \times (\underline{n} \times \underline{f})\, dS \right] \qquad \text{(A.119)} \end{aligned}$$

(6) As a general comment, we note that the left-hand side of Eq. (A.117) and Eq. (A.119) is independent of the choice of the origin of $\underline{x}$, so must be the right-hand side. Namely, if we remove $\underline{x}$ from the right-hand side of these equations, the remaining integrals must vanish. Then Eq. (A.117) can be

written as, with adding a constant vector $\underline{x}_0$,

$$\int_V \underline{f}\, dV = -\int_V (\underline{x} - \underline{x}_0)\, (\nabla \cdot \underline{f})\, dV + \oint_S (\underline{x} - \underline{x}_0)\, (\underline{n} \cdot \underline{f})\, dS \quad \text{(A.120)}$$

A.4 Curvilinear Orthogonal Coordinates

We will have need for the expressions of several vector differential operators in terms of curvilinear orthogonal coordinates. [6] Suppose x_1, x_2, x_3 are mutually orthogonal curvilinear coordinates.

A.4.1 Line element

(1) When the line-element vector in the orthogonal system is expressed in terms of a scalar multiple, the scalar multiple is usually written h_i and is called a scale factor:

$$d\underline{s} = (h_1\, dx_1, h_2\, dx_2, h_3\, dx_3) \quad \text{(A.121)}$$

where

$$h_1 = h_1(x_1, x_2, x_3) = \left|\frac{\partial \underline{s}}{\partial x_1}\right| = \left\{\left(\frac{\partial s_1}{\partial x_1}\right)^2 + \left(\frac{\partial s_2}{\partial x_1}\right)^2 + \left(\frac{\partial s_3}{\partial x_1}\right)^2\right\}^{1/2}, \text{etc.} \quad \text{(A.122)}$$

(2) The base vectors, $\dfrac{\partial \underline{s}}{\partial x_i}$, is then expressed in terms of the scale factor and a unit vector, e.g.

$$\frac{\partial \underline{s}}{\partial x_1} = h_1(x_1, x_2, x_3)\ \underline{e}_1(x_1, x_2, x_3) \quad \text{(A.123)}$$

(3) For example, if we take spherical coordinates $x_1 = r, x_2 = \theta$, and $x_3 = \phi$ where ϕ is the azimuthal angle about the axis $\theta = 0$, the line element is

[6] For example, expressions for the related common differentials in spherical, cylindrical and polar coordinate systems are found in Batchelor, G. K. (1967), *An Introduction to Fluid Dynamics*, Cambridge University Press, Cambridge, pp. 598–603.

$d\underline{s} = (dr,\, r\,d\theta,\, r\sin\theta\,d\phi)$; hence the scale factors $h_1 = 1, h_2 = r, h_3 = r\sin\theta$.

(4) If we take cylindrical coordinates $x_1 = \rho, x_2 = \phi$, and $x_3 = z$ where ϕ is the azimuthal angle about the axis $\rho = 0$, the line element is $d\underline{s} = (d\rho, \rho\,d\phi, dz)$; hence the scale factors $h_1 = 1, h_2 = \rho, h_3 = 1$.

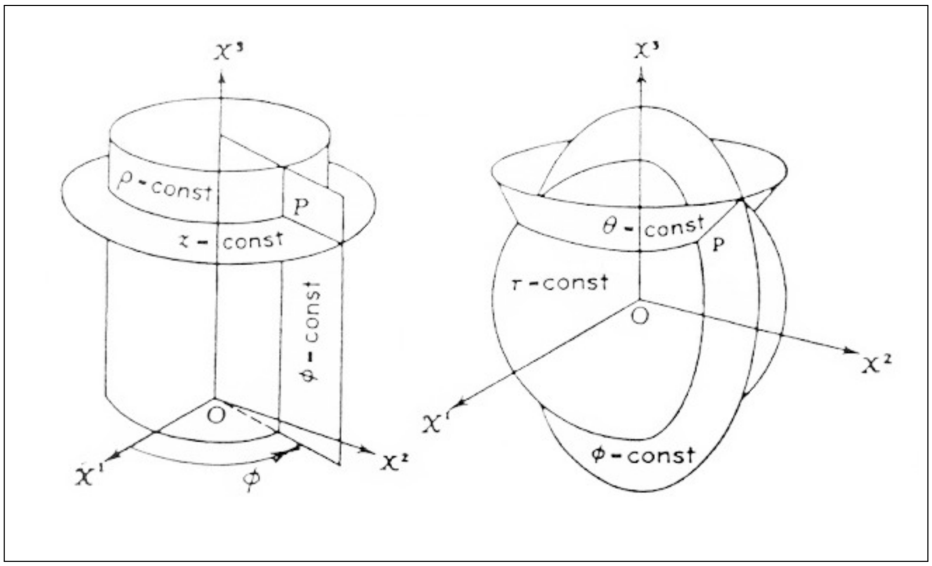

Figure A.2 Cylindrical and spherical coordinate systems. (From Brockett 1988, p. 1-30a)

(5) The scalar differential arc length, denoted by ds is determined from

$$\begin{aligned} ds^2 &= d\underline{s}\cdot d\underline{s} = \left(\frac{\partial \underline{s}}{\partial x_i}\,dx_i\right)\cdot\left(\frac{\partial \underline{s}}{\partial x_j}\,dx_j\right) \\ &= h_i\,h_j\,dx_i\,dx_j\,\underline{e}_i\cdot\underline{e}_j \end{aligned} \tag{A.124}$$

When the unit base vectors are orthogonal, this expression reduces to the simple form

$$ds^2 = h_1^2\,dx_1^2 + h_2^2\,dx_2^2 + h_3^2\,dx_3^2 \tag{A.125}$$

(6) By the triple scalar product, the volume element can be obtained from the elemental arc length vectors:

$$dV = \pm\left(\frac{\partial \underline{s}}{\partial x_1}dx_1\right)\cdot\left(\frac{\partial \underline{s}}{\partial x_2}\times\frac{\partial \underline{s}}{\partial x_3}\,dx_2\,dx_3\right) \tag{A.126}$$

where the $\pm$ sign is necessary to provide a positive element of volume.

For an orthogonal coordinate system, with $\frac{\partial \underline{s}}{\partial x_1} = h_1 \, \underline{e}_1$, etc, the volume element is

$$dV = h_1 \, h_2 \, h_3 \, dx_1 \, dx_2 \, dx_3 \tag{A.127}$$

since $\underline{e}_1 \cdot (\underline{e}_2 \times \underline{e}_3) = \pm 1$. Multiplication of the scale factors corresponds to the Jacobian $J = h_1 \, h_2 \, h_3$.

A.4.2 Gradient (∇u)

(1) We have the formula $du = d\underline{s} \cdot \nabla u$, which is completely general. Also in any coordinate system, we have

$$du = \frac{\partial u}{\partial x_1} \, dx_1 + \frac{\partial u}{\partial x_2} \, dx_2 + \frac{\partial u}{\partial x_3} \, dx_3 \tag{A.128}$$

(2) Equating these two relations gives

$$\frac{\partial u}{\partial x_1} \, dx_1 + \frac{\partial u}{\partial x_2} \, dx_2 + \frac{\partial u}{\partial x_3} \, dx_3 = h_1 \, dx_1 \, (\nabla u)_1 + h_2 \, dx_2 \, (\nabla u)_2 + h_3 \, dx_3 \, (\nabla u)_3 \tag{A.129}$$

(3) Now dx_1, dx_2, dx_3 are completely arbitrary; hence this equation can be true only if their coefficients are equal. Thus

$$\nabla u = \left(\frac{1}{h_1} \frac{\partial u}{\partial x_1}, \frac{1}{h_2} \frac{\partial u}{\partial x_2}, \frac{1}{h_3} \frac{\partial u}{\partial x_3} \right) \tag{A.130}$$

A.4.3 Divergence ($\nabla \cdot \underline{v}$)

(1) For this operator we return to the original definition; thus, denoting by v_1, v_2, v_3 the components of $\underline{v}$ in the 1, 2, 3 directions at any point,

$$\begin{aligned}
\nabla \cdot \underline{v} \;\approx\;& (h_1\, h_2\, h_3 \,\triangle x_1 \,\triangle x_2 \,\triangle x_3)^{-1} \\
& \Big\{ -v_1\, h_2\, h_3 \,\triangle x_2 \,\triangle x_3 - v_2\, h_3\, h_1 \,\triangle x_3 \,\triangle x_1 - v_3 h_1 h_2 \triangle x_1 \triangle x_2 \\
& \quad + \left[v_1\, h_2\, h_3 + \frac{\partial}{\partial x_1}(v_1\, h_2\, h_3)\,\triangle x_1 \right] \triangle x_2 \,\triangle x_3 \\
& \quad + \left[v_2\, h_3\, h_1 + \frac{\partial}{\partial x_2}(v_2\, h_3\, h_1)\,\triangle x_2 \right] \triangle x_3 \,\triangle x_1 \\
& \quad + \left[v_3\, h_1\, h_2 + \frac{\partial}{\partial x_3}(v_3\, h_1\, h_2)\,\triangle x_3 \right] \triangle x_1 \,\triangle x_2 \Big\} \\
=\;& \frac{1}{h_1\, h_2\, h_3} \left\{ \frac{\partial}{\partial x_1}(h_2\, h_3\, v_1) + \frac{\partial}{\partial x_2}(h_3\, h_1\, v_2) + \frac{\partial}{\partial x_3}(h_1\, h_2\, v_3) \right\}
\end{aligned} \tag{A.131}$$

A.4.4 Curl ($\nabla \times \underline{v}$)

(1) Apply Stokes' theorem to one face of the element of a cube, say $y = \text{const}$ face:

$$\begin{aligned}
& \int_S \underline{n} \cdot \nabla \times \underline{v}\; dS = \oint_C \underline{v} \cdot d\underline{\ell} \\
=\;& v_1\, h_1 \,\triangle x_1 - v_3\, h_3 \,\triangle x_3 \\
& + \left[v_3\, h_3 + \frac{\partial}{\partial x_1}(v_3\, h_3)\,\triangle x_1 \right] \triangle x_3 - \left[v_1\, h_1 + \frac{\partial}{\partial x_3}(v_1\, h_1)\,\triangle x_3 \right] \triangle x_1 \\
=\;& \left[\frac{\partial}{\partial x_1}(h_3\, v_3) - \frac{\partial}{\partial x_3}(h_1\, v_1) \right] \triangle x_1 \,\triangle x_3
\end{aligned} \tag{A.132}$$

But also

$$\int_S \underline{n} \cdot \nabla \times \underline{v}\; dS \approx -h_1\, h_3 \triangle x_1 \,\triangle x_3\, (\nabla \times \underline{v})_2 \tag{A.133}$$

(2) Thus, by cyclic substitution,

$$
\begin{aligned}
(\nabla \times \underline{v})_2 &= \frac{1}{h_3\, h_1}\left[\frac{\partial}{\partial x_3}(h_1\, v_1) - \frac{\partial}{\partial x_1}(h_3\, v_3)\right] \\
(\nabla \times \underline{v})_3 &= \frac{1}{h_1\, h_2}\left[\frac{\partial}{\partial x_1}(h_2\, v_2) - \frac{\partial}{\partial x_2}(h_1\, v_1)\right] \\
(\nabla \times \underline{v})_1 &= \frac{1}{h_2\, h_3}\left[\frac{\partial}{\partial x_2}(h_3\, v_3) - \frac{\partial}{\partial x_3}(h_2\, v_2)\right]
\end{aligned}
\tag{A.134}
$$

or, symbolically

$$
\nabla \times \underline{v} = \frac{1}{h_1\, h_2\, h_3}\begin{vmatrix} h_1\, \underline{i}_1 & h_2\, \underline{i}_2 & h_3\, \underline{i}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ h_1\, v_1 & h_2\, v_2 & h_3\, v_3 \end{vmatrix}
\tag{A.135}
$$

(3) For example, if we take spherical coordinates $x_1 = r$, $x_2 = \theta$, and $x_3 = \alpha$ where α is the azimuthal angle about the axis $\theta = 0$,

$$
\begin{aligned}
\nabla \times \underline{v} \;=\; & \frac{\underline{e}_r}{r \sin\theta}\left\{\frac{\partial(v_\alpha \sin\theta)}{\partial\theta} - \frac{\partial v_\theta}{\partial\alpha}\right\} \\
& + \frac{\underline{e}_\theta}{r}\left\{\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\alpha} - \frac{\partial(r\, v_\alpha)}{\partial r}\right\} \\
& + \frac{\underline{e}_\alpha}{r}\left\{\frac{\partial(r\, v_\theta)}{\partial r} - \frac{\partial(r\, v_r)}{\partial\theta}\right\}
\end{aligned}
\tag{A.136}
$$

A.4.5 Laplacian ($\nabla^2 u$)

(1) For $\nabla^2 u$, we simply employ Eqs. (A.130) and (A.131) above:

$$
\nabla^2 u = \nabla \cdot (\nabla u) = \frac{1}{h_1\, h_2\, h_3}\left\{\frac{\partial}{\partial x_1}\left(\frac{h_2\, h_3}{h_1}\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_3\, h_1}{h_2}\frac{\partial u}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{h_1\, h_2}{h_3}\frac{\partial u}{\partial x_3}\right)\right\}
\tag{A.137}
$$

(2) The most convenient way to write out $\nabla^2 \underline{v}$ is by use of expansion formula

Eq. (A.78):

$$\nabla^2 \underline{v} = \nabla(\nabla \cdot \underline{v}) - \nabla \times (\nabla \times \underline{v}) \tag{A.138}$$

which can be expanded by use of formulas, Eqs. (A.130), (A.131), and (A.135).

A.4.6 Convection term ($\underline{u} \cdot \nabla \underline{v}$)

(1) This useful vector appears in the Navier-Stokes equation when we write the time rate of flow momentum in Eulerian description sense. Performing very complicated procedure but straightforward manipulation, we arrive at the following result:

$$\begin{aligned}(\underline{u} \cdot \nabla \underline{v})_1 \;=\;& \frac{1}{h_1}\Big[u_1 \frac{\partial v_1}{\partial x_1} + u_2 \frac{\partial v_2}{\partial x_1} + u_3 \frac{\partial v_3}{\partial x_1} \\ &+\frac{1}{h_2}(u_1 v_2 - u_2 v_1) \frac{\partial h_1}{\partial x_2} + \frac{1}{h_3}(u_1 v_3 - u_3 v_1) \frac{\partial h_1}{\partial x_3}\Big] \\ &-\frac{u_2}{h_1\,h_2}\left[\frac{\partial(h_2\,v_2)}{\partial x_1} - \frac{\partial(h_1\,v_1)}{\partial x_2}\right] \\ &+\frac{u_3}{h_3\,h_1}\left[\frac{\partial(h_1\,v_1)}{\partial x_3} - \frac{\partial(h_3\,v_3)}{\partial x_1}\right]\end{aligned} \tag{A.139}$$

$$\begin{aligned}(\underline{u} \cdot \nabla \underline{v})_2 \;=\;& \frac{1}{h_2}\Big[u_1 \frac{\partial v_1}{\partial x_2} + u_2 \frac{\partial v_2}{\partial x_2} + u_3 \frac{\partial v_3}{\partial x_2} \\ &+\frac{1}{h_3}(u_2\,v_3 - u_3\,v_2) \frac{\partial h_2}{\partial x_3} + \frac{1}{h_1}(u_2\,v_1 - u_1\,v_2) \frac{\partial h_2}{\partial x_1}\Big] \\ &-\frac{u_3}{h_2\,h_3}\left[\frac{\partial(h_3\,v_3)}{\partial x_2} - \frac{\partial(h_2\,v_2)}{\partial x_3}\right] \\ &+\frac{u_1}{h_1\,h_2}\left[\frac{\partial(h_2\,v_2)}{\partial x_1} - \frac{\partial(h_1\,v_1)}{\partial x_2}\right]\end{aligned} \tag{A.140}$$

$$
\begin{aligned}
(\underline{u}\cdot\nabla\underline{v})_3 &= \frac{1}{h_3}\Big[u_1\,\frac{\partial v_1}{\partial x_3} + u_2\,\frac{\partial v_2}{\partial x_3} + u_3\,\frac{\partial v_3}{\partial x_3} \\
&\quad + \frac{1}{h_1}(u_3\,v_1 - u_1\,v_3)\,\frac{\partial h_3}{\partial x_1} + \frac{1}{h_2}(u_3\,v_2 - u_2\,v_3)\,\frac{\partial h_3}{\partial x_2}\Big] \\
&\quad - \frac{u_1}{h_3\,h_1}\left[\frac{\partial(h_1\,v_1)}{\partial x_3} - \frac{\partial(h_3\,v_3)}{\partial x_1}\right] \\
&\quad + \frac{u_2}{h_2\,h_3}\left[\frac{\partial(h_3\,v_3)}{\partial x_2} - \frac{\partial(h_2\,v_2)}{\partial x_3}\right] \qquad \text{(A.141)}
\end{aligned}
$$

A.5 Tensors of Second Order

(1) For example, let us consider a stress tensor that is a key quantity in continuum mechanics. [7] A stress is a force per unit area, in which force and an element of area are vectors. The area element have to specify both its magnitude and the direction of its normal.

(2) If $\underline{F}$ denotes the force and $\underline{S}$ is the area element, the stress tensor $\underline{\underline{T}}$ might be thought of as $\underline{F}/\underline{S}$. This quotient of two vectors cannot be defined, but rather we can define $\underline{F}$ as $\underline{S}\cdot\underline{\underline{T}}$. The stress tensor at a point $\underline{\underline{T}}$ becomes a newly physical quantity associated with two directions.

(3) In fact, it needs 9 numbers to specify the stress tensor in a reference system corresponding to the 9 possible combinations of 2 base vectors.

(4) A <u>second-order tensor</u> is a set of nine numbers τ_{ij}, having the property that when transferred from the x_1, x_2, x_3 system to the x'_1, x'_2, x'_3 system the corresponding quantities are given by

$$
\tau'_{ij} = \sum_{k=1}^{3}\sum_{\ell=1}^{3} a_{ki}\,a_{\ell j}\,\tau_{k\ell}, \quad \text{for} \quad i,j = 1,2,3 \qquad \text{(A.142)}
$$

A.5.1 Dyadic Products

(1) Much of our work can be simplified if we extend our definitions of vector multiplication to include the *dyadic product* $\underline{u}\ \underline{v}$.

[7] See Aris, R. (1962), *Vectors, Tensors and the Basic Equations of Fluid Mechanics*, Prentice Hall, p. 5.

(2) For our purpose, this need only be defined by the relations

$$\begin{aligned}(\underline{u}\,\underline{v}) \cdot \underline{w} &\equiv \underline{u}\,(\underline{v} \cdot \underline{w}) \\ \underline{w} \cdot (\underline{u}\,\underline{v}) &\equiv (\underline{w} \cdot \underline{u})\,\underline{v}\end{aligned} \tag{A.143}$$

(3) Actually the dyadic product $\underline{u}\ \underline{v}$ is a special form of *second-order tensor*; it can easily be seen to satisfy the definition of such a tensor. This definition may be stated as follows, with reference to the x_i and x'_i coordinate systems.

(4) In the case of $\underline{u}\ \underline{v}$, of course, the nine numbers involved are the products $u_i v_j\ (i, j = 1, 2, 3)$.

(5) Let us consider some examples:

(a) For $\nabla(\underline{u} \cdot \underline{v})$, using dyadic notation, $\nabla(\underline{u} \cdot \underline{v}) = (\nabla \underline{u}) \cdot \underline{v} + (\nabla \underline{v}) \cdot \underline{u}$.

(b) Laplacian $\nabla^2 \underline{v} = \nabla \cdot (\nabla \underline{v})$.

(c) When we define $(\underline{u}\ \underline{v}) \times \underline{w} \equiv \underline{u}(\underline{v} \times \underline{w})$ and $\underline{w} \times (\underline{u}\ \underline{v}) \equiv (\underline{w} \times \underline{u})\underline{v}$, these are obviously dyadics.

(d) If ϕ is any dyadic product, $\phi \cdot (\underline{a} \times \underline{b}) = (\phi \times \underline{a}) \cdot \underline{b}$.

(e) Let us look at the more important example. Let u_i be a vector, and consider the set of nine numbers $\partial u_i / \partial x_j$. This is easily shown to be a second-order tensor. It might be represented by the symbol *grad* $\underline{u}$ or $\nabla \underline{u}$.

A.5.2 Gradient of a Vector

(1) Now, consider the gradient of a vector, $\nabla \underline{u}$, which is involved into the convection and the diffusion terms of the Navier-Stokes equations.

(2) The velocity change at a point $d\underline{u}$ is

$$d\underline{u} = (d\underline{x} \cdot \nabla)\underline{u} \tag{A.144}$$

(3) The gradient of a vector is defined by, in a similar fashion to the gradient of a scalar,

$$\nabla \underline{u} = \lim_{V \to 0} \frac{1}{V} \oint_S \underline{n}\ \underline{u}\ dS = \frac{\partial u_j}{\partial x_i} \tag{A.145}$$

(4) In a rectangular Cartesian coordinate system, the gradient of a vector $\underline{u} = u_1\ \underline{i} + u_2\ \underline{j} + u_3\ \underline{k}$ is

$$\nabla \underline{u} = \frac{\partial u_1}{\partial x_1} \underline{i}\ \underline{i} + \frac{\partial u_2}{\partial x_1} \underline{i}\ \underline{j} + \frac{\partial u_3}{\partial x_1} \underline{i}\ \underline{k} + \cdots \text{ similar 6 terms} \tag{A.146}$$

(5) In general orthogonal curvilinear coordinates, the gradient of a vector $\underline{u} = u_1\ \underline{e}_1 + u_2\ \underline{e}_2 + u_3\ \underline{e}_3$, is [8]

$$\begin{aligned}\nabla \underline{u} \ &= \ \frac{1}{h_1}\left(\frac{\partial u_1}{\partial x_1} + \frac{u_2}{h_2}\frac{\partial h_1}{\partial x_2} + \frac{u_3}{h_3}\frac{\partial h_1}{\partial x_3}\right) \underline{e}_1\ \underline{e}_1 + \frac{1}{h_1}\left(\frac{\partial u_2}{\partial x_1} - \frac{u_1}{h_2}\frac{\partial h_1}{\partial x_2}\right) \underline{e}_1\ \underline{e}_2 \\ &\quad + \frac{1}{h_1}\left(\frac{\partial u_3}{\partial x_1} - \frac{u_1}{h_3}\frac{\partial h_1}{\partial x_3}\right) \underline{e}_1\ \underline{e}_3 + \cdots \text{ (similar 6 terms more)}\end{aligned} \tag{A.147}$$

(6) If the vector $\underline{v}$ is a velocity vector in the field of fluid mechanics, this is often resolved into a symmetric and antisymmetric form:

$$\begin{aligned}\nabla \underline{v} \ &= \ \frac{1}{2}\left[(\nabla \underline{v} + \nabla \underline{v}^T) + (\nabla \underline{v} - \nabla \underline{v}^T)\right] \\ &= \ \frac{1}{2} def(\underline{v}) + \frac{1}{2} rot(\underline{v})\end{aligned} \tag{A.148}$$

where, if we consider a second-order tensor to be a 3×3 matrix, the superscript T stand for transpose of the matrix which is the operation described by interchanging the rows and columns of the matrix.

(7) The first term is called the *strain rate tensor*, having 6 independent components. It represents (i) normal strain rate and (ii) shear strain rate which cause stress in fluid.
The second term is called the *spin tensor* or *vorticity tensor* $\underline{\underline{\Omega}}$, having only off-diagonal components. It represents rigid body rotation of a fluid

[8] For details, see Milne-Thomson, L. M. (1968), *Theoretical Hydrodynamics*, 5th edition, Macmillan, London, pp. 62–66, and Batchelor, G. K. (1967), *An Introduction to Fluid Dynamics*, Cambridge University Press, Cambridge, pp. 598–603.

element.

A.6 Reynolds Transport Theorem

A.6.1 Mathematical Derivation of Transport Theorem

(1) We will have need for the rate of change of an integral taken over a volume moving through a field

$$\frac{d}{dt}\int_{V(t)} F(\underline{x}, t)\; dV \tag{A.149}$$

where $F(\underline{x}, t)$ may be a scalar, vector or tensor variable.

(2) We assume the path of points in $V(t)$ are known:

$$\underline{x} = \underline{x}(\underline{\xi}, t) \tag{A.150}$$

where $\underline{\xi}$ is the initial point of $\underline{x}$.

(3) Hence we can invert the integral to the $\underline{\xi}$ variable:

$$\int_{V(t)} F(\underline{x}, t)\; dV \;=\; \int_{V(0)} F^*(\underline{\xi}, t)\; J\; d\xi_1\, d\xi_2\, d\xi_3 \tag{A.151}$$

where Jacobian J is written as

$$J = \frac{\partial(x_1,\, x_2,\, x_3)}{\partial(\xi_1,\, \xi_2,\, \xi_3)} = \epsilon_{ijk}\, \frac{\partial x_1}{\partial \xi_i}\, \frac{\partial x_2}{\partial \xi_j}\, \frac{\partial x_3}{\partial \xi_k} \tag{A.152}$$

$$F^*(\underline{\xi}, t) = F\left\{\underline{x}(\underline{\xi}, t), t\right\} \tag{A.153}$$

(4) Hence

$$\frac{d}{dt}\int_{V(0)} F^*(\underline{\xi}, t)\; J\; d\xi_1\, d\xi_2\, d\xi_3 \;=\; \int_{V(0)} \left(\frac{\partial F^*}{\partial t} J + F^* \frac{\partial J}{\partial t}\right)\; d\xi_1\, d\xi_2\, d\xi_3 \tag{A.154}$$

(5) Now

$$\frac{\partial J}{\partial t} = \epsilon_{ijk} \frac{\partial}{\partial t} \left(\frac{\partial x_1}{\partial \xi_i} \frac{\partial x_2}{\partial \xi_j} \frac{\partial x_3}{\partial \xi_k} \right) \tag{A.155}$$

$$\frac{\partial}{\partial t} \frac{\partial x_1}{\partial \xi_i} = \frac{\partial}{\partial \xi_i} \frac{\partial x_1}{\partial t} = \frac{\partial v_1}{\partial \xi_i} \tag{A.156}$$

(6) If $v_1 = v_1(x_1,\, x_2,\, x_3)$

$$\frac{\partial v_1}{\partial \xi_i} = \frac{\partial v_1}{\partial x_j} \frac{\partial x_j}{\partial \xi_i} \tag{A.157}$$

Since $\epsilon_{ijk} \dfrac{\partial v_1}{\partial x_2} \dfrac{\partial x_2}{\partial \xi_i} \dfrac{\partial x_2}{\partial \xi_j} \dfrac{\partial x_3}{\partial \xi_k}$ and similar terms are zero, the non-zero terms

$$\epsilon_{ijk} \left(\frac{\partial v_1}{\partial x_1} \frac{\partial x_1}{\partial \xi_i} \frac{\partial x_2}{\partial \xi_j} \frac{\partial x_3}{\partial \xi_k} + \frac{\partial v_2}{\partial x_2} \frac{\partial x_1}{\partial \xi_i} \frac{\partial x_2}{\partial \xi_j} \frac{\partial x_3}{\partial \xi_k} + \frac{\partial v_3}{\partial x_3} \frac{\partial x_1}{\partial \xi_i} \frac{\partial x_2}{\partial \xi_j} \frac{\partial x_3}{\partial \xi_k} \right) \tag{A.158}$$

remain. So

$$\frac{\partial J}{\partial t} = \left(\nabla \cdot \underline{v} \right) J \tag{A.159}$$

where $\underline{v}$ is the velocity of the point $\underline{x}$.

(7) Hence

$$\begin{aligned} &\int_{V(0)} \left(\frac{\partial F^*}{\partial t} + F^* \, \nabla \cdot \underline{v} \right) J \; d\xi_1 \, d\xi_2 \, d\xi_3 \\ &= \int_{V(t)} \left[\left(\frac{\partial F^*}{\partial t} \right)_{\underline{\xi}=const} + F^* \, \nabla \cdot \underline{v} \right] dV \end{aligned} \tag{A.160}$$

The Jacobian can be interpreted as the ratio of the volume occupied by a small piece of the fluid at time t to the volume occupied by this piece when $t = 0$. The divegence of the velocity can be interpreted as the rate of change of volume per unit volume of the moving piece of fluid. (See Kaplan, W., *Advanced Mathematics for Engineers*, pp. 589–591, 1981.)

(8) Now

$$\left.\frac{\partial F^*}{\partial t}\right|_{(\underline{\xi}=const)} = \left.\frac{F\left\{\underline{x}(\underline{\xi},t),t\right\}}{\partial t}\right|_{\underline{\xi}} = \frac{\partial F}{\partial t} + \frac{\partial \underline{x}}{\partial t}\cdot\nabla F = \frac{\partial F}{\partial t} + \underline{v}\cdot\nabla F \tag{A.161}$$

(9) Hence

$$\frac{d}{dt}\int_{V(t)} F\,dV = \int_V \left[\frac{\partial F}{\partial t} + \nabla\cdot(\underline{v}\,F)\right] dV \tag{A.162}$$

or

$$\boxed{\frac{d}{dt}\int_{V(t)} F\,dV = \int_{V(t)} \frac{\partial F}{\partial t}\,dV + \oint_{S(t)} \underline{n}\cdot(\underline{v}\,F)\,dS} \tag{A.163}$$

(10) We can apply this relation at any instant in time. The first integral implies rate of change in volume and the second one rate of change associated with motion of surface bounding volume. [9]

(11) It is noted that this is similar to Leibnitz's rule for an integral over one dimensional region: (For proof, see Kaplan, W., *Advanced Mathematics for Engineers*, pp. 520–521, 1981, or Kaplan, W., *Advanced Calculus*, pp. 220–221, 1952)

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\,dx = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}\,dx + f[b(t),t]\,b'(t) - f[a(t),t]\,a'(t) \tag{A.164}$$

(12) We can apply extensively the transport theorem to the case that there is a discontinuity interface Σ within a volume V. [10] The volume V is considered to be composed of two volumes V_1 and V_2 divided by an internal surface Σ. V is a material volume but as Σ moves with arbitrary velocity $\underline{u}_\Sigma$ and across it F suffers a discontinuity, F_1 and F_2 being its values on either side. If $\underline{n}_\Sigma$ is the normal to Σ in the direction form V_1 to V_2,

[9] See Newman, J. N. (1977), *Marine Hydrodynamics*, MIT Press, for depicted interpretation.
[10] Refer to Aris, R. (1962), *Vectors, Tensors and the Basic Equations of Fluid Mechanics*, Prentice Hall, p. 86.

Eq. (A.163) may be generalized to

$$\frac{d}{dt}\int_{V(t)} F\, dV = \int_{V(t)} \frac{\partial F}{\partial t}\, dV + \oint_{S(t)} \underline{n}\cdot(\underline{v}\, F)\, dS + \oint_{\Sigma(t)} \underline{n}_\Sigma \cdot (\underline{u}_\Sigma\, F)\, dS \tag{A.165}$$

A.6.2 Alternative Derivation of Transport Theorem

(1) General volume integral with boundary $S(t)$, and its difference:

$$I(t) = \int_{V(t)} F(\underline{x}, t)\, dV \tag{A.166}$$

$$\begin{aligned} \triangle I &= I(t+\triangle t) - I(t) \\ &= \int_{V(t+\triangle t)} F(\underline{x}, t+\triangle t)\, dV - \int_{V(t)} F(\underline{x}, t)\, dV \end{aligned} \tag{A.167}$$

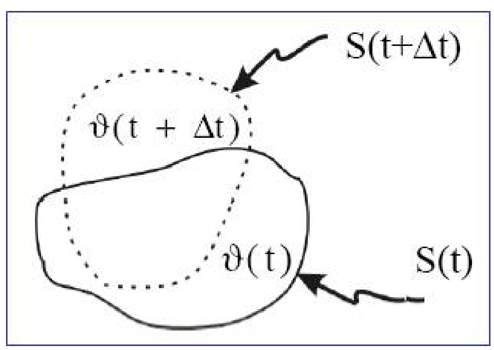

Figure A.3 Change of material volume in transport of physical quantity.

(2) First-order difference in Taylor series expansion:

$$F(\underline{x},\, t+\triangle t) \simeq F(\underline{x}, t) + \triangle t\, \frac{\partial F(\underline{x}, t)}{\partial t} \tag{A.168}$$

$$
\begin{aligned}
\triangle I &= \int_{V+\triangle V}\left(F+\triangle t\,\frac{\partial F}{\partial t}\right)dV-\int_V F\,dV\\
&= \triangle t\int_V \frac{\partial F}{\partial t}\,dV+\int_{\triangle V}F\,dV+O\left[(\triangle t)^2\right]\\
&= \triangle t\int_V \frac{\partial F}{\partial t}\,dV+\oint_S (U_n\triangle t)F\,dV+O\left[(\triangle t)^2\right] \quad \text{(A.169)}
\end{aligned}
$$

(3) Transport theorem for a volume $V(t)$

$$
\frac{dI}{dt}=\int_V \frac{\partial F}{\partial t}\,dV+\oint_S U_nF\,dV \qquad \text{(A.170)}
$$

For material $V(t)$, with same normal velocity as the fluid

$$
\begin{aligned}
\frac{d}{dt}\int_{V(t)}F(\underline{x},t)\,dV &= \int_{V(t)}\frac{\partial F}{\partial t}\,dV+\oint_{S(t)}(u_i\,n_i)F(\underline{x},t)\,dV\\
&= \int_{V(t)}\left[\frac{\partial F}{\partial t}+\frac{\partial}{\partial x_i}(Fu_i)\right]dV \qquad \text{(A.171)}
\end{aligned}
$$

A.7 Moving Coordinate Systems

A.7.1 Velocity due to Rigid Body Rotation

(1) Suppose a rigid body rotates about an axis through the origin of a coordinate system with an angular velocity $\underline{\omega}=\omega\,\underline{n}$, where the direction of the axis is given by a unit vector $\underline{n}$ and ω is the magnitude of the angular velocity (see Figure A.4). [11]

[11] The description herein is based on Aris, R. (1962), *Vectors, Tensors and the Basic Equations of Fluid Mechanics*, Prentice Hall, p. 17.

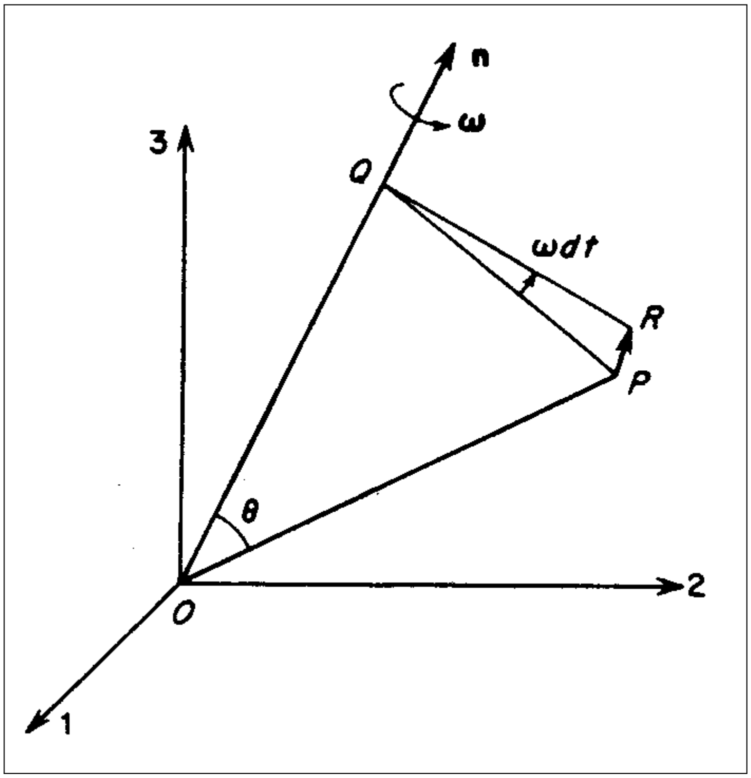

Figure **A.4** Rotation of a rigid body. (From Aris 1962, p. 17)

(2) Let P be any point in the body at position $\underline{x}$. Then $\underline{n} \times \underline{x}$ is a vector in the direction of PR of which magnitude is $|\underline{x}| \sin\theta$. However, $|\underline{x}| \sin\theta = PQ$ is the perpendicular distance from P to the axis of rotation.

(3) In a small interval of time δt, the radius PQ moves through an angle $\omega\ \delta t$ and hence P moves through a distance $(PQ)\,\omega\,\delta t$.

(4) It follows that the small short distance PR is a vector $\delta\underline{x}$ perpendicular to the plane of OP and the axis of rotation:

$$\delta\underline{x} = (\underline{n} \times \underline{x})\,\omega\,\delta t = (\underline{\omega} \times \underline{x})\,\delta t \tag{A.172}$$

(5) Dividing both sides by δt and taking the limit $\delta t \to 0$ provide the velocity of the point P. Thus the linear velocity $\underline{v}$ of the point $\underline{x}$ due to a rotation $\underline{\omega}$ is

$$\underline{v} = \underline{\omega} \times \underline{x} \tag{A.173}$$

This result can be directly applied to moving coordinate systems. Details are given in the following subsection.

A.7.2 Transformations of Moving Coordinates

(1) Let us introduce two coordinate systems: one system fixed to space and the other moving relative to the space-fixed system. The moving (the unprimed) coordinate system is supposed to be in motion of both translation and rotation relative to the space-fixed (the unprimed) system.

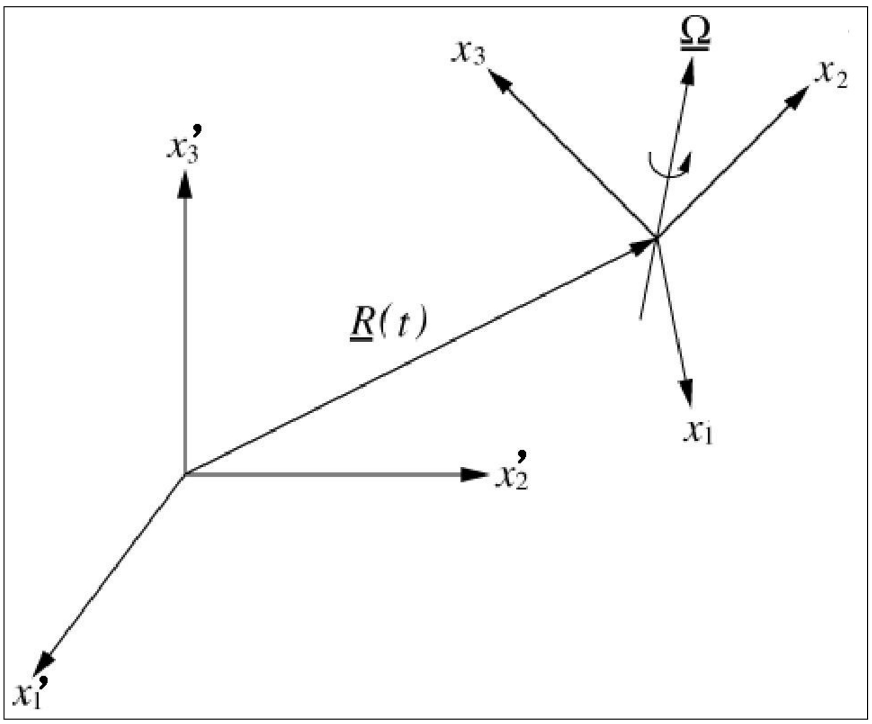

Figure A.5 Moving coordinate system.

(2) Then the position vector $\underline{x}'$ defined in the space-fixed system is related to the position vector $\underline{x}$ defined in the moving system as follows:

$$\underline{x}' = \underline{x} + \underline{R} \tag{A.174}$$

where $\underline{R}$ is the distance vector between two coordinate systems. (See figure A.5).

(3) Because of the relative motion, time-derivative will appear different to observers in the two coordinate systems. For example, a vector that is constant in either system would seem to vary with time to an observer fixed in the other system. We can write the relationship between the

derivative(d'/dt) observed in the space-fixed system and the derivative(d/dt) observed in the moving system, for an arbitrary vector:

$$\frac{d\underline{A}'}{dt} = \frac{d\underline{A}}{dt} + \underline{\Omega} \times \underline{A} \tag{A.175}$$

where $\underline{\Omega}$ is the vector angular velocity of the moving system. The last term in Eq. (A.175) implies a rotation of a rigid body. [12]

(4) If this formula is applied to the special case of the position vector $\underline{x}$ given in Eq. (A.174), we have the velocity:

$$\underline{q}' = \underline{q} + \underline{\Omega} \times \underline{x} + \underline{\dot{R}} \tag{A.176}$$

where $\underline{\dot{R}}$ represents the translation velocity of the moving frame. Therefore this equation implies that the absolute velocity is the sum of the velocity($\underline{q}$) measured by an observer in the moving system and the frame velocity of the moving system ($\underline{\Omega} \times \underline{x} + \underline{\dot{R}}$).

(5) In a similar manner, we can obtain the relation between acceleration vectors by making use of the general rule Eq. (A.175):

$$\underline{a}' \equiv \frac{d'^2\underline{x}'}{dt^2} = \underline{a} + 2\,\underline{\Omega} \times \underline{q} + \frac{d\underline{\Omega}}{dt} \times \underline{x} + \underline{\Omega} \times (\underline{\Omega} \times \underline{x}) + \underline{\ddot{R}} \tag{A.177}$$

Here we have written $d\underline{\Omega}/dt$ instead of $d'\underline{\Omega}/dt$ because $\underline{\Omega}$ is a vector that is always the same in both systems.

(6) The first term of Eq. (A.177) ($\underline{a}$) is the acceleration viewed in the moving system. The second is the *Coriolis acceleration*, which depends on the velocity in the moving system. The meaning of the third term is not clear. The fourth term is the generalized centripetal acceleration, since

$$|\underline{\Omega} \times (\underline{\Omega} \times \underline{x})| = \Omega^2\, \underline{x}\, \sin(\underline{\Omega}, \underline{x}) \tag{A.178}$$

(7) It is noted that, if we consider the self-rotation of earth with constant angular speed, this term becomes a form of gradient of a scalar function and

[12] See 김 형 종 (1999), 미적분학, 총 2권, 서울대학교 출판부, pp. 317–318, and Aris, R. (1962), *Vectors, Tensors and the Basic Equations of Fluid Mechanics*, Prentice Hall, p. 17.

its effect was already included in gravitational acceleration for treatment as a body force term of the momentum equations.

A.8 Mathematical Identities

A.8.1 Green's Scalar Identity

(1) If $\underline{u} = \psi\nabla\phi$ in Eq. (A.96), we obtain Green's first identity:

$$\int_V \left[\psi\nabla^2\phi + \nabla\psi \cdot \nabla\phi\right]\, dV = \oint_S \psi\, \underline{n} \cdot \nabla\phi\, dS \tag{A.179}$$

And if $\underline{u} = \phi\nabla\psi$, use Eq. (A.96) and add the result to Green's first identity, we obtain Green's second(scalar) identity:

$$\int_V \left[\psi\nabla^2\phi - \phi\nabla^2\psi\right]\, dV = \oint_S \left[\psi\frac{\partial\phi}{\partial n} - \phi\frac{\partial\psi}{\partial n}\right]\, dS \tag{A.180}$$

where $\underline{n} \cdot \nabla\phi = \dfrac{\partial\phi}{\partial n}$.

(2) For these relations to be valid, ϕ and ψ must be continuous in the volume and on the surface and the second derivatives must be continuous within the volume while on the surface only the first derivatives need be continuous. [13]

(3) As an practical application, an arbitrary scalar field defined in a volume V can be represented in terms of integrals over the enclosing surfaces plus an integral of $\nabla^2\phi$ over the volume. We will show this fact as the following derivation:

(4) From the expansion formulas, we see that $\dfrac{1}{|\underline{x}|} = \dfrac{1}{|\underline{r}|} = \dfrac{1}{r}$ satisfies Laplace's equation: $\nabla^2\left(\dfrac{1}{r}\right) = 0$ if $\underline{r} \neq 0$. Similarly $\nabla^2\left(\dfrac{1}{|\underline{y} - \underline{x}|}\right) = 0$ for $\underline{y}$ a

[13] More detailed explanation can be found in mathematical texts, e.g., Kreyszig, E. (1993), *Advanced Engineering Mathematics*, Seventh ed.,Wiley, pp. 553–554.

constant vector. Since $\nabla^2\left(\frac{1}{|\underline{y}-\underline{x}|}\right)$ does not exist at $\underline{x}=\underline{y}$, we exclude this point from the volume by surrounding it with a sphere.

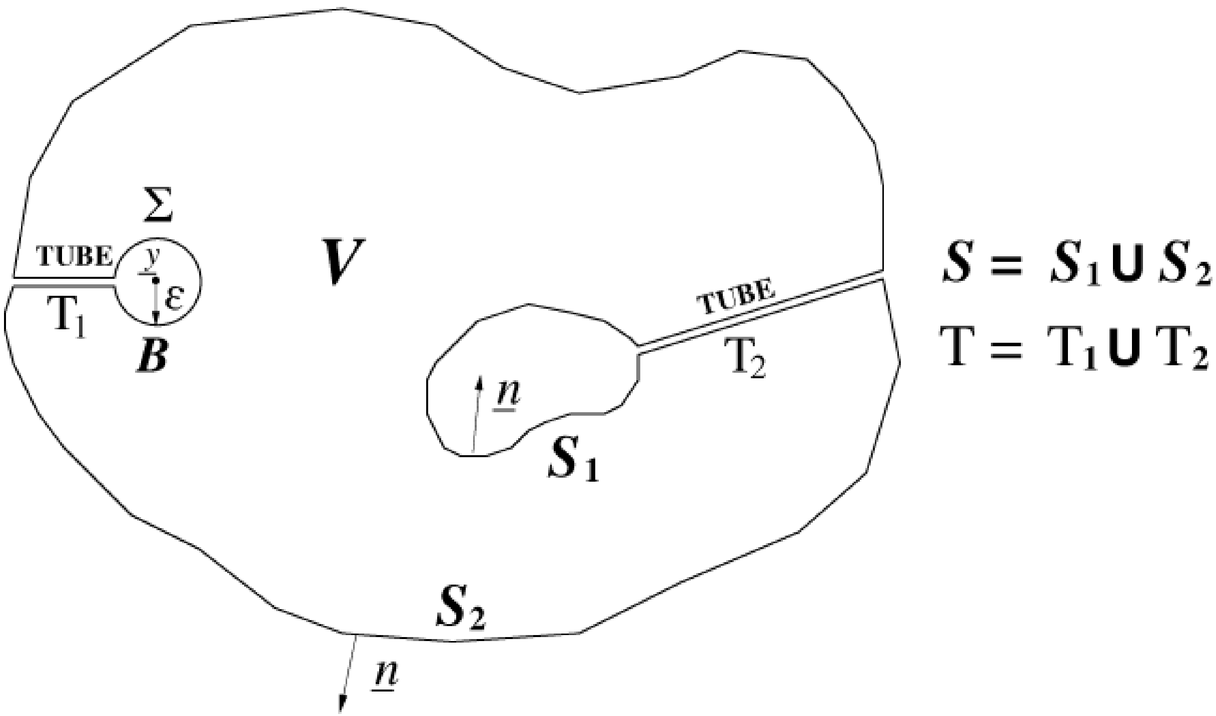

Figure A.6 Two-dimensional drawing of a simply connected region for deriving the scalar identity.

(5) Hence if we take $\psi=\frac{1}{|\underline{y}-\underline{x}|}$, Green's second identity becomes:

$$\oint_{S+\sum(\underline{y},\epsilon)}\left[\frac{\underline{n}\cdot\nabla\phi}{|\underline{y}-\underline{x}|}-\phi\,\underline{n}\cdot\nabla\frac{1}{|\underline{y}-\underline{x}|}\right]dS=\int_{V-B(\underline{y},\epsilon)}\left[\frac{1}{|\underline{y}-\underline{x}|}\nabla^2\phi\right]dV \tag{A.181}$$

where $B(\underline{y},\epsilon)$ is a sphere of radius ϵ centered at $\underline{y}$ and bounded by Σ. In this application, the surface is in three-dimensional space and the integration variable is $\underline{x}$.

(6) We illustrate the situation with a two-dimensional drawing as shown in Figure A.6. Integrations over the small tubes joining Σ and S_2, and S_1 and S_2 vanish by continuity of ϕ.

(7) On the surface Σ surrounding the point $\underline{y}$, as shown in Figure A.7 for an

enlarged view, we have

$$\underline{y} - \underline{x} = -\epsilon\, \underline{e}_r \tag{A.182}$$

$$\underline{n} = -\underline{e}_r \tag{A.183}$$

$$dS = (\epsilon\, d\theta)(\epsilon\, \sin\theta\, d\alpha) \tag{A.184}$$

$$\phi(\underline{x}) = \phi(\underline{y}) + \epsilon \left.\frac{\partial \phi}{\partial r}\right|_{\underline{y}} + \cdots \tag{A.185}$$

$$\nabla \frac{1}{|\underline{y} - \underline{x}|} = \frac{(\underline{y} - \underline{x})}{|\underline{x} - \underline{y}|^3} = -\frac{\epsilon\, \underline{e}_r}{\epsilon^3} \tag{A.186}$$

where $\underline{e}_r$ is the unit vector in the radial direction.

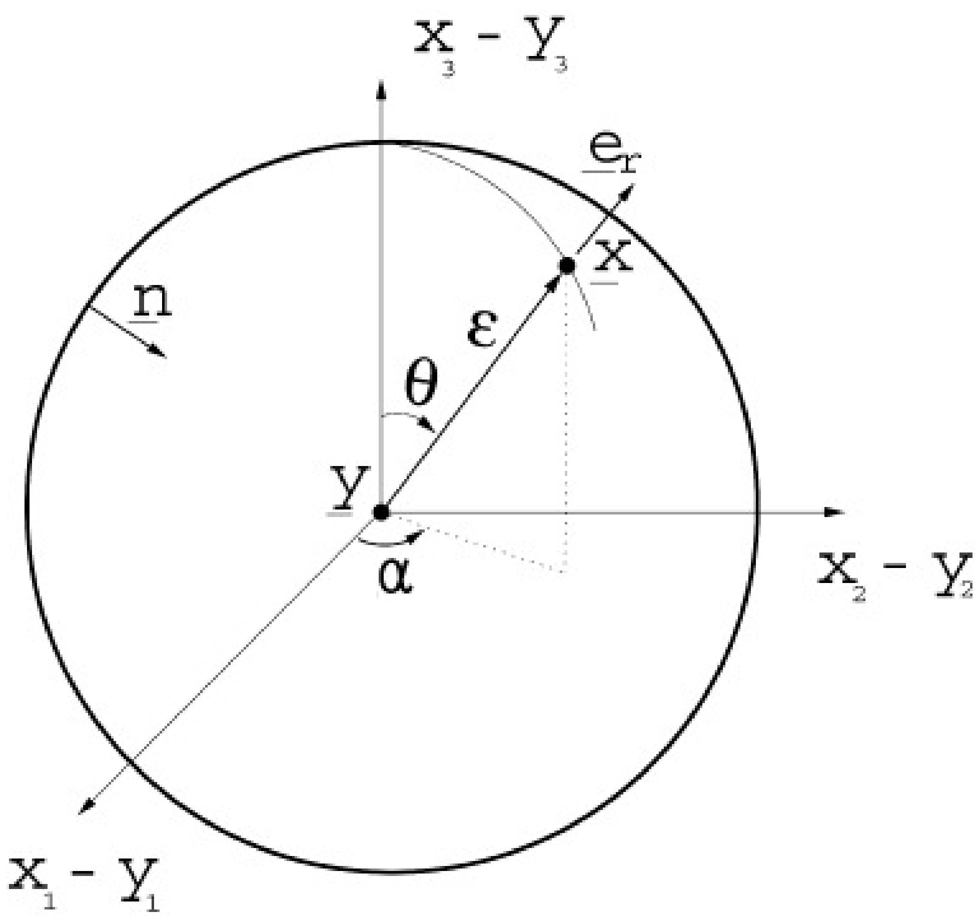

Figure A.7 Small sphere region containing a singular point.

(8) Hence the integration for the surface Σ and the small ball B becomes,

respectively,

$$\oint_{\Sigma}\left[\frac{\underline{n}\cdot\nabla\phi}{|\underline{y}-\underline{x}|}-\phi\,\underline{n}\cdot\nabla\frac{1}{|\underline{y}-\underline{x}|}\right]\,dS$$
$$=-\phi(\underline{y})\int_0^{2\pi}d\alpha\int_0^{\pi}\left[\epsilon^2\,\frac{\underline{e}_r\cdot(\epsilon\,\underline{e}_r)}{\epsilon^3}\sin\theta\right]\,d\theta+O(\epsilon)$$
$$=-4\pi\,\phi(\underline{y})+O(\epsilon) \tag{A.187}$$

and

$$\int_B\left[\nabla^2\phi\frac{1}{|\underline{y}-\underline{x}|}\right]\,dV=\nabla^2\phi\big|_{\underline{y}}\,\left(O(\epsilon^2)\right) \tag{A.188}$$

(9) Hence, taking the limit as $\epsilon\to 0$, we find

$$\boxed{\phi(\underline{y})=\frac{1}{4\pi}\oint_S\left[\frac{\underline{n}\cdot\nabla\phi}{|\underline{y}-\underline{x}|}-\phi\,\frac{\underline{n}\cdot(\underline{y}-\underline{x})}{|\underline{y}-\underline{x}|^3}\right]\,dS-\frac{1}{4\pi}\int_V\frac{\nabla^2\phi}{|\underline{y}-\underline{x}|}\,dV} \tag{A.189}$$

If the point $\underline{y}$ had been outside V, the left-hand side would have been zero.

(10) For a two-dimensional field, $\psi=\ln\dfrac{1}{\sqrt{x_1^2+x_2^2}}$ in Green's second identity and a similar expression is obtained.

A.8.2 Uniqueness of Scalar Identity

(1) Let us consider the uniqueness of this integral representation. If another scalar field, say $\phi'(\underline{x})$ had the same value of $\nabla^2\phi$ in V and the same value of ϕ or $\underline{n}\cdot\nabla\phi$ on S, then we could construct a third solution which had $\nabla^2\phi''=0$ in V, and either $\phi''=0$ or $\underline{n}\cdot\nabla\phi''$ on S.

(2) If $\phi=\psi=\phi''$ in Green's first identity, then

$$\int_V\left[\,\phi''\,\nabla^2\phi''+\nabla\phi''\cdot\nabla\phi''\,\right]\,dV=\oint_S\left[\phi''\,\underline{n}\cdot\nabla\phi''\right]\,dS \tag{A.190}$$

and this reduces to only

$$\int_V \nabla\phi'' \cdot \nabla\phi''\, dV = 0 \tag{A.191}$$

(3) Since $(\nabla\phi)^2$ is always greater than or equal to zero, the only solution is

$$\nabla\phi'' \cdot \nabla\phi'' = 0 \tag{A.192}$$

This requires that ϕ'' be at most a constant. If ϕ were specified on the boundary, the constant is zero. If $\underline{n} \cdot \nabla\phi$ is specified on the boundary, ϕ is uniquely determined by the integral to within a constant.

(4) It is important to recognize that our expression for ϕ is in terms of ϕ and $\underline{n} \cdot \nabla\phi$ and the above consideration shows we need specify only one of these on the boundary. Hence to find the unknown on the boundary, one must first solve an integral equation.

(5) Also we have assumed that the field boundaries are fixed. If they were to depend on the field, then special conditions must be specified to insure the solution is unique. In addition to this uniqueness, we should also consider the far-field behavior of ϕ as the distance r goes to infinity. [14]

A.8.3 Type of Boundary Conditions

(1) Dirichlet boundary condition (1st type)

- The Dirichlet (or first type) boundary condition is perhaps the easiest one to understand. When we solve a differential equation, we put specified values on the boundary of the domain where a solution needs to take.
- For example, when Poisson equation such as $\nabla^2\psi = -\omega$ for stream function ψ and vorticity ω is satisfied in a domain Ω, the Dirichlet boundary condition takes the form $\psi(\underline{x}) = f(\underline{x})$ on the boundary $\partial\Omega$, where $f(\underline{x})$ is a known function defined on the boundary.

[14] Detailed consideration may be found in Batchelor, G. K. (1967), *An Introduction to Fluid Dynamics*, Cambridge University Press, Cambridge.

(2) Neumann boundary condition (2nd type)

- The Neumann (or second type) boundary condition specifies the values that the derivative of a solution is to take on the boundary of the domain, when imposed on an ordinary or a partial differential equation.
- For example, for Laplace equation $\nabla^2\phi = 0$ which we will present later on, the Neumann boundary condition takes the form $\dfrac{\partial\phi(\underline{x})}{\partial n} = g(\underline{x})$. Here, n denotes the (typically exterior) normal to the boundary and g is a given scalar function.

(3) Robin boundary condition (3rd type)

- The Robin (or third type) boundary condition is a type of hybrid boundary condition; it is a linear combination of Dirichlet and Neumann boundary conditions, namely, it is a specification of a linear combination of the values of a function and the values of its derivative on the boundary of the domain.
- Robin boundary conditions are a weighted combination of Dirichlet boundary conditions and Neumann boundary conditions, such as $a\,\phi + b\,\dfrac{\partial\phi}{\partial n} = h(\underline{x})$ where a and b are non-zero constants or functions more generally.
- Robin boundary conditions are commonly used in solving Sturm-Liouville problems. (See Stakgold, 1986 for details). These boundary conditions should not be confused with mixed boundary conditions, which are boundary conditions of different types specified on different subsets of the boundary.

(4) Mixed boundary condition

- The mixed boundary condition for a partial differential equation implies that different types of boundary condition are used on different parts of the boundary.
- For example, in partial sheet cavity problems for a hydrofoil, if ϕ is a solution to Laplace equation on a fluid domain and the boundary is

divided into two portions of cavity and non-cavity, one would impose a Dirichlet boundary condition on the cavity portion and a Neumann boundary condition on the non-cavity portion.

(5) Cauchy boundary condition

- A Cauchy boundary condition imposed on an ordinary differential equation or a partial differential equation specifies both the values a solution of a differential equation is to take on the boundary of the domain and the normal derivative at the boundary. It corresponds to imposing both a Dirichlet and a Neumann boundary condition.
- Cauchy boundary conditions can be understood from the theory of second order, ordinary differential equations, where to have a particular solution one has to specify the value of the function and the value of the derivative at a given initial or boundary point.
- For a second order partial differential equation, we now need to know the value of the function at the boundary, and its normal derivative in order to solve the partial differential equation. When the variable is specially time, Cauchy conditions can also be called initial value conditions.

A.8.4 Vector Identity

(1) Another identity involving vectors can be constructed from divergence theorems for a vector and a dyadic. In the third divergence theorem given by Eq. (A.97), let the vector be $\underline{u} \times \underline{v}$, then

$$\int_V \left[\nabla \times (\underline{u} \times \underline{v})\right] \, dV = \oint_S \left[\underline{n} \times (\underline{u} \times \underline{v})\right] \, dS \tag{A.193}$$

(2) According to the expansion formula on vector triple products, we know

$$\begin{aligned} \underline{n} \times (\underline{u} \times \underline{v}) &= (\underline{n} \times \underline{u}) \times \underline{v} + (\underline{v} \times \underline{n}) \times \underline{u} \\ &= (\underline{n} \times \underline{u}) \times \underline{v} - \underline{v}\,(\underline{n} \cdot \underline{u}) + \underline{n}\,(\underline{u} \cdot \underline{v}) \end{aligned} \tag{A.194}$$

Hence

$$\oint_S [\underline{n} \times (\underline{u} \times \underline{v})]\, dS = \oint_S [(\underline{n} \times \underline{u}) \times \underline{v} - (\underline{n} \cdot \underline{u})\, \underline{v} + \underline{n}\, (\underline{u} \cdot \underline{v})]\ dS \tag{A.195}$$

(3) These integrals can be rearranged by the divergence theorem:

$$\oint_S [(\underline{n} \times \underline{u}) \times \underline{v}]\ dS = \int_V [\nabla \times (\underline{u} \times \underline{v}) + \nabla \cdot (\underline{u}\ \underline{v}) - \nabla(\underline{u} \cdot \underline{v})]\ dV \tag{A.196}$$

(4) Now adding the results of the divergence theorem for a dyadic $\underline{u}\,\underline{v}$ to both sides:

$$\begin{aligned}&\oint_S (\underline{n} \times \underline{u}) \times \underline{v} + (\underline{n} \cdot \underline{u})\, \underline{v}]\ dS \\ &= \int_V [\nabla \times (\underline{u} \times \underline{v}) + 2\, \underline{v}\, (\nabla \cdot \underline{u}) + 2\, \underline{u}\ \cdot \nabla \underline{v} - \nabla(\underline{u} \cdot \underline{v})]\ dV\end{aligned} \tag{A.197}$$

(5) Using the expansion formulas

$$\nabla \times (\underline{u} \times \underline{v}) = \underline{v} \cdot \nabla \underline{u} + \underline{u}\, (\nabla \cdot \underline{v}) - \underline{v}\, (\nabla \cdot \underline{u}) - \underline{u} \cdot \nabla \underline{v} \tag{A.198}$$

$$\nabla(\underline{u} \cdot \underline{v}) = \underline{v} \cdot \nabla \underline{u} + \underline{v} \times (\nabla \times \underline{u}) + \underline{u} \times (\nabla \times \underline{v}) + \underline{u} \cdot \nabla \underline{v} \tag{A.199}$$

and subtracting one from the other, we obtain

$$\begin{aligned}\nabla \times (\underline{u} \times \underline{v}) - \nabla(\underline{u} \cdot \underline{v}) &= \underline{u}\, (\nabla \cdot \underline{v}) - \underline{v}\, (\nabla \cdot \underline{u}) - 2\, \underline{u} \cdot \nabla \underline{v} \\ &\quad - \underline{v} \times (\nabla \times \underline{u}) - \underline{u} \times (\nabla \times \underline{v})\end{aligned} \tag{A.200}$$

(6) Hence

$$\oint_S \left[(\underline{n} \times \underline{u}) \times \underline{v} + (\underline{n} \cdot \underline{u})\, \underline{v}\right]\, dS$$
$$= \int_V \left[\underline{v}\,(\nabla \cdot \underline{u}) + \underline{u}\,(\nabla \cdot \underline{v}) - \underline{u} \times (\nabla \times \underline{v}) - \underline{v} \times (\nabla \times \underline{u})\right]\, dV \tag{A.201}$$

This is called vector identity.

(7) An arbitrary vector field can be represented by this vector identity by choosing

$$\underline{v} = \nabla \frac{1}{|\underline{y} - \underline{x}|} = \frac{(\underline{y} - \underline{x})}{|\underline{y} - \underline{x}|^3} \quad \text{for } \underline{y} \text{ fixed} \tag{A.202}$$

For which, we have

$$\left.\begin{aligned} \nabla \times \underline{v} &= 0 \\ \nabla \cdot \underline{v} &= 0 \end{aligned}\right\} \quad \text{for } \underline{x} \neq \underline{y} \tag{A.203}$$

(8) Hence for $\underline{y}$ not in V, Eq. (A.201) becomes, without any restriction,

$$\oint_S \left[(\underline{n} \times \underline{u}) \times \frac{(\underline{y} - \underline{x})}{|\underline{y} - \underline{x}|^3} + (\underline{n} \cdot \underline{u})\, \frac{(\underline{y} - \underline{x})}{|\underline{y} - \underline{x}|^3}\right]\, dS$$
$$= \int_V \left[\frac{(\underline{y} - \underline{x})}{|\underline{y} - \underline{x}|^3}\,(\nabla \cdot \underline{u}) - \frac{(\underline{y} - \underline{x})}{|\underline{y} - \underline{x}|^3} \times (\nabla \times \underline{u})\right]\, dV \tag{A.204}$$

(9) For the case when $\underline{y}$ is in V, $\left(\dfrac{1}{|\underline{y} - \underline{x}|}\right)$ becomes singular as $\underline{y}$ tends to $\underline{x}$. The point $\underline{y}$ can be excluded from V by surrounding it with a sphere of radius ϵ centered at $\underline{y}$, as shown in Figure **A.6**. This sphere plus any other surfaces inside V can be connected to the exterior surface by small tubes to make all the surfaces continuous and the region remains simply connected, in the same manner as for the scalar identity.

(10) The vector identity applies to the region V as defined with the exclusions:

$$\oint\limits_{S+T+\sum(\underline{y},\epsilon)} \left[(\underline{n}\times\underline{u})\times\frac{(\underline{y}-\underline{x})}{|\underline{y}-\underline{x}|^3}+(\underline{n}\cdot\underline{u})\,\frac{(\underline{y}-\underline{x})}{|\underline{y}-\underline{x}|^3}\right]\,dS$$
$$= \int\limits_{V-B}\left[\frac{(\underline{y}-\underline{x})}{|\underline{y}-\underline{x}|^3}\,(\nabla\cdot\underline{u})-\frac{(\underline{y}-\underline{x})}{|\underline{y}-\underline{x}|^3}\times(\nabla\times\underline{u})\right]\,dV \qquad \text{(A.205)}$$

Integrations over the small tubes T_1 and T_2 vanish by continuity as they become increasingly small.

(11) On the surface Σ surrounding the point $\underline{y}$ (see Figure A.7):

$$\underline{y}-\underline{x} = -\epsilon\,\underline{e}_r \qquad \text{(A.206)}$$
$$\underline{n} = -\underline{e}_r \qquad \text{(A.207)}$$
$$dS = (\epsilon\,d\theta)\,(\epsilon\,\sin\theta\,d\phi) \qquad \text{(A.208)}$$
$$\frac{(\underline{y}-\underline{x})}{|\underline{x}-\underline{y}|^3} = \frac{-\epsilon\,\underline{e}_r}{\epsilon^3} \qquad \text{(A.209)}$$

where $\underline{e}_r$ is the unit vector in the radial direction. Furthermore,

$$\underline{u}|_{\sum} = \underline{u}(\underline{y})+(\underline{x}-\underline{y})\cdot\nabla\underline{u}+\cdots = \underline{u}(\underline{y})+O(\epsilon) \qquad \text{(A.210)}$$
$$(\underline{n}\times\underline{u})\times(\underline{y}-\underline{x}) = \epsilon\,(-\underline{e}_r\times\underline{u})\times(-\underline{e}_r) = \epsilon\,\{\underline{u}-\underline{e}_r(\underline{u}\cdot\underline{e}_r)\} \qquad \text{(A.211)}$$
$$(\underline{n}\cdot\underline{u})\,(\underline{y}-\underline{x}) = \epsilon\,(\underline{e}_r\cdot\underline{u})\,\underline{e}_r \qquad \text{(A.212)}$$

(12) Hence,

$$\oint_{\sum}\left[(\underline{n}\times\underline{u})\times\frac{(\underline{y}-\underline{x})}{|\underline{y}-\underline{x}|^3}+(\underline{n}\cdot\underline{u})\,\frac{(\underline{y}-\underline{x})}{|\underline{y}-\underline{x}|^3}\right]\,dS$$
$$= \underline{u}(\underline{y})\int_0^{2\pi}d\alpha\int_0^{\pi}\frac{\epsilon^3\sin\theta\,d\theta}{\epsilon^3}+O(\epsilon)$$
$$= 4\pi\,\underline{u}(\underline{y})+O(\epsilon) \qquad \text{(A.213)}$$

And for $\epsilon \to 0$,

$$4\pi\, \underline{u}(\underline{y}) = -\oint_S \left[(\underline{n} \times \underline{u}) \times \frac{(\underline{y}-\underline{x})}{|\underline{y}-\underline{x}|^3} + (\underline{n}\cdot\underline{u})\,\frac{(\underline{y}-\underline{x})}{|\underline{y}-\underline{x}|^3} \right] dS_x$$
$$+ \lim_{\epsilon\to 0} \int_{V-B(\underline{y},\epsilon)} \left[\frac{(\underline{y}-\underline{x})}{|\underline{y}-\underline{x}|^3} (\nabla\cdot\underline{u}) - \frac{(\underline{y}-\underline{x})}{|\underline{y}-\underline{x}|^3} \times (\nabla\times\underline{u}) \right] dV_x \tag{A.214}$$

This is a representation of $\underline{u}$ in terms of both components on the boundary, the normal component $\underline{n}\cdot\underline{u}$, and the tangential component, $\underline{n}\times\underline{u}$, plus the divergence and the curl integrated over the field.

(13) If $\underline{u}$ is divided into two components after interchanging the variables $\underline{x}$ and $\underline{y}$, Eq. (A.214) is rewritten as

$$4\pi\, \underline{u} = \underline{u}_1 + \underline{u}_2 \tag{A.215}$$

$$\boxed{\begin{aligned} \underline{u}_1(\underline{x}) &= +\,⨍_V \frac{(\underline{x}-\underline{y})}{|\underline{x}-\underline{y}|^3}(\nabla\cdot\underline{u})\, dV_y - ∯_S (\underline{n}\cdot\underline{u})\frac{(\underline{x}-\underline{y})}{|\underline{x}-\underline{y}|^3}\, dS_y \qquad \text{(A.216)} \\ \underline{u}_2(\underline{x}) &= -\,⨍_V \frac{(\underline{x}-\underline{y})}{|\underline{x}-\underline{y}|^3} \times (\nabla\times\underline{u})\, dV_y - ∯_S (\underline{n}\times\underline{u}) \times \frac{(\underline{x}-\underline{y})}{|\underline{x}-\underline{y}|^3}\, dS_y \qquad \text{(A.217)} \end{aligned}}$$

where the bar through the integral sign indicates the limit integration.

A.8.5 Integral Expression of Helmholtz Decomposition

(1) For a vector field $\underline{u}$ given in a domain V, we define a vector $\underline{F}$ by

$$\underline{F}(\underline{x}) = -\int_V G(\underline{x}-\underline{y})\,\underline{u}(\underline{y})\, dV_y \tag{A.218}$$

where $G(\underline{r})$ is the fundamental solution (Green function) of Poisson equation

$$\nabla^2 G(\underline{r}) = \delta(\underline{r}). \tag{A.219}$$

For example, $G(\underline{r}) = -\dfrac{1}{4\pi|\underline{r}|}$ in 3-D dimesional free space.

(2) By Eqs. (A.218) and (A.219) and the definition of the Dirac delta function, we have

$$-\nabla^2 \underline{F} = -\int_V \delta(\underline{x} - \underline{y})\, \underline{u}(\underline{y})\, dV_y = \underline{u}(\underline{x}) \tag{A.220}$$

(3) According to Eq. (A.78),

$$\underline{u}(\underline{x}) = -\nabla^2 \underline{F} = -\nabla(\nabla \cdot \underline{F}) + \nabla \times (\nabla \times \underline{F}) \tag{A.221}$$

(4) By comparing this expression with the Helmholtz decomposition form $\underline{u} = \nabla\phi + \nabla \times \underline{A}$, the scalar and the vector potentials are simply given by

$$\phi = -\nabla \cdot \underline{F}, \quad \underline{A} = \nabla \times \underline{F} \tag{A.222}$$

(5) We can then perform the integration of Eq. (A.218) to yield

$$\begin{aligned}
\phi = -\nabla \cdot \underline{F} &= \int_V \nabla \cdot \left\{G(\underline{x} - \underline{y})\, \underline{u}(\underline{y})\right\}\, dV_y \\
&= \int_V \nabla G(\underline{x} - \underline{y}) \cdot \underline{u}(\underline{y})\, dV_y \\
&= -\int_V \nabla_y G(\underline{x} - \underline{y}) \cdot \underline{u}(\underline{y})\, dV_y \\
&= -\int_V \{\nabla_y \cdot (G\, \underline{u}) - G\, \nabla_y \cdot \underline{u}\}\, dV_y \\
&= -\oint_S G\, \underline{n} \cdot \underline{u}\, dS_y + \int_V G\, \theta\, dV_y
\end{aligned} \tag{A.223}$$

and

$$\underline{A} = \nabla \times \underline{F} = -\int_V \nabla \times \left\{ G(\underline{x}-\underline{y})\, \underline{u}(\underline{y}) \right\}\, dV_y$$
$$= -\int_V \nabla G(\underline{x}-\underline{y}) \times \underline{u}(\underline{y})\, dV_y$$
$$= \int_V \nabla_y G(\underline{x}-\underline{y}) \times \underline{u}(\underline{y})\, dV_y$$
$$= \int_V \left\{ \nabla_y \times (G\, \underline{u}) - G\, \nabla_y \times \underline{u} \right\}\, dV_y$$
$$= \oint_S G\, \underline{n} \times \underline{u}\, dS_y - \int_V G\, \underline{\omega}\, dV_y \qquad \text{(A.224)}$$

Here we denote the gradient operator with respect to the integration variables $\underline{y}$ by ∇_y so that $\nabla G = -\nabla_y G$.

(6) Equations (A.223) and (A.224) provide the mathematical background of the Helmholtz decomposition for any vector field. Therefore the irrotational vector $\nabla\phi$ and the solenoidal vector $\nabla \times \underline{A}$ can be expressed in terms of dilatation and vorticity, respectively:

$$\nabla\phi = -\oint_S (\underline{n} \cdot \underline{u})\, \nabla G\, dS_y + \int_V \theta\, \nabla G\, dV_y \qquad \text{(A.225)}$$
$$\nabla \times \underline{A} = -\oint_S (\underline{n} \times \underline{u}) \times \nabla G\, dS_y + \int_V \underline{\omega} \times \nabla G\, dV_y \qquad \text{(A.226)}$$

Note that we have dropped the subscript y in ∇G for brevity, and hence it denotes the operator with respect to the integration variables $\underline{y}$. This result is the same as the expression of the vector identity Eqs. (A.216) and (A.217) derived in the previous subsection.

A.8.6 Uniqueness of Vector Identity

(1) To examine uniqueness of the solution as before, suppose that vectors $\underline{u}_1$ and $\underline{u}_2$ satisfy $\nabla \cdot \underline{u}_1 = \nabla \cdot \underline{u}_2$ and $\nabla \times \underline{u}_1 = \nabla \times \underline{u}_2$ in V. Then the difference vector $\underline{u}_3 = \underline{u}_1 - \underline{u}_2$ satisfies $\nabla \cdot \underline{u}_3 = 0$ and $\nabla \times \underline{u}_3 = 0$ in V.

(2) The condition that the curl and divergence of $\underline{u}_3$ are both zero is necessary

and sufficient to establish that $\underline{u}$ is the gradient of a scalar function P which satisfies Laplace's equation:

$$\underline{u}_3 = \nabla P \tag{A.227}$$
$$\nabla^2 P = 0 \tag{A.228}$$

(3) Green's first identity, Eq. (A.179),

$$\int_V \left[\psi \nabla^2 \phi + \nabla \psi \cdot \nabla \phi\right] \, dV = \oint_S \psi \, \underline{n} \cdot \nabla \phi \, dS \tag{A.229}$$

with $\psi = \phi = P$ reduces to

$$\int_V \underline{u}_3 \cdot \underline{u}_3 \, dV = \oint_S P \, \underline{n} \cdot \underline{u}_3 \, dS \tag{A.230}$$

(4) If the normal component of the two solution vector is specified equal on the boundary, then $\underline{n} \cdot \underline{u}_3 = 0$ on S and hence

$$\int_V \underline{u}_3 \cdot \underline{u}_3 \, dV = 0 \tag{A.231}$$

Since $\underline{u}_3 \cdot \underline{u}_3$ is always greater than or equal to zero, the only possible solution is

$$\underline{u}_3 = 0 \tag{A.232}$$

(5) When the boundary condition uniquely defines the normal component of the vector, Eq. (A.215) represents a unique representation of an arbitrary vector and no information need be given about the tangential component of the vector.

A.8.7 Improper Integrals

(1) Proper integrals in physics: a limit of a Riemann sum.

(2) Two general types of improper integrals:

(a) a range of integration that tends to infinity and

(b) integrands that are singular at points within the range of integration.

(3) Example of the improper integral,

$$\int_a^b f(x)\,dx = \lim_{a_1,b_1,c_1\to 0}\left[\int_{a+a_1}^{x_0-b_1} f(x)\,dx + \int_{x_0+c_1}^{b} f(x)\,dx\right] \quad \text{(A.233)}$$

(4) The improper integral $\int_1^\infty \frac{dx}{x}$:

$$\lim_{R\to\infty}\left[\int_1^R \frac{dx}{x}\right] = \lim_{R\to\infty}[\ln(R)] \to \infty \quad \text{(A.234)}$$

(5) The integrand of $\int_0^1 \frac{dx}{\sqrt{x}}$ is singular at $x = 0$, the integral is convergent improper:

$$\lim_{\epsilon\to 0}\int_\epsilon^1 \frac{dx}{\sqrt{x}} = \lim_{\epsilon\to 0}\left[2-\sqrt{\epsilon}\right] = 2 \quad \text{(A.235)}$$

(6) Principal Value Integrals
Define with some aspect of symmetry:

$$(P.V.)\int_{-\infty}^{\infty} f(x)\,dx \equiv \fint_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx \quad \text{(A.236)}$$

(7) Also at the point x_0 such that $\lim_{x\to x_0} f(x) \to \infty$,

$$\begin{aligned}(P.V.)\int_a^b f(x)\,dx &\equiv \fint_a^b f(x)\,dx \\ &= \lim_{\epsilon\to 0}\left[\int_a^{x_0-\epsilon} f(x)\,dx + \int_{x_0+\epsilon}^{b} f(x)\,dx\right]\end{aligned} \quad \text{(A.237)}$$

(8) Cauchy Principal Value Integral: A specific form with a well-behaved numerator and singular denominator.
In application, such integrals is derived for the case that a field point approaches the body surface.

A form of the general solution for the flow about a body is obtained with sources, sinks,dipoles or vortices distributed over the body surface.

A.8.8 Green Functions

When other surfaces can be included in the problem of the Laplace equation (more generally other partial differential equations, not necessarily the Laplace equation) that governs flow fields, additional boundary conditions are imposed. Then the Green function is often taken instead of the elementary function for computational advantage.

(1) Green fucntion is defined as an elementary singularity plus another non-singular component that satisfies Laplace equation as well as boundary contions on the other surfaces.

(2) What is left unsatisfied is boundary conditions on a body.

(3) Scalar (velocity potential) at $\underline{x}$ in terms of a distribution of elementary singularities $\psi = \dfrac{1}{|\underline{x} - \underline{y}|}$. When we add a function (say $H(\underline{x}, \underline{y})$) that also satisfies the Laplace equation and is not singular within the field to ψ, identity is unchanged except that we have a modified singularity element. It is necessary but not easy to find a function H with the following properties.

(4) If there were surfaces near a body, construct new singularity element $G(\underline{x}, \underline{y})$ with $\nabla^2 G = 0$ and such that

   (a) G satifies given boundary conditions on non-body surfaces

   (b) G contains elementary singularity element (say $\dfrac{1}{|\underline{x} - \underline{y}|}$) to give the field point value $\phi(\underline{x})$

   (c) G results in integral equation over only the body surface.

(5) The formulation is as follows

$$G(\underline{x}, \underline{y}) = \frac{1}{|\underline{x} - \underline{y}|} + H(\underline{x}, \underline{y}) \tag{A.238}$$

where $H(\underline{x}, \underline{y})$ is non-singular for all $\underline{x} \in V$, $\nabla^2 H = 0$ and

$$\phi\, \underline{n} \cdot \nabla G - G\, \underline{n} \cdot \nabla \phi = 0 \quad \text{on} \quad \mathrm{S} \neq \mathrm{S_B} \tag{A.239}$$

(6) For a simple example, if a wall is aligned with onset flow, H is image of elementary singularity.