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- 보 텍 스 방 법 -

COMPUTATIONAL MARINE HYDRODYNAMICS
-VORTEX METHODS-

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Suh, Jung-Chun

서 정 천

Seoul National Univ., Dept. NAOE

서울대학교 공과대학조선해양공학과

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BASIS OF FLUID FLOWS

2.1 Introduction

In this chapter, basic concepts and analysis for fluid flow are listed below and will not be presented in detail. Detailed and fundamental explanation can be found in some hydrodynamics texts.

2.1.1 Basic definitions

While solid can be in stable equilibrium under shear stress oblique to the surface separating any two parts, fluid cannot be in stationary equilibrium. [1] Resistance to rate of shear deformation from viscosity gives rise to drag for bodies. We can easily recognize that such shear stresses do exist in fluids: e.g., consider how the fluid in a rotating circular vessel takes on the rotating motion of the vessel eventually.
Other observed properties of fluids are:

(1) resistance to volumetric compression and tension in general,

(2) no shape or preferred orientation,

[1] For detailed information on difference and similarity among various fields in continuum mechanics and their historical background, see the article: 이승준 (1992), "재료역학과 고체역학: 유체역학자의 관점에서", *대한조선학회지*, 제29권, 제3호.

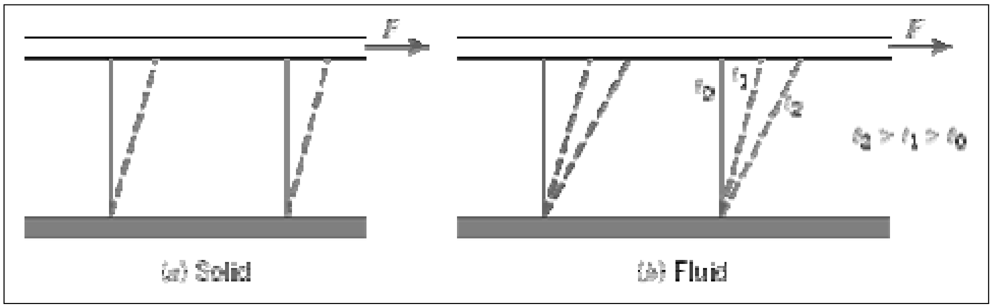

Figure 2.1 Behavior of a solid and a fluid, under the action of constant shear force. (left) solid; (right) fluid. From Fox et al. (2004).

(3) homogeneous matter in general, and

(4) has mass.

There are two kinds of fluids depending on bulk elasticity(compressibility): [2]

(1) liquid forms a free surface(density $\rho \approx 0$ above free surface), and

(2) gas expands to fill container.

With the principal types of fluid flow and their associated phenomena, it is possible to make up practically any flow combination in nature, even the complex system around a moving ship: potential flow pattern, viscosity of fluid, turbulent flow, separation of flow, cavitation, wave making, vortex motion and flow-induced sound.

Such flow phenomena can be characterized by several principal effects which constitute the basis for important relationships in the form of non-dimensional numbers: velocity effects, acceleration effects, force effects, inertia effects, gravity effects, viscosity, elastic effects, surface tension effects.

2.1.2 Assumptions and axioms

We assume that the fluid is continuous and homogeneous in structure. Actually this is not so since matter is ultimately made up of molecules and atoms, but in

[2] On the mechanism of formation of liquid and vapor, see Brennen, C. (1995), *Cavitation and Bubble Dynamics*, Oxford University Press, pp. 1–6.

many applications the dimensions we are concerned with are large compared to the molecular structure, and the smallest sample of fluid that concerns us contains a very great number of molecules(i.e., number of about $2.687 \times 10^7/\mu^3$). In such cases, the properties of any sample are the average values over many molecules, and the approximation of a continuum is found to be acceptable and useful.

Nevertheless, results obtained on the assumption of a continuum may be erroneous whenever the molecular structure dimensions are relatively large. For example, at very high altitudes (low pressures), the molecular spacing is so great that air is not even approximately a continuum in its contact with a body the size of an airplane wing. Failures of the continuum assumption occur probably in the cases of that body size compares with molecular dimensions (e.g., a very small body in a fluid) or with distances between molecules (e.g., a body in a rarefied gas).

Other acceptable and useful assumptions are those as follows:

(1) that physical laws are independent of the coordinate system used to express them (frame indifference).

(2) that natural laws are independent of the dimensions of physical quantities that occur in the expressions (dimensional homogeneity),

(3) that derivations of physical quantities with respect to space and time exist to the required order (smoothness of quantities), and

(4) that the present motion is a function of its history and not the future (memory of history).

Newton's laws of motion are derived from rigid body mechanics. Our use of these laws are based on continuum hypothesis. We postulate that mass, momentum and energy are conserved: Conservation of mass, Conservation of momentum, Conservation of energy. Since these notes tend to deal mostly with incompressible flows, we do not examine the conservation of energy.

2.1.3 Description of fluid motion

Although one of our assumption on a fluid is that it is a continuum and does not consist of discrete particles, we introduce the term "fluid particles," such as, "velocity of a particle," etc, to identify simply an infinitesimal portion or sample of the fluid by mathematically tagging it. There are two common ways of representing equations to describe a fluid flow.

2.1.3.1 Lagrangian description

We may take the tag to be the initial position, denoted by $\underline{\xi}(a, b, c)$. Let a, b, c denote the coordinates of any fluid particle at the time $t = 0$. Let x, y, z denote the coordinates of the same particle at time t. Then the flow geometry is completely specified if we know $x = x(a, b, c, t)$, $y = y(a, b, c, t)$, $z = z(a, b, c, t)$. These give the trajectories of various particles.

The pathline of a particle is the curve $\underline{x} = \underline{x}(\underline{\xi}, t)$, where $\underline{x}$ is the position vector. The velocity is $\underline{q}(a, b, c, t) = \partial\underline{x}/\partial t$ and the acceleration is $\partial\underline{q}/\partial t = \partial^2\underline{x}/\partial t^2$. Any other physical quantities would be given by a function, say, $f = f(a, b, c, t)$. This description is called Lagrangian, material, or convective description of motion.

2.1.3.2 Eulerian description

Instead of following individual particles as above, in Eulerian description we fix our attention on a point in space, x, y, z. Consider any property of the fluid, for example, the density ρ, and calculate its differential:

$$\rho = \rho(x, y, z, t) \tag{2.1}$$

$$d\rho = \frac{\partial\rho}{\partial x}\,dx + \frac{\partial\rho}{\partial y}\,dy + \frac{\partial\rho}{\partial z}\,dz + \frac{\partial\rho}{\partial t}\,dt = d\underline{\ell}\cdot\nabla\rho + \frac{\partial\rho}{\partial t}\,dt \tag{2.2}$$

For any given particle as it moves along, dx, dy, dz are not independent; in fact, $dx = u\,dt, dy = v\,dt$, and $dz = w\,dt$, i.e., $d\underline{\ell} = \underline{q}\,dt$, where $\underline{q}(x, y, z, t)$ is the velocity. Thus, the rate of change of the density of a particle is

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = \frac{\partial \rho}{\partial t} + \underline{q} \cdot \nabla \rho \tag{2.3}$$

2.1.4 Particle tracing lines

In the previous subsection, the material and spatial descriptions of the flow were described. Below we list some additional prerequisites.

(1) Local derivative
The time rate of change of a flow quantity at a fixed point $\underline{x}$ is given by

$$\left.\frac{\partial}{\partial t}\right|_{\underline{x}=\text{const}} \tag{2.4}$$

The flow is then called <u>steady</u> if the first term vanishes, that is, it does not vary with time.

(2) Material derivative
We use the symbol $\frac{D}{Dt}$ for this type of derivative, sometimes called the "convective or material derivative":

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \underline{q} \cdot \nabla \tag{2.5}$$

The time rate of change of a flow quantity following a particle is given by

$$\left.\frac{\partial}{\partial t}\right|_{\underline{\xi}=\text{const}} \equiv \frac{D}{Dt} \tag{2.6}$$

The velocity of a particle is the material derivative of the position vector of the particle:

$$\underline{q}^*(\underline{\xi}, t) = \left.\frac{\partial \underline{x}}{\partial t}\right|_{\underline{\xi}} = \frac{D\underline{x}}{Dt} = \underline{q}(\underline{x}, t) \tag{2.7}$$

This can be applied to any fluid property including vector properties. The acceleration of a particle, for example, is

$$\frac{D\underline{q}}{Dt} \equiv \frac{\partial \underline{q}}{\partial t} + \underline{q} \cdot \nabla \underline{q} \tag{2.8}$$

A similar description for the evolution of the material line element [3] is

$$\frac{D(d\underline{\xi})}{Dt} = d\underline{q} = dx_j \frac{\partial \underline{q}}{\partial x_j} = d\underline{x} \cdot \nabla \underline{q}. \tag{2.9}$$

If $F(\underline{x}, t)$ is some property of the flow field, then

$$\left.\frac{\partial F}{\partial t}\right|_{\underline{\xi}} = \left.\frac{\partial F}{\partial t}\right|_{\underline{x}} + \underline{q} \cdot \nabla F \tag{2.10}$$

(3) Streamlines

A streamline is defined as a line everywhere parallel to velocity $\underline{q}$. Namely, the tangent of the streamline at each point is parallel to the fluid velocity at that point. We can produce a streamline by taking a short time exposure picture of a flow for which numerous particles have been tagged. We try to trace out curves on the photograph such that each curve is tangent to the velocity vector at a point.

Let the fluid velocity be denoted by the vector $\underline{q}$; then $\underline{q} = \underline{q}(x, y, z, t) = (u, v, w)$. Differential equations for streamlines are

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}. \tag{2.11}$$

If $\underline{x}(\sigma)$ (σ parameter) describes the position vector of a streamline, then $\dfrac{d\underline{x}}{d\sigma}$ is tangent to a streamline and parallel to the velocity at $\underline{x}(\sigma)$. Hence we can express the differential equation for streamlines in terms of the parameter σ:

$$\frac{d\underline{x}}{d\sigma} \times \underline{q}(\underline{x}(\sigma), t) = 0, \quad \text{or} \quad \frac{d\underline{x}}{d\sigma} \propto \underline{q}(\underline{x}(\sigma), t) \tag{2.12}$$

[3] A material line is a line composed of the same fluid particles in a moving fluid. Similarly a material surface and a material volume are, respectively, a surface and a volume composed of the same particles. A material surface may be a bounding surface and every impenetrable bounding surface must be a materialsurface.

(4) Streaklines

At time t, a streakline through a fixed point $\underline{y}$ is the curve traced out by particles each of which have gone through $\underline{y}$ since time $t_0 < t$. (Typically $t_0 = 0$.) Physically we construct a streakline by making (or tagging) all particles that pass a point, e.g., by continuously emitting dye at that point. The dye trail marks the streakline.

A particle is on the streakline at time of observation t if it had been at $\underline{y}$ at time s where s lies in the inteval $t_0 \le s \le t$. The material coordinates fo the particle that went through $\underline{y}$ at s are $\underline{\xi} = \underline{\xi}(\underline{y}, t)$. At time t, the particle is at the spatial position

$$\underline{x} = \underline{x}\left(\underline{\xi}(\underline{y}, s), t\right) \tag{2.13}$$

where $\underline{y}$ and t are to be assigned and s varies from t_0 to t to trace out the streakline.

For steady flows, a pathline, a streamline and a streakline coincide.

2.1.4.1 Example of particle tracing lines

(1) Velocity field

The concepts of various flow lines may be illustrated by the 2-D case for which the particle velocity is considered to be

$$\underline{q}^*(\underline{\xi}, t) = \xi_1\, \underline{i} + \xi_2\, e^t\, \underline{j} \tag{2.14}$$

This means that at the initial time $t_0 = 0$ the particle velocity is equal to the position vector: $\underline{q}^*(\underline{\xi}, 0) = \underline{\xi}$, and as time proceeds from $t = 0$, the horizontal component of the velocity remains unchanged but the vertical velocity component grows exponentially with time.

(2) Pathlines

The pathline of the particle that was initially at $\underline{\xi}$ is the curve

$$\underline{x} = \underline{\xi} + \int_0^t \underline{q}^*(\underline{\xi}, t)\, dt = \xi_1(1+t)\, \underline{i} + \xi_2\, e^t\, \underline{j} \tag{2.15}$$

Spatial coordinates and material coordinates can be related:

$$x_1 = \xi_1\,(1+t), \qquad x_2 = \xi_2\,e^t \tag{2.16}$$

This is the parametric representation of the pathline. Eliminate the parameter t from the equation to find the pathline in the (x_1, x_2) plane:

$$x_2 = \xi_2\,e^{(x_1/\xi_1 - 1)} \tag{2.17}$$

The inverse of the pathline is the relation obtained by solving for $\underline{\xi}(\underline{x}, t)$

$$\underline{\xi} = \frac{x_1}{(1+t)}\,\underline{i} + \frac{x_2}{e^t}\,\underline{j} \tag{2.18}$$

With the inverse of the pathlines known, the spatial description of the velocity vector can be constructed:

$$\begin{aligned} \underline{q}(\underline{x}, t) &= \underline{q}^*(\underline{\xi}(x, t), t) \\ &= \frac{x_1}{(1+t)}\,\underline{i} + \frac{x_2}{e^t}\,e^t\,\underline{j} \\ &= \frac{x_1}{(1+t)}\,\underline{i} + x_2\,\underline{j} \end{aligned} \tag{2.19}$$

If the spatial description of the velocity vector were given, the differential equation of the particle pathline would be

$$\frac{\partial \underline{x}\left(\underline{\xi}, t\right)}{\partial t} = \underline{q}\left(\underline{x}(\underline{\xi}, t), t\right) \tag{2.20}$$

and, if solved, would give the same expressions as above.

(3) Streamlines
We can also use the spatial description of the velocity field to the find the position vector of a streamline, $\underline{x}(\sigma, t)$:

$$\begin{aligned} \left.\frac{\partial \underline{x}}{\partial \sigma}\right|_t &= \underline{q}\left(\underline{x}(\sigma), t)\right) \\ &= \frac{x_1(\sigma)}{(1+t)}\,\underline{i} + x_2(\sigma)\,\underline{j} \end{aligned} \tag{2.21}$$

From which we obtain

$$x_1(\sigma) = c_1\, e^{\frac{\sigma}{(1+t)}} \tag{2.22}$$

$$x_2 = c_2\, e^{\sigma} \tag{2.23}$$

If we eliminate the parameter σ from these two equations, then in the (x_1, x_2) plane the streamlines are the curves:

$$x_2 = c_2\,(x_1/c_1)^{(1+t)} \tag{2.24}$$

Note that $x_2 = k\,x_1$ at $t = 0$.

(4) Streaklines

The streaklines are determined by finding the material coordinates of a particle that was a spatial position $\underline{y}$ at some time s. We use the inverse relations for the pathline to define the relationship:

$$\underline{\xi} = \frac{y_1}{(1+s)}\,\underline{i} + \frac{y_2}{e^s}\,\underline{j} \tag{2.25}$$

Hence the streakline is

$$\underline{x}(s) = \frac{y_1}{(1+s)}(1+t)\,\underline{i} + \frac{y_2}{e^s}\,e^t\,\underline{j} \tag{2.26}$$

At $s = t$, these relations give $\underline{x} = \underline{y}$, so that is the location of the particle just passing through the spatial point $\underline{y}$. At $s = 0$, the particle that was previously at $\underline{y}$ for $t = 0$ is to be found. To find the streakline definition for any time, we solve the i component for the relationship between s and the other variables:

$$s = \left(\frac{y_1}{x_1}\right)(1+t) - 1 \tag{2.27}$$

and from the second equation:

$$x_2 = y_2\, e^{t-(1+t)(y_1/x_1)+1} \tag{2.28}$$

Thus for any particular time t this equation gives the equation of the streakline through the point $\underline{y}$.

Typical flow patterns are illustrated in Figure 2.2 .

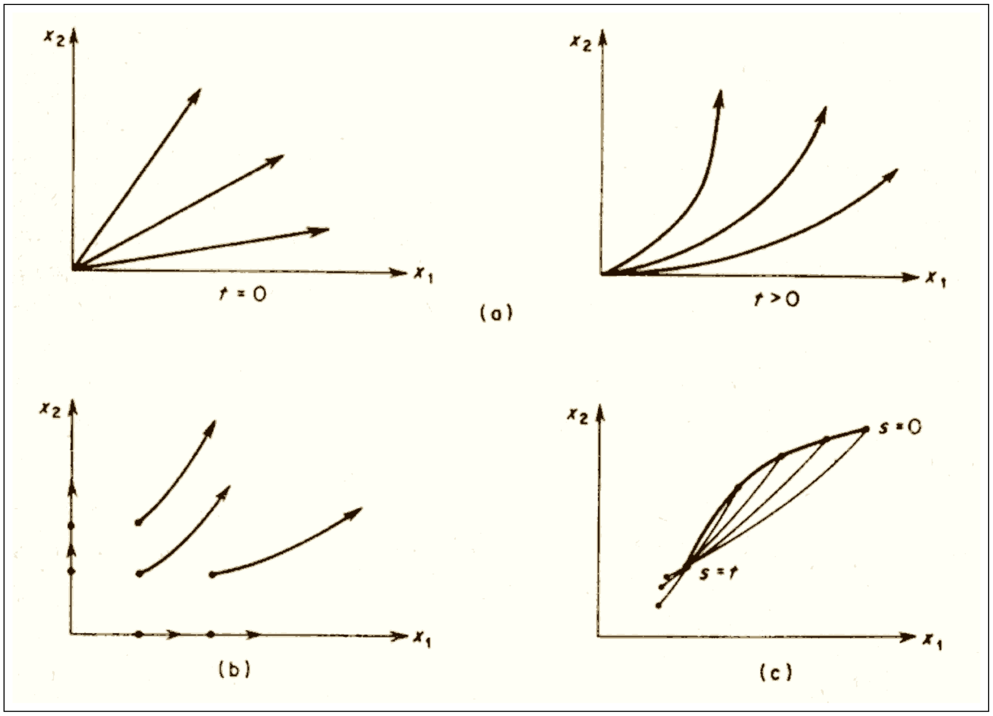

Figure 2.2 Example of various flow lines. (a) streamlines at $t = 0$ and $t > 0$; (b) path lines; (c) streakline. From Aris (1962), p. 82.

2.2 Kinematics

2.2.1 Continuity

Consider an arbitrary volume $V(t)$ enclosed in a material surface $S(t)$. A material surface is always composed of the same fluid particles. As the volume moves through space it experiences deformation although the mass within the volume remains constant. The mass enclosed within $V(t)$ is given by, in an integral form for density ρ,

$$\int_{V(t)} \rho \, dV \tag{2.29}$$

where the integration is over the region of space occupied by V at time t. Since the mass of the material volume is constant, the time derivation of this expression is zero:

$$\frac{d}{dt}\int_{V(t)} \rho \, dV = 0 \tag{2.30}$$

Using the (Reynolds) transport theorem, one obtains

$$\int_{V(t)} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \, \underline{q})\right] dV = 0. \tag{2.31}$$

This is the integral form of the continuity equation. Since the volume taken is arbitrary, the integrand must be zero at all points within V:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \, \underline{q}) = 0. \tag{2.32}$$

This is the spatial or Eulerian description of the continuity equation. The above derivation of the continuity equation was from the system analysis point of view for which the mass within a deformable bounding surface is constant. Meanwhile, it is common to also use control volume analysis, for which one consider an arbitrary fixed volume V enclosed in a surface S. Let $\underline{n}$ be the outward unit normal vector. The mass of fluid in V is $\int_V \rho \, dV = m$, say. If m increases it means that fluid has entered through S:

$$\frac{dm}{dt} = -\int_S \rho \, \underline{n} \cdot \underline{q} \, dS \tag{2.33}$$

and by the "divergence theorem", this surface integral is equal to

$$-\int_V \nabla \cdot (\rho \, \underline{q}) \, dV, \tag{2.34}$$

V being a fixed volume, we can write

$$\frac{dm}{dt} = \int_V \frac{\partial \rho}{\partial t} \, dV \tag{2.35}$$

Hence, for arbitrary choice of V, we have

$$\int_V \frac{\partial \rho}{\partial t} \, dV = -\int_V \nabla \cdot (\rho \, \underline{q}) \, dV. \tag{2.36}$$

The only way that these integrals can be equal for any and every choice of V is that their integrands be equal; thus we obtain the *General Equation of Continuity*:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\,\underline{q}) = 0 \tag{2.37}$$

Noting that $\nabla\cdot(\rho\,\underline{q}) = \underline{q}\cdot\nabla\rho + \rho\nabla\cdot\underline{q}$, this equation can be expressed, in an alternative form, as

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\underline{q} = 0 \tag{2.38}$$

There are two important special cases:

(1) Steady motion
Since, for steady motion, all partial derivatives $\partial(\ )/\partial t$ vanish, Eq. (2.37) becomes

$$\nabla\cdot(\rho\,\underline{q}) = 0 \tag{2.39}$$

(2) Incompressible flow
If the density of every particle is constant, $D\rho/Dt = 0$, and Eq. (2.38) gives us

$$\nabla\cdot\underline{q} = 0 \tag{2.40}$$

Vector fields with this property are called solenoidal. Most of our work will deal with incompressible fluid. It is to be noted that this is correct whether the fluid is steady or not, and moreover it applies to the case of an inhomogeneous fluid, such as a stratified liquid, in which ρ varies throughout the fluid, provided each particle is incompressible.

2.2.2 Vorticity, circulation, and velocity potential

2.2.2.1 Vorticity

The vector function $\nabla\times\underline{q}$, where $\underline{q}(x, y, z, t)$ is the velocity of the fluid, is called the vorticity. Its components are occasionally represented by the symbols

ξ, η, ζ; namely, in rectangular Cartesian coordinates

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \quad \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \quad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{2.41}$$

To give a physical feature of the meaning of vorticity, it is often said that $\nabla \times \underline{q}$ is twice the angular-velocity vector of the fluid particle. Since the particle is being deformed continually, perhaps we should say the average angular velocity at a point.

2.2.2.2 Vortex line and vortex tube

A vortex line is a curve which is tangent at each point to the vorticity at the point. It is analogous to the stream line. Its differential equation is $dx/\xi = dy/\eta = dz/\zeta$ where the Cartesian component of $\underline{\omega}$ are ξ, η, ζ.

Since the divergence of any curl of a vector must be zero, a continuity equation $\nabla \cdot \underline{\omega}$ for $\underline{\omega}$ must be invoked especially in the case that the vorticity field is itself to be sought with independence of the velocity field. The condition $\nabla \cdot \underline{\omega} = 0$ can be thought of as meaning that vortex lines do not begin nor end in the fluid. We call a tube whose walls are made up of vortex lines a vortex tube. (The analogous tube made up of streamlines would be called a stream tube.)

2.2.2.3 Circulation and vorticity flux

We classify flows as irrotational and rotational, depending on whether $\nabla \times \underline{q}$ is or is not everywhere zero. The irrotational type will be found to be rather common, for sound physical reasons, and will occupy a considerable portion of our time.

The line integral

$$\Gamma = \oint_C \underline{q} \cdot d\underline{\ell} \tag{2.42}$$

where $\underline{q}$ is the fluid velocity, taken about any closed curve C in space, is called the circulation about the contour C.

By Stokes theorem, it is clear that the circulation and vorticity are related, for

$$\Gamma = \oint_C \underline{q} \cdot d\underline{\ell} = \int_S \underline{n} \cdot (\nabla \times \underline{q})\, dS = \int_S \underline{n} \cdot \underline{\omega}\, dS \tag{2.43}$$

The transformation is only permissible, of course, when $\underline{q}$ is finite and has continuous partial derivatives at each point of S; we may encounter some cases where certain singularities have to be excluded from such processes.

Obviously, if the flow is wholly irrotational, Γ will be zero for every contour. In any case, Γ is zero if C encloses only irrotational portions of the flow.

2.2.2.4 Vortex strength

Again consider the application of Stokes theorem to a cross-section of the vortex tube:

$$\int_\Sigma \underline{n} \cdot \underline{\omega}\, dS = \Gamma = \text{constant along tube} \tag{2.44}$$

Thus the average vorticity in the cross-section varies inversely as the cross-sectional area. The vorticity becomes very small if the tube spreads out. This is the result of viscosity, for example; the vorticity is dissipated over a wide region.

Suppose, on the other hand, that the tube is necked down; this makes the vorticity large. In the extreme case, we imagine that the tube is contracted to a line. Then the vorticity at this line becomes infinite, but the circulation is still the same, Γ. This is called a <u>vortex</u> <u>filament</u>, or briefly a "vortex", and Γ is its <u>strength</u>.

It is a kind of mathematical approximation to the case where all the vorticity is confined to a tube of relatively small cross-section, as often occurs in nature – for example in a tornado. Outside the core of a tornado, the air is in practically irrotational motion.

The irrotational concentric flow represents the case of a long, straight vortex filament; the singularity at the center is the filament, and there the vorticity is infinite, as predicted. Clearly, a vortex tube or filament, consisting of vortex lines, cannot begin nor end in the fluid. It can double back on itself in a ring or terminate at a boundary of the fluid.

2.2.2.5 Velocity potential

In regions where the flow is irrotational, the line integral around an entire closed path is the circulation and is zero because the flow is irrotational. This implies that the open line integral $\int \underline{q} \cdot d\underline{\ell}$ is independent of the path within the regions, but only dependent of the end points of the path. Therefore, choosing A as a fixed point and B as a varying point,

$$\int_{A}^{B(x,y,z)} \underline{q} \cdot d\underline{\ell} = \phi(x, y, z), \tag{2.45}$$

and

$$d\phi = \underline{q} \cdot d\underline{\ell} \tag{2.46}$$

Now we see that $d\phi = d\underline{\ell} \cdot \nabla\phi$ from Eq. (1.61) and hence $d\underline{\ell} \cdot \nabla\phi = \underline{q} \cdot d\underline{\ell}$ for arbitrary choice of $d\underline{\ell}$. This means that

$$\underline{q} = \nabla\phi \tag{2.47}$$

By retracing these steps you will see immediately that this result has nothing to do with the physical meaning of $\underline{q}$. That is, the result Eq. (2.47) will follow for every vector function $\underline{q}$ whose curl is zero. Moreover, the condition $\nabla \times \underline{q} = 0$ is necessary, as well as sufficient, for the result $\underline{q} = \nabla\phi$, because the curl of every gradient is identically zero.

In the case considered here, where $\underline{q}(\underline{x}, t)$ is the fluid velocity, $\phi(\underline{x}, t)$ is called the velocity potential. The surfaces $\phi =$ constant are called *equipotential* surface; thus $\underline{q}$ is the vector perpendicular to these surfaces at every point, and its magnitude is that of derivative $\partial\phi/\partial n$ in the normal direction. These statements are verified by using the relation $d\phi = d\underline{\ell} \cdot \nabla\phi$.

2.2.3 Helmholtz decomposition of a velocity field

The Helmholtz decomposition theorem states that an arbitrary continuously differentiable velocity field can be represented as a combination of solenoidal and irrotational velocity field. [4] Thus for any finite continous velocity field which vanishes at infinity we may find a scalar function (velocity potential) ϕ and a vector potential function (vector stream function) $\underline{A}$ such that

$$\underline{q} = \nabla\phi + \nabla \times \underline{A} \tag{2.48}$$

To prove this decomposition, we first need the solution of Possion's equation

$$\nabla^2\phi = f(\underline{x}) \tag{2.49}$$

where $f(\underline{x}) = \nabla \cdot \underline{q}$. We can consider an unsteady velocity field, if necessary, just by adding the time variable.
The solution is provided by the integral [5]

$$\phi(\underline{x}) = -\frac{1}{4\pi}\int_V \frac{f(\underline{\xi})}{r}\, dV_\xi \tag{2.50}$$

where $r = |\underline{x} - \underline{\xi}|$ is the distance from the volumetric element dV_ξ to the point x, y, z. The integration is carried throughout the entire fluid. If f is only defined in a certain region, we may set it equal to zero outside, and if it is defined everywhere we require that it should tend to zero towards infinity.

Now let us return the Helmholtz decomposition form. We have started with the equation $\nabla \cdot \underline{q} = \nabla^2\phi$ by taking the divegence of the orginal decomposion form, and have derived the solution given by, again,

$$\phi(\underline{x}) = -\frac{1}{4\pi}\int_V \frac{\nabla \cdot \underline{q}}{r}\, dV_\xi \tag{2.51}$$

Consequently, $\underline{q} - \nabla\phi$ is a solenoidal since $\nabla \cdot (\underline{q} - \nabla\phi) = 0$ from $\nabla \cdot \underline{q} = \nabla^2\phi$.

[4] See Aris, R. (1962), *Vectors, Tensors and the Basic Equations of Fluid Mechanics*, Prentice Hall, p. 70.
[5] See Batchelor, G. K. (1967), *An Introduction to Fluid Dynamics*, Cambridge University Press, Cambridge.

Hence we can construct a vector potential function $\underline{A}$ such that

$$\underline{q} - \nabla\phi = \nabla \times \underline{A} \tag{2.52}$$

which reaches to the Helmholtz decomposition form. In the next subsection, we will deal with the vector potential $\underline{A}$.

Proof of Eq. (2.50)

Consider the gradient of ϕ with respect to $\underline{x}$

$$\nabla_x\phi = +\frac{1}{4\pi}\int_V \frac{f(\underline{\xi})}{r^3}(\underline{x} - \underline{\xi})\, dV_\xi \tag{2.53}$$

Now take integral of $\nabla^2\phi$, where ϕ is given by Eq. (2.50), over an arbitrary volume V enclosed in a closed surface S, and use the divergence theorem:

$$\begin{aligned}
\int_V \nabla^2\phi\, dV_x &= \oint_S \underline{n}\cdot\nabla\phi\, dS_x \\
&= \oint_S \underline{n}\cdot\left\{\frac{1}{4\pi}\int_V \frac{f(\underline{\xi})}{r^3}(\underline{x}-\underline{\xi})\, dV_\xi\right\} dS_x \\
&= \int_V f(\underline{\xi})\, dV_\xi \oint_S \frac{\underline{n}\cdot(\underline{x}-\underline{\xi})}{4\pi r^3}\, dS_x
\end{aligned} \tag{2.54}$$

Here we have changed the order of integration, since $f(\underline{\xi})$ is the value at the element dV_ξ and is therefore independent of the integration over S. Since $\dfrac{(\underline{x}-\underline{\xi})}{r}$ is a unit vector, $\dfrac{\underline{n}\cdot(\underline{x}-\underline{\xi})\, dS}{r^3}$ is just the solid angle subtended at the point $\underline{\xi}$ by the surface element at the integration point $\underline{x}$. (See Figure **2.3** .)

Now if $\underline{\xi}$ is outside of the volumetric region V, the integral of this solid angle is zero because the contribution from the surface element at $\underline{x}$ is equal and opposite to one from the surface element that is projected extensively on the opposite side of the closed surface. However if $\underline{\xi}$ is inside of the volumetric region V, then the integral is the total solid angle for the closed surface which is equal to 4π.

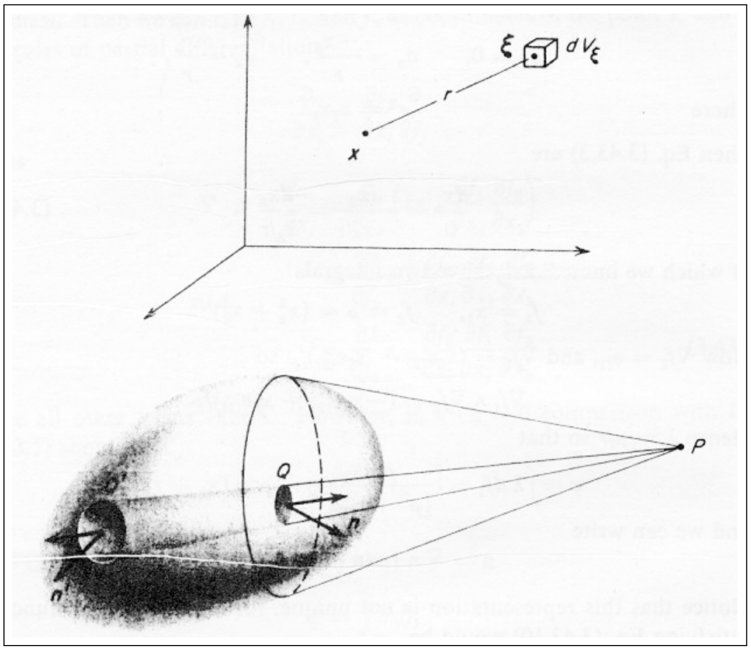

Figure **2.3** Integration region for Poisson's solution of vector fields. From Aris (1962), p. 70.

It follows that

$$\oint_S \frac{\underline{n} \cdot (\underline{x} - \underline{\xi})}{4\pi r^3}\, dS_x = \begin{cases} 1 & \text{if } \underline{\xi} \text{ is inside } V \\ 0 & \text{if } \underline{\xi} \text{ is ouside } V \end{cases} \tag{2.55}$$

Thus the last integral over the whole space of $\underline{\xi}$ has zero integrand outside of V and so may be regarded as the ontegral over V only. Then

$$\int_V \nabla^2 \phi \, dV_x = \int_V f(\underline{x})\, dV_x \tag{2.56}$$

Since the volume V was arbitrary, this equation reduces to the Poisson's equation (Eq. (2.49)), and Eq. (2.50) gives its solution.

■■■

2.2.4 Velocity field of a vortex: Biot-Savart integral

In order to determine the velocity field of a vortex in an incompressible fluid, we begin with a more general case of rotational flow and later specialize for a vortex filament. Consider incompressible rotational flow in general. The velocity potential does not exist, but, as will be seen later, it may be possible to determine a *vector-potential function* $\underline{A}(x, y, z, t)$, such that

$$\underline{q} = \nabla \times \underline{A} \tag{2.57}$$

This form has the advantage of satisfying the incompressible equation of continuity identically, for the divergence of every curl is zero. Thus $\underline{A}$ is related to a stream function.

Now we shall try to determine $\underline{A}(x, y, z, t)$ for any given distribution of vorticity $\underline{\omega}(x, y, z, t)$, for then we shall have $\underline{q}(x, y, z, t)$ in terms of the vorticity—a sort of inverse of the relation $\underline{\omega} = \nabla \times \underline{q}$. The relation between $\underline{\omega}$ and $\underline{A}$ is, using Eq. (1.72)

$$\underline{\omega} = \nabla \times (\nabla \times \underline{A}) = \nabla(\nabla \cdot \underline{A}) - \nabla^2 \underline{A} \tag{2.58}$$

This is a differential equation for $\underline{A}$, for given $\underline{\omega}$, and our aim is to obtain a particular integral. We can now assume that $\nabla \cdot \underline{A} = 0$; this does not sacrifice any generality, for we are trying to calculate $\underline{A}$ for given $\underline{\omega}$. If we can succeed in calculating it with this restriction, the problem will be solved. However, we shall have to check our result to verify that the divergence vanishes. With this assumption,

$$\underline{\omega} = -\nabla^2 \underline{A} \tag{2.59}$$

Now Eq. (2.59) is Poisson's equation, and its solution is [6]

$$\underline{A}(x, y, z, t) = \frac{1}{4\pi} \int_V \frac{\underline{\omega}}{r} \, dV \tag{2.60}$$

where $r = |\underline{x} - \underline{\xi}|$ is the distance from the element dV to the point x, y, z, and the integration is carried throughout the entire fluid.

Consequently, the velocity induced by the vorticity distribution is given by

$$\underline{q}(\underline{x}, t) = \nabla \times \underline{A} = \frac{1}{4\pi} \int_V \underline{\omega} \times \frac{(\underline{x} - \underline{\xi})}{|\underline{x} - \underline{\xi}|^3} \, dV \tag{2.61}$$

where the integration variable is $\underline{\xi}$.

[6] See Batchelor, G. K. (1967), *An Introduction to Fluid Dynamics*, Cambridge University Press, Cambridge.

Equation (2.61) is called the *Biot-Savart integral*, by analogy with the expression for the magnetic flux due to a conductor carrying a current. This analogy also leads to the name ‘induced velocity’ for $\underline{q}$.

To illustrate the use of this result Eq. (2.60), let us calculate the velocity in the field of a vortex filament. That is, let us assume that the vorticity is concentrated in a tube of very small cross-sectional area δS and circulation $\Gamma = \omega\,\delta S$. Then

$$\underline{A} = \frac{1}{4\pi}\int_C \frac{\underline{\omega}}{r}\,\delta S\,d\ell = \frac{1}{4\pi}\int_C \frac{\Gamma\,\underline{e}_\ell}{r}\,d\ell = \frac{\Gamma}{4\pi}\int_C \frac{d\underline{\ell}}{r} \tag{2.62}$$

where $d\ell$ is an element of length along the filament, $\underline{e}_\ell$ is a unit vector in the direction of the filament, and $d\underline{\ell}$ denotes $\underline{e}_\ell d\ell$. The velocity at $P(x, y, z)$, due to the particular element $d\ell$ is

$$d\underline{q} = \frac{\Gamma}{4\pi}\nabla\times\left(\frac{d\underline{\ell}}{r}\right) = \frac{\Gamma}{4\pi}\nabla\left(\frac{1}{r}\right)\times d\underline{\ell} = -\frac{\Gamma}{4\pi}\frac{\underline{r}\times d\underline{\ell}}{r^3} \tag{2.63}$$

In other words, the velocity due to the filament $d\underline{\ell}$ is directed normal to the plane of $d\underline{\ell}$ and $\underline{r}$, and its magnitude is

$$dq = \frac{\Gamma}{4\pi r^2}\sin\theta\,d\ell \tag{2.64}$$

where θ is defined by the angle between $d\underline{\ell}$ and $\underline{r}$.

Proof of Eq. (2.60)

Consider the integral of $\nabla^2\underline{A}$, where $\underline{A}$ is given by Eq. (2.60), through an arbitrary volume V enclosed in a surface S:

$$\int_V \nabla^2\underline{A}\,dV = \oint_S \underline{n}\cdot\nabla\underline{A}\,dS = \oint_S \underline{n}\cdot\nabla\left(\frac{1}{4\pi}\int_V \frac{\underline{\omega}}{r}\,dV\right)dS \tag{2.65}$$

Since $\underline{\omega}$ is the value at the element dV and is therefore independent of the integration over S, the contribution of the element dV to this integral is

$$\frac{1}{4\pi}\underline{\omega}\,dV\oint_S \underline{n}\cdot\nabla\left(\frac{1}{r}\right)dS = -\frac{1}{4\pi}\underline{\omega}\,dV\oint_S \frac{\underline{n}\cdot\underline{r}}{r^3}\,dS \tag{2.66}$$

The last integral is either zero or 4π, depending on whether dV is inside or outside S, according to the divergence theorem. Thus the contribution of dV is zero if $dV$is outside V and is $-\underline{\omega}\, dV$ if dV is within V. Consequently, when the integration is taken throughout the fluid, the result is

$$\int_V \nabla^2 \underline{A}\, dV = -\int_V \underline{\omega}\, dV \tag{2.67}$$

and since V is arbitrary, the integrands must be equal.

■■■

Proof of divergence-free $\nabla \cdot \underline{A} = 0$

Next, we must prove that Eq. (2.60) is divergence-free $\nabla \cdot \underline{A} = 0$. Equation (2.60) makes $\underline{A}$ a solution of Eq. (2.59), but not a solution of the differential equation we are trying to solve, Eq. (2.58), unless $\nabla \cdot \underline{A}$ is zero, as has already been mentioned. Therefore take the divergence of Eq. (2.60):

$$\nabla \cdot \underline{A} = \frac{1}{4\pi} \int_V \underline{\omega} \cdot \nabla \left(\frac{1}{r}\right) dV \tag{2.68}$$

But, using an obvious notation, $r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$ and $\nabla_x \left(\frac{1}{r}\right) = -\nabla_\xi \left(\frac{1}{r}\right)$, where ∇_x denotes $\underline{i}\,\frac{\partial}{\partial x} + \cdots$ etc. and ∇_ξ denotes $\underline{i}\,\frac{\partial}{\partial \xi} + \cdots$ etc. Moreover,

$$\underline{\omega} \cdot \nabla_\xi \left(\frac{1}{r}\right) = \nabla_\xi \cdot \left(\frac{\underline{\omega}}{r}\right) - \frac{1}{r} \nabla_\xi \cdot \underline{\omega} = \nabla_\xi \cdot \left(\frac{\underline{\omega}}{r}\right) \tag{2.69}$$

Thus, the integral in Eq. (2.68) can be changed to a surface integral of $\underline{n}\cdot\underline{\omega}/r$, by the divergence theorem, and the surface integrated over is the surface enclosing all the areas of rotational flow. But this will be the outer walls of vortex tubes, and on these $\underline{n} \cdot \underline{\omega} = 0$. Hence the divergence is zero as required. [7]

■■■

[7] See Lamb, H. (1932), *Hydrodynamics*, Sixth Ed., Dover.

2.3 Dynamics

2.3.1 Forces

Two types of net forces act in flow problems: (1) body forces and (2) surface forces.

2.3.1.1 Body forces

From external source the body force acts throughout volume from afar (e.g., gravity, magnetic attraction). It is convenient to define the net body force as

$$\underline{F}_b(t) = \int_V \underline{F}_B(\underline{x}, t)\, dV \tag{2.70}$$

where $\underline{F}_B(\underline{x}, t)$ is the body force per unit volume acting at a point $\underline{x}$. For gravity $\underline{F}_B = -\rho\, g\, \underline{e}_3$ where $\underline{e}_3$ is the unit vector directed along the upward vertical. Often the body force is defined as the body force per unit mass:

$$\underline{F}_b(t) = \int_V \rho\, \underline{f}(\underline{x}, t)\, dV \tag{2.71}$$

e.g., for gravity $\underline{f}(\underline{x}, t) = -g\, \underline{e}_3$. The torque due to the body force about the spatial point $\underline{x}_0$ is

$$\underline{Q}(t) = \int_V (\underline{x} - \underline{x}_0) \times \underline{F}_B(\underline{x}, t)\, dV \tag{2.72}$$

We will consider only conservative body forces, for which the body force is derived from a scalar potential

$$\underline{F}_B(\underline{x}, t) = -\nabla\Omega(\underline{x}, t) \tag{2.73}$$

For the body force due to gravity, $\Omega = \rho\, g\, x_3$.

2.3.1.2 Surface forces

Internal sources that cancel except at bounding surfaces that have no continuation volume provide an equal but oppositely directed force. Surface force is defined in terms of a stress distribution on the bounding surface

$$\underline{F}_S = \oint_S \underline{\tau}(\underline{x}_s, t)\, dS \tag{2.74}$$

where S bounds V, $\underline{\tau}(\underline{x}_s, t)$ is the stress vector at a point $\underline{x}_s$ on the surface S, with three components. The torque about a field point $\underline{x}_0$ due to the surface force is

$$\underline{Q}(t) = \oint_S (\underline{x}_s - \underline{x}_0) \times \underline{\tau}(\underline{x}_s, t)\, dS \tag{2.75}$$

2.3.1.3 Stress and stress tensor

The stress vector $\underline{\tau}(\underline{x}_s, t)$ is associated with a normal vector to the surface upon which it acts in the sense that if the stresses were in local equilibrium (i.e, no acceleration or other surface forces act), the stress on the one side of a surface(the side denoted by the normal) is equal and oppositely directed as that on the other side (see Figure **2.4**).

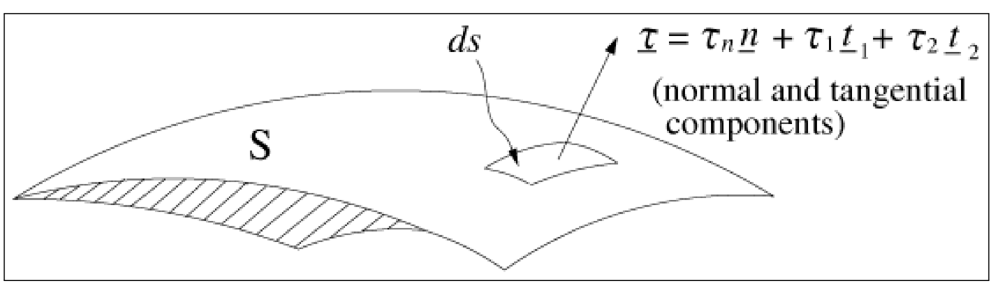

Figure 2.4 Stress vector at surface.

Let $\underline{n}$ be the normal pointing out of the volume(exterior normal), then $-\underline{n}$ is pointing into the volume(the interior normal), and $\underline{\tau}_{(\underline{n})} = -\underline{\tau}_{(-\underline{n})}$. This fact leads one to an expressing for the local stress vector as the dot product of the normal(so the equal and opposite property is satisfied) and a dyadic quantity called the stress tensor of 2nd order

$$\underline{\tau} = \underline{n} \cdot \underline{\underline{\tau}} \tag{2.76}$$

where $\underline{\underline{\tau}}$ is the stress tensor represented with 3 by 3 elements. [8] In three-dimensional Euclidian space, we can write the stress tensor in the dyadic form

$$\underline{\underline{\tau}} = \sum_{i=1}^{3} \sum_{j=1}^{3} \tau_{ij} \; \underline{e}_i \; \underline{e}_j \tag{2.77}$$

where $\underline{e}_i$ and $\underline{e}_j$ are unit base vectors and the scalars τ_{ij} are the physical components of the tensor. Our expression has been with Cartesian coordinates but the same concepts apply to other curvilinear coordinate systems just as well, and definition of the stress tensor components in a system compatible with the geometry is desired.

A fluid is defined as a material that cannot be in stationary equilibrium with applied shear stress. A Newtonian fluid has a resistance to shear deformation that is linearly proportional to the rate of deformation(i.e., proportional to the gradient of velocity), while an ideal(perfect) fluid has no resistance to shear deformation. This linearity can be applied to elastic solids that follow Hooke's law.

Four motions of a fluid particle element are possible: (i) translation, (ii) rotation, (iii) volumetric change, and (iv) squeeze motion. Among them, (iii) and (iv) cause stress in a fluid, while (i) and (ii) represent only rigid body motion for which there will be no stress developed. If there were no deformation(also including rigid body motion), then only a static pressure acts normal to the surface of the volume of interest. The stress vector is simply $\underline{\tau} = -p \; \underline{n}$. Thus the stress tensor $\underline{\underline{\mathbf{T}}} = -p \; \underline{\underline{\mathbf{I}}} = -p \; \delta_{ij}$. The pressure diagram of a fluid is shown in Figure 2.5 .

Combining the above statements leads one to, for a Newtonian fluid,

$$\underline{\underline{\tau}} \equiv \tau_{ij} = -p \, \underline{\underline{\mathbf{I}}} + \mu \left[\nabla \underline{q} + (\nabla \underline{q})^T \right] \tag{2.78}$$

[8] Tensors of second order are denoted by using the double under bar $\underline{\underline{\cdot}}$.

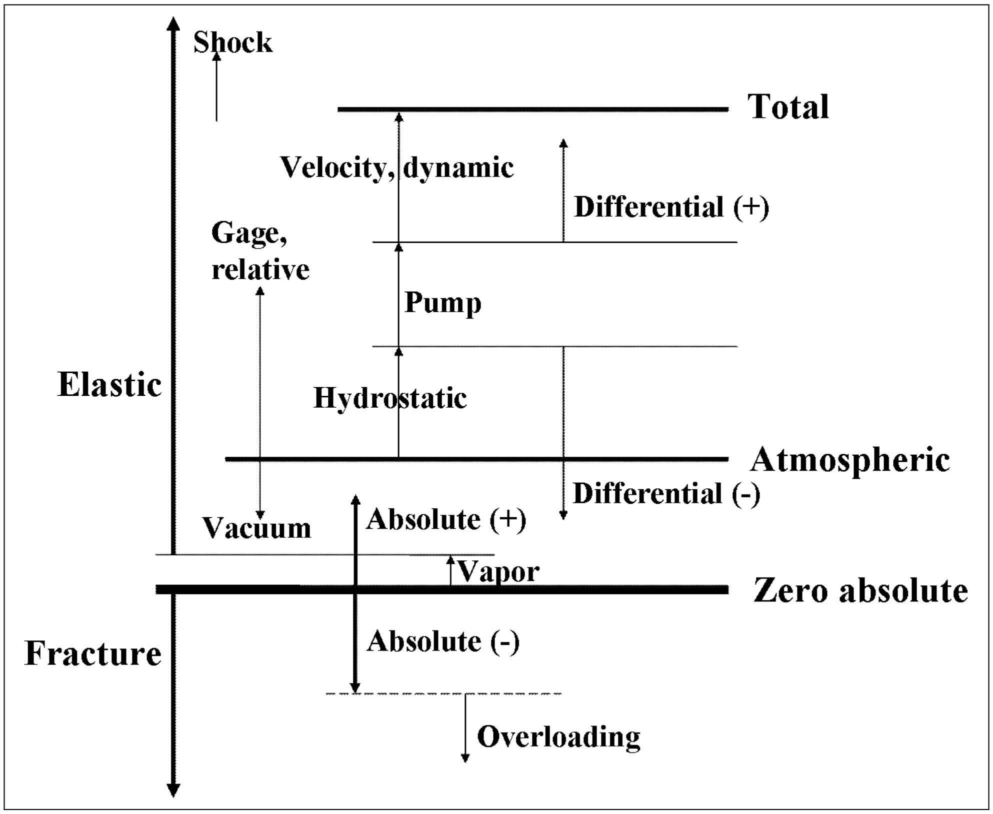

Figure 2.5 Pressure diagram of a fluid.

Here, the proportionality constant μ is called the viscosity coefficient of the

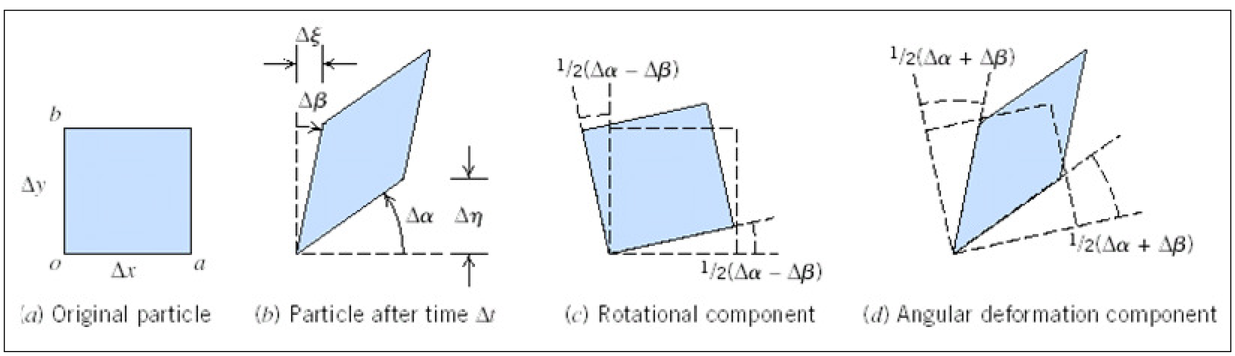

Figure 2.6 Deformation of fluid element in 2-D flows. From Fox, McDonald & Pritchard (2004)

fluid, and the second term is called the viscous stress tensor. The stress tensor is symmetric, i.e., $\tau_{ij} = \tau_{ji}$, because otherwise there will be an unreasonable motion with an infinite speed due to the resultant unbalanced forces acting on an infinitesimal fluid element. Symmetry means that only 6 components are independent (not fully 9). The diagonal terms consist of the divergence of the velocity vector and (averaged) static pressures, and hence, for an incompressible fluid, are associated with volumetric changes. The off-diagonal terms are associated with the squeeze-like motion.

2.3.2 Example: Stress tensors for low Reynolds number flows

As an example of stress tensors, let us consider low-Reynolds number flow (Stokes flow). [9]

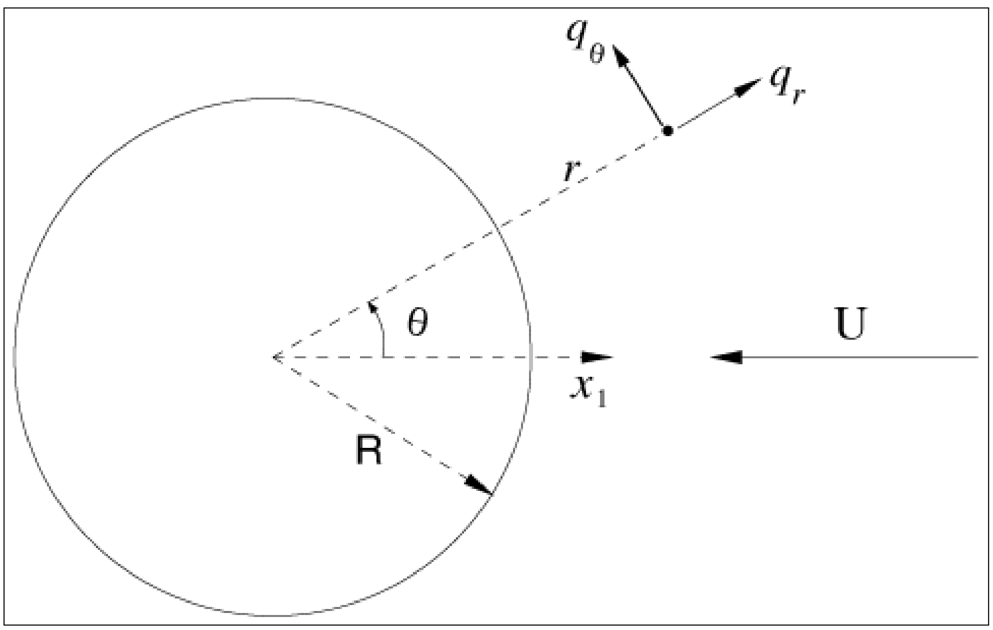

Figure 2.7 Notation for a spherical bubble in uniform flow.

2.3.2.1 Velocity field

The governing equations for such a fluid are, neglecting the inertia terms of the Navier-Stokes equations,

$$\nabla \cdot \underline{u} = 0 \tag{2.79}$$

$$\mu \nabla^2 \underline{u} = \nabla p \tag{2.80}$$

[9] See, e.g., Brennen, C. E. (1995), *Cavitation and Bubble Dynamics*, Oxford University Press. and Ton Tran-Cong and J.R. Blake (1984), "General solutions of the Stokes flow equations," *J. of Mathematical Analysis and Applications*, vol. 92, pp. 72–84.

where $\underline{u}$ is the disturbed velocity about a sphere moving in x-axis direction in otherwise fluid at rest. The second equation above physically implies that the pressure gradient balances the viscous force.

The general solution to these equations is given in a form of

$$\underline{u} = \nabla(\underline{r} \cdot \underline{B} + B_o) - 2\underline{B} \tag{2.81}$$

$$p = 2\mu\,(\nabla \cdot \underline{B}) \tag{2.82}$$

where $\underline{B}$ and B_0 should satisfy the following conditions, respectively,

$$\nabla^2 \underline{B} = 0, \quad \nabla^2 B_o = 0 \tag{2.83}$$

For Reynolds number values of O(1), the solution for the disturbed velocity field is known to be, by setting $B_x = -3UR/4r, B_o = UR^3x/4r^3, B_y = B_z = 0$ where U is the moving speed and R is the radius of a sphere,

$$\underline{u} = \left(\frac{3R}{4r} + \frac{R^3}{4r^3}\right) U\,\underline{i} - \left(-\frac{3Rx}{4r^3} + \frac{3R^3x}{4r^5}\right) U\underline{r} \tag{2.84}$$

where $x = r\cos\theta, \underline{i} = \underline{e}_r \cos\theta - \underline{e}_\theta \sin\theta, \underline{r} = r\,\underline{e}_r$ (see Figure **2.7**).

By introducing a moving frame fixed to the sphere, we can consider equivalently a stream of viscous fluid flows at speed U slowly about a stationary sphere of radius R. Then the relative velocity components for the moving coordinate system are given by, i.e.,

$$\underline{q} = -U\,\underline{i} + \underline{u} \tag{2.85}$$

or

$$q_r = -U\cos\theta + 2\left(\frac{C}{r^3} + \frac{D}{r}\right)\cos\theta \tag{2.86}$$

$$q_\theta = U\sin\theta + \left(\frac{C}{r^3} - \frac{D}{r}\right)\sin\theta \tag{2.87}$$

$$q_\alpha = 0 \tag{2.88}$$

where $C = -UR^3/4$ and $D = +3UR/4$.

The vorticity is

$$\underline{\omega} = \omega\, \underline{e}_\alpha = \left(\frac{1}{r}\frac{\partial(r\, q_\theta)}{\partial r} - \frac{1}{r}\frac{\partial q_r}{\partial \theta}\right)\underline{e}_\alpha = \frac{3}{2}UR\,\frac{\sin\theta}{r^2}\underline{e}_\alpha \tag{2.89}$$

The pressure can be obtained from the momentum equation $\nabla p = \mu\,\nabla^2 \underline{q}$, i.e., $\nabla p = -\mu\,\nabla\times\underline{\omega}$, using Eqs. (1.176) and (2.89)

$$\frac{\partial p}{\partial r} = -\frac{3\mu\, UR\cos\theta}{r^3}, \quad \frac{1}{r}\frac{\partial p}{\partial \theta} = -\frac{3\mu\, UR\sin\theta}{2r^3}, \tag{2.90}$$

Integrating with respect to either r or θ, the solution for pressure is known to be

$$p = p_0 + \frac{3}{2}\mu\, R\, U\,\frac{\cos\theta}{r^2} \tag{2.91}$$

where p_0 is a reference pressure at infinity.

2.3.2.2 Stream function approach

Alternatively, we can obtain the same results by introducing the stream function. In terms of spherical polar coordinates (r, θ, α) where α is the azimuth angle about the axis $\theta = 0$ (see Figure **2.7**), the flow is of axi-symmetry and then the continuity equation becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 q_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(q_\theta\,\sin\theta) = 0 \tag{2.92}$$

Now, we define the stream function $\underline{\Psi} = (0, 0, \psi/r\sin\theta)$ to satisfy the continuity equation automatically, such that $\underline{q} = \nabla\times\underline{\Psi} = \nabla\times\left(\dfrac{\psi\,\underline{e}_\alpha}{r\sin\theta}\right)$, i.e.,

$$q_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}, \; q_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} \tag{2.93}$$

Now we get the equation for the vorticity $\underline{\omega}$ and then for the stream function ψ as follows:

Take the curl of Eq. (2.80) using the expansion formula $\nabla\times(\nabla\times\underline{u}) = \nabla(\nabla\cdot$

$\underline{u}) - \nabla^2 \underline{u}$, we have

$$\nabla \times (\nabla \times (\nabla \times \underline{u})) = 0, \quad \text{i.e.,} \quad \nabla \times (\nabla \times \underline{\omega}) = 0 \tag{2.94}$$

Consequently, it reduces to $\nabla^2(\nabla^2 \underline{\Psi}) = 0$ where we have used the relation $\nabla^2 \underline{\Psi} = -\underline{\omega}$. Namely, this equation becomes a scalar equation for ψ

$$\left\{ \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \right) \right\}^2 \psi = 0 \tag{2.95}$$

Taking the separation of variables for the resulting equation, the stream function would be of a form

$$\psi = \sin^2\theta \left(\frac{C}{r} + D\, r + E\, r^2 + F\, r^4 \right) \tag{2.96}$$

Applying the boundary conditions on the sphere surface ($q_r = q_\theta = 0$) and at infinity ($\psi_\infty = -U r^2 \sin^2\theta / 2$), we obtain $C = -U R^3/4$, $D = +3\, U R/4$, $E = -U/2$, and $F = 0$. The first term and the third term represent the inviscid flow past a sphere, while the second term corresponds to the viscous correction.

2.3.2.3 Stress tensor and drag

From these expressions, the stress tensor is related to rate of strain tensor in a spherical coordinate system. Only 6 components are expressed as:

$$\tau_{rr} = -p + 2\mu \frac{\partial q_r}{\partial r} \tag{2.97}$$

$$\tau_{r\theta} = \mu \left(\frac{1}{r} \frac{\partial q_r}{\partial \theta} + \frac{\partial q_\theta}{\partial r} - \frac{q_\theta}{r} \right) \tag{2.98}$$

$$\tau_{r\alpha} = \mu \left(\frac{\partial q_\alpha}{\partial r} + \frac{1}{r\sin\theta} \frac{\partial q_r}{\partial \alpha} - \frac{q_\alpha}{r} \right) \tag{2.99}$$

$$\tau_{\theta\theta} = -p + 2\mu\left(\frac{1}{r}\frac{\partial q_\theta}{\partial \theta} + \frac{q_r}{r}\right) \quad (2.100)$$

$$\tau_{\theta\alpha} = \mu\left(\frac{1}{r}\frac{\partial q_\alpha}{\partial \theta} - \frac{q_\alpha}{r}\cot\theta + \frac{1}{r\sin\theta}\frac{\partial q_\theta}{\partial \alpha}\right) \quad (2.101)$$

$$\tau_{\alpha\alpha} = -p + 2\mu\left(\frac{1}{r\sin\theta}\frac{\partial q_\alpha}{\partial \alpha} + \frac{q_r}{r} + \frac{q_\theta}{r}\cot\theta\right) \quad (2.102)$$

With the solution for the velocity field, the values of these components are evaluated on the surface of the sphere(i.e., on $r = R$):

$$\tau_{rr} = \tau_{\theta\theta} = \tau_{\alpha\alpha} = -p(R, \theta) \quad (2.103)$$

$$\tau_{r\theta} = \frac{3U}{2R}\mu\sin\theta, \quad \tau_{r\alpha} = \tau_{\theta\alpha} = 0 \quad (2.104)$$

On the surface of the sphere, the normal vector $\underline{n} = \underline{e}_r$(pointing into the fluid from the surface) is taken to find forces acting on the sphere by the fluid). Then, the surface stresses become $\underline{\tau} = \tau_{rr}\ \underline{e}_r + \tau_{r\theta}\ \underline{e}_\theta$ where $\underline{e}_\theta = \sin\theta\ \underline{i} + \cos\theta\ \underline{j}$. The surface force is composed of two components due to the normal and the tangential stress:

$$\underline{F}_S^{(n)} = \oint_S \tau_{rr}\,\underline{e}_r\,dS = 2\pi\mu\,UR\,\underline{i} \quad (2.105)$$

$$\underline{F}_S^{(t)} = \oint_S \tau_{r\theta}\,\underline{e}_\theta\,dS = 4\pi\mu\,UR\,\underline{i} \quad (2.106)$$

Herein we have used the surface element $dS = 2\pi R^2\sin\theta\,d\theta$ for the actual integrations. The total drag of the sphere becomes $D = 6\pi\mu\,UR$ which corresponds to the drag coefficient $C_D = D/(0.5\,\rho\,U^2\pi R^2) = 24/Re$ where Re is the Reynolds number based on the sphere diameter and the speed of the onset flow.

2.3.3 Surface tension

On an interface surface between two fluids (i.e., stratified fluids), the surface tension should be included to satisfy the continuity of the stress across the in-

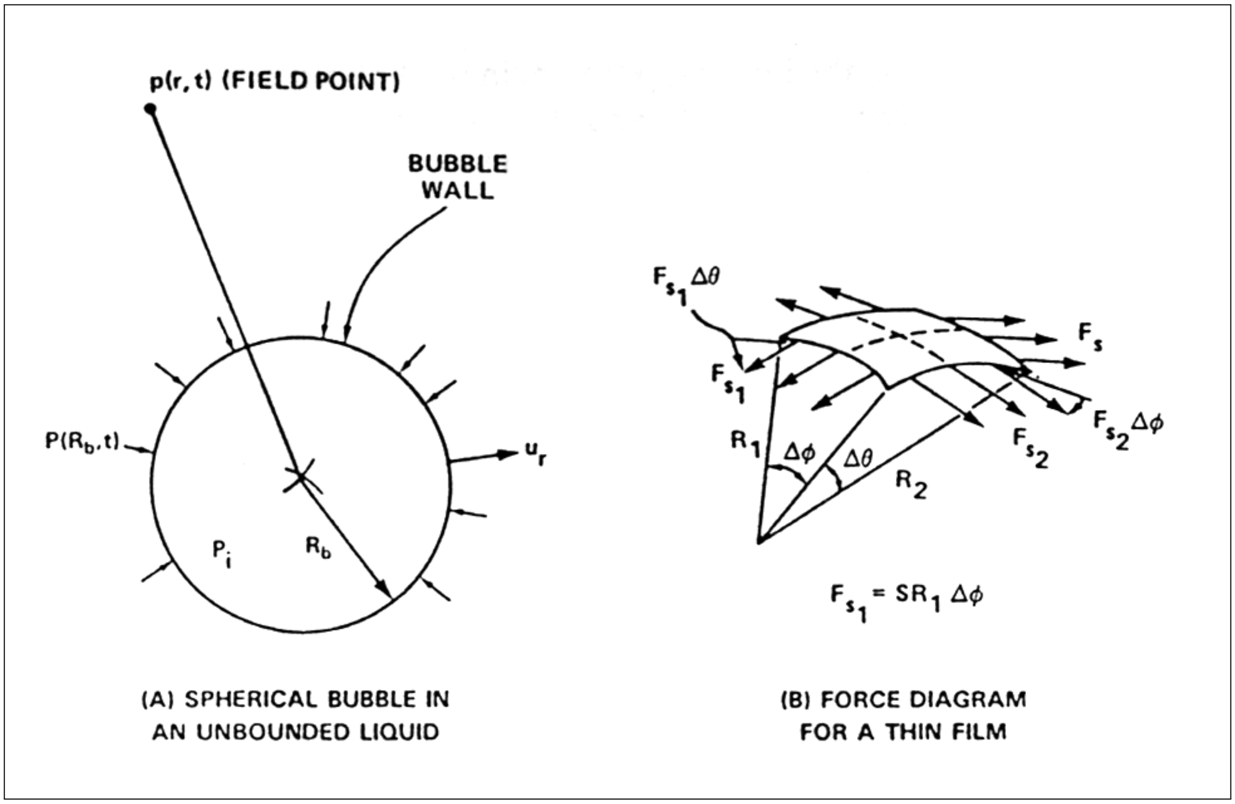

Figure 2.8 Force diagram for a spherical bubble with surface tension.

terface:

$$\underline{n} \cdot (\underline{\underline{\tau}}_i - \underline{\underline{\tau}}_o) = \sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \underline{n} \tag{2.107}$$

where σ is called the surface tension(whose unit is given by force per length), R_1 and R_2 are the principal radii of curvature of the interface, $\underline{n}$ is the normal vector at the interface, and the subscripts *i* and *o* refer the two fluid sides of the interface. (See Figure 2.8).

When the two fluids are stationary, only the pressure terms remain in the above relation. From the force equilibrium in the normal direction for a small element of interface,

$$(p_i - p_o)\,(R_1 \triangle\phi)\,(R_2 \triangle\theta) = F_{s_1} \triangle\theta + F_{s_2} \triangle\phi \tag{2.108}$$

where $F_{s_1} = \sigma\, R_1 \triangle\phi$ and $F_{s_2} = \sigma\, R_2 \triangle\theta$. Then,

$$p_i - p_o = \sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \tag{2.109}$$

As a special example, for a stationary spherical droplet(or bubble) of radius R, $p_i - p_o = 2\sigma/R$.

2.3.4 Equations of motion: Navier-Stokes equations

Newton's second law states that the time rate of change of the linear momentum is equal to the applied forces. This statement is also appropriate for continuum matter. Hence for a moving volume $V(t)$ bounded by a material surface $S(t)$, we have (for detailed derivations, refer to texts dealing with fluid mechanics.)

$$\frac{d}{dt}\int_V \rho\,\underline{q}\,dV = \int_V \underline{F}_B\,dV + \int_S \underline{n}\cdot\underline{\underline{\tau}}\,dS \tag{2.110}$$

For incompressible Newtonian fluids, the corresponding differential form becomes the so-called Navier-Stokes equations:

$$\rho\frac{D\underline{q}}{Dt} = -\nabla p + \underline{F}_B + \mu\,\nabla^2\underline{q} \tag{2.111}$$

Alternate forms of the non-linear convective and the viscous term of the Navier-Stokes equations are listed in Gresho (1991). [10]

(1) Alternate form of the convective term, $\underline{q}\cdot\nabla\underline{q}$

(a) Divergence form : $\nabla\cdot(\underline{q}\,\underline{q}) = \underline{q}\cdot\nabla\underline{q} + \underline{q}\,(\nabla\cdot\underline{q})$

(b) Advective/convective form : $\underline{q}\cdot\nabla\underline{q} = \frac{1}{2}\nabla q^2 - \underline{q}\times(\nabla\times\underline{q})$

(c) Rotational form : $\underline{\omega}\times\underline{q}$

(d) Skew-symmetric (transpose of a matrix equals minus the matrix):
$\frac{1}{2}\left[\nabla\cdot(\underline{q}\,\underline{q}) + \underline{q}\cdot\nabla\underline{q}\right] = \underline{q}\cdot\nabla\underline{q} + \frac{1}{2}\underline{q}(\nabla\cdot\underline{q})$

(2) Alternate form of the viscous term,$\nabla^2\underline{q}$

(a) Stress-divergence form : $\nabla\cdot\left[(\nabla\underline{q}) + (\nabla\underline{q})^T\right] = \nabla^2\underline{q} + \nabla(\nabla\cdot\underline{q})$

[10] Gresho, P. M. (1991), "Incompressible fluid dynamics: some fundamental formulation issues", *Annual Review of Fluid Mechanics*, vol. 23, pp. 413–453.

(b) Div-curl form : $\nabla^2 \underline{q} = \nabla(\nabla \cdot \underline{q}) - \nabla \times (\nabla \times \underline{q})$

(c) Curl form : $-\nabla \times (\nabla \times \underline{q}) = -\nabla \times \underline{\omega}$

2.3.5 Bernoulli equation

The equations of motion for inviscid fluids are called Euler's equations, with dropping the last term of Eq. (2.111). The term $\underline{q} \cdot \nabla \underline{q}$, which occurs in the equations can be transformed by the vector expansion formula:

$$\nabla \left(q^2\right) = \nabla(\underline{q} \cdot \underline{q}) = 2\,\underline{q} \cdot \nabla \underline{q} + 2\,\underline{q} \times (\nabla \times \underline{q}) \tag{2.112}$$

Thus

$$\underline{q} \cdot \nabla \underline{q} = \frac{1}{2} \nabla \left(q^2\right) - \underline{q} \times \underline{\omega} \tag{2.113}$$

Eventually an alternate form of the Euler's equation is

$$\frac{\partial \underline{q}}{\partial t} + \underline{\omega} \times \underline{q} = -\frac{1}{\rho} \nabla p + \frac{1}{\rho} \underline{F}_B - \nabla \left(\frac{q^2}{2}\right) \tag{2.114}$$

It is often assumed that the body force $\underline{F}_B$ is derivable from a potential; that is, that it is a conservative force, such as gravity. Then we can write $\underline{F}_B = -\nabla \Omega$, and the equations appear in

$$\frac{\partial \underline{q}}{\partial t} - \underline{q} \times \underline{\omega} = -\nabla \left(\frac{q^2}{2} + \Omega\right) - \frac{1}{\rho} \nabla p \tag{2.115}$$

This is about as far as we can go with complete generality, but the equations can be simplified still further if the fluid is barotropic; that is, if the density ρ depends on the pressure p only: $\rho = \rho(p)$. [11]

Example of this state of affairs are compressible fluids flowing adiabatically ($p \sim \rho^k$) or isothermally ($p \sim \rho$), or, of course, incompressible fluids ($\rho =$ constant). In these cases the term $\frac{1}{\rho} \nabla p$ can also be expressed as the gradient of

[11] We call a fluid baroclinic if the density does not depend on the pressure.

a function, for consider

$$d\underline{\ell} \cdot \frac{\nabla p}{\rho(p)} = \frac{dp}{\rho(p)} = d\int \frac{dp}{\rho(p)} = d\underline{\ell} \cdot \nabla \int \frac{dp}{\rho(p)} \tag{2.116}$$

Since $d\underline{\ell}$ is arbitrary, $\dfrac{\nabla p}{\rho(p)} = \nabla \displaystyle\int \frac{dp}{\rho(p)}$, and the Euler's equations of motion are reduced to

$$\frac{\partial \underline{q}}{\partial t} - \underline{q} \times \underline{\omega} = -\nabla \left(\frac{q^2}{2} + \Omega + \int \frac{dp}{\rho} \right) \tag{2.117}$$

There are several important cases in which the Euler's equations of motion can be integrated directly.

(1) Irrotational barotropic flow
In this type of flow $\underline{q}$ is $\nabla\phi$. The left-hand side of Eq. (2.117) becomes simply $\dfrac{\partial}{\partial t}(\nabla\phi)$, and since the time and space derivatives are independent and can be exchanged in order, this is equal to $\nabla\left(\dfrac{\partial\phi}{\partial t}\right)$. Thus

$$\nabla \left(\frac{\partial\phi}{\partial t} + \frac{q^2}{2} + \Omega + \int \frac{dp}{\rho} \right) = 0 \tag{2.118}$$

But when the gradient of a function is zero throughout a region, the function must certainly be constant throughout the region– or rather, since the gradient involves space derivatives only, the function must be constant throughout the region at any instant, but may vary with time. The integrated form is therefore

$$\frac{\partial\phi}{\partial t} + \frac{q^2}{2} + \Omega + \int \frac{dp}{\rho} = C(t) \tag{2.119}$$

Remember that the term $\displaystyle\int \frac{dp}{\rho}$ is just a function of ρ (or p) whose form is known as soon as the particular barotropic law $\rho = \rho(p)$ is specified. For example, the simplest law is that of the incompressible fluid: $\rho =$constant. Hence the integrated equation for <u>incompressible</u>, frictionless, irrotational, unsteady flow is

$$\frac{\partial \phi}{\partial t} + \frac{q^2}{2} + \Omega + \frac{p}{\rho} = C(t) \tag{2.120}$$

(2) Steady barotropic motion
For this case we need not assume irrotational flow, and therefore we return to the original form of the Euler's equation of motion, but again assume $\rho = \rho(p)$. The equations then read, for steady flow,

$$\underline{q} \cdot \nabla \underline{q} = -\nabla \left(\int \frac{dp}{\rho} + \Omega \right) \tag{2.121}$$

We shall now show that this can be integrated along individual streamlines; that is, we shall obtain an integral that will tell how the quantities behave along a streamline, but not how they change from streamline to streamline. Let an orthogonal curvilinear coordinate system be defined so that s is measured along a streamline, and r and t normal to it. Then $\underline{q} = (q, 0, 0)$, and $\underline{q} \cdot \nabla \underline{q} = \left(q \frac{\partial q}{\partial s}, \cdots, \cdots \right)$ (as may be verified by reference to the formulas in general curvilinear orthogonal coordinates).

Let us substitute this into Eq. (2.121) and then multiply both sides by $\cdot d\underline{s}$:

$$q \frac{\partial q}{\partial s} ds = -\frac{\partial}{\partial s} \left(\int \frac{dp}{\rho} + \Omega \right) ds \tag{2.122}$$

and, integrating along the streamline

$$\frac{1}{2} q^2 + \int \frac{dp}{\rho} + \Omega = C_s \tag{2.123}$$

where C_s is the constant of integration, and we give it the subscript s to emphasize that the constant may vary from streamline to streamline.

Since Eqs. (2.119) and (2.123) must yield the same result in cases of steady, irrotational barotropic flow, we see that the irrotational assumption is equivalent to taking the same constant, C_s, for all streamlines:

$$\frac{1}{2} q^2 + \int \frac{dp}{\rho} + \Omega = \text{constant} \tag{2.124}$$

and finally if this is also incompressible,

$$\frac{1}{2}\rho q^2 + p + \rho\Omega = \text{constant} \tag{2.125}$$

This will be recognized as Bernoulli's equation, which is an energy equation, although we obtained it by integration of momentum equations. In fact, Eqs. (2.119), (2.120), and (2.121) are also sometimes called generalized forms of Bernoulli's equation.

2.3.6 Kelvin's theorem

In any flow of a barotropic inviscid fluid, the circulation about any closed path does not vary with time if the contour is imagined to move with the fluid, that is, always to be made up of the same particles. [12] We give here a different proof, which offered more generality. We begin by considering the contour integral $\Gamma = \oint_C \underline{q} \cdot d\underline{\ell}$ where $\underline{q}$ is any vector quantity and C (material contour) is carried by the fluid. Now consider the time derivative of the circulation for a closed curve following the motion:

$$\frac{d\Gamma}{dt} = \frac{d}{dt}\oint_{C(t)} \underline{q}(\underline{x}, t) \cdot \underline{s}(\underline{x}, t)\, d\ell \tag{2.126}$$

where $\underline{s}(\underline{x}, t)$ is the unit tangent vectors along the integration path of the contour C. This differential is similar to the starting point in our derivation of the Reynolds transport theorem in Chapter 1, but for a moving curve instead of a moving volume. We make the same transformation from spatial $(\underline{x})$ to initial

[12] See Lamb, H. (1932), *Hydrodynamics*, sixth ed., Dover.

coordinates $(\underline{\xi})$:

$$
\begin{aligned}
\frac{d\Gamma}{dt} &= \oint_{C(0)} \frac{\partial}{\partial t}\left[\underline{q}^*(\underline{\xi},t)\cdot\frac{d\underline{x}^*(\underline{\xi},t)}{d\ell}d\ell\right] \\
&= \oint_{C(0)}\left[\left.\frac{\partial \underline{q}^*}{\partial t}\right|_{\underline{\xi}}\cdot\frac{d\underline{x}^*}{d\ell}+\underline{q}^*\cdot\frac{\partial}{\partial t}\left(\frac{\partial \underline{x}^*(\underline{\xi},t)}{\partial \xi_j}\frac{d\xi_j}{d\ell}\right)\right]d\ell \\
&= \oint_{C(0)}\left[\left.\frac{\partial \underline{q}^*}{\partial t}\right|_{\underline{\xi}}\cdot\frac{d\underline{x}^*}{d\ell}+\underline{q}^*\cdot\left(\frac{\partial \underline{q}^*(\underline{\xi},t)}{\partial \xi_j}\frac{d\xi_j}{d\ell}\right)\right]d\ell \\
&= \oint_{C(t)}\left[\frac{D\underline{q}}{Dt}\cdot\underline{s}+\underline{q}\cdot\left\{(\underline{s}\cdot\nabla)\underline{q}\right\}\right]d\ell \\
&= \oint_{C(t)}\left[\frac{D\underline{q}}{Dt}\cdot\underline{s}+\frac{\partial}{\partial \ell}\left(\frac{1}{2}q^2\right)\right]d\ell \\
&= \oint_{C(t)}\frac{D\underline{q}}{Dt}\cdot\underline{s}\,d\ell \\
&= \oint_{C(t)}\underline{a}\cdot d\underline{\ell}
\end{aligned}
\tag{2.127}
$$

Now the total (material) derivative term is the LHS of the momentum equation allowing us to put the RHS of the momentum equation into the integral:

$$\frac{d\Gamma}{dt} = \oint_{C(t)}\left(-\frac{\nabla p}{\rho}+\frac{\underline{F}_B}{\rho}+\nu\nabla^2\underline{q}\right)\cdot d\underline{\ell} \tag{2.128}$$

For barotropic fluids and conservative body forces, the time rate of change of the circulation reduces to an integral containing only viscous terms since the integral of a gradient about a closed curve is zero:

$$\frac{d\Gamma}{dt} = \nu\oint_{C(t)}\nabla^2\underline{q}\cdot d\underline{\ell} \tag{2.129}$$

If the fluid is baroclinic, the circulation can be modified because of the baroclinic generation of vorticity as

$$\frac{d\Gamma}{dt} = \nu\oint_{C(t)}\nabla^2\underline{q}\cdot d\underline{\ell}+\int_S\frac{1}{\rho^2}\left(\nabla\rho\times\nabla p\right)\cdot d\underline{\ell} \tag{2.130}$$

where the surface integral on the right-hand side is performed over the area bounded by material line. [13]

Every flow that can be produced (without friction) in a barotropic fluid initially at rest, initially in a uniform stream, or initially in any irrotational state, must be an irrotational flow.

2.3.6.1 Viscous diffussion

If we use the vector expansion Eq. (1.72) in Chapter 1 with a solenoidal vector $\underline{q}$ of an incompressible fluid:

$$\nabla^2 \underline{q} = \nabla(\nabla \cdot \underline{q}) - \nabla \times (\nabla \times \underline{q}) = -\nabla \times \underline{\omega} \tag{2.131}$$

the time derivative of the circulation can be written as:

$$\frac{d\Gamma}{dt} = -\nu \oint_{C(t)} (\nabla \times \underline{\omega}) \cdot d\underline{\ell} \tag{2.132}$$

In fact, the viscous term on the right-hand side represents the vorticity diffusion that activates the circulation change. We consider a sufficiently thin vortex tube with outward unit normal $\underline{n}$, in which the vortex lines are all parallel. At a point $\underline{x}$ on the side surface S of the tube, let $\underline{e}_2 = \underline{\omega}/\omega$ be the unit vector along the vortex line through $\underline{x}$ such that $\underline{e}_1 = \underline{e}_2 \times \underline{n}$ defines a unit vector tangent to S. Let $\underline{x}$ move along the $\underline{e}_1$-direction around the tube to form a closed line C; see the sketch of Figure **2.9** . $\underline{e}_1$ is actually along the direction of shear stress $\underline{\tau} = \mu\, \underline{\omega} \times \underline{n}$.

[13] For details, see Milne-Thomson, L. M. (1968), *Theoretical Hydrodynamics*, fifth edition, Macmillan, London, p. 84.

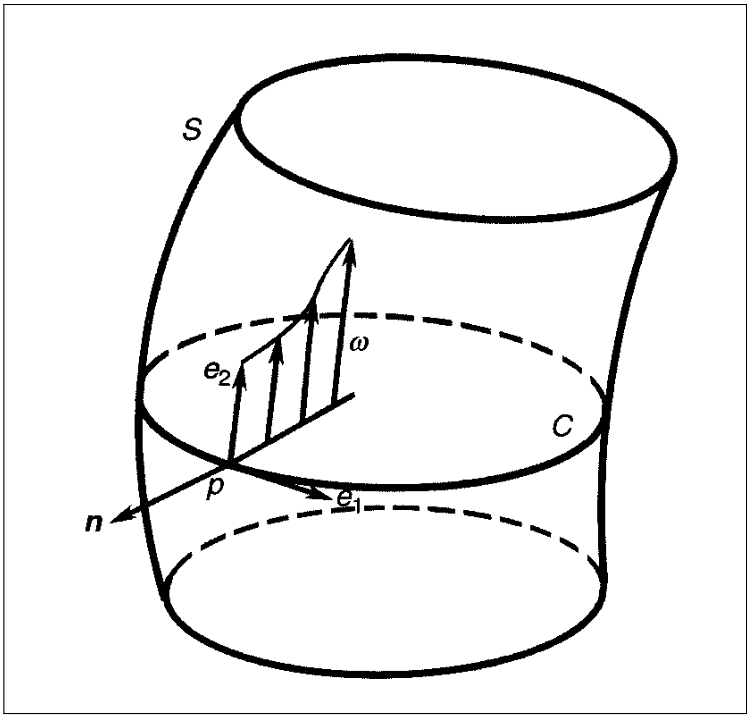

Figure 2.9 Intrinsic frame $\underline{e}_1, \underline{e}_2, \underline{n}$ on the side surface S of a vortex tube. From Wu et al. (2006), p. 139.

For this thin tube, the curvature of S along the tube direction is negligible. It can then be shown that

$$(\nabla \times \underline{\omega}) \cdot d\underline{\ell} = -\underline{e}_2 \cdot \frac{\partial \underline{\omega}}{\partial n} \, d\ell \tag{2.133}$$

and hence Eq. (2.132) yields

$$\frac{d\Gamma}{dt} = \nu \oint_{C(t)} (\underline{\sigma} \cdot \underline{e}_2) \, d\ell \tag{2.134}$$

where $\underline{\sigma} \equiv \nu \, \underline{n} \cdot \nabla \underline{\omega} = \nu \, \dfrac{\partial \underline{\omega}}{\partial n}$. In a 2-D viscous flow, Eq. (2.132) simply becomes

$$\frac{d\Gamma}{dt} = \nu \oint_{C(t)} \frac{\partial \omega}{\partial n} \, d\ell \tag{2.135}$$

where $\nu \, \dfrac{\partial \omega}{\partial n}$ is the vorticity diffusion flux across the contour. $\underline{n}$ is the unit normal outward from the contour.

2.3.6.2 Cases of inviscid flow

Thus when either the kinematic viscosity or the gradient of the vorticity is small, the circulation will be preserved about a curve moving with the fluid. Note that the circulation is not influenced by the pressure or conservative body forces and that if the fluid were inviscid the equation irrespective of the field vorticity. For such flows,

$$\frac{D\Gamma}{Dt} = 0 \tag{2.136}$$

as the theorem states.

Applying this is to the contours that enclose vortex filaments, we see immediately that such vortices do not vary in strength as they move about in any barotropic inviscid fluid. From the theorem above, it becomes clear that if a fluid particle in this type of fluid once has zero vorticity it will always have zero vorticity. For consider a contour surrounding a very small sample of fluid; if ω is zero for this sample, Γ is also zero, and according to the theorem must remain so. But this certainly implies that ω remains zero, since the statement is true for every contour that surround any part of the sample. [14]

2.4 Potential Flows

2.4.1 Laplace equation

We shall devote considerable attention to the study of irrotational motions of incompressible fluids. As we have seen, the irrotational approximation is likely to be valid throughout much of the flow.

Since the equation of continuity is $\nabla \cdot \underline{q}$ and $\underline{q}$ is the gradient of the velocity potential ϕ, the differential equation satisfied by ϕ is Laplace equation:

$$\nabla \cdot (\nabla \phi) = \nabla^2 \phi = 0 \tag{2.137}$$

[14] See Kuethe, A. M. and Chow, C.-Y. (1976), *Foundations of Aerodynamics: Bases of Aerodynamic Design*, Wiley, pp. 53–54.

The pressure-velocity relation is given by the integrated dynamical equation:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}q^2 + \frac{p}{\rho} + \Omega = C(t) \tag{2.138}$$

In a more general case, even if we restrict ourselves to barotropic fluids, we have five dependent variables: u, v, w, p, ρ; and five equations to solve for them:

3 equations of motion, 1 equation of continuity, and 1 equation of state: $\rho = p(\rho)$

We see now that an extreme simplification has been achieved in the irrotational incompressible case, for the equation of state has degenerated to $\rho =$ constant, and we have replaced u, v, w by the velocity potential ϕ, leaving only two unknowns (ϕ and p) and two equations, Eqs. (2.137) and (2.138).

Moreover, Eq. (2.138) has been integrated, and constitutes a formula for calculation of p when Eq. (2.137) has been solved. The only mathematical problem that remains is the solution of Laplace's equation Eq. (2.137), with the appropriate boundary conditions. The most surprising result is that the dynamical equations do not impose any restrictions on the flow. Any solution of the equation of continuity Eq. (2.137) is a possible flow pattern, for some set of boundary conditions. Another statement of this situation is that every *kinematically possible flow* is *dynamically possible*.

It is also important to notice that Eq. (2.137) does not involve t. In a case of unsteady flow, the boundary conditions will vary with time. All that is required is that we solve Laplace's equation with the instantaneous boundary conditions. Another statement of this is that *every unsteady flow pattern* is a *possible steady flow pattern* (and vice versa). Of course, the corresponding pressure will depend on whether the flow is steady or not.

If f satisfies Laplace's equation in a region, then f has no maxima or minima in that region. For any volume V, enclosed in a surface S, lying entirely inside the region:

$$0 = \int_V \nabla^2 f \, dV = \int_V \nabla \cdot \nabla f \, dV = \int_S \underline{n} \cdot \nabla f \, dS = \int_S \frac{\partial f}{\partial n} \, dS \tag{2.139}$$

But if f has a maxima at any point P, we can surely obtain a negative value of $\int_S \frac{\partial f}{\partial n} dS$ by taking V to enclose P and making it small enough. Similarly we can obtain a positive value by integrating $\frac{\partial f}{\partial n}$ around a minimum. Consequently there can be neither.

2.4.2 Kinematic boundary condition

In order to determine the velocity field of a potential flow, we need the boundary condition for velocity on the body surface. For general formulation, we consider a moving boundary here. $\underline{x}$ denotes the position vector of a point on the moving surface, where we take $\underline{\xi}$ as the initial position vector of the point. The velocity of the moving surface is then $\underline{u} = \left.\frac{\partial \underline{x}}{\partial t}\right|_{\underline{\xi}}$. The boundary condition on the moving boundary is that the normal component of the fluid velocity must equal the normal velocity of the moving boundary: $\underline{u} \cdot \underline{n} = \underline{U}_B \cdot \underline{n}$.

2.4.2.1 Alternative form

When the moving boundary is specified by a function, $F(\underline{x}, t) = 0$, the boundary condition can be written in an alternate form. Along the path of motion $\underline{x} = \underline{x}(\underline{\xi}, t)$ and the moving surface $F\left\{\underline{x}(\underline{\xi}, t), t\right\} = 0$, the particles are always located at the material surface: $\left.\frac{\partial F}{\partial t}\right|_{\underline{\xi}} = 0$. It implies that

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x_i}\left.\left(\frac{\partial x_i}{\partial t}\right)\right|_{\underline{\xi}} = 0 \qquad (2.140)$$

or

$$\frac{\partial F}{\partial t} + \underline{U}_B \cdot \nabla F = 0 \qquad (2.141)$$

The normal vector is defined from the function F as [15]

$$\underline{n} = \frac{\nabla F}{|\nabla F|} \tag{2.142}$$

Then,

$$\frac{\partial F}{\partial t} + \underline{U}_{B} \cdot \underline{n}\ |\nabla F| = 0 \tag{2.143}$$

Use the condition for the normal components of the velocities of both fluid and surface $\underline{u} \cdot \underline{n} = \underline{U}_{B} \cdot \underline{n}$, to obtain

$$\frac{\partial F}{\partial t} + \underline{u} \cdot \underline{n}\ |\nabla F| = 0. \tag{2.144}$$

And we find

$$\frac{\partial F}{\partial t} + \underline{u} \cdot \nabla F = 0 \tag{2.145}$$

or

$$\frac{DF}{Dt} = 0 \tag{2.146}$$

This expression is valid for all material surfaces and for any flow conditions, e.g., for unsteady compressible viscous fluids. If F is independent of time, the expression reduces to the simple one: $\underline{u} \cdot \underline{n} = 0$.

But, if we use a relative coordinate system fixed to a moving body to describe the flow field, the influence of the frame velocity of the moving coordinate system should be added. The detailed formulation is given in Chapter 4.

2.4.3 Dynamic boundary condition: Free surface condition

For inviscid fluids in the absence of surface tension, the pressure is continuous across any interface between two fluids. Perhaps the most familiar case is the liquid free surface under the atmospheric air. The boundary condition is that in

[15] For scalar field $F(\underline{x})$, we consider a curve $\underline{x}(\sigma)$ on a surface of $F(\underline{x}) = constant$, which is specified with the parameter σ. Then $\frac{dF(\underline{x})}{d\sigma} = \frac{d\underline{x}(\sigma)}{d\sigma} \cdot \nabla F(\underline{x})$ Because F is constant along the curve $\underline{x}(\sigma)$, $\frac{dF}{d\sigma} = 0$. Since $\frac{d\underline{x}(\sigma)}{d\sigma}$ is tangent to $\underline{x}(\sigma)$, it requires either that $\nabla F = 0$ or that ∇F is perpendicular to $\frac{d\underline{x}(\sigma)}{d\sigma}$ (namely, to $\underline{x}(\sigma)$). Therefore non-zero ∇F is perpendicular to the surface $F = constant$.

the liquid just below the free surface, the pressure is the same as in the air just above the free surface. The atmospheric pressure p_a can generally be taken as a constant. For a liquid free surface that is at rest far ahead of a body whose motion creates disturbances on the free surface, the free surface boundary conditions are $x_3 = \zeta(x_1, x_2, t)$ and $p(x_1, x_2, \zeta, t) = p_a$. The first condition is the kinematic condition which describes the free surface height from $x_3 = 0$ and the second one is dynamic for imposing the atmospheric pressure on the free surface.

For upstream we can expect disturbance-free condition. Hence Bernoulli's equation on the free surface reduces to the expression:

$$\frac{(p - p_a)}{\rho} + g\,\zeta(x_1, x_2, t) + \frac{1}{2}\,(\nabla\phi)^2 + \frac{\partial\phi}{\partial t} = 0 \tag{2.147}$$

With $p = p_a$ in this equation, we have a suitable form to derive explicitly the free surface elevation if the velocity potential ϕ is known:

$$\zeta(x_1, x_2, t) = -\frac{1}{g}\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}\,(\nabla\phi)^2\right) \tag{2.148}$$

Note that the vortex sheet trailing behind lifting surfaces is another case, about which we will describe in detail in Chapter 4

2.4.4 Examples

2.4.4.1 Flow past a sphere

Let us first consider the steady irrotational flow past a sphere. After the doublet strength μ is replaced in terms of the stream speed U and the radius R of the sphere, as the student can easily verify, the flow is described by

$$\left.\begin{aligned} \psi &= \frac{1}{2}U\left(r^2 - \frac{R^3}{r}\right)\sin^2\theta; \quad \phi = \frac{1}{2}U\left(2r + \frac{R^3}{r^2}\right)\cos\theta \\ q_r &= U\left(1 - \frac{R^3}{r^3}\right)\cos\theta; \quad q_\theta = -\frac{1}{2}U\left(2 + \frac{R^3}{r^3}\right)\sin\theta \end{aligned}\right\} \tag{2.149}$$

On the sphere, the velocity is given by

$$q_\theta = -\frac{3}{2}U\sin\theta \tag{2.150}$$

so that there are two stagnation points at $\theta = 0, \pi$, and the maximum local speed is 50% greater than the stream speed. The pressure is given by

$$\frac{p - p_0}{\frac{1}{2}\rho U^2} = 1 - \frac{9}{4}\sin^2\theta \tag{2.151}$$

Consequently there can be no force on the sphere.

In describing real-fluid flows, the boundary layer separates from the surface just forward of the equator $\theta = \pi/2$ (in agreement with viscous-fluid theory), and from there back the flow loses its resemblance to perfect-fluid flow. It is clear that the boundary-layer separation is ultimately responsible for the appreciable drag of the sphere.

2.4.4.2 Flow around a circular cylinder

We shall proceed to the consideration of the plane steady irrotational flow around a circular cylinder. In terms of the radius of the cylinder, R, the formulas for this case are

$$\left.\begin{aligned} \psi = U\left(r - \frac{R^2}{r}\right)\sin\theta; &\quad \phi = U\left(r + \frac{R^2}{r}\right)\cos\theta \\ q_r = U\left(1 - \frac{R^2}{r^2}\right)\cos\theta; &\quad q_\theta = -U\left(1 + \frac{R^2}{r^2}\right)\sin\theta \end{aligned}\right\} \tag{2.152}$$

Thus the maximum surface speed is $2U$ in this case, and the local pressures correspondingly lower than on the sphere. Once more the pressure is distributed symmetrically fore-and-aft, and there is no force on the cylinder.

The boundary conditions satisfied by equation Eq. (2.152) are

$$q_r = 0 \;\text{ when }\; r = R, \;\text{ and }\; \underline{q} \to U\underline{i} \;\text{ as }\; r \to \infty. \tag{2.153}$$

But these would be just as well satisfied if we were to superimpose a plane vortex flow about the origin, of any desired strength; namely,

$$\psi = U\left(r - \frac{R^2}{r}\right)\sin\theta - \frac{\Gamma}{2\pi}\ln r \tag{2.154}$$

$$\phi = U\left(r + \frac{R^2}{r}\right)\cos\theta + \frac{\Gamma}{2\pi}\theta \quad (0 \le \theta < 2\pi) \tag{2.155}$$

$$q_r = U\left(1 - \frac{R^2}{r^2}\right)\cos\theta \tag{2.156}$$

$$q_\theta = -U\left(1 + \frac{R^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r} \tag{2.157}$$

With circulation, the local velocity at the surface becomes $-2U\sin\theta + \Gamma/2\pi R$. Thus the stagnation points have moved to

$$\theta = \sin^{-1}\frac{\Gamma}{4\pi UR} \tag{2.158}$$

provided that $|\Gamma| \le 4\pi UR$. (If $|\Gamma|$ has a greater value, the stagnation points merge and occur in the flow outside the cylinder.)

The fluid pressure on the cylinder is now

$$p = p_0 + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho\left(2U\sin\theta - \frac{\Gamma}{2\pi R}\right)^2 \tag{2.159}$$

and again, we see that there is no force component in the x direction, i.e. no drag. The force component in the y direction is easily computed:

$$Y = -\int_0^{2\pi} p\, R\sin\theta\, d\theta = -\rho U\Gamma \tag{2.160}$$

We see that there is lift on a circular cylinder with circulation. This is obviously related to the so-called 'Magnus effect', which produces lift on a rotating cylinder – in a real fluid, the circulation is produced by the action of viscosity near the spinning cylinder.