교 과 목 414.519 강의록

수 치 선 박 유 체 역 학

- 보 텍 스 방 법 -

COMPUTATIONAL MARINE HYDRODYNAMICS

-VORTEX METHODS-

2019년 6월 27일

Suh, Jung-Chun

서 정 천

Seoul National Univ., Dept. NAOE

서울대학교 공과대학조선해양공학과

5

ANALYTICAL EVALUATION OF BOUNDARY INTEGRALS

5.1 Introduction

The fundamental problem of fluid mechanics for inviscid incompressible flow is to determine velocity potential ϕ in simply connected fluid domain V bounded by the boundary S (S is composed of a body surface and non-realistic surface). The governing equation of the velocity potential becomes the Laplace equation,

$$\nabla^2\phi = 0, \tag{5.1}$$

satisfying certain proper conditions on S.

With the Green's scalar identity, the potential ϕ within the domain V is expressed in terms of the proper value of ϕ and its normal derivative $\underline{n}\cdot\nabla\phi$ on the boundary S;

$$\phi(\underline{x}_p) = -\frac{1}{4\pi}\left\{\int_S \frac{1}{r}\,\underline{n}\cdot\nabla\phi - \phi\,\underline{n}\cdot\nabla\left(\frac{1}{r}\right)\,dS\right\}. \tag{5.2}$$

Here r is a distance between an integration point $\underline{x}_\xi$ on S and a field point $\underline{x}_p$ located in V. Namely, $\underline{r} = \underline{x}_\xi - \underline{x}_p$.

The first surface integral is interpreted as the potential by surface distribution of source-type singularities with density $\sigma \equiv \underline{n}\cdot\nabla\phi$, the second surface integral as the potential by surface distribution of doublet-type singularities, $\mu \equiv -\phi$. [1]

[1]Recall that the doublet strength is defined as $\mu \equiv \phi$ in the preceding chapters.

Such a singularity method can be applied for solution of this problem. In numerical implementation, the integration over the bounding surface S is approximately performed by summing up each contribution in terms of the proper value of ϕ and its normal derivative $\underline{n} \cdot \nabla\phi$ on each panel element of the discretized boundary surface S;

$$\phi(\underline{x}_p) = -\frac{1}{4\pi} \sum_j \int_{S_j} \left\{ \frac{1}{r} \underline{n} \cdot \nabla\phi - \phi\, \underline{n} \cdot \nabla \left(\frac{1}{r} \right) \right\} dS \tag{5.3}$$

Applying the boundary condition at collocation points to this equation results in a linear system of algebraic equations for unknown doublet strengths on each panel. [2]

Evaluations of the associated integrals at the collocation points should be performed to obtain the matrix elements. The fast and accurate computation of these elements is very important in the numerical solution.

The velocity components can be derived by differentiation of Eq. (5.3) with respect to the coordinates of the field point. We may take without loss of generality one planar panel as the integration region concerned herein, which can be regarded as a part of the discretized boundary surface.

5.2 Transformation of the Surface Integrals to Contour Integrals

As will be shown in the followings, using the integral theorems, the surface integrals of the singularity method can be transformed into contour integrals for planar facets. Furthermore, for a planar polygon element with the uniform or linear or bilinear or higher-order density distribution of singularities, the analytical evaluation is possible. The numerical integration is then very precise at less calculation cost.

This section is especially prepared to show all the mathematical derivations

[2] The basic idea of the singularity method has been introduced by Hess, J. L. and Smith A. M. O. (1966), "Calculation of potential flow about arbitrary bodies," *Progress in Aeronautical Science Series*, vol. 8, pp. 1–138.

and proofs of the related equations. A few of test calculations will show the superiority of these analytic evaluations to numerical integrations. A subroutine program based on the analysis is also provided in Appendix C for computations of the influence coefficients in applications of the panel method.

Cantaloube & Rehbach (1986) [3] show that the surface integrals for constant and/or linear distributions of sources and doublets over a planar facet can be transformed into line integrals along contour of the panel: [4]

(1) for source distributions,

$$\begin{aligned}\phi^{(\sigma)} &= -\frac{1}{4\pi}\left[\underline{n}\cdot\oint_C \sigma\,\frac{\underline{r}}{r}\times d\underline{l} - (\underline{n}\cdot\underline{r})\oint_C \sigma\,\underline{A}\cdot d\underline{l} + (\underline{n}\cdot\underline{r})(\underline{n}\cdot\underline{e})\,\underline{n}\cdot\right.\\ &\quad\left.\cdot\left\{\nabla\sigma\times\oint_C \ln(r+\underline{e}\cdot\underline{r})\,d\underline{l}\right\} - \underline{n}\cdot\left(\nabla\sigma\times\oint_C r\,d\underline{l}\right)\right] \qquad (5.4)\\ \underline{q}^{(\sigma)} &= -\frac{1}{4\pi}\left[\underline{n}\oint_C \sigma\,\underline{A}\cdot d\underline{l} + \underline{n}\times\oint_C \frac{\sigma}{r}\,d\underline{l} - \underline{n}(\underline{n}\cdot\underline{e})(\underline{n}\times\nabla\sigma)\cdot\right.\\ &\quad\left.\cdot\oint_C \ln(r+\underline{e}\cdot\underline{r})\,d\underline{l} + \nabla\sigma\left\{\underline{n}\cdot\oint_C\frac{\underline{r}\times d\underline{l}}{r} - (\underline{n}\cdot\underline{r})\oint_C \underline{A}\cdot d\underline{l}\right\}\right] \qquad (5.5)\end{aligned}$$

(2) for doublet distributions,

$$\begin{aligned}\phi^{(\mu)} &= -\frac{1}{4\pi}\left\{-\oint_C \mu\underline{A}\cdot d\underline{l} + (\underline{n}\cdot\underline{e})(\underline{n}\times\nabla\mu)\cdot\oint_C \ln(r+\underline{e}\cdot\underline{r})\,d\underline{l}\right\} \qquad (5.6)\\ \underline{q}^{(\mu)} &= -\frac{1}{4\pi}\left\{\oint_C \mu\,\nabla\left(\frac{1}{r}\right)\times d\underline{l} - \nabla\mu\oint_C \underline{A}\cdot d\underline{l} - (\underline{n}\times\nabla\mu)\times\right.\\ &\quad\left.\times\left(\underline{n}\times\oint_C\frac{d\underline{l}}{r}\right)\right\} \qquad (5.7)\end{aligned}$$

[3] Cantaloube, B. and Rehbach, C. (1986), "Calcul des Integrales de la Methode des Singularites," *Recherche Aerospatiale*, n^o 1, pp. 15–22, English Title: "Calculation of the Integrals of the Singularity Method," *Aerospace Research*, no. 1, pp. 15–22.

[4] See Appendix B for derivation in detail, and also Suh, J.-C. (1990b), *Review of the Paper; Calculation of the Integrals of the Singularity Method by Cantaloube and Rehbach*, KRISO Propulsor Technology Laboratory Report, 22-90. Suh, J.-C. (1990c) *Analytic Evaluations of the Induction-Integrals for Distributions of Sources and Doublets over a Planar Polygon Element*, KRISO Propulsor Technology Laboratory Report, 23-90.

where the vector

$$\underline{A} = \frac{\underline{e} \times \underline{r}}{r(r + \underline{e} \cdot \underline{r})}, \tag{5.8}$$

is introduced by Guiraud (1978), [5] the distance vector $\underline{r} = \underline{x}_\xi - \underline{x}_p$ and $\underline{e} = \pm \underline{n}$. Note that the distance vector is the position vector of the source point relative to the field point, whose direction is opposite to direction of conventionally defined position vectors. The contour integrals are performed along the perimeter of the element in counterclockwise sense.

The details on the derivation of the transformation of the surface integral is given in Appendix B.

For a planar polygon element, the line integrals can be reduced to closed-form expressions. Derivation of these analytic evaluations is the main scope of Appendix B. The computer program based on the analytic evaluation of the contour integrals as outlined in the following section is provided in Appendix C.

5.3 Constant Density Distributions over a Planar Polygon

5.3.1 Source distribution: Potential

The potential at a field point $\underline{x}_p(x, y, z)$ induced by a distribution of sources with unit density (i.e. $\sigma = 1$) over a planar element is, from Eq. (5.4),

$$\phi^{(\sigma)} = -\frac{1}{4\pi} \left\{ \underline{n} \cdot \oint_C \frac{\underline{r}}{r} \times d\underline{l} - (\underline{n} \cdot \underline{r}) \oint_C \underline{A} \cdot d\underline{l} \right\}. \tag{5.9}$$

Rearrange Eq. (5.9) to yield

$$\begin{aligned} \phi^{(\sigma)} &= -\frac{1}{4\pi} \left\{ \underline{n} \cdot \oint_C \frac{\underline{r}}{r} \times d\underline{l} - (\underline{n} \cdot \underline{r}) \oint_C \left(\frac{1}{r} - \frac{1}{r + \underline{e} \cdot \underline{r}} \right) \frac{\underline{e} \times \underline{r}}{\underline{e} \cdot \underline{r}} \cdot d\underline{l} \right\} \\ &= -\frac{1}{4\pi} \oint_C \frac{\underline{n} \times \underline{r}}{r + \underline{e} \cdot \underline{r}} \cdot d\underline{l} = -\frac{1}{4\pi} \oint_C \frac{\underline{r} \cdot (d\underline{l} \times \underline{n})}{r + \underline{e} \cdot \underline{r}}. \end{aligned} \tag{5.10}$$

[5] Guiraud, J. P. (1978), "Potential of Velocities Generated by a Localized Vortex Distribution," *Aerospace Research*, English Translation-ESA-TT-560, pp. 105–107.

Because $\underline{r} \cdot (d\underline{l} \times \underline{n})/dl$ and $\underline{e} \cdot \underline{r}$ are constant for the respective side lines (each of which is a straight-line), it can be written as

$$\phi^{(\sigma)} = -\frac{1}{4\pi} \sum_{i=1}^{N_s} b_i \int_{C_i} \frac{1}{r+a}\, dl, \tag{5.11}$$

where N_s is the number of sides of the polygon element, for example, $N_s = 3$ for triangular elements, $a = \underline{e} \cdot \underline{r}$ is a positive constant value for all sides, and $b_i = \underline{r} \cdot (\underline{e}_{l_i} \times \underline{n})$ is constant for one side whose directional vector $\underline{e}_{l_i} = d\underline{l}/dl$ is chosen counterclockwise direction. The vertices composed of the element and the side of directional vector $\underline{e}_{l_i}$ are numbered in counterclockwise order. The field point is at an arbitrary position except on the side lines. It is seen that each integral for the respective side is related to the relative position of the field point.

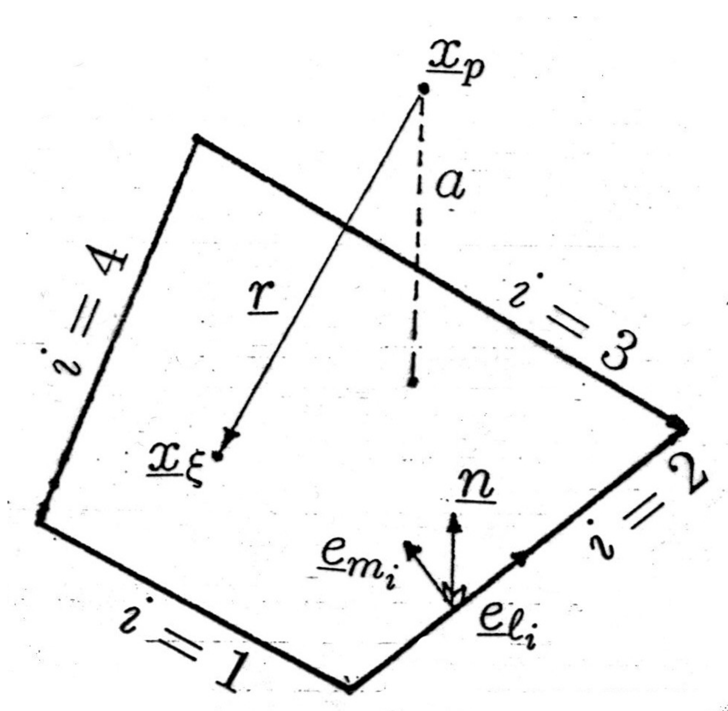

Figure 5.1 Schematic diagram of a planar element.

In the self-induction case that the field point is just above or below the element, we take $\underline{n} \cdot \underline{r} = 0$ in Eq. (5.9) and then the second term of Eq. (5.9) vanishes:

$$\phi^{(\sigma)} = -\frac{1}{4\pi} \underline{n} \cdot \oint_C \frac{\underline{r}}{r} \times d\underline{l}. \tag{5.12}$$

It is reduced to, by setting $a = 0$ in Eq. (5.11)

$$\phi^{(\sigma)} = -\frac{1}{4\pi}\sum_{i=1}^{N_s} b_i \int_{C_i} \frac{1}{r}\, dl. \tag{5.13}$$

This expression is also applied for the case that the field point is on the extension plane of the planar element.

5.3.2 Source distribution: Velocity

The velocity at a field point $\underline{x}_p(x, y, z)$ induced by a distribution of sources with unit density (i.e. $\sigma = 1$) over a planar element is, from Eq. (5.5),

$$\underline{q}^{(\sigma)} = -\frac{1}{4\pi}\left\{\underline{n}\oint_C \underline{A}\cdot d\underline{l} + \underline{n}\times\oint_C \frac{1}{r}d\underline{l}\right\}. \tag{5.14}$$

Rearrange this equation to yield

$$\begin{aligned}
\underline{q}^{(\sigma)} &= -\frac{1}{4\pi}\left\{\underline{n}\oint_C \frac{\underline{e}\times\underline{r}}{r(r+\underline{e}\cdot\underline{r})}\cdot d\underline{l} + \underline{n}\times\oint_C \frac{1}{r}\, d\underline{l}\right\} \\
&= -\frac{1}{4\pi}\left\{\underline{n}(\underline{n}\cdot\underline{e})\oint_C \frac{\underline{n}\times\underline{r}}{r(r+\underline{e}\cdot\underline{r})}\cdot d\underline{l} + \underline{n}\times\oint_C \frac{1}{r}\, d\underline{l}\right\} \\
&= -\frac{1}{4\pi}\left\{\underline{n}(\underline{n}\cdot\underline{e})\oint_C \frac{\underline{r}\cdot(d\underline{l}\times\underline{n})}{r(r+\underline{e}\cdot\underline{r})} + \underline{n}\times\oint_C \frac{1}{r}\, d\underline{l}\right\} \\
&= -\frac{1}{4\pi}\left\{\underline{n}(\underline{n}\cdot\underline{e})\sum_{i=1}^{N_s} b_i \int_{C_i} \frac{1}{r(r+a)}\, dl + \sum_{i=1}^{N_s} \underline{e}_{m_i}\int_{C_i} \frac{1}{r}\, dl\right\},
\end{aligned} \tag{5.15}$$

where $\underline{e}_{m_i} = \underline{n}\times\underline{e}_{l_i}$.

In the self-induction case that the field point is just above the element, the first integral of Eq. (5.14) is reduced to, with $\underline{e} = -\underline{n}$ (representing the approach of the field point toward the upper surface) and $a = 0$,

$$\oint_C \underline{A}\cdot d\underline{l} = \oint_C \frac{-\underline{n}\times\underline{r}}{r^2}\cdot d\underline{l} = \oint_C \frac{-\underline{n}\times r\,\underline{e}_r}{r^2}\cdot(\underline{e}_r\, dr + r\,\underline{e}_\theta\, d\theta) = -2\pi \tag{5.16}$$

Here we have introduced the unit vectors $\underline{e}_r$ and $\underline{e}_\theta$ of the local polar coordinates,

$\underline{x}_p$ being its origin, to define the line segment $d\underline{l}$. Therefore Eq. (5.15) is reduced to

$$\underline{q}^{(\sigma)} = \frac{1}{2}\,\underline{n} - \frac{1}{4\pi}\sum_{i=1}^{N_s} \underline{e}_{m_i} \int_{C_i} \frac{1}{r}\,dl \tag{5.17}$$

On the other hand, when $\underline{x}_p$ approaches toward the lower surface of the element, the sign of the first term of Eq. (5.17) is opposite. If $\underline{x}_p$ is on the (outside) extension plane of the planar panel, the first term in Eq. (5.17) vanishes.

5.3.3 Doublet distribution: Potential

The potential at a field point $\underline{x}_p(x, y, z)$ induced by a distribution of doublets with unit density (i.e. $\mu = 1$) over a planar element is, from Eq. (5.6),

$$\phi^{(\mu)} = +\frac{1}{4\pi}\oint_C \underline{A}\cdot d\underline{l}, \tag{5.18}$$

where μ is defined as $\mu \equiv -\phi$. Rearrange Eq. (5.18) to yield

$$\begin{aligned}
\phi^{(\mu)} &= +\frac{1}{4\pi}\oint_C \frac{\underline{e}\times\underline{r}}{r(r+\underline{e}\cdot\underline{r})}\cdot d\underline{l} = \frac{1}{4\pi}\oint_C \left(\frac{1}{r} - \frac{1}{r+a}\right)\frac{\underline{e}\times\underline{r}}{\underline{e}\cdot\underline{r}}\cdot d\underline{l} \\
&= \frac{1}{4\pi}\frac{1}{\underline{n}\cdot\underline{r}}\oint_C \left(\frac{1}{r} - \frac{1}{r+a}\right)\underline{r}\cdot(d\underline{l}\times\underline{n}) \\
&= \frac{1}{4\pi}(\underline{n}\cdot\underline{e})\sum_{i=1}^{N_s} b_i \int_{C_i} \frac{1}{r(r+a)}\,dl
\end{aligned} \tag{5.19}$$

In the self-induction case that the field point is just above the element, we take $\underline{e} = -\underline{n}$ and the same manner as in derivation of Eq. (5.16) for $\oint_C \underline{A}\cdot d\underline{l}$. It follows that

$$\phi^{(\mu)} = -\frac{\mu}{2} \tag{5.20}$$

For the case that the field point is on the (outside) extension plane of the planar element, this expression is replaced by $\phi^{(\mu)} = 0$.

5.3.4 Doublet distribution: Velocity

The velocity at a field point $\underline{x}_p(x, y, z)$ induced by a distribution of doublets with unit density (i.e. $\mu = 1$) over a planar element is, from Eq. (5.7),

$$\underline{q}^{(\mu)} = -\frac{1}{4\pi} \oint_C \nabla(\frac{1}{r}) \times d\underline{l} \tag{5.21}$$

Rearrange this equation to yield

$$\underline{q}^{(\mu)} = +\frac{1}{4\pi} \oint_C \frac{\underline{r}}{r^3} \times d\underline{l} = +\frac{1}{4\pi} \sum_{i=1}^{N_s} \underline{d}_i \int_{C_i} \frac{1}{r^3}\, dl, \tag{5.22}$$

where $\underline{d}_i = \underline{r} \times \underline{e}_{l_i}$.

Either in the self-induction case that the field point $\underline{x}_p$ is just above the element or in the case that $\underline{x}_p$ is on the (outside) extension plane of the planar element, Eq. (5.22) becomes, with $\underline{d}_i = d_i \underline{n}$,

$$\underline{q}^{(\mu)} = +\frac{1}{4\pi} \underline{n} \sum_{i=1}^{N_s} d_i \int_{C_i} \frac{1}{r^3}\, dl \tag{5.23}$$

5.3.5 Basic integrals

In the preceding sections, we have derived closed-form expressions for the simpler cases, in terms of only the geometric parameters for each side of a polygon i.e., Eqs. (5.11), (5.15), (5.19), (5.22) for field points off the element surface and Eqs. (5.13), (5.17), (5.20), (5.23) for field points on the element surface (for the self-induction cases). There are four types of basic integrals to be evaluated; $\int_{C_i} \frac{1}{r}\, d\xi$, $\int_{C_i} \frac{1}{r+a}\, d\xi$, $\int_{C_i} \frac{1}{r(r+a)}\, d\xi$ and $\int_{C_i} \frac{1}{r^3}\, d\xi$. For the purpose of these evaluations, we define local relative coordinates (X', Z'), as shown in Figure **5.2** , for each side of the polygon in the plane through the field point $\underline{x}_p$ and that side, such that without loss of generality the side line corresponds to the X'-axis and the path of the line integral becomes the positive X'-direction.

First the end points of the side of length l are denoted with $Q_1(x_1, y_1, z_1)$ and

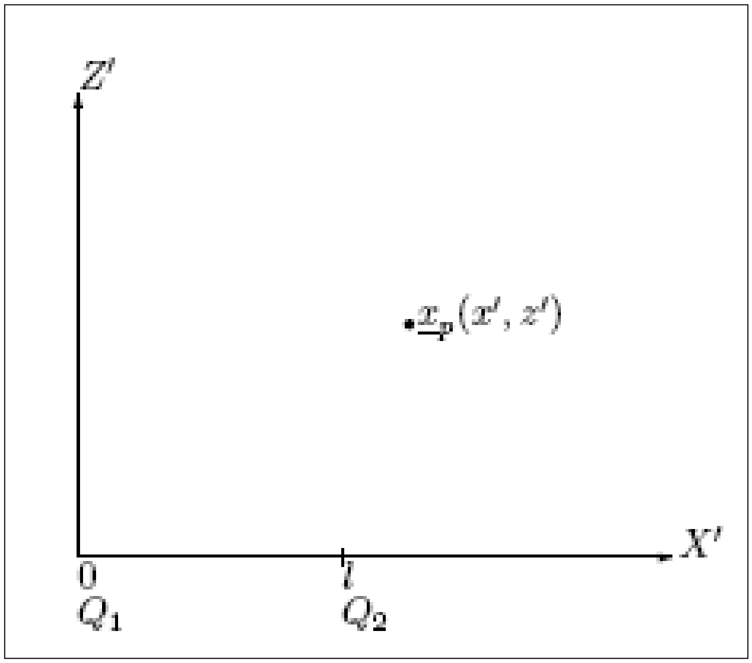

Figure 5.2 A local plane coordinate system for integral over the respective side of a panel. Q_1 and Q_2 denote two vertices of the side with length l.

$Q_2(x_2, y_2, z_2)$ in the global coordinate system, or with $Q_1(0, 0)$ and $Q_2(l, 0)$ in the local plane coordinate system. The field point is also defined by $\underline{x}_p(x_p, y_p, z_p)$ or $\underline{x}_p(x', z')$. The local coordinates (x', z') are related to the global coordinates as follows. The vectors $\underline{Q_1Q_2}$ and $\underline{Q_1x_p}$ are written as

$$\underline{Q_1Q_2} = (x_2 - x_1)\,\underline{i} + (y_2 - y_1)\,\underline{j} + (z_2 - z_1)\,\underline{k} \tag{5.24}$$

$$\underline{Q_1x_p} = (x_p - x_1)\,\underline{i} + (y_p - y_1)\,\underline{j} + (z_p - z_1)\,\underline{k}. \tag{5.25}$$

Then magnitude of cross product of the two vectors is given by

$$|\underline{Q_1Q_2} \times \underline{Q_1x_p}| = |\underline{Q_1Q_2}|\,|\underline{Q_1x_p}|\,|\sin(\angle x_pQ_1Q_2)| = |\underline{Q_1Q_2}|\,|z'|. \tag{5.26}$$

Thus

$$|z'| = |d_i| = \frac{|\underline{Q_1Q_2} \times \underline{Q_1x_p}|}{|\underline{Q_1Q_2}|} = |\underline{e}_l \times \underline{Q_1x_p}|. \tag{5.27}$$

Similarly, from dot product of the two vectors

$$\underline{Q_1Q_2} \cdot \underline{Q_1x_p} = |\underline{Q_1Q_2}|\,|\underline{Q_1x_p}|\,\cos(\angle x_pQ_1Q_2) = |\underline{Q_1Q_2}|\,x'. \tag{5.28}$$

Then

$$x' = \underline{e}_l \cdot \underline{Q_1 x_p}. \tag{5.29}$$

Now the basic integrals can be expressed in terms of x' and z'. For simplicity, we drop the prime $(')$ in x' and z' in the following analysis. [6]

(1) For the integral $I_1 = \displaystyle\int_0^l \frac{1}{r}\, d\xi$,

$$\begin{aligned}
I_1 &= \int_0^l \frac{1}{\sqrt{(x-\xi)^2+z^2}}\, d\xi = \int_0^l \frac{1}{\sqrt{\xi^2-2x\xi+x^2+z^2}}\, d\xi \\
&= \ln\left\{2\sqrt{(\xi-x)^2+z^2}+2(\xi-x)\right\}\Big|_0^l \\
&= \ln\frac{\sqrt{(l-x)^2+z^2}+l-x}{\sqrt{x^2+z^2}-x} \qquad (5.30)\\
&\left(\text{or,}\quad I_1 = \ln\frac{\sqrt{x^2+z^2}+x}{\sqrt{(l-x)^2+z^2}-(l-x)}\right)
\end{aligned}$$

This expression also includes the cases of $z = 0$.

(2) For the integral $I_2 = \displaystyle\int_0^l \frac{1}{r+a}\, d\xi = \int_0^l \frac{1}{\sqrt{(x-\xi)^2+z^2}+a}\, d\xi$, change the integration variable ξ into $t = \sqrt{(x-\xi)^2+z^2}+a$. Then $d\xi = \dfrac{\sqrt{(x-\xi)^2+z^2}}{\xi-x}\, dt$. Consider three cases for the sign of the denominator $\xi - x$.

(a) When $\xi - x \geq 0$ in the entire interval $[0, l]$ (i.e., $x \leq 0$), $d\xi = \dfrac{t-a}{\sqrt{(t-a)^2-z^2}}\, dt$. With the new integration limits $A = \sqrt{x^2+z^2}+a$ and $B = \sqrt{(l-x)^2+z^2}+a$,

$$\begin{aligned}
I_2 &= \int_A^B \frac{t-a}{t\sqrt{t^2-2at+a^2-z^2}}\, dt \\
&= \int_A^B \frac{1}{\sqrt{t^2-2at+a^2-z^2}}\, dt - a\int_A^B \frac{1}{t\sqrt{t^2-2at+a^2-z^2}}\, dt
\end{aligned}$$

[6] For integral formulas, refer to Gradshteyn, I. S. and Ryzhik, I. M. (1965), *Table of Integrals, Series and Products*, Academic Press, Inc., New York and London, pp. 68, 81, 84.

Namely,

$$
\begin{aligned}
I_2 &= \ln\left\{2\sqrt{(t-a)^2-z^2}+2(t-a)\right\}\Big|_A^B \\
&\quad -a\frac{1}{\sqrt{z^2-a^2}}\arcsin\left\{\frac{2(a^2-z^2)-2\,a\,t}{t\sqrt{4\,z^2}}\right\}\Big|_A^B,
\end{aligned}
$$

or

$$
\begin{aligned}
I_2 &= \ln\left\{\frac{\sqrt{(l-x)^2+z^2}+(l-x)}{\sqrt{x^2+z^2}-x}\right\}-\frac{a}{\sqrt{z^2-a^2}}\cdot \\
&\quad \cdot\left[\arcsin\left\{\frac{-z^2-a\sqrt{(l-x)^2+z^2}}{|z|(\sqrt{(l-x)^2+z^2}+a)}\right\}\right. \\
&\quad \left.-\arcsin\left\{\frac{-z^2-a\sqrt{x^2+z^2}}{|z|(\sqrt{x^2+z^2}+a)}\right\}\right]
\end{aligned} \tag{5.31}
$$

(b) When $\xi-x\leq 0$ in the entire interval $[0,l]$ (i.e., $x\geq l$), $d\xi = -\dfrac{t-a}{\sqrt{(t-a)^2-z^2}}\,dt$.

$$
\begin{aligned}
I_2 &= -\int_A^B \frac{t-a}{t\sqrt{t^2-2at+a^2-z^2}}\,dt \\
&= -\int_A^B \frac{1}{\sqrt{t^2-2at+a^2-z^2}}\,dt+\int_A^B \frac{a}{t\sqrt{t^2-2at+a^2-z^2}}\,dt
\end{aligned}
$$

$$
\begin{aligned}
I_2 &= \ln\left\{\frac{\sqrt{(l-x)^2+z^2}+(l-x)}{\sqrt{x^2+z^2}-x}\right\}+\frac{a}{\sqrt{z^2-a^2}}\cdot \\
&\quad \cdot\left[\arcsin\left\{\frac{-z^2-a\sqrt{(l-x)^2+z^2}}{|z|(\sqrt{(l-x)^2+z^2}+a)}\right\}\right. \\
&\quad \left.-\arcsin\left\{\frac{-z^2-a\sqrt{x^2+z^2}}{|z|(\sqrt{x^2+z^2}+a)}\right\}\right]
\end{aligned} \tag{5.32}
$$

(c) When $0<x<l$, the integration interval is divided into two parts to

apply the above two cases:

$$
\begin{aligned}
I_2 &= \int_0^l \frac{1}{r+a}\, d\xi \\
&= \int_0^x \frac{1}{\sqrt{(x-\xi)^2+z^2}+a}\, d\xi + \int_x^l \frac{1}{\sqrt{(x-\xi)^2+z^2}+a}\, d\xi
\end{aligned}
$$

$$
\begin{aligned}
I_2 = & -\Bigg[\ln\left\{2\sqrt{(t-a)^2-z^2}+2\,(t-a)\right\}\Bigg|_A^{|z|+a} \\
& -a\frac{1}{\sqrt{z^2-a^2}}\arcsin\left\{\frac{2(a^2-z^2)-2\,a\,t}{t\sqrt{4z^2}}\right\}\Bigg|_A^{|z|+a}\Bigg] \\
& +\ln\left\{2\sqrt{(t-a)^2-z^2}+2(t-a)\right\}\Bigg|_{|z|+a}^{A} \\
& -a\frac{1}{\sqrt{z^2-a^2}}\arcsin\left\{\frac{2(a^2-z^2)-2\,a\,t}{t\sqrt{4z^2}}\right\}\Bigg|_{|z|+a}^{A}
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
I_2 = & \ln\left\{\frac{\sqrt{(l-x)^2+z^2}+(l-x)}{\sqrt{x^2+z^2}-x}\right\} - \frac{a}{\sqrt{z^2-a^2}}\cdot \\
& \cdot\left[\pi+\arcsin\left\{\frac{-z^2-a\sqrt{(l-x)^2+z^2}}{|z|(\sqrt{(l-x)^2+z^2}+a)}\right\}\right. \\
& \left.+\arcsin\left\{\frac{-z^2-a\sqrt{x^2+z^2}}{|z|(\sqrt{x^2+z^2}+a)}\right\}\right]
\end{aligned}
\tag{5.33}
$$

If $z^2=a^2$, the integral should be performed by the simplified form;

$$
\begin{aligned}
I_2 = & \ln\left\{\frac{\sqrt{(l-x)^2+z^2}+(l-x)}{\sqrt{x^2+z^2}-x}\right\} \\
& -\left\{\frac{l-x}{\sqrt{(l-x)^2+z^2}+a}+\frac{x}{\sqrt{x^2+z^2}+a}\right\}
\end{aligned}
\tag{5.34}
$$

(3) For the integral $I_3 = \int_0^l \frac{1}{r(r+a)}\,d\xi$, take partial fraction to use the preceding results:

$$I_3 = \frac{1}{a}\int_0^l \left(\frac{1}{r} - \frac{1}{r+a}\right) d\xi = \frac{1}{a}(I_1 - I_2) \tag{5.35}$$

But if $a = 0$ for which this expression is not valid, another form should be performed;

$$\begin{aligned} I_3 &= \int_0^l \frac{1}{(x-\xi)^2 + z^2}\,d\xi = \int_0^l \frac{1}{\xi^2 - 2x\xi + x^2 + z^2}\,d\xi \\ &= \frac{1}{|z|}\arctan\left(\frac{\xi - x}{|z|}\right)\Big|_0^l = \frac{1}{|z|}\left(\arctan\frac{l-x}{|z|} + \arctan\frac{x}{|z|}\right) \end{aligned} \tag{5.36}$$

Furthermore if $a = 0$ and $z = 0$, then

$$I_3 = \int_0^l \frac{1}{(x-\xi)^2}\,d\xi = \frac{1}{x-l} - \frac{1}{x}. \tag{5.37}$$

(4) For the integral $I_4 = \int_0^l \frac{1}{r^3}\,d\xi$,

$$\begin{aligned} I_4 &= \int_0^l \frac{1}{\sqrt{(x-\xi)^2 + z^2}^3}\,d\xi = \frac{\xi - x}{z^2\sqrt{(x-\xi)^2 + z^2}}\Bigg|_0^l \\ &= \frac{1}{z^2}\left\{\frac{l-x}{\sqrt{(l-x)^2 + z^2}} + \frac{x}{\sqrt{x^2 + z^2}}\right\} \end{aligned} \tag{5.38}$$

If $z = 0$, it should be replaced by

$$I_4 = \int_0^l \frac{1}{|x-\xi|^3}\,d\xi = \begin{cases} +\frac{1}{2}\left\{\frac{1}{(l-x)^2} - \frac{1}{x^2}\right\} & \text{when } x > l \\ -\frac{1}{2}\left\{\frac{1}{(l-x)^2} - \frac{1}{x^2}\right\} & \text{when } x < 0 \end{cases} \tag{5.39}$$

When the inverse trigonometric functions are implemented in the computational algorithm, their values are evaluated in the interval $[-\pi/2, \pi/2]$ without considering the separate arguments of the functions.

5.3.6 Test calculations for constant distributions

A planar rectangular element of 12×1 is adopted for test calculations herein. It may be assumed that the element is in the plane $z = 0$ with the four vertices located at $(0, 0, 0)$,$(1, 0, 0)$, $(1, 12, 0)$,$(0, 12, 0)$, respectively. To check a sensitivity of the calculation, we take various field points in the vicinity of the element surface or one vertex. The coordinates of the field points are, with labelling, $P1(0.5, 6, 0)$, $P2(0.5, 6, +0.00001)$, $P3(0.5, 6, -0.00001)$, $P4(0, -0.00001, 0)$, $P5(0, 0, +0.00001)$ and $P6(0, 0, -0.00001)$. The points $P1, P2$ and $P3$ are on, just above and below, respectively, of the centroid of the element and $P4, P5$ and $P6$ are points very near one vertex of the origin. The constant densities of source and doublet distributions are taken with 1, (i.e., $\sigma = 1, \mu = -\phi = 1$).

First we will compare numerical integrations and analytical evaluations of the basic integrals described in the preceding subsection. At the field point E, the evaluations are compared in Table **5.1**. The influences of the basic integrals at the field points by the respective sides of the element are listed. The side 1 denotes the line between the vertices $(0, 0, 0)$ and $(1, 0, 0)$ and the other sides are numbered in a counterclockwise order in a similar way. In the numerical calculations, the Gaussian quadrature is used with various quadrature-base points to show numerical convergence. It is seen that for a field point having a numerically singular behavior in line integral, a large number of quadrature-base points are required to reach the same order as the analytical evaluations. It results in large computing time undesirably.

The numerical and analytical evaluations for the induced velocities and potentials at the selected field points due to the uniform source and doublet distributions over the rectangular element are compared in Table **5.2**.

Table 5.1 Comparison of the Basic Integrals by Analytic and Numerical Calculation at Point $P5(0.0,\ 0.0,\ +0.00001)$.

	Side	Gaussian-Quadrature Points, $N =$				Analytic
		20	100	500	2500	
$\int \frac{1}{r}\, d\xi$	1	.7195E+01	.1037E+02	.1224E+02	.1221E+02	.1221E+02
	2	.3180E+01	.3180E+01	.3180E+01	.3180E+01	.3180E+01
	3	.8324E-01	.8324E-01	.8324E-01	.8324E-01	.8324E-01
	4	.7195E+01	.1037E+02	.1356E+02	.1469E+02	.1469E+02
$\int \frac{1}{(r+a)}\, d\xi$	1	.7187E+01	.1018E+02	.1121E+02	.1121E+02	.1121E+02
	2	.3180E+01	.3180E+01	.3180E+01	.3180E+01	.3180E+01
	3	.8324E-01	.8324E-01	.8324E-01	.8324E-01	.8324E-01
	4	.7195E+01	.1036E+02	.1320E+02	.1369E+02	.1369E+02
$\int \frac{1}{r(r+a)}\, d\xi$	1	.8378E+03	.1893E+05	.1025E+06	.1000E+06	.1000E+06
	2	.1488E+01	.1488E+01	.1488E+01	.1488E+01	.1488E+01
	3	.6928E-02	.6928E-02	.6928E-02	.6928E-02	.6928E-02
	4	.6998E+02	.1674E+04	.3636E+05	.9987E+05	.1000E+06
$\int \frac{1}{r^3}\, d\xi$	1	.2211E+06	.1266E+09	.1084E+11	.1000E+11	.1000E+11
	2	.9965E+00	.9965E+00	.9965E+00	.9965E+00	.9965E+00
	3	.5767E-03	.5767E-03	.5767E-03	.5767E-03	.5767E-03
	4	.1536E+04	.8856E+06	.5284E+09	.1004E+11	.1000E+11

Table 5.2 Comparison of Potentials and Velocities by Analytic and Numerical Calculation at Point $P2(0.5,\ 6.0,\ +0.00001)$.

	Gaussian-Quadrature Points, $N =$				Analytic
	20	100	500	2500	
$\phi^{(\sigma)}$	-0.6583E+00	-0.6650E+00	-0.6650E+00	-0.6650E+00	-0.6650E+00
$q_x^{(\sigma)}$	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
$q_y^{(\sigma)}$	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
$q_z^{(\sigma)}$	0.4681E+00	0.5000E+00	0.5000E+00	0.5000E+00	0.5000E+00
$\phi^{(\mu)}$	-0.4681E+00	-0.5000E+00	-0.5000E+00	-0.5000E+00	-0.5000E+00
$q_x^{(\mu)}$	0.5371E-21	0.2791E-22	0.5166E-21	0.3339E-21	-0.2264E-21
$q_y^{(\mu)}$	-0.5143E-25	0.2006E-25	0.2288E-25	0.2600E-24	0.2309E-24
$q_z^{(\mu)}$	0.5366E+00	0.6388E+00	0.6388E+00	0.6388E+00	0.6388E+00

5.3.7 Extension to linear distributions

The preceding analysis can be extended to include a linear distribution of sources and dipoles on each panel. In order to determine the distribution function uniquely, we take only three points (that are not collinear) of a polygon. Therefore we consider a triangular element of a linear source distribution herein. For the doublet distribution, the following procedure can be applied in a similar manner. A form of the linear varying source strength is specified as

$$\sigma(x, y, z) = A\,x + B\,y + C\,z + D \tag{5.40}$$

The coefficients A, B, C and D are determined from the singularity strength values at the vertices.

Define the unit directional vectors of the respective sides (of length l_i) of the element by $\underline{e}_{l_i}, (i = 1, 2, 3)$ and the vertex positions by (x_i, y_i, z_i). From elementary geometry for a corresponding linear source (or doublet) distribution, we can form a linear equation system for A, B and C;

$$\nabla\sigma\cdot\underline{e}_{l_i} \equiv \frac{\sigma_{i+1} - \sigma_i}{l_i} = \frac{A(x_{i+1} - x_i) + B(y_{i+1} - y_i) + C(z_{i+1} - z_i)}{l_i}, \quad i = 1, 2, 3 \tag{5.41}$$

Here the vertices and sides are defined in a counterclockwise sense and the index 4 corresponds to 1. By the Cramer's rule, A, B, C and D are determined ;

$$A = \text{det}\ (\sigma_i - \sigma_{i+1}, y_i - y_{i+1}, z_i - z_{i+1})/\triangle, \tag{5.42}$$

$$B = \text{det}\ (x_i - x_{i+1}, \sigma_i - \sigma_{i+1}, z_i - z_{i+1})/\triangle, \tag{5.43}$$

$$C = \text{det}\ (x_i - x_{i+1}, y_i - y_{i+1}, \sigma_i - \sigma_{i+1})/\triangle, \tag{5.44}$$

$$D = \sigma_1 - (A\,x_1 + B\,y_1 + C\,z_1) \tag{5.45}$$

where det $(\cdots)$ denotes the determinant of a matrix and $\triangle =$ det $(x_i - x_{i+1}, y_i - y_{i+1}, z_i - z_{i+1})$. For a given distribution shape, we can sum up the contribution of the associated line integrals for each side, as in the constant distribution cases. In the linear variation cases, additionally there are integral forms to be evaluated:

$$\int_0^l \frac{\xi}{r}\,d\xi, \quad \int_0^l \frac{\xi}{r+a}\,d\xi, \quad \int_0^l \frac{\xi}{r(r+a)}\,d\xi, \quad \int_0^l \frac{\xi}{r^3}\,d\xi,$$
$$\int_0^l r\,d\xi, \quad \int_0^l \ln r\,d\xi, \quad \int_0^l \ln(r+a)\,d\xi. \tag{5.46}$$

These integrals are performed without explicit representation in the following manner, by referring to the integrals described in the preceding section and to some of integral formulas in Gradshteyn & Ryzhik (1965). [7]

(1) For $J_1 = \displaystyle\int_0^l \frac{\xi}{r}\,d\xi$,

$$J_1 = \int_0^l \frac{\xi - x + x}{\sqrt{(x-\xi)^2 + z^2}}\,d\xi = \sqrt{(\xi - x)^2 + z^2}\Big|_0^l + x\,I_1 \tag{5.47}$$

(2) For $J_2 = \displaystyle\int_0^l \frac{\xi}{r+a}\,d\xi$,

$$\begin{aligned} J_2 &= \int_0^l \frac{\xi - x + x}{\sqrt{(x-\xi)^2 + z^2} + a}\,d\xi \\ &= \sqrt{(x-\xi)^2 + z^2} - a\ln\left\{\sqrt{(x-\xi)^2 + z^2} + a\right\}\Big|_0^l + x\,I_2 \end{aligned} \tag{5.48}$$

Here we have changed the integration variable: $t = \sqrt{(x-\xi)^2 + z^2} + a$.

(3) For $J_3 = \displaystyle\int_0^l \frac{\xi}{r(r+a)}\,d\xi$,

$$\begin{aligned} J_3 &= \frac{1}{a}\int_0^l \left(\frac{\xi}{r} - \frac{\xi}{r+a}\right)d\xi = \frac{1}{a}(J_1 - J_2) \\ &= \ln\left\{\sqrt{(x-\xi)^2 + z^2} + a\right\}\Big|_0^l + x\,I_3 \end{aligned} \tag{5.49}$$

[7] Gradshteyn, I. S. and Ryzhik, I. M. (1965), *Table of Integrals, Series and Products*, Academic Press, Inc., New York and London.

But if $a = 0$ for which this expression is not valid, other form should be performed;

$$
\begin{aligned}
J_3 &= \int_0^l \frac{\xi}{r^2}\, d\xi = \int_0^l \frac{\xi - x + x}{(x-\xi)^2 + z^2}\, d\xi \\
&= \frac{1}{2} \ln \left\{ (\xi - x)^2 + z^2 \right\} \Big|_0^l + x \int_0^l \frac{1}{r^2}\, d\xi
\end{aligned}
\tag{5.50}
$$

The last integral has been already considered in the preceding section as the special case of the integral I_3.

(4) For $J_4 = \displaystyle\int_0^l \frac{\xi}{r^3}\, d\xi$,

$$
J_4 = \int_0^l \frac{\xi - x + x}{\sqrt{(x-\xi)^2 + z^2}^{\,3}}\, d\xi = -\frac{1}{\sqrt{(x-\xi)^2 + z^2}} \Bigg|_0^l + x\, I_4 \tag{5.51}
$$

(5) For $J_5 = \displaystyle\int_0^l r\, d\xi$,

$$
J_5 = \int_0^l \sqrt{(x-\xi)^2 + z^2}\, d\xi = \frac{1}{2} (\xi - x)\sqrt{(\xi - x)^2 + z^2} \Big|_0^l + \frac{1}{2} z^2\, I_1 \tag{5.52}
$$

(6) For $J_6 = \displaystyle\int_0^l \ln r\, d\xi$,

$$
\begin{aligned}
J_6 &= \frac{1}{2} \int_0^l \ln \left\{ (x-\xi)^2 + z^2 \right\}\, d\xi \\
&= \frac{1}{2} \left[(\xi - x) \ln \left\{ (\xi - x)^2 + z^2 \right\} - 2\xi + 2|z| \arctan \left(\frac{\xi - x}{|z|} \right) \right] \Bigg|_0^l
\end{aligned}
\tag{5.53}
$$

(7) For $J_7 = \displaystyle\int_0^l \ln(r + a)\, d\xi$, take an integration by parts and then the same procedure as I_2:

$$\begin{aligned} J_7 &= \int_0^l \ln\left\{\sqrt{(x-\xi)^2+z^2}+a\right\} d\xi \\ &= (\xi - x)\ \ln(\sqrt{(x-\xi)^2+z^2}+a) - \xi \\ &\quad +a\ \ln(\xi - x + \sqrt{(x-\xi)^2+z^2}) \\ &\quad + \sqrt{z^2-a^2}\ \arcsin\left\{\frac{-z^2 - a\sqrt{(\xi-x)^2+z^2}}{|z|(\sqrt{(\xi-x)^2+z^2}+a)}\right\}\Bigg|_0^l \end{aligned} \tag{5.54}$$

5.4 Bilinear Source and Doublet Distribution

5.4.1 Introduction

This section deals with evaluations of the surface integrals in the potential-based panel method, associated with bilinear density distributions of source and/or doublet singularities over a planar panel. The surface integrals can be transformed into contour integrals by using Stokes' formulas after simple manipulation on the integrands. We also present the closed-forms for obtaining the induced potentials and velocities due to those singularity distributions over a polygon panel.

5.4.2 Transformation of the surface integrals for Stokes' theorem

Without loss of generality we will consider the domain of one planar panel for the integration region as a part of the discretized boundary surface in Eq. (5.3). We take an orthogonal coordinate system (ξ, η, ζ) to specify a bilinear form, such that the panel is in the plane $\zeta = 0$ and the direction of ζ-axis is the same as that of the unit normal vector ($\underline{n}$) of the panel, as shown in Figure **5.3** . The unit vectors in the direction of ξ-axis and η-axis are denoted by $\underline{e}_\xi$ and $\underline{e}_\eta$, respectively. These two axes may be chosen arbitrarily in the directions but lying on the panel surface. The coordinates (x, y, z) of the field point $\underline{x}_p$ are relatively measured from the origin of the coordinate system.

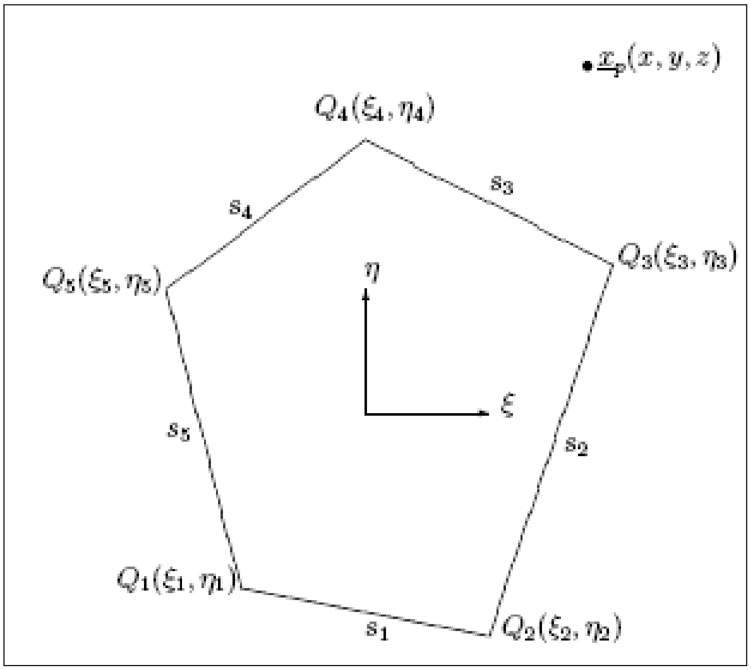

Figure 5.3 A planar panel defined in a local coordinate system. The present derivation can be applied to arbitrary polygons. A pentagon is taken with no loss of generality.

The potentials and the velocities at a field point $\underline{x}_p(x,y,z)$ induced by a bilinear source distribution $\sigma = a_0 + a_1\,\xi + a_2\,\eta + a_3\,\xi\,\eta$ and by a doublet distribution $\mu = b_0 + b_1\,\xi + b_2\,\eta + b_3\,\xi\,\eta$, respectively, can be written as,

$$\begin{aligned}
\phi^{(\sigma)} &= -\frac{1}{4\pi}\Bigg[(a_0 + a_1\,x + a_2\,y + a_3\,x\,y)\int_S \frac{1}{r}\,dS \\
&\quad +(a_1 + a_3\,y)\int_S \frac{\xi - x}{r}\,dS + (a_2 + a_3\,x)\int_S \frac{\eta - y}{r}\,dS \\
&\quad +a_3\int_S \frac{(\xi - x)\,(\eta - y)}{r}\,dS\Bigg] && (5.55)
\end{aligned}$$

$$\begin{aligned}
\underline{q}^{(\sigma)} &= +\frac{1}{4\pi}\Bigg[(a_0 + a_1\,x + a_2\,y + a_3\,x\,y)\int_S \nabla\left(\frac{1}{r}\right)\,dS \\
&\quad +(a_1 + a_3\,y)\int_S (\xi - x)\nabla\left(\frac{1}{r}\right)\,dS \\
&\quad +(a_2 + a_3\,x)\int_S (\eta - y)\nabla\left(\frac{1}{r}\right)\,dS \\
&\quad +a_3\int_S (\xi - x)\,(\eta - y)\nabla\left(\frac{1}{r}\right)\,dS\Bigg] && (5.56)
\end{aligned}$$

$$
\begin{aligned}
\phi^{(\mu)} &= -\frac{1}{4\pi}\underline{n}\cdot\Bigg[(b_0+b_1\,x+b_2\,y+b_3\,x\,y)\int_S\nabla\left(\frac{1}{r}\right)\,dS \\
&\qquad +(b_1+b_3\,y)\int_S(\xi-x)\nabla\left(\frac{1}{r}\right)\,dS \\
&\qquad +(b_2+b_3\,x)\int_S(\eta-y)\nabla\left(\frac{1}{r}\right)\,dS \\
&\qquad +b_3\int_S(\xi-x)\,(\eta-y)\nabla\left(\frac{1}{r}\right)\,dS\Bigg] \qquad (5.57)\\
\underline{q}^{(\mu)} &= -\frac{1}{4\pi}\left\{\oint_C\mu\nabla\left(\frac{1}{r}\right)\times d\underline{l}_\xi+\int_S(\underline{n}\times\nabla\mu)\times\nabla\left(\frac{1}{r}\right)\,dS)\right\}\\
&= -\frac{1}{4\pi}\Bigg[\oint_C(b_0+b_1\,\xi+b_2\,\eta+b_3\,\xi\,\eta)\nabla\left(\frac{1}{r}\right)\times d\underline{l}\\
&\qquad +\int_S\left[\{b_1+b_3\,y+b_3\,(\eta-y)\}\,\underline{e}_\eta\right.\\
&\qquad\quad \left.-\{b_2+b_3\,x+b_3\,(\xi-x)\}\,\underline{e}_\xi\right]\times\nabla\left(\frac{1}{r}\right)\,dS\Bigg] \qquad (5.58)
\end{aligned}
$$

Here Eq. (5.58) has been expressed in the form of the vortex distribution equivalent to the doublet distribution (Lee, J. T. (1987), Brockett (1988)). [8]

In the case of a bilinear singularity distribution, eight different integrands are involved in the surface integrals in Eq. (5.3), which will be described later on. For use of Stokes' formulas, all integrands are changed into equivalent ones either in curl-forms of a vector or in cross product-forms of a vector with the normal $\underline{n}$ as follows:

$$\frac{1}{r} = \underline{e}\cdot(\nabla\times\underline{B}), \quad \text{with} \quad \underline{B}=\frac{\underline{e}\times\underline{r}}{r+\underline{e}\cdot\underline{r}} \qquad (5.59)$$

$$\frac{\xi-x}{r} = \underline{e}_\eta\cdot(\underline{n}\times\nabla r) \qquad (5.60)$$

$$\frac{\eta-y}{r} = -\underline{e}_\xi\cdot(\underline{n}\times\nabla r) \qquad (5.61)$$

$$\frac{(\xi-x)\,(\eta-y)}{r} = \underline{e}_\eta\cdot[\underline{n}\times\nabla\{(\eta-y)\,r\}] \qquad (5.62)$$

[8] Lee, J. T. (1987), *A Potential Based Panel Method for the Analysis of Marine Propellers in Steady Flow*, PhD. thesis, Department of Ocean Engineering, MIT, Report no. 87-13.
Brockett, T. E. (1988), *NA 520 Lecture Notes*, (unpublished), Department of Naval Architecture and Marine Engineering, University of Michigan.

$$\nabla\left(\frac{1}{r}\right) = -\nabla\times\underline{A}, \quad \text{with} \quad \underline{A}=\frac{\underline{e}\times\underline{r}}{r(r+\underline{e}\cdot\underline{r})} \tag{5.63}$$

$$\begin{aligned}(\xi-x)\nabla\left(\frac{1}{r}\right) &= \left[\underline{e}_\eta\cdot\left\{\underline{n}\times\nabla\left(\frac{\xi-x}{r}\right)\right\}-\frac{1}{r}\right]\underline{e}_\xi \\ &\quad -\left[\underline{e}_\xi\cdot\left\{\underline{n}\times\nabla\left(\frac{\xi-x}{r}\right)\right\}\right]\underline{e}_\eta \\ &\quad -z\left[\underline{e}_\eta\cdot\left\{\underline{n}\times\nabla\left(\frac{1}{r}\right)\right\}\right]\underline{n}\end{aligned} \tag{5.64}$$

$$\begin{aligned}(\eta-y)\nabla\left(\frac{1}{r}\right) &= \left[\underline{e}_\eta\cdot\left\{\underline{n}\times\nabla\left(\frac{\eta-y}{r}\right)\right\}\right]\underline{e}_\xi \\ &\quad -\left[\underline{e}_\xi\cdot\left\{\underline{n}\times\nabla\left(\frac{\eta-y}{r}\right)\right\}+\frac{1}{r}\right]\underline{e}_\eta \\ &\quad +z\left[\underline{e}_\xi\cdot\left\{\underline{n}\times\nabla\left(\frac{1}{r}\right)\right\}\right]\underline{n}\end{aligned} \tag{5.65}$$

$$\begin{aligned}(\xi-x)\,(\eta-y)\nabla\left(\frac{1}{r}\right) &= -\underline{e}_\xi\cdot\left[\underline{n}\times\nabla\left\{\frac{(\xi-x)^2}{r}\right\}\right]\underline{e}_\xi \\ &\quad -\underline{e}_\eta\cdot\left[\underline{n}\times\nabla\left\{\frac{(\eta-y)^2}{r}\right\}\right]\underline{e}_\eta \\ &\quad +z\,\underline{e}_\xi\cdot\left[\underline{n}\times\nabla\left\{\frac{(\xi-x)}{r}\right\}\right]\underline{n}\end{aligned} \tag{5.66}$$

Equations (5.59) and (5.63) have been introduced by Suh (1992) and Guiraud (1978), respectively, [9] which can be also derived by direct manipulation with starting from the right sides. Of course, these two relations can be simply used if one wants to compute the induced potentials and velocities due to the constant source and doublet distributions.

In the constant distribution cases, although the results are consistent, the present approach using these key relations is considered simpler than those presented by Newman (1986) and by Cantaloube & Rehbach (1986). [10] The re-

[9] Suh, J.-C. (1992a), "Analytical evaluation of the surface integral in the singularity methods," *Trans. Soc. Naval Arch. Korea*, vol. 29, no. 1, pp. 1–17.
Guiraud, J. P. (1978), "Potential of velocities generated by a localized vortex distribution," *Aerospace Research*, English Translation-ESA-TT-560, pp. 105–107.

[10] Newman, J. N. (1986), "Distributions of sources and normal dipoles over a quadrilateral panel," *J. Eng. Math.*, vol. 20, pp. 113–126.
Cantaloube, B. and Rehbach, C. (1986), "Calcul des Integrales de la Methode des Singularites," *Recherche Aerospatiale*, n^o 1, pp. 15–22, English Title: "Calculation of the integrals of the singularity method," *Aerospace*

maining equations have been derived by a similar deduction, under hypothesis of planarity of the panel. The distance vector $\underline{r}$ is defined as $\underline{x}_\xi - \underline{x}_p$ where the subscripts ξ and p refer to the source point and the field point respectively. While Eqs. (5.59) and (5.63) hold for any $\underline{e}$ independent of the integration point $\underline{x}_\xi$ more generally, the unit vector $\underline{e}$ is taken as $\pm\underline{n}$ for application of Stokes' transformation, where the sign is chosen such that $\underline{e} \cdot \underline{r}$ is not negative.

Using Stokes' formulas of the surface integrals with the alternative forms for a polygon panel, we can express the surface integrals as a sum of the associated line integrals for each side of the panel with independent treatment of the contribution from the side. Each contribution can be written as closed-forms in term of only the geometrical parameters of the side as described in the following section.

5.4.3 Induced potential due to source distribution

The potential at a field point $\underline{x}_p(x, y, z)$ induced by a bilinear source distribution $\sigma = a_0 + a_1\xi + a_2\eta + a_3\xi\eta$, Eq. (5.55) can be written as,

$$\begin{aligned} \phi^{(\sigma)} &= -\frac{1}{4\pi} \int_S \frac{\sigma}{r}\, dS \\ &= -\frac{1}{4\pi} \int_S \left\{ c_1 \frac{1}{r} + c_2 \frac{\xi - x}{r} + c_3 \frac{\eta - y}{r} + c_4 \frac{(\xi - x)(\eta - y)}{r} \right\} dS \end{aligned} \tag{5.67}$$

where for shortness of expressions we have defined the constants $c_1 = a_0 + a_1 x + a_2 y + a_3 x y, c_2 = a_1 + a_3 y, c_3 = a_2 + a_3 x, c_4 = a_3$. Using Eq. (5.59) through Eq. (5.63) for the corresponding integrands and then performing Stokes' transformations, we can write Eq. (5.67) as, in terms of line integrals,

$$\begin{aligned} \phi^{(\sigma)} = -\frac{1}{4\pi} \Bigg[& c_1 \oint_C \frac{\underline{r} \cdot (d\underline{l} \times \underline{n})}{r + \underline{e} \cdot \underline{r}} + c_2\, \underline{e}_\eta \cdot \oint_C r\, d\underline{l} - c_3\, \underline{e}_\xi \cdot \oint_C r\, d\underline{l} \\ & + c_4\, \underline{e}_\eta \cdot \oint_C (\eta - y)\, r\, d\underline{l} \Bigg] \end{aligned} \tag{5.68}$$

Research, no. 1, pp. 15–22.

The term $\underline{r} \cdot (d\underline{l} \times \underline{n})/dl$ represents the projection of the distance vector $\underline{r}$ onto the vector perpendicular to both $d\underline{l}$ and $\underline{n}$. Because it is constant for each side of a straight-line and $\underline{e} \cdot \underline{r}$ ($\equiv a$, that is, the normal distance of the field point from the panel) is a non-negative constant for all sides of the planar panel, Eq. (5.68) can be written as

$$\begin{aligned}\phi^{(\sigma)} &= -\frac{1}{4\pi}\sum_{i=1}^{N_S}\left[c_1\,t_i\int_{C_i}\frac{1}{r+a}\,dl + c_2\,v_i\int_{C_i} r\,dl - c_3\,u_i\int_{C_i} r\,dl\right.\\ &\qquad\left. +c_4\,v_i\int_{C_i}(\eta-y)\,r\,dl\right]\end{aligned} \tag{5.69}$$

The index i denotes the integer for identification of the side concerned, N_S is the number of sides of the polygon panel (e.g., $N_S = 3$ for triangular panels), $t_i = \underline{r}\cdot(\underline{e}_{l_i}\times\underline{n})$, $u_i = \underline{e}_\xi\cdot\underline{e}_{l_i}$ and $v_i = \underline{e}_\eta\cdot\underline{e}_{l_i}$. The directional vector $\underline{e}_{l_i} = d\underline{l}/dl$ is chosen in a counterclockwise direction as the convention of the contour integral. Rewriting the last integral in Eq. (5.69) in terms of the local coordinates of the nodes, we finally obtain the expression for the source-induced potential:

$$\boxed{\begin{aligned}\phi^{(\sigma)} &= -\frac{1}{4\pi}\sum_{i=1}^{N_S}\left[c_1\,t_i\int_{C_i}\frac{1}{r+a}\,dl + c_2\,v_i\int_{C_i} r\,dl - c_3\,u_i\int_{C_i} r\,dl\right.\\ &\qquad\left. +c_4\,v_i\,(\eta_i-y)\int_{C_i} r\,dl + c_4\,v_i^2\int_{C_i} l\,r\,dl\right]\end{aligned} \qquad (5.70)}$$

Here l is the integral variable representing the arc-length along the straight-line of each integration path C_i. The vertices composed of the panel (ξ_i, η_i) and the sides are also defined in a counterclockwise order. It is seen that the integral term for each side is related to the relative position of the field point from the side. Each integral, as will be shown, depends only on the coordinates of the two end points of the corresponding side. Equation (5.70) can be directly used even in the cases of that the field points are just at the panel surface (i.e., in the self-induction cases), by setting $a = 0$ in Eq. (5.70) since $\underline{n}\cdot\underline{r} = 0$. Furthermore, when the field point is just at the side of the panel, the first term vanishes because t_i decays faster than the integral with r approaches zero, while the other terms have finite values.

5.4.4 Induced velocity due to source distribution

The source-induced velocity at the field point, Eq. (5.56) is expressed as

$$\begin{aligned} \underline{q}^{(\sigma)} &= +\frac{1}{4\pi}\int_S \sigma\,\nabla\left(\frac{1}{r}\right)\,dS \\ &= \frac{1}{4\pi}\left[c_1\int_S \nabla\left(\frac{1}{r}\right)\,dS + c_2\int_S(\xi-x)\nabla\left(\frac{1}{r}\right)\,dS\right. \\ &\quad \left. +c_3\int_S(\eta-y)\nabla\left(\frac{1}{r}\right)\,dS + c_4\int_S(\xi-x)\,(\eta-y)\nabla\left(\frac{1}{r}\right)\,dS\right] \end{aligned} \tag{5.71}$$

We rearrange the integrand of the first integral in Eq. (5.71):

$$\nabla\left(\frac{1}{r}\right) = \underline{n}\left\{\underline{n}\cdot\nabla\left(\frac{1}{r}\right)\right\} - \underline{n}\times\left\{\underline{n}\times\nabla\left(\frac{1}{r}\right)\right\} \tag{5.72}$$

Like Eq. (5.67), using Eq. (5.63) through Eq. (5.66) and rearranging the resulting expressions give us the expression for the source-induced velocity:

$$\boxed{\begin{aligned} \underline{q}^{(\sigma)} &= \frac{1}{4\pi}\sum_{i=1}^{N_S}\left[-c_1\left\{\underline{n}(\underline{n}\cdot\underline{e})\,t_i\int_{C_i}\frac{1}{r(r+a)}\,dl + \underline{e}_{m_i}\int_{C_i}\frac{1}{r}\,dl\right\}\right. \\ &\quad +c_2\,\underline{e}_\xi\left\{v_i\left\{(\xi_i-x)\int_{C_i}\frac{1}{r}\,dl + u_i\int_{C_i}\frac{l}{r}\,dl\right\} - t_i\int_{C_i}\frac{1}{r+a}\,dl\right\} \\ &\quad +c_2\,\underline{e}_\eta(-u_i)\left\{(\xi_i-x)\int_{C_i}\frac{1}{r}\,dl + u_i\int_{C_i}\frac{l}{r}\,dl\right\} + c_2\,\underline{n}v_i(-z)\int_{C_i}\frac{1}{r}\,dl \\ &\quad +c_3\,\underline{e}_\xi\,v_i\left\{(\eta_i-y)\int_{C_i}\frac{1}{r}\,dl + v_i\int_{C_i}\frac{l}{r}\,dl\right\} + c_3\,\underline{n}u_i z\int_{C_i}\frac{1}{r}\,dl \\ &\quad +c_3\,\underline{e}_\eta\left\{-u_i\left\{(\eta_i-y)\int_{C_i}\frac{1}{r}\,dl + v_i\int_{C_i}\frac{l}{r}\,dl\right\} - t_i\int_{C_i}\frac{1}{r+a}\,dl\right\} \\ &\quad +c_4\,\underline{e}_\xi(-u_i)\left\{(\xi_i-x)^2\int_{C_i}\frac{1}{r}\,dl + 2(\xi_i-x)u_i\int_{C_i}\frac{l}{r}\,dl + u_i^2\int_{C_i}\frac{l^2}{r}\,dl\right\} \\ &\quad +c_4\,\underline{e}_\eta\,v_i\left\{(\eta_i-y)^2\int_{C_i}\frac{1}{r}\,dl + 2(\eta_i-y)v_i\int_{C_i}\frac{l}{r}\,dl + v_i^2\int_{C_i}\frac{l^2}{r}\,dl\right\} \\ &\quad \left.+c_4\,\underline{n}\,u_i\,z\left\{(\xi_i-x)\int_{C_i}\frac{1}{r}\,dl + u_i\int_{C_i}\frac{l}{r}\,dl\right\}\right] \end{aligned}} \tag{5.73}$$

5.4.5 Induced potential and velocity due to doublet distribution

For a bilinear doublet distribution $\mu = b_0 + b_1\,\xi + b_2\,\eta + b_3\,\xi\,\eta$, the induced potentials and velocities can be obtained in a straight-forward manner similar to one in the case of the source distribution. The final results can be written as, for the induced potentials,

$$\boxed{\begin{aligned}\phi^{(\mu)} &= -\frac{1}{4\pi}\sum_{i=1}^{N_S}\left[-d_1(\underline{n}\cdot\underline{e})\,t_i\int_{C_i}\frac{1}{r(r+a)}\,dl + d_2\,v_i(-z)\int_{C_i}\frac{1}{r}\,dl\right.\\ &\left.\quad +d_3\,u_i\,z\int_{C_i}\frac{1}{r}\,dl + d_4\,u_i\,z\left\{(\xi_i - x)\int_{C_i}\frac{1}{r}\,dl + u_i\int_{C_i}\frac{l}{r}\,dl\right\}\right] \qquad (5.74)\end{aligned}}$$

and, for the induced velocities,

$$\boxed{\begin{aligned}\underline{q}^{(\mu)} &= \frac{1}{4\pi}\sum_{i=1}^{N_S}\left[\left(\underline{r}\times\underline{e}_{l_i}\right)\left\{b_0\int_{C_i}\frac{1}{r^3}\,dl + b_1\left(\xi_i\int_{C_i}\frac{1}{r^3}\,dl + u_i\int_{C_i}\frac{l}{r^3}\,dl\right)\right.\right.\\ &\quad +b_2\left(\eta_i\int_{C_i}\frac{1}{r^3}\,dl + v_i\int_{C_i}\frac{l}{r^3}\,dl\right)\\ &\quad \left. +b_3\left(\xi_i\,\eta_i\int_{C_i}\frac{1}{r^3}\,dl + (\xi_i\,v_i + \eta_i\,u_i)\int_{C_i}\frac{l}{r^3}\,dl + u_i\,v_i\int_{C_i}\frac{l^2}{r^3}\,dl\right)\right\}\\ &\quad +d_2\left\{\underline{e}_\xi(\underline{n}\cdot\underline{e})\,t_i\int_{C_i}\frac{1}{r(r+a)}\,dl + (\underline{e}_\eta\times\underline{e}_{m_i})\int_{C_i}\frac{1}{r}\,dl\right\}\\ &\quad +d_3\left\{\underline{e}_\eta(\underline{n}\cdot\underline{e})\,t_i\int_{C_i}\frac{1}{r(r+a)}\,dl - (\underline{e}_\xi\times\underline{e}_{m_i})\int_{C_i}\frac{1}{r}\,dl\right\}\\ &\quad +d_4\left\{\underline{n}\left\{(v_i(\eta_i - y) - u_i(\xi_i - y))\int_{C_i}\frac{1}{r}\,dl + \left(v_i^2 - u_i^2\right)\int_{C_i}\frac{l}{r}\,dl\right\}\right.\\ &\quad \left.\left. -\underline{e}_\xi\,u_i\,z\int_{C_i}\frac{1}{r}\,dl + \underline{e}_\eta\,v_i\,z\int_{C_i}\frac{1}{r}\,dl\right\}\right] \qquad (5.75)\end{aligned}}$$

For shortness of expressions, we have also defined the constants $d_1 = b_0 + b_1\,x + b_2\,y + b_3\,x\,y$, $d_2 = b_1 + b_3\,y$, $d_3 = b_2 + b_3\,x$, $d_4 = b_3$. It is easily found that the expression for the induced potential $\phi^{(\mu)}$ has the same form as the normal component of $\underline{q}^{(\sigma)}$ except notation of the singularity distribution.

5.4.6 Closed-forms of the basic integrals

In the preceding subsections, we have expressed the induced potentials and velocities in forms of a sum of the more simplified line integral given in Eqs. (5.70), (5.73), (5.74) and (5.75). We will derive here closed-forms of the following line integrals involved in those expressions:

$$\begin{aligned}
&\mathrm{I1}_i = \int_{C_i} \frac{1}{r}\,dl, \quad \mathrm{I2}_i = \int_{C_i} \frac{1}{r+a}\,dl, \quad \mathrm{I3}_i = \int_{C_i} \frac{1}{r(r+a)}\,dl, \quad \mathrm{I4}_i = \int_{C_i} \frac{1}{r^3}\,dl \\
&\mathrm{J1}_i = \int_{C_i} \frac{l}{r}\,dl, \quad \mathrm{J2}_i = \int_{C_i} \frac{l}{r^3}\,dl, \quad \mathrm{J3}_i = \int_{C_i} r\,dl, \\
&\mathrm{K1}_i = \int_{C_i} \frac{l^2}{r}\,dl, \quad \mathrm{K2}_i = \int_{C_i} lr\,dl, \quad \mathrm{K3}_i = \int_{C_i} \frac{l^2}{r^3}\,dl
\end{aligned} \tag{5.76}$$

The line integrals for each side of the polygon can be treated independently by the geometric parameters of that side. It is sufficient, therefore, to consider only one side of the panel, say $i = 1$, for the purpose of these evaluations. For simplicity of the presentation, we drop the subscript i used for identifying the side. We take, without loss of generality, a local plane coordinate system (x', z') in the plane through the field point $\underline{x}_p$ and the side concerned, such that the side lies on the x'-axis, one end point of the side is at the origin and the integration path is performed along the positive x'-axis, as shown in Figure **5.2** . Then the local coordinates (x', z') can be expressed, in terms of the global coordinates, as $|z'| = |\underline{e}_l \times \underline{Q_1 x_p}|$. and $x' = \underline{e}_l \cdot \underline{Q_1 x_p}$.

In the following development, we define the distances between the end points and the field point by $R_1 \equiv |\underline{Q_1 x_p}| = \sqrt{x'^2 + z'^2}$ and $R_2 \equiv |\underline{Q_2 x_p}| = \sqrt{(\ell - x)'^2 + z'^2}$. Expressing the integrals in terms of the local coordinates x' and z' and performing the integration (Gradshteyn & Ryzhik 1965), [11] we

[11] Gradshteyn, I. S. and Ryzhik, I. M. (1965), *Table of Integrals, Series and Products*, Academic Press, Inc., New York and London.

get the following results for the integrals.

$$
\begin{aligned}
\mathrm{I1} &= \ln \frac{R_2+\ell-x'}{R_1-x'} && (5.77)\\
\mathrm{I2} &= \mathrm{I1} - \frac{a}{\sqrt{z'^2-a^2}}\ \sin^{-1}(H) && (5.78)\\
&\quad \text{with}\ \ H = \frac{\sqrt{z'^2-a^2}\ \left\{z^2\,\ell + a\,(\ell-x')\,R_1 + a\,x'\,R_2)\right\}}{z'^2(R_1+a)\,(R_2+a)}\\
\Bigg(\mathrm{I2} &= \mathrm{I1} - \frac{a}{\sqrt{z'^2-a^2}}\left\{\pi - \sin^{-1}(H)\right\},\\
&\text{if}\ \ (R_1+a)^2\,(z'^2+aR_2)^2 + (R_2+a)^2\,(z'^2+a\,R_1)^2 \le z'^2\,(R_1+a)^2\,(R_2+a)^2\Bigg)\\
\mathrm{I3} &= \frac{1}{a}(\mathrm{I1}-\mathrm{I2}) && (5.79)\\
\mathrm{I4} &= \frac{1}{z'^2}\left\{\frac{\ell-x'}{R_2} + \frac{x'}{R_1}\right\} && (5.80)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{J1} &= R_2 - R_1 + x'\,\mathrm{I1} && (5.81)\\
\mathrm{J2} &= \frac{1}{R_1} - \frac{1}{R_2} + x'\,\mathrm{I4} && (5.82)\\
\mathrm{J3} &= \frac{1}{2}\left\{(\ell-x')\,R_2 + x'\,R_1 + z'^2\,\mathrm{I1}\right\} && (5.83)\\
\mathrm{K1} &= \frac{1}{2}\left\{(\ell-x')\,R_2 + x'\,R_1 - z'^2\,\mathrm{I1}\right\} + 2x'\,\mathrm{J1} - x'^2\,\mathrm{I1} && (5.84)\\
\mathrm{K2} &= \frac{1}{3}\left(R_2^3 - R_1^3\right) + x'\mathrm{J3} && (5.85)\\
\mathrm{K3} &= -\left(\frac{\ell-x'}{R_2} + \frac{x'}{R_1}\right) + \mathrm{I1} + 2x'\,\mathrm{J2} - x'^2\,\mathrm{I1} && (5.86)
\end{aligned}
$$

In the cases of $a = 0$ and/or $z = 0$, we can take the limit forms of the above expressions.

The closed-forms of Eq. (5.70) through Eq. (5.75) can be written as, in terms of the basic integrals retaining the index i for the side and vertex:

$$\phi^{(\sigma)} = -\frac{1}{4\pi}\sum_{i=1}^{N_S}\left[c_1\, t_i\, \mathrm{I2}_i + \{c_2\, v_i - c_3\, u_i + c_4\, v_i\,(\eta_i - y)\}\, \mathrm{J3}_i + c_4\, v_i^2\, \mathrm{K2}_i\right] \tag{5.87}$$

$$\begin{aligned} \underline{q}^{(\sigma)} = \frac{1}{4\pi}\sum_{i=1}^{N_S} & \left[-c_1\left\{\underline{n}(\underline{n}\cdot\underline{e})\, t_i\, \mathrm{I3}_i + \underline{e}_{m_i}\mathrm{I1}_i\right\}\right. \\ & + c_2\left\{-\underline{e}_\xi\, t_i\, \mathrm{I2}_i + (\underline{e}_\xi\, v_i - \underline{e}_\eta\, u_i)\left\{(\xi_i - x)\,\mathrm{I1}_i + u_i\,\mathrm{J1}_i\right\} - \underline{n}\, v_i\, z\, \mathrm{I1}_i\right\} \\ & + c_3\left\{-\underline{e}_\eta\, t_i\, \mathrm{I2}_i + (\underline{e}_\xi\, v_i - \underline{e}_\eta\, u_i)\left\{(\eta_i - y)\,\mathrm{I1}_i + v_i\,\mathrm{J1}_i\right\} + \underline{n}\, u_i\, z\mathrm{I1}_i\right\} \\ & + c_4\left\{\underline{e}_\xi(-u_i)\left\{(\xi_i - x)^2\,\mathrm{I1}_i + 2(\xi_i - x)\, u_i\,\mathrm{J1}_i + u_i^2\,\mathrm{K1}_i\right\}\right. \\ & \quad + \underline{e}_\eta\, v_i\left\{(\eta_i - y)^2\,\mathrm{I1}_i + 2(\eta_i - y)\, v_i\,\mathrm{J1}_i + v_i^2\,\mathrm{K1}_i\right\} \\ & \quad \left.\left. + \underline{n}\, u_i\, z\left\{(\xi_i - x)\,\mathrm{I1}_i + u_i\,\mathrm{J1}_i\right\}\right\}\right] \end{aligned} \tag{5.88}$$

$$\begin{aligned} \phi^{(\mu)} = \frac{1}{4\pi}\sum_{i=1}^{N_S} & [d_1(\underline{n}\cdot\underline{e})\, t_i\, \mathrm{I3}_i + d_2\, v_i\, z\, \mathrm{I1}_i - d_3\, u_i\, z\, \mathrm{I1}_i \\ & - d_4\, u_i\, z\left\{(\xi_i - x)\,\mathrm{I1}_i + u_i\,\mathrm{J1}_i\right\}] \end{aligned} \tag{5.89}$$

$$\begin{aligned} \underline{q}^{(\mu)} = \frac{1}{4\pi}\sum_{i=1}^{N_S} & \left[(\underline{r}\times\underline{e}_{l_i})\left\{b_0\,(\xi_i\,\mathrm{I4}_i + u_i\,\mathrm{J4}_i) + b_1\,(\eta_i\,\mathrm{I4}_i + v_i\,\mathrm{J4}_i)\right.\right. \\ & \qquad \left. + b_2\left\{\xi_i\,\eta_i\,\mathrm{I4}_i + (\xi_i\, v_i + \eta_i\, u_i)\,\mathrm{J4}_i + u_i\, v_i\,\mathrm{K3}_i\right\} + b_3\,\mathrm{I4}_i\right\} \\ & + d_2\left\{\underline{e}_\xi(\underline{n}\cdot\underline{e})\, t_i\,\mathrm{I3}_i + (\underline{e}_\eta\times\underline{e}_{m_i})\,\mathrm{I1}_i\right\} \\ & + d_3\left\{\underline{e}_\eta(\underline{n}\cdot\underline{e})\, t_i\,\mathrm{I3}_i - (\underline{e}_\xi\times\underline{e}_{m_i})\,\mathrm{I1}_i\right\} \\ & + d_4\left\{\underline{n}\left\{v_i\,(\eta_i - y)\,\mathrm{I1}_i + v_i^2\,\mathrm{J1}_i - u_i\,(\xi_i - y)\,\mathrm{I1}_i - u_i^2\,\mathrm{J1}_i\right\}\right. \\ & \quad \left.\left. - \underline{e}_\xi\, u_i\, z\,\mathrm{I1}_i + \underline{e}_\eta\, v_i\, z\,\mathrm{I1}_i\right\}\right] \end{aligned} \tag{5.90}$$

We found the newly useful relation of Eq. (5.59) which can be applied directly to calculation of the volumetric integral of vorticity distributions given by the Biot-Savart integral. This integral would often require to be evaluated when the vorticity-velocity formulation is used in inviscid rotational flow problems involving shear-flow interaction. For piecewise constant vorticity distribution within a volumetric element with planar faces, we can first transform the volume integral into the surface integrals on the enclosed faces by using Gauss theorem. The integrand of the transformed surface integrals becomes $1/r$ and then Eq. (5.59) (with $\underline{e} = \pm\underline{n}$) can be used to transform each surface integral into the line integrals expressed in a form analogous to the first integral term in Eq. (5.70). The evaluation of the Biot-Savart integral is presented in Appendix D.