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COMPUTATIONAL MARINE HYDRODYNAMICS

-VORTEX METHODS-

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INTEGRATION FOR SINGULARITY DISTRIBUTIONS

Using Stokes' formulas, Cantaloube & Rehbach (1986) show that the surface integrals of the singularity method can be transformed into contour integrals for planar facets. The numerical integration is then very precise at less calculation cost.

This Appendix is especially prepared to show all the mathematical derivations and proofs of the equations in the original paper by Cantaloube & Rehbach. A subroutine program based on the analysis is also provided in the Appendix C for computations of the influence coefficients in applications of the panel method.

B.1 Introduction

The fundamental problem of fluid mechanics for inviscid incompressible flow is to determine velocity potential ϕ, whose governing equation becomes the Laplace equation,

$$\nabla^2 \phi = 0, \tag{B.1}$$

satisfying certain proper conditions on the boundary S.

The singularity method is applied for solution of this problem. This basic idea of the singularity method has been introduced by Hess & Smith (1966), using the surface distribution of sources. With the Green's scalar identity, the potential ϕ within the domain V is expressed in terms of the proper value of ϕ and its normal derivative $\underline{n} \cdot \nabla \phi$ on the boundary S;

$$\phi = -\frac{1}{4\pi} \left(\int_S \frac{1}{r} \left(\underline{n} \cdot \nabla \phi \right) \, dS_\xi - \int_S \phi \, \underline{n} \cdot \nabla \left(\frac{1}{r} \right) \, dS_\xi \right). \tag{B.2}$$

Here r is a distance between an integration point ξ on S and a field point p located in V. The first surface integral is interpreted as the potential by surface distribution of source-type singularities with density $\sigma \equiv \underline{n} \cdot \nabla \phi$, the second surface integral as the potential by surface distribution of doublet-type singularities, $\mu \equiv -\phi$. [1]

For a planar polygon element with the uniform or linear density distributions of singularities, the closed-forms for obtaining the influence coefficients in the

[1] We follow herein the definition given in the original paper: $\mu \equiv -\phi$.

panel method are derived.

The analytic evaluations of the associated integrals may improve a solution accuracy in the panel method with much reduced computing time. A few of test calculations show the superiority of these analytic evaluations to numerical integrations.

B.1.1 Related work for closed-form expressions

The closed-form expressions of the surface integrals for constant source distributions over flat quadrilateral panels have been introduced by Hess & Smith (1966). [2] They expressed the surface integrals as a superposition of line integrals for each side of the panels, with independent treatment of the contribution from the side.

Webster (1975) [3] has extended the Hess and Smith analysis to a triangular panel in order to eliminate the discontinuity problem for a flat quadrilateral source panel by allowing a linear variation of the source strength across the triangular panel. These two approaches are concerned with only the source distributions and the resultant expressions are considerably complicated to employ a computer code.

A simpler and more unified derivation has been provided by Newman (1986) [4] for computing the potential due to a constant doublet or source distribution. His analysis is based on the elementary plane geometry related to the solid angle of a panel. He defined four infinite sectors (for a quadrilateral panel), bounded by semi-infinite extensions of the two adjacent sides of the panel with respect to the corresponding vertices, such that the difference between the domains of the four sectors is the domain of the panel. Then the surface integral over each infinite sector is evaluated in terms of the included angle of the corresponding vertex projected onto the unit sphere with center at the field point. He has also

[2] Hess, J. L. and Smith A. M. O. (1966), "Calculation of Potential Flow about Arbitrary Bodies," *Progress in Aeronautical Science Series*, vol. 8, Pergamon Press, pp. 1–138.

[3] Webster, W. C. (1975), "The flow about arbitrary, three-dimensional smooth bodies," *J. Ship Res.*, vol 19, no. 4, pp. 206–218.

[4] Newman, J. N. (1986), "Distributions of sources and normal dipoles over a quadrilateral panel," *J. Eng. Math.*, vol. 20, pp. 113–126.

described the more general recursive scheme for computing the potential due to a source or doublet distribution of linear, bilinear or higher order form, using the base results for the case of the constant distribution.

Another elegant approach based on mathematical formulations has been presented by Cantaloube & Rehbach (1986), [5] by which they introduced more explicit expressions of the surface integrals for the source or doublet distribution. With vector operations of the integrands for using Stokes' formulas, they show that the surface integrals for the constant or linear distributions of sources and doublets over a planar facet can be transformed into line integrals along the contour of the panel. The major advantages of their study are that the formulations are valid for a planar curve-sided panel and that the resultant equations are expressed in a global coordinate system while the aforementioned analysis requires the transformation of the local coordinate system. Thus the expressions derived by Cantaloube & Rehbach may be regarded as a more computer-oriented form.

They have proposed the use of direct numerical integrations of the line integrals by an integration quadrature (e.g. Simpson rule or Gaussian quadrature), illustrating the numerical consistency and accuracy for a linear doublet distribution on a quadrilateral panel. However when a field point is very close to the sides or vertices of a panel, a large number of the quadrature base points and considerable effort to choose these points suitably would be needed in order to achieve good comparisons with the known values. Such numerical implementation in a computer code may lead to a large amount of extra-computer time. Any attempt for finding closed form expressions of the line integrals for a polygon panel does not appear in their study.

Suh (1992a) [6] obtained, as an extension of Cantaloube & Rehbach's work (with some corrections in sign), the closed-forms for computing the induced potentials and velocities due to constant and/or linear distributions of the singularities. He expressed them as a sum of contribution from each side of the

[5] Cantaloube, B. and Rehbach, C. (1986), "Calcul des Integrales de la Methode des Singularites," *Recherche Aerospatiale*, n^o 1, pp. 15–22, English Title: "Calculation of the Integrals of the Singularity Method," *Aerospace Research*, no. 1, pp. 15–22.

[6] Suh, J. C. (1992a), "Analytical evaluation of the surface integral in the singularity methods," *Trans. Soc. Naval Arch. Korea*, vol. 29, no. 1, pp. 1–17.

panel, in terms of appropriate basic integrals.

As an another extension but by a different approach the present section deals with a *bilinear* distribution over *a planar polygonal panel*. In numerical implementation of the potential-based panel method for solving the potential flow around a lifting body, the trailing wake sheet is represented approximately as the doublet distribution of potential jump. One possible way to include the effect of the local variation of these doublet strengths is with the use of a bilinear distribution over each wake panel (which is uniquely determined from imposed potential jump values at its four vertices). The use of the bilinear distribution over quadrilateral panels (or the linear distribution over triangular panels) eliminates the discontinuity problem for the piecewise constant distribution. Then the singularity strength will be chosen to vary bilinearly (or linearly) across the panel. The main scope of this section is therefore to derive explicit and elegant closed-forms of the induced potential and velocity due to a bilinear distribution. The bilinear distribution case includes, of course, both the constant and the linear distribution cases.

In order to transform the associated surface integrals into line integrals along contour of the panel by using Stokes' formulas, alternative forms of the associated integrands for the bilinear distribution of sources and doublets over a planar panel are presented. For a planar polygon panel, the derived line integrals can be reduced to closed-form expressions for the potential and velocity. The closed-form expressions of the line integrals for the induced potential and velocity are presented. They are expressed compactly as a sum of contribution from each side of the panel, in terms of appropriate basic integrals. It will be shown that each contribution depends on the relative position of a field point from the side.

B.1.2 Stokes' theorem

The general form of Stokes' formulae is

$$\int_S (d\underline{S} \times \nabla) X = \int_C d\underline{l}\, X \tag{B.3}$$

where S is the surface enclosed by a curve C, $d\underline{S} \equiv \underline{n}\,dS$ is the oriented surface element and $d\underline{l} \equiv \underline{t}\,dl$ is the integration element along the curve C. X is a scalar or vector function of space coordinates.

If we choose X as scalar f, then it becomes

$$\int_S \underline{n} \times \nabla f\, dS = \oint_C f\, d\underline{l}. \tag{B.4}$$

Identifying X as vector $\underline{f}$ reduces it to

$$\int_S (d\underline{S} \times \nabla) \cdot \underline{f} = \oint_C \underline{f} \cdot d\underline{l}, \tag{B.5}$$

or vector transformation of the first part gives

$$\int_S (\nabla \times \underline{f}) \cdot d\underline{S} = \oint_C \underline{f} \cdot d\underline{l}. \tag{B.6}$$

B.1.3 Basic vector operations

For the purpose of derivations of some relations, often-used vector expansion formulas are presented as follows.

$$\underline{a} \cdot (\underline{b} \times \underline{c}) = \underline{b} \cdot (\underline{c} \times \underline{a}) = \underline{c} \cdot (\underline{a} \times \underline{b}) \tag{B.7}$$

$$\underline{a} \times (\underline{b} \times \underline{c}) = \underline{b}\,(\underline{a} \cdot \underline{c}) - \underline{c}\,(\underline{a} \cdot \underline{b}) \tag{B.8}$$

$$\nabla \times (\underline{a} \times \underline{b}) = \underline{a}\,(\nabla \cdot \underline{b}) + (\underline{b} \cdot \nabla)\,\underline{a} - \underline{b}\,(\nabla \cdot \underline{a})\,(\underline{a} \cdot \nabla)\,\underline{b} \tag{B.9}$$

$$\nabla(\underline{a} \cdot \underline{b}) = (\underline{a} \cdot \nabla)\,\underline{b} + (\underline{b} \cdot \nabla)\,\underline{a} + \underline{a} \times (\nabla \times \underline{b}) + \underline{b} \times (\nabla \times \underline{a}) \tag{B.10}$$

$$\nabla \cdot (\phi\,\underline{a}) = \underline{a} \cdot \nabla\phi + \phi\,\nabla \cdot \underline{a} \tag{B.11}$$

$$\nabla \times (\phi\,\underline{a}) = (\nabla\phi) \times \underline{a} + \phi\,\nabla \times \underline{a} \tag{B.12}$$

$$\nabla r = \frac{\underline{r}}{r} \tag{B.13}$$

$$\nabla\left(\frac{1}{r}\right) = \frac{-\underline{r}}{r^3} \tag{B.14}$$

$$\nabla \cdot \underline{r} = 3 \tag{B.15}$$

$$\nabla \times \underline{r} = 0 \tag{B.16}$$

B.2 Induced Potential Due to Source Distribution

The potential and velocity induced by surface distribution of source density σ over the surface S are

$$\phi = -\frac{1}{4\pi}\int_S \frac{\sigma}{r}\, dS_\xi \tag{B.17}$$

with $\underline{r} = \underline{x}_\xi - \underline{x}_p$, $r \equiv |\underline{r}|$.

For the indices of variables of differentiation and integration, the reciprocal relation holds: $\nabla_p(\frac{1}{r}) = -\nabla(\frac{1}{r})$. For the integration variable $\underline{x}_\xi$, the distribution surface S is represented hypothetically as collection of planar surfaces. Then $\underline{n}$ is independent of $\underline{x}_\xi$. Equation (B.17) for the potential is reduced to,

$$\begin{aligned}\phi \;=\; & -\frac{1}{4\pi}\left[\int_S \underline{n}\cdot\left\{\nabla\times\left(\sigma\,\underline{n}\times\frac{\underline{r}}{r}\right)\right\} dS + \int_S(\underline{n}\cdot\underline{r})\left\{(\sigma\,\underline{n})\cdot\nabla\left(\frac{1}{r}\right)\right\} dS\right.\\ & \left. -\int_S \underline{n}\cdot\left\{\nabla\sigma\times\left(\underline{n}\times\frac{\underline{r}}{r}\right)\right\} dS\right]\end{aligned} \tag{B.18}$$

<u>Proof of Eq. (B.18)</u>

The detailed derivation is performed reversely from Eq. (B.18) into Eq. (B.17) as follows:

(1) The integrand of the first surface integral becomes

$$\begin{aligned} I_1 \;=\; & \underline{n}\cdot\nabla\times\left(\sigma\,\underline{n}\times\frac{\underline{r}}{r}\right) \quad : \text{using Eq. (B.9)}\\ =\; & \underline{n}\cdot\left[\sigma\,\underline{n}\left\{\nabla\cdot\left(\frac{\underline{r}}{r}\right)\right\} + \left(\frac{\underline{r}}{r}\cdot\nabla\right)(\sigma\,\underline{n}) - \frac{\underline{r}}{r}\{\nabla\cdot(\sigma\,\underline{n})\} - (\sigma\,\underline{n}\cdot\nabla)\frac{\underline{r}}{r}\right]\\ =\; & \underline{n}\cdot\left[\sigma\,\underline{n}\left\{\underline{r}\cdot\nabla\left(\frac{1}{r}\right) + \frac{1}{r}\nabla\cdot\underline{r}\right\} \quad : \text{using Eq. (B.9)}\right.\\ & \left. +\underline{n}\left(\frac{\underline{r}}{r}\cdot\nabla\sigma\right) - \frac{\underline{r}}{r}(\underline{n}\cdot\nabla\sigma) - (\sigma\,\underline{n}\cdot\nabla)\frac{\underline{r}}{r}\right] \quad : \text{since } \underline{n} \text{ is constant}\end{aligned} \tag{B.19}$$

Here the last term is rearranged by tensor operation,

$$\begin{aligned}\underline{n}\cdot(\sigma\,\underline{n}\cdot\nabla)\frac{\underline{r}}{r} &= n_i\left\{\sigma\,n_j\,\frac{\partial}{\partial\xi_j}\left(\frac{\xi_i-x_i}{r}\right)\right\}\\ &= n_i\frac{1}{r}\sigma\,n_j\,\frac{\partial}{\partial\xi_j}(\xi_i-x_i)+n_i\,(\xi_i-x_i)\,\sigma\,n_j\,\frac{\partial}{\partial\xi_j}\left(\frac{1}{r}\right)\\ &= n_i\,\frac{1}{r}\,\sigma\,n_j\,\delta_{ij}+n_i\,(\xi_i-x_i)\,\sigma\,n_j\,\frac{-(\xi_j-x_j)}{r^3}\\ &= \frac{\sigma}{r}-\frac{\sigma\,(\underline{n}\cdot\underline{r})^2}{r^3}\end{aligned}\tag{B.20}$$

Then,

$$I_1=\frac{2}{r}\sigma+\frac{\underline{r}}{r}\cdot\nabla\sigma-\frac{(\underline{n}\cdot\underline{r})}{r}(\underline{n}\cdot\nabla\sigma)-\frac{\sigma}{r}+\frac{\sigma(\underline{n}\cdot\underline{r})^2}{r^3}\tag{B.21}$$

(2) The integrand of the second surface integral in Eq. (B.18) becomes:

$$\begin{aligned}I_2 &= (\underline{n}\cdot\underline{r})\left\{\sigma\,\underline{n}\cdot\nabla\left(\frac{1}{r}\right)\right\}=(\underline{n}\cdot\underline{r})\left\{\sigma\,\underline{n}\cdot\left(\frac{-\underline{r}}{r^3}\right)\right\}\quad:\ \text{using Eq. (B.14)}\\ &= -\frac{\sigma\,(\underline{n}\cdot\underline{r})^2}{r^3}\end{aligned}\tag{B.22}$$

(3) The use of Eq. (B.8) reduces the integrand of the third surface integral to

$$\begin{aligned}I_3 &= \underline{n}\cdot\left\{\nabla\sigma\times\left(\underline{n}\times\frac{\underline{r}}{r}\right)\right\}\\ &= \underline{n}\cdot\left\{\underline{n}\left(\nabla\cdot\frac{\underline{r}}{r}\right)-\frac{\underline{r}}{r}\,(\nabla\sigma\cdot\underline{n})\right\}\\ &= \nabla\sigma\cdot\frac{\underline{r}}{r}-\frac{\underline{n}\cdot\underline{r}}{r}\,(\nabla\sigma\cdot\underline{n})\end{aligned}\tag{B.23}$$

Therefore by combining the three items above, Eq. (B.18) can be reduced to Eq. (B.17):

$$\begin{aligned}\phi &= -\frac{1}{4\pi}\int_S (I_1 + I_2 - I_3)\,dS \\ &= -\frac{1}{4\pi}\int_S \left\{\frac{\sigma}{r} + \frac{\underline{r}}{r}\cdot\nabla\sigma - \frac{(\underline{n}\cdot\underline{r})}{r}(\underline{n}\cdot\nabla\sigma) + \frac{\sigma(\underline{n}\cdot\underline{r})^2}{r^3}\right. \\ &\quad \left. -\frac{\sigma(\underline{n}\cdot\underline{r})^2}{r^3} - \nabla\sigma\cdot\frac{\underline{r}}{r} + \frac{(\underline{n}\cdot\underline{r})}{r}(\nabla\sigma\cdot\underline{n})\right\} dS \\ &= -\frac{1}{4\pi}\int_S \frac{\sigma}{r}\,dS \end{aligned} \tag{B.24}$$

■■■

B.2.1 Transformation of Eq. (B.18) into line integrals

Now the surface integrals of Eq. (B.18) can be transformed as follows:

(1) The first one becomes, using Eq. (B.5)

$$\begin{aligned}\int_S \underline{n}\cdot\left\{\nabla\times\left(\sigma\,\underline{n}\times\frac{\underline{r}}{r}\right)\right\}dS &= \oint_C \sigma\left(\frac{\underline{n}\times\underline{r}}{r}\right)\cdot d\underline{l} \\ &= \underline{n}\cdot\oint_C \sigma\,\frac{\underline{r}\times d\underline{l}}{r}\end{aligned} \tag{B.25}$$

(2) For the second one, let us introduce the relation [7]

$$\nabla\left(\frac{1}{r}\right) = -\nabla\times\underline{A} \tag{B.26}$$

with

$$\underline{A} = \frac{\underline{e}\times\underline{r}}{r\,(r + \underline{e}\cdot\underline{r})} \tag{B.27}$$

where $\underline{e}$ is a unit vector, being a function of $\underline{x}_p$, chosen such that the de-

[7] Proof is given below and see also Guiraud, J. P. (1978), “Potential of velocities generated by a localized vortex distribution,” *Aerospace Research*, English Translation-ESA-TT-560, pp. 105–107.

nominator is not zero, and use the vector operation

$$\sigma\,\nabla\left(\frac{1}{r}\right) = -\sigma\,\nabla\times\underline{A} = -\nabla\times(\sigma\,\underline{A}) - \underline{A}\times\nabla\sigma. \tag{B.28}$$

Then, by Eq. (B.5)

$$\begin{aligned}
&\int_S (\underline{n}\cdot\underline{r})\left\{(\sigma\,\underline{n})\cdot\nabla\left(\frac{1}{r}\right)\right\}\,dS \\
&= \int_S (\underline{n}\cdot\underline{r})\,\underline{n}\cdot\{-\nabla\times(\sigma\,\underline{A}) - \underline{A}\times\nabla\sigma\}\,dS \\
&= -(\underline{n}\cdot\underline{r})\oint_C \sigma\,\underline{A}\cdot dl + (\underline{n}\cdot\underline{r})\,\underline{n}\cdot\int_S \nabla\sigma\times\underline{A}\,dS
\end{aligned} \tag{B.29}$$

(3) The third integral becomes

$$-\int_S \underline{n}\cdot\left\{\nabla\sigma\times\left(\underline{n}\times\frac{\underline{r}}{r}\right)\right\}\,dS = -\int_S \underline{n}\cdot\{\nabla\sigma\times(\underline{n}\times\nabla r)\}\,dS. \tag{B.30}$$

Consequently, the expression (B.18) is replaced by

$$\begin{aligned}
\phi \;=\; -\frac{\underline{n}}{4\pi}\cdot\Bigg\{&\oint_C \sigma\,\frac{\underline{r}}{r}\times d\underline{l} - \underline{r}\oint_C \sigma\,\underline{A}\cdot d\underline{l} + (\underline{n}\cdot\underline{r})\int_S \nabla\sigma\times\underline{A}\,dS \\
&- \int_S \nabla\sigma\times(\underline{n}\times\nabla r)\,dS\Bigg\}
\end{aligned} \tag{B.31}$$

Now the double integral in Eq. (B.31) can be transformed into contour integral if we suppose $\nabla\sigma$ to be constant over S and if we choose $\underline{e} = \pm\,\underline{n}$:

$$\begin{aligned}
\phi \;=\; -\frac{1}{4\pi}\Bigg[&\underline{n}\cdot\oint_C \sigma\,\frac{\underline{r}}{r}\times d\underline{l} - (\underline{n}\cdot\underline{r})\oint_C \sigma\,\underline{A}\cdot d\underline{l} \\
&+ (\underline{n}\cdot\underline{r})\,(\underline{n}\cdot\underline{e})\,\underline{n}\cdot\left\{\nabla\sigma\times\oint_C \ln(r+\underline{e}\cdot\underline{r})d\underline{l}\right\} \\
&- \underline{n}\cdot\left(\nabla\sigma\times\oint_C r\,d\underline{l}\right)\Bigg]
\end{aligned} \tag{B.32}$$

For this transformation taking account of Eq. (B.4), another form for the vector function $\underline{A}$, has been used $\underline{A} = \underline{e}\times\nabla R$ with $R = \ln(r+\underline{e}\cdot\underline{r})$. Namely the second integral in Eq. (B.31) can be written as, with $\nabla\sigma = \text{const.}$ and $\underline{e} = \pm\,\underline{n}$,

$$\begin{aligned}(\underline{n}\cdot\underline{r})\int_S \nabla\sigma\times\underline{A}\,dS &= (\underline{n}\cdot\underline{r})\nabla\sigma\times\int_S(\underline{n}\cdot\underline{e})\,(\underline{n}\times\nabla R)\;dS \\ &= (\underline{n}\cdot\underline{r})\,(\underline{n}\cdot\underline{e})\nabla\sigma\times\oint_C \ln(r+\underline{e}\cdot\underline{r})\,d\underline{l} \end{aligned} \quad \text{(B.33)}$$

Accordingly, the following relation holds for $\underline{e} = \pm\,\underline{n}$,

$$\int_S \underline{A}\,dS = (\underline{n}\cdot\underline{e})\oint_C \ln(r+\underline{e}\cdot\underline{r})\,d\underline{l}. \quad \text{(B.34)}$$

Proof of Eq (B.26)

Now let's prove Eq. (B.26) by showing that the right-hand side reduces to the left hand side. For simplicity, dropping out the subscript ξ of the operator ∇,

$$\begin{aligned}&\nabla\times\underline{A} = \nabla\times\left\{\frac{1}{r(r+\underline{e}\cdot\underline{r})}(\underline{e}\times\underline{r})\right\} \\ &= \nabla\left\{\frac{1}{r(r+\underline{e}\cdot\underline{r})}\right\}\times(\underline{e}\times\underline{r}) + \frac{1}{r(r+\underline{e}\cdot\underline{r})}\nabla\times(\underline{e}\times\underline{r}) \quad : \text{using (B.12)} \\ &= \frac{-\nabla\left\{r(r+\underline{e}\cdot\underline{r})\right\}}{r^2(r+\underline{e}\cdot\underline{r})^2}\times(\underline{e}\times\underline{r}) + \frac{1}{r(r+\underline{e}\cdot\underline{r})}\nabla\times(\underline{e}\times\underline{r}) \end{aligned} \quad \text{(B.35)}$$

Knowing that $\underline{e}$ is chosen as a function of $\underline{x}_p$, independent of $\underline{x}_\xi$ and using (B.10), (B.13) and (B.16),

$$\begin{aligned}&\nabla\left\{r(r+\underline{e}\cdot\underline{r})\right\} = (r+\underline{e}\cdot\underline{r})\nabla r + r\nabla(r+\underline{e}\cdot\underline{r}) \\ &= (r+\underline{e}\cdot\underline{r})\frac{\underline{r}}{r} + r\left\{\frac{\underline{r}}{r} + (\underline{e}\cdot\nabla)\underline{r} + (\underline{r}\cdot\nabla)\underline{e} + \underline{e}\times(\nabla\times\underline{r}) + \underline{r}\times(\nabla\times\underline{e})\right\} \\ &= \left(2+\frac{\underline{e}\cdot\underline{r}}{r}\right)\underline{r} + r\underline{e} \end{aligned} \quad \text{(B.36)}$$

Recall that the following relation is used while deriving the above expression:

$$(\underline{e}\cdot\nabla)\,\underline{r} = e_i\,\frac{\partial x_j}{\partial x_i} = e_i\,\delta_{ij} = e_j = \underline{e}. \quad \text{(B.37)}$$

Then use Eq. (B.8) for the triple vector product

$$
\begin{aligned}
& \nabla \left\{ r(r + \underline{e} \cdot \underline{r}) \right\} \times (\underline{e} \times \underline{r}) \\
= & \left(2 + \frac{\underline{e} \cdot \underline{r}}{r}\right) \left\{ r^2 \underline{e} - \underline{r}(\underline{e} \cdot \underline{r}) \right\} + r \left\{ (\underline{e} \cdot \underline{r}) \underline{e} - \underline{r} \right\} \\
= & \left\{ 2r^2 + 2r(\underline{e} \cdot \underline{r}) \right\} \underline{e} - \left\{ \left(2 + \frac{\underline{e} \cdot \underline{r}}{r}\right) (\underline{e} \cdot \underline{r}) + r \right\} \underline{r} \\
= & \; 2r \left\{ r + (\underline{e} \cdot \underline{r}) \right\} \underline{e} - \left\{ r + (\underline{e} \cdot \underline{r}) \right\}^2 \frac{\underline{r}}{r}
\end{aligned}
\tag{B.38}
$$

Therefore, one can derive Eq. (B.26):

$$
\begin{aligned}
\nabla \times \underline{A} &= -\frac{2\underline{e}}{r(r + \underline{e} \cdot \underline{r})} + \frac{\underline{r}}{r^3} \\
&\quad + \frac{1}{r(r + \underline{e} \cdot \underline{r})} \left\{ \underline{e} \, (\nabla \cdot \underline{r}) + (\underline{r} \cdot \nabla) \underline{e} - \underline{r}(\nabla \cdot \underline{e}) - (\underline{e} \cdot \nabla) \underline{r} \right\} \\
&= -\frac{2\underline{e}}{r(r + \underline{e} \cdot \underline{r})} + \frac{\underline{r}}{r^3} + \frac{1}{r(r + \underline{e} \cdot \underline{r})} (3\underline{e} - \underline{e}) \\
&= \frac{\underline{r}}{r^3} = -\nabla \left(\frac{1}{r}\right)
\end{aligned}
\tag{B.39}
$$

Remark: The vector $\underline{A}$ is evidently related to an explicit expression for the velocity potential for the volumetric distribution of vorticity. We define the solid angle ψ_p subtended at a point $\underline{x}_p$ by the surface S (is not necessarily a plane) (Milne-Thomson (1968)):

$$
\begin{aligned}
\psi_p &= \int_S \underline{n} \cdot \nabla \left(\frac{1}{r}\right) dS \\
&= -\oint_C \frac{\underline{e} \times \underline{r}}{r(r + \underline{e} \cdot \underline{r})} \cdot d\underline{l}
\end{aligned}
\tag{B.40}
$$

■■■

B.3 Induced Velocity Due to Source Distribution

Expression for the velocity induced by the sources distribution is given by

$$\underline{q} = \nabla_p \phi = \frac{1}{4\pi} \int_S \sigma \, \nabla \left(\frac{1}{r} \right) \, dS_\xi \tag{B.41}$$

Equation (B.41)) is transformed into, applying the triple vector product (Eq. (B.9)) to $\underline{n} \times \left\{ \underline{n} \times \nabla \left(\frac{1}{r} \right) \right\}$,

$$\underline{V}_p = \frac{1}{4\pi} \left[\int_S \sigma \left\{ \underline{n} \cdot \nabla \left(\frac{1}{r} \right) \right\} \underline{n} \, dS - \int_S \sigma \, \underline{n} \times \left\{ \underline{n} \times \nabla \left(\frac{1}{r} \right) \right\} \, dS \right]. \tag{B.42}$$

For the first surface integral, use Eq. (B.28)

$$\int_S \sigma \left\{ \underline{n} \cdot \nabla \left(\frac{1}{r} \right) \right\} \underline{n} \, dS = - \int_S \left[\underline{n} \cdot \{ \nabla \times (\sigma \, \underline{A}) \} \, \underline{n} + \{ \underline{n} \cdot (\underline{A} \times \nabla \sigma) \} \, \underline{n} \right] \, dS \tag{B.43}$$

and then apply Eq. (B.6) for a plane surface S

$$- \, \underline{n} \oint_C \sigma \, \underline{A} \cdot d\underline{l} - \underline{n} \int_S \{ \underline{n} \cdot (\underline{A} \times \nabla \sigma) \} \, dS. \tag{B.44}$$

The second surface integral of Eq. (B.42) is decomposed into two parts:

$$\begin{aligned} - \int_S \sigma \, \underline{n} \times \left\{ \underline{n} \times \nabla \left(\frac{1}{r} \right) \right\} \, dS \;\; &= \;\; - \int_S \underline{n} \times \left\{ \underline{n} \times \nabla \left(\frac{\sigma}{r} \right) \right\} \, dS \\ &\quad + \int_S \frac{\underline{n}}{r} \times (\underline{n} \times \nabla \sigma) \, dS. \end{aligned} \tag{B.45}$$

For the plane S, apply Eq. (B.6) for the first part to yield:

$$- \, \underline{n} \times \oint_C \frac{\sigma}{r} \, d\underline{l} + \int_S \frac{\underline{n}}{r} \times (\underline{n} \times \nabla) \, dS \tag{B.46}$$

Either Eq. (B.41) or Eq. (B.42) is replaced by

$$\begin{aligned} \underline{q} &= \frac{1}{4\pi}\left\{-\underline{n}\oint_C \sigma\,\underline{A}\cdot d\underline{l} - \underline{n}\times\oint_C \frac{\sigma}{r}\,d\underline{l} \right. \\ &\qquad \left. -\,\underline{n}\int_S \underline{n}\cdot(\underline{A}\times\nabla\sigma)\,dS + \int_S \frac{\underline{n}}{r}\times(\underline{n}\times\nabla\sigma)\,dS\right\} \end{aligned} \tag{B.47}$$

The two surface integrals can be simplified if we suppose $\nabla\sigma$ constant over S and if we pretend $\underline{e} = \pm\,\underline{n}$. The first one becomes, by Eq. (B.34):

$$\begin{aligned} -\,\underline{n}\int_S \underline{n}\cdot(\underline{A}\times\nabla\sigma)\,dS &= -\underline{n}\,(\nabla\sigma\times\underline{n})\cdot\int_S \underline{A}\,dS \quad : \text{using (B.7)} \\ &= -\underline{n}\,(\underline{n}\cdot\underline{e})\,(\nabla\sigma\times\underline{n})\cdot\oint_C \ln(r+\underline{e}\cdot\underline{r})\,d\underline{l} \end{aligned} \tag{B.48}$$

and the second one becomes, using Eq. (B.8) and the fact that $\underline{n}\cdot\nabla\sigma = 0$ over S,

$$\begin{aligned} \int_S \frac{\underline{n}}{r}\times(\underline{n}\times\nabla\sigma)\,dS &= \{\underline{n}\,(\underline{n}\cdot\nabla\sigma) - \nabla\sigma\,(\underline{n}\cdot\underline{n})\}\int_S \frac{1}{r}\,dS \\ &= -\nabla\sigma\int_S \frac{1}{r}\,dS \end{aligned} \tag{B.49}$$

Now applying Eqs. (B.17) and (B.31) for the case of σ = const., it becomes

$$-\,\nabla\sigma\left\{\underline{n}\cdot\oint_C \frac{\underline{r}\times d\underline{l}}{r} - (\underline{n}\cdot\underline{r})\oint_C \underline{A}\cdot d\underline{l}\right\} \tag{B.50}$$

Therefore, the final result is

$$\begin{aligned} \underline{q} &= -\frac{1}{4\pi}\left[\underline{n}\oint_C \sigma\,\underline{A}\cdot d\underline{l} + \underline{n}\times\oint_C \frac{\sigma}{r}d\underline{l}\right. \\ &\qquad -\underline{n}(\underline{n}\cdot\underline{e})\,(\underline{n}\times\nabla\sigma)\cdot\oint_C \ln(r+\underline{e}\cdot\underline{r})\,d\underline{l} \\ &\qquad \left. +\nabla\sigma\left\{\underline{n}\cdot\oint_C \frac{\underline{r}\times d\underline{l}}{r} - (\underline{n}\cdot\underline{r})\oint_C \underline{A}\cdot d\underline{l}\right\}\right] \end{aligned} \tag{B.51}$$

B.4 Induced Potential Due to Doublet Distribution

The potential induced by a surface distribution of doublets with density $\mu(\equiv -\phi)$ is written as

$$\phi = -\frac{1}{4\pi}\int_S \mu\,\underline{n}\cdot\nabla\left(\frac{1}{r}\right)\,dS \tag{B.52}$$

A variation for expression of doublet-potential (B.52), using Eq. (B.26) and the relation $\nabla\times(\mu\,\underline{A}) = \mu\nabla\times\underline{A} - \underline{A}\times\nabla\mu$ by Eq. (B.12), can be performed:

$$\begin{aligned}
&\mu\,\underline{n}\cdot\nabla\left(\frac{1}{r}\right) = \mu\,\underline{n}\cdot(-\nabla\times\underline{A})\\
=&\ -\underline{n}\cdot\mu\nabla\times\underline{A} = -\underline{n}\cdot\nabla\times(\mu\,\underline{A}) - \underline{n}\cdot(\underline{A}\times\nabla\mu)\\
=&\ -\underline{n}\cdot\nabla\times(\mu\,\underline{A}) + (\underline{n}\times\nabla\mu)\cdot\underline{A}
\end{aligned} \tag{B.53}$$

Consequently,

$$\phi = -\frac{1}{4\pi}\left[-\int_S \underline{n}\cdot\{\nabla\times(\mu\,\underline{A})\}\,dS + \int_S(\underline{n}\times\nabla\mu)\cdot\underline{A}\,dS\right] \tag{B.54}$$

becomes, with transformation of the first surface integral by Eq. (B.6),

$$\phi = -\frac{1}{4\pi}\left\{-\oint_C \mu\,\underline{A}\cdot d\underline{l} + \int_S(\underline{n}\times\nabla\mu)\cdot\underline{A}\,dS\right\}. \tag{B.55}$$

For constant $\nabla\mu$ over a plane surface S and $\underline{e} = \pm\,\underline{n}$, the surface integral is transformed into the contour integral (by Eq. (B.34)).

$$\phi = -\frac{1}{4\pi}\left\{-\oint_C \mu\,\underline{A}\cdot d\underline{l} + (\underline{n}\cdot\underline{e})\,(\underline{n}\times\nabla\mu)\cdot\oint_C \ln(r+\underline{e}\cdot\underline{r})\,d\underline{l}\right\}. \tag{B.56}$$

B.5 Induced Velocity Due to Doublet Distribution

Differentiation of Eq. (B.52) with respect to $\underline{x}_p$ yields

$$\underline{q} = -\frac{1}{4\pi}\int_S \mu\nabla_p\left\{\underline{n}\cdot\nabla\left(\frac{1}{r}\right)\right\}\,dS \tag{B.57}$$

or, alternatively by using a lengthy transformation (see Lee, J. T. (1987) and Brockett (1988))

$$\underline{q} = -\frac{1}{4\pi}\left\{\oint_C \mu\,\nabla\left(\frac{1}{r}\right)\times d\underline{l} + \int_S (\underline{n}\times\nabla\mu)\times\nabla\left(\frac{1}{r}\right)dS\right\} \tag{B.58}$$

In this form showing the correspondence presented by Hess (1969) for the first part, the velocity can be considered as one induced by two distributions of vorticity:

(1) a first due to a concentrated vorticity $\mu\, d\underline{l}$ over the contour C of the surface cap S, and

(2) a second due to a surface distribution of vorticity density, $\underline{\gamma} \equiv \underline{n}\times\nabla\mu$ over S.

Surface integral, one component of expression (Eq. (B.58)) for the velocity induced by the doublet distribution is transformed to, with the identity (B.8) applied on scalar $\dfrac{1}{r}$:

$$\begin{aligned}\int_S \underline{\gamma}\times\nabla\left(\frac{1}{r}\right)dS \;&=\; \int_S \underline{\gamma}\times\left[\underline{n}\left\{\underline{n}\cdot\nabla\left(\frac{1}{r}\right)\right\}\right]dS \\ &\quad -\int_S \underline{\gamma}\times\left[\underline{n}\times\left\{\underline{n}\times\nabla\left(\frac{1}{r}\right)\right\}\right]dS.\end{aligned} \tag{B.59}$$

For constant γ for a plane surface S, the first integral is reduced to, by Eqs. (B.6) and (B.26)

$$\begin{aligned}&\int_S \underline{\gamma}\times\left[\underline{n}\left\{\underline{n}\cdot\nabla\left(\frac{1}{r}\right)\right\}\right]dS = \int_S \underline{\gamma}\times\left[\underline{n}\left\{\underline{n}\cdot(-\nabla\times\underline{A})\right\}\right]dS \\ &= -\underline{\gamma}\times\underline{n}\oint_C \underline{A}\cdot d\underline{l} = -(\underline{n}\times\nabla\mu)\times\underline{n}\oint_C \underline{A}\cdot d\underline{l} \\ &= \left\{\underline{n}\,(\underline{n}\cdot\nabla\mu)-\nabla\mu\right\}\oint_C \underline{A}\cdot d\underline{l} = -\nabla\mu\oint_C \underline{A}\cdot d\underline{l},\end{aligned} \tag{B.60}$$

and the second one becomes, by Eq. (B.5)

$$-\int_S \underline{\gamma} \times \left[\underline{n} \times \left\{\underline{n} \times \nabla \left(\frac{1}{r}\right)\right\}\right] dS = -\underline{\gamma} \times \left(\underline{n} \times \oint_C \frac{d\underline{l}}{r}\right). \qquad \text{(B.61)}$$

The first expression represents a velocity component tangent to S and the second one a velocity component normal to S.

The velocity induced by a doublet distribution characterized by constant $\nabla\mu$ over a plane surface S is written as

$$\begin{aligned} \underline{q} \;=\; & -\frac{1}{4\pi}\left\{\oint_C \mu\nabla\left(\frac{1}{r}\right) \times d\underline{l} - \nabla\mu \oint_C \underline{A} \cdot d\underline{l} \right. \\ & \left. - (\underline{n} \times \nabla\mu) \times \left(\underline{n} \times \oint_C \frac{d\underline{l}}{r}\right)\right\}. \end{aligned} \qquad \text{(B.62)}$$