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-VORTEX METHODS-

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EVALUATION OF THE BIOT-SAVART INTEGRAL

D.1 Introduction

In this appendix, a computational method is described for evaluating the Biot-Savart integral. The approach emphasizes the transformation of the involved integrand into suitable forms, from which integral theorems can be used to reduce the volume integral into line integrals. This method is applied to the case where the density of vorticity distributed over a volumetric element bounded by planar surfaces (straight lines in 2-D) is constant and/or linear. The resulting expressions for the volume integral involve closed-form expressions for line integrals along the edges of the element. The evaluation of the line integrals is treated independently for each of the edges as opposed to direct numerical integration. The closed-form formulas are expressed in terms of geometric parameters of the element edges. Vector mathematical identities involving an integral of singularities distributed over a surface and a field can be employed to define field values of a vector variable of interest at a point within a field. For example, the field values of an irrotational and solenoidal vector can be obtained from the integrals over the sole surfaces bounding the field. In boundary-integral methods which were inspired by the work of Hess and Smith (1964, 1966, 1969) for potential flow problems of an incompressible fluid, the surface integrals involved may be evaluated on the boundary by assuming that the bounding surfaces are composed of a set of discrete panels and assuming a certain variation in the boundary values of the dependent variable in space (over the panels) and time.

For other problems related to rotational and solenoidal vector fields, a volume integral exists, the so-called Biot-Savart integral. It is well known that the Biot-Savart integral represents a formula in electromagnetic field theory that relates a field distribution of electric current to the induced magnetic field (see e.g., Bodner (1992)). In a manner analogous with the magnetic field induced by the given distribution of current, this induction law has been also applied to hydro- and aerodynamics by many workers: a distribution of vorticity in a field induces the velocity field whose curl becomes the given value of vorticity everywhere (Batchelor 1967, Saffman 1992).

In vortex methods for viscous flow analyses— especially in the vorticity-velocity integro-differential formulations (see e.g., Gresho (1991)), the Biot-Savart integral must be evaluated at appropriate field points within the discretized fluid domain. With N elements used in discretizing the fluid domain over which vorticity is distributed, $O(N^2)$ evaluations of the Biot-Savart integral may be required in order to calculate the velocity field. The evaluation of the Biot-Savart integral is, therefore, an important task in the numerical implementations associated with computational electromagnetics and fluid mechanics.

D.1.1 Integral representation

For a distributed vorticity field, $\underline{\omega}$, in a fluid region V, the general form of the Biot-Savart law is

$$\underline{q} = \int_V \underline{\omega} \times \nabla G \, dV, \tag{D.1}$$

where $\underline{q}$ is the induced velocity (magnetic) field and G the fundamental function, defined by

$$G = \begin{cases} \dfrac{1}{4\pi r} & \text{in 3-dimensions,} \\ \\ -\dfrac{1}{2\pi} \ln r & \text{in 2-dimensions.} \end{cases} \tag{D.2}$$

Hereafter, ∇ denotes the gradient, divergence, and curl differential operator with respect to integration variables $\underline{\xi}$, and r the distance between a field point $\underline{x}$ and an integration point $\underline{\xi}$.

In this appendix, efficient numerical analysis schemes for a linear distribution of vorticity over a surface in two-dimensions or over a volume in three-dimensions are presented on the basis of transformations of the integrals. It will be shown that the induced velocity field due to a vorticity distribution with linear strength can be derived from a sum of line integrals along the edges of a subdivided element. The derivation used here employs Stokes's and/or Gauss's theorem, by which the velocity field can be expressed in terms which are dependent only on the properties of each edge: namely, the terms of the position of a field point relative to each edge. In this manner, an analysis associated with direct calculation of the triple (double in 2-d) integral over the element can be avoided. An additional feature of the present derivation is that it is valid for an arbitrary element bounded by planar surfaces (straight lines in 2-D).

D.2 Biot-Savart Integral in 2-D

D.2.1 Transformation of integral

A quadrilateral element is, without loss of generality, taken for the present analysis. The complete induced field is constructed by superposing the field contributions due to the individual elements. For any polygon, we can easily deduce the corresponding results from the expression, Eq. (D.6) below, by taking into account the number of sides of the polygon in the summation of the contributions for each side. The vertices with coordinates (ξ_i, η_i) are denoted by $\underline{\xi}_i$, as shown in Figure **D.1**, where each vertex is indicated by the index i. The induced velocity ($\underline{q}$) at an arbitrary field point $P(\underline{x})$ with coordinates (x, y) due to a distribution of vorticity over the domain of the element S is

$$\underline{q} = -\frac{\underline{k}}{2\pi} \times \int_S \omega \, \nabla(\ln r) \, dS, \tag{D.3}$$

where $r = |\underline{r}| = |\underline{\xi} - \underline{x}|$ and ω is the scalar plane component of the vorticity vector, $\underline{\omega}$ ($\equiv \omega \, \underline{k}$).

The integrand can be transformed into, through simple vector operations,

$$\omega \, \nabla(\ln r) = \nabla(\omega \, \ln r) - \frac{1}{2} \left\{ \nabla \cdot (\underline{r} \, \ln r) - 1 \right\} \nabla\omega. \tag{D.4}$$

For a vorticity distribution of linear-variation density, we can convert the surface integral in Eq. (D.3) into line integral terms, by applying the Gauss theorem with the transformed integrand given in Eq. (D.4):

$$\int_S \omega \, \nabla(\ln r) \, dS = \frac{1}{2} \oint_C \underline{n} \, \omega \, (\ln r^2 + 1) \, dl - \frac{1}{2} \nabla\omega \oint_C (\underline{n} \cdot \underline{r}) \, \ln r \, dl. \tag{D.5}$$

Here the contour integrals are performed along the perimeter (C) of the element in a counter-clockwise direction, and $\underline{n}$ is the unit normal vector on the boundary of the element in the sense of a right-handed rule, i.e., $\underline{n} = \underline{s} \times \underline{k}$ where $\underline{s}$ is the unit directional vector of the contour integral path. Then $\underline{k}$, $\underline{n}$ and $\underline{s}$ constitute a right-handed triple of orthogonal unit vectors.

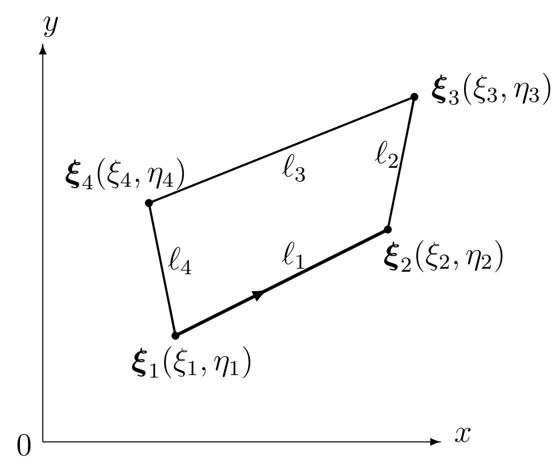

Figure D.1 Definition of a quadrilateral element.

D.2.2 Analytic form of integrals

The resulting expressions for the velocity field include the line integrals only along the boundary contour of the element. Let the value of the line integral along each straight edge of the element be $\underline{I}_i$. It then follows that

$$\underline{q} = -\frac{\underline{k}}{2\pi} \times \left(\sum_{i=1}^{4} \underline{I}_i \right), \tag{D.6}$$

where, with the side of length ℓ_i,

$$\underline{I}_i = \frac{1}{2} \underline{n}_i \int_0^{\ell_i} \omega \left(\ln r^2 + 1 \right) dl - \frac{1}{4} \nabla\omega \left(\underline{n}_i \cdot \underline{r} \right) \int_0^{\ell_i} \ln r^2 \, dl. \tag{D.7}$$

It is seen that the line integral for each side can be treated independently. It is sufficient, therefore, to consider only one side of the polygon for the purpose of integration. The essential task is to evaluate the line integrals along a straight segment from $\boldsymbol{\xi}_i$ to $\boldsymbol{\xi}_{i+1}$ with linear variation of ω over it.

For the evaluation of the associated integrals, we take a local coordinate system (x', y') in the plane through the field point $\underline{x}$ and the side concerned, such that the side lies on the x'-axis and one end point of the side is at the origin of the coordinates (see Figure **D.2**). The integration is performed along the positive x'-axis. The reason for choosing the local coordinate system as such is because the integration is more compact and systematic than that for the case of the global coordinate system, even though both procedures, in fact, produce identical results. Of course the coordinates of the field point in the global coordinate system must be transformed into the local coordinate systems of the respective sides, and the computed field components must then be defined in the global coordinate system to superpose the contributions due to the respective sides.

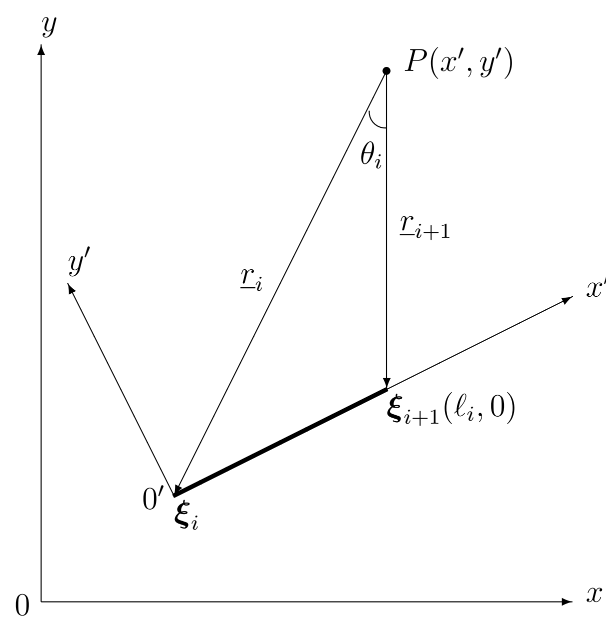

Figure D.2 Definition sketch of the local coordinate system (x', y').

The local coordinates are related to the vectors defined in the global coordinate system as: $x' = -\underline{r}_i \cdot \underline{s}_i$ and $y' = (\underline{r}_i \times \underline{s}_i) \cdot \underline{k}$. This transformation implies the projections of distance vectors between the field point P and the end points of the segment on the x'- and y'-axis. Let us denote the distances between the two end points of the side and the field point by $r_i = \sqrt{x'^2 + y'^2}$

and $r_{i+1} = \sqrt{(\ell_i - x')^2 + y'^2}$, respectively. After a substantial amount of algebraic manipulations (see Gradshteyn and Rhyzik 1980, pp. 81-84) for integral formulae), the following result for $\underline{I}_i$ can be obtained:

$$\underline{I}_i = \frac{1}{2}\underline{n}_i \left\{ \omega_i \left(\ell_i + I^{(1)} \right) + (\nabla\omega \cdot \underline{s}_i) \left(\frac{1}{2}\ell_i^2 + I^{(2)} \right) \right\} - \frac{1}{4}\nabla\omega(\underline{n}_i \cdot \underline{r}) I^{(1)}, \quad \text{(D.8)}$$

where ω_i denotes the vorticity value at the i-th vertex,

$$I^{(1)} = (\ell_i - x')\,\ln r_{i+1}^2 + x'\,\ln r_i^2 - 2\,\ell_i + 2\,|y'|\,\theta_i, \quad \text{(D.9)}$$

$$I^{(2)} = \frac{1}{2}\left(r_{i+1}^2 \ln r_{i+1}^2 - r_i^2 \ln r_i^2\right) - \frac{\ell_i^2}{2} + \ell_i\,x' + x'\,I^{(1)}, \quad \text{(D.10)}$$

and

$$\theta_i = \tan^{-1} \frac{|y'|\,\ell_i}{r_i^2 - \ell_i\,x'}. \quad \text{(D.11)}$$

Here the pair of arctangents appearing in this evaluation have been combined by using the trigonometric formulae. Eventually it is seen that θ_i denotes the included angle between distance vectors of the segment end points as viewed from the field point P (see Figure **D.2**). Thus, the included angle is uniquely measured as a value between 0 and π without considering the separate arguments of the arctangent function, since the numerator of the argument of the arctangent is non-negative. Note that the terms $I^{(1)}$ and $I^{(2)}$ given by Eqs. (D.9) and (D.10) are determinate when the field point is on the extensions of the side. For example, if the field point approaches one of the end points of the side, we have finite values according to L'Hospital's rule (for the indeterminate form $0 \cdot \infty$).

D.3 Biot-Savart Integral in 3-D

D.3.1 Transformation of integral

The induced velocity due to a vorticity distribution over an element whose boundary is composed of planar panels, can be expressed in a volume form analogous to Eq. (D.3):

$$\begin{aligned} \underline{q} &= \frac{1}{4\pi}\int_V \underline{\omega}\times\nabla\left(\frac{1}{r}\right)\,dV \\ &= \frac{1}{4\pi}\int_V \left\{\frac{1}{r}\left(\nabla\times\underline{\omega}\right) - \nabla\times\left(\frac{1}{r}\underline{\omega}\right)\right\}\,dV. \end{aligned} \tag{D.12}$$

The vorticity distribution is assumed to be linear so that $(\nabla\times\underline{\omega})$ is constant. By using the divergence (Gauss) theorem, Eq. (D.12) can be reduced to

$$4\pi\,\underline{q} = (\nabla\times\underline{\omega})\int_V \frac{1}{r}\,dV - \oint_S \underline{n}\times\left(\frac{1}{r}\underline{\omega}\right)\,dS, \tag{D.13}$$

where S is the surfaces bounding the volume V and $\underline{\mathrm{n}}$ is the outward normal unit vector on the bounding surfaces.

In order to evaluate the volume integral term in Eq. (D.13), we use here Green's second identity for a scalar function ϕ such that $\nabla^2\phi = 1$;

$$\int_V \frac{1}{r}\,dV = -\alpha\,\phi(\underline{x}) - \oint_S \left\{\phi\,\underline{n}\cdot\nabla\left(\frac{1}{r}\right) - \frac{\underline{n}\cdot\nabla\phi}{r}\right\}\,dS, \tag{D.14}$$

where α is constant. When $\underline{x}$ is inside the volumetric region V, α is 4π. If $\underline{x}$ is on the boundary of V, it is 2π. For $\underline{x}$ outside the volume, this value is zero. Equation (D.13) can then be expressed as a sum of integrals over the bounding planar surfaces as:

$$\begin{aligned} 4\pi\underline{q} &= -(\nabla\times\underline{\omega})\left[\alpha\,\phi(\underline{x}) + \oint_S \left\{\phi\,\underline{n}\cdot\nabla\left(\frac{1}{r}\right) - \frac{\underline{n}\cdot\nabla\phi}{r}\right\}\,dS\right] \\ &\quad - \oint_S \underline{n}\times\left(\frac{1}{r}\underline{\omega}\right)\,dS, \\ &= -(\nabla\times\underline{\omega})\,\alpha\,\phi(\underline{x}) - \sum_{j=1}^{6}\left\{(\nabla\times\underline{\omega})\,K_j + \underline{L}_j\right\}. \end{aligned} \tag{D.15}$$

Here the upper limit 6 in the summation denotes the number of faces of the volumetric cell element taken. Let us consider the surface integral term over one planar panel since the corresponding integral terms for other panels can be evaluated in the same manner. We drop the subscript j in K_j and $\underline{L}_j$ for simplicity of notation. The integral K represents induced potentials due to dipole distributions of the second order in density and source distributions with linearly varying density over the bounding surfaces. The integral has been evaluated in various manners by numerous researchers. Bai and Yeung (1974) have set up the basic framework for treating the potential and the normal potential induced by a source density distribution which varies linearly over a triangular patch element (see also (Webster 1975, Newman 1986). Herein on the basis of Bai & Yeung's procedure, we take the approach described in literature Suh et al. (1992) for consistency with the present work. The analysis schemes are based on transformation of the associated integrals.

Let us take, for example, $\phi = 0.5x^2$ as a simple choice of ϕ in Eq. (D.14). In order to specify the second order variation of dipole density μ and the linear variation of source density σ over the respective planar panels of the bounding surfaces, we take a local coordinate system (ξ, η, ζ) such that the integration surface is in the plane $\zeta = 0$ and the direction of the ζ-axis is the same as that of the normal vector $\underline{n}$. The other two axes are on the surface and their directional unit vectors $(\underline{e}_\xi, \underline{e}_\eta)$ with the normal vector $(\underline{n})$ form a right-handed triple of orthogonal unit vectors. We can specify the dipole distribution as $\mu = 0.5\{x_0 + \xi(\underline{e}_\xi \cdot \underline{i})\}^2$ and the source distribution as $\sigma = \{x_0 + \xi(\underline{e}_\xi \cdot \underline{i})\}(\underline{n} \cdot \underline{i})$, where x_0 is the x-coordinate of the origin of the local coordinate system and $\underline{e}_\xi = \underline{n} \times (\underline{i} \times \underline{n})/|\underline{i} \times \underline{n}|$. The integrands involved in Eq. (D.14) can now be transformed into either the curl form of a vector or the cross product of a vector with the normal $\underline{n}$, as follows (Guiraud 1978, Suh 1992):

$$\underline{n} \cdot \nabla \left(\frac{1}{r}\right) = -\underline{n} \cdot (\nabla \times \underline{A}), \tag{D.16}$$

$$(\xi - x_r)\, \underline{n} \cdot \nabla \left(\frac{1}{r}\right) = -z_r \left\{\underline{e}_\eta \cdot \underline{n} \times \nabla \left(\frac{1}{r}\right)\right\}, \tag{D.17}$$

$$(\xi - x_r)^2\, \underline{n} \cdot \nabla \left(\frac{1}{r}\right) = z_r \left\{\frac{1}{r} - \underline{e}_\xi \cdot \nabla \left(\frac{\xi - x_r}{r}\right)\right\}, \tag{D.18}$$

$$\frac{1}{r} = \underline{e}_n \cdot (\nabla \times \underline{B}), \tag{D.19}$$

$$\frac{\xi - x_r}{r} = \underline{e}_\eta \cdot (\underline{n} \times \nabla r), \tag{D.20}$$

with

$$\underline{A} = \frac{\underline{e}_n \times \underline{r}}{r(r + \underline{e}_n \cdot \underline{r})}, \qquad \underline{B} = \frac{\underline{e}_n \times \underline{r}}{(r + \underline{e}_n \cdot \underline{r})}, \tag{D.21}$$

where the coordinates (x_r, y_r, z_r) of the field point are measured with respect to the origin of this local coordinate system, and $\underline{e}_n$ is a constant unit vector, which is independent of the integration variables of the surface integral. Note that Eqs. (D.17), (D.18) and (D.20) have been derived under the hypothesis of planarity of the surfaces. While Eq. (D.19) holds for any $\underline{e}_n$ independent of the integration variables, the unit vector $\underline{e}_n$ is conveniently taken as $\pm\underline{n}$ in order to use Stokes's theorem for Eqs. (D.16) and (D.33) where the sign is chosen such that the term $\underline{e}_n \cdot \underline{r}$ in the numerator of $\underline{A}$ and $\underline{B}$ is non-negative.

D.3.2 Specific line integrals

The integral K can then be written as, with the constants $a_0 = x_0 + x_r(\underline{e}_\xi \cdot \underline{i})$ and $a_1 = \underline{e}_\xi \cdot \underline{i}$ for shortness of expressions,

$$K = (\underline{n} \cdot \underline{i})(a_0\, \phi_\sigma^{(0)} + a_1\, \phi_\sigma^{(1)}) + 0.5\, a_0^2\, \phi_\mu^{(0)} + a_0\, a_1\, \phi_\mu^{(1)} + 0.5\, a_1^2\, \phi_\mu^{(2)}, \tag{D.22}$$

where

$$\begin{aligned}
\phi_\sigma^{(0)} &= -\sum_{i=1}^{4} b_i\, K^{(1)}, \quad \phi_\sigma^{(1)} = -\sum_{i=1}^{4} s_{i\eta}\, K^{(2)}, \\
\phi_\mu^{(0)} &= -\sum_{i=1}^{4} b_i\, (\underline{n} \cdot \underline{e}_n)\, \frac{E - K^{(1)}}{e}, \quad \phi_\mu^{(1)} = -z_r \sum_{i=1}^{4} s_{i\eta}\, E, \\
\phi_\mu^{(2)} &= -z_r \left[\phi_\sigma^{(0)} + \sum_{i=1}^{4} \left\{ \underline{e}_\xi \cdot (\underline{s}_i \times \underline{n})\, K^{(3)} \right\} \right],
\end{aligned} \tag{D.23}$$

and the upper limit 4 in the summation denotes the number of sides of the panel. Similar to the 2-D cases, the associated line integrals for the sides of the quadrilateral planar surface can be treated independently by using the geometric parameters of each side. Taking the local coordinate system (x', y'), as shown in Figure **D.2** for the evaluation of the line integrals, the following closed-form expressions of the associated integrals can be obtained by using the integral formulae (Grashteyn and Rhyzik 1980, pp. 81-84):

$$
\begin{aligned}
K^{(1)} &= \int_0^{\ell_i} \frac{1}{\sqrt{(x'-\xi)^2+y'^2}+e}\, d\xi = E - \frac{e}{\sqrt{y'^2-e^2}}\,\beta, && \text{(D.24)}\\
K^{(2)} &= \int_0^{\ell_i} \sqrt{(x'-\xi)^2+y'^2}\, d\xi = \frac{1}{2}\left\{(\ell_i - x')\, r_{i+1} + x'\, r_i + y'^2\, E\right\}, && \text{(D.25)}\\
K^{(3)} &= (\xi_i - x_r)\, E + s_{i\xi}\,(r_{i+1} - r_i + x'\, E), && \text{(D.26)}\\
E &= \ln \frac{r_{i+1} + \ell_i - x'}{r_i - x'}, && \text{(D.27)}\\
\beta &= \begin{cases} \sin^{-1} H & \text{if } F > 0,\\ \pi - \sin^{-1} H & \text{if } F \le 0, \end{cases} && \text{(D.28)}\\
H &= \frac{\sqrt{y'^2-e^2}\left\{y'^2\,\ell_i + e\,(\ell_i - x')\, r_i + e\, x'\, r_{i+1}\right\}}{y'^2(r_i+e)(r_{i+1}+e)}, && \text{(D.29)}\\
F &= \left(\frac{y'^2 + e\, r_i}{r_i + e}\right)^2 + \left(\frac{y'^2 + e\, r_{i+1}}{r_{i+1}+e}\right)^2 - y'^2, && \text{(D.30)}\\
b_i &= (\underline{n} \times \underline{r}) \cdot \underline{s}_i, \quad s_{i\xi} = \underline{s}_i \cdot \underline{e}_\xi, \quad s_{i\eta} = \underline{s}_i \cdot \underline{e}_\eta, \quad e = \underline{e}_n \cdot \underline{r}. && \text{(D.31)}
\end{aligned}
$$

Recall that $\underline{s}_i$ denotes the unit directional vector along the path of integration. In certain cases, some evaluations require special treatment. While the term $K^{(2)}$ is bounded, the term $K^{(1)}$ might be indeterminate if the field point lies on the same plane as the panel or on one of the lines defining the panel edge. In this respect, let us investigate the behavior of the term $b_i K^{(1)}$ in the vicinity of the panel sides. If $|y'|$ is equal to e, we have $K^{(1)} = E - \dfrac{x'}{r_i + e} - \dfrac{\ell_i - x'}{r_{i+1} + e}$ but the factor b_i vanishes and, hence, the term $b_i K^{(1)}$ also vanishes. Furthermore, when y' is very small (accordingly the factor e approaches zero), b_i and $b_i K^{(1)}$ vanish in the same limit. When the field point approaches one of the vertices (i.e., as $x' \to 0$ and $y' \to 0$) $K^{(1)}$ is logarithmically infinite, but $b_i K^{(1)}$ vanishes. Thus the integral K has a finite value even, if the field point is on the same plane as the panel.

Next we will evaluate the second integral term $\underline{L}$ in Eq. (D.15):

$$\underline{L} = -\int_S \underline{n} \times \left(\frac{1}{r} \underline{\omega} \right) dS = \int_S \frac{\gamma \, \underline{t}}{r} dS, \tag{D.32}$$

where $\gamma \, \underline{t} = -\underline{n} \times \underline{\omega}$.

Similar to the integral K, Eq. (D.32) has the same form as the expression for the induced potential due to a source distribution over a surface. For the cases of distributions of vorticity with linearly varying densities within an element domain, γ has a linear variation over the surface being an integration region. With a specified linear distribution $\gamma \, \underline{t} = c_0 \, \underline{t}_0 + c_1 \, (\xi - x_r) \, \underline{t}_1 + c_2 \, (\eta - y_r) \, \underline{t}_2$, we have

$$\underline{L} = c_0 \, \underline{t}_0 \int_S \frac{1}{r} dS + c_1 \, \underline{t}_1 \int_S \frac{\xi - x_r}{r} dS + c_2 \, \underline{t}_2 \int_S \frac{\eta - y_r}{r} dS. \tag{D.33}$$

Herein the vectors $\underline{t}_0$, $\underline{t}_1$ and $\underline{t}_2$ are brought outside the integral, because they are the constant vectors which are uniquely determined from the linearly varying distribution of vorticity density over the panel. The integrands in Eq. (D.33) can now be transformed, as given in Eqs. (D.19) and (D.20), and

$$\frac{\eta - y_r}{r} = -\underline{e}_\xi \cdot (\underline{n} \times \nabla r). \tag{D.34}$$

Consequently Eq. (D.33) can be written as

$$\underline{L} = \sum_{i=1}^{4} \left\{ c_0 \, \underline{t}_0 \, b_i \, K^{(1)} + (c_1 \, \underline{t}_1 \, s_{i\eta} - c_2 \, \underline{t}_2 \, s_{i\xi}) \, K^{(2)} \right\}. \tag{D.35}$$

For constant distributions of vorticity, we need only the term $-\sum_{i=1}^{6} \underline{L}_j$ without the first and the second term in Eq (D.15).