Chapter 7. Linear Algebra

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서울대학교

조선해양공학과

서 유 택

※ 본 강의 자료는 이규열, 장범선, 노명일 교수님께서 만드신 자료를 바탕으로 일부 편집한 것입니다.

Contents

7.1 Matrices, Vectors : Addition and Scalar Multiplication

Matrices

Matrix : a rectangular array of numbers (or functions) enclosed in brackets.

Entries (성분)
(or sometimes the elements (원소))

Examples)

$$\begin{array}{cc} \begin{bmatrix} 0.3 & 1 & -5 \\ 0 & -0.2 & 16 \end{bmatrix}, & \text{rows} \end{array}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

columns

$$\begin{bmatrix} e^{-x} & 2x^2 \\ e^{6x} & 4x \end{bmatrix}$$

square matrices

(no. of rows = no. of columns)

Matrices

We shall denote matrices by ***capital boldface letters*** **A, B, C,** ··· **, or by writing the general entry in brackets ; thus** $\mathbf{A}=[a_{jk}]$**, and so on. By an** m **x** n **matrix (read** m **by** n **matrix) we mean a matrix with** m **rows and** n **columns – rows come always first!**

m **x** n **is called the size of the matrix. Thus an** m **x** n **matrix is of the form**

$$\mathbf{A}=[a_{jk}]=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \cdots (2).$$

Matrices

$$\mathbf{A} = [a_{jk}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

If $m = n$, we call A an n x n square matrix (정방행렬).

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

main diagonal

If $m \neq n$, we call A a rectangular matrix (구형 또는 장방행렬).

Vectors

Matrix : a rectangular array of numbers (or functions) enclosed in brackets.

Vector : a matrix with only one row or column.

We shall denote vectors by *lowercase boldface letters* $\mathbf{a}$, $\mathbf{b}$, $\cdots$ or by its general component in brackets, $\mathbf{a}=[a_i]$, and so on.

Examples)

row vector

$$\mathbf{a} = [a_1 \;\; a_2 \;\; a_3],$$

Components (entries)

column vector

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Matrices

Ex 1) Linear Systems, a Major Application of Matrices

In a system of linear equations, briefly called a linear system, such as

$$
\begin{aligned}
4x_1 + 6x_2 + 9x_3 &= 6 \\
6x_1 \qquad\;\; - 2x_3 &= 20 \\
5x_1 - 8x_2 + \;\; x_3 &= 10
\end{aligned}
$$

the coefficients of the unknowns x_1, x_2, x_3 are the entries of the coefficient matrix (계수행렬), call it A,

$$\mathbf{A} = \begin{bmatrix} 4 & 6 & 9 \\ 6 & 0 & -2 \\ 5 & -8 & 1 \end{bmatrix}.$$

The matrix

$$\tilde{\mathbf{A}} = \left[\begin{array}{ccc|c} 4 & 6 & 9 & 6 \\ 6 & 0 & -2 & 20 \\ 5 & -8 & 1 & 10 \end{array}\right]$$

is called augmented matrix (첨가행렬) of the system.

We shall discuss this in great detail, beginning in Sec. 7.3

Matrices

Ex 2) Sales Figures in Matrix Form

Sales Figures for Three products I, II, III in a store on Monday (M), Tuesday (T), ··· may for each week be arrange in a matrix.

$$A = \begin{array}{cc} & \begin{array}{cccccc} M & T & W & Th & F & S \end{array} \\ & \left[\begin{array}{cccccc} 400 & 330 & 810 & 0 & 210 & 470 \\ 0 & 120 & 780 & 500 & 500 & 960 \\ 100 & 0 & 0 & 270 & 430 & 780 \end{array}\right] \begin{array}{c} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \end{array} \end{array}.$$

If the company has ten stores, we can set up ten such matrices, on for each store. Then by adding corresponding entries of these matrices we can get a matrix showing the total sales of each product on each day.

Matrix Addition and Scalar Multiplication

Equality of Matrices

Two matrices $\mathbf{A}=[a_{jk}]$ and $\mathbf{B}=[b_{jk}]$ are equal, written $\mathbf{A}=\mathbf{B}$, if and only if they have the same size and the corresponding entries are equal, that is, $a_{11}=b_{11}$, $a_{12}=b_{12}$, and so on. Matrices that are not equal are called different. Thus, matrices of different sizes are always different.

Ex3) Equality of Matrices

$$\text{Let} \quad \mathbf{A}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } \mathbf{B}=\begin{bmatrix} 4 & 0 \\ 3 & -1 \end{bmatrix}$$

$$\text{Then} \quad \mathbf{A}=\mathbf{B} \text{ if and only if } \begin{matrix} a_{11}=4, & a_{12}=0, \\ a_{21}=3, & a_{22}=-1. \end{matrix}$$

Matrix Addition and Scalar Multiplication

Addition of Matrices

The sum of two matrices $\mathrm{A}=[a_{jk}]$ and $\mathrm{B}=[b_{jk}]$ of the same size is written $\mathrm{A+B}$ and has the entries $a_{jk}+b_{jk}$ obtained by adding the corresponding entries of A and B. Matrices of different sizes cannot be added.

Ex4) Addition of Matrices and Vectors

$$\text{If} \quad \mathbf{A}=\begin{bmatrix} -4 & 6 & 3 \\ 0 & 1 & 2 \end{bmatrix} \text{ and } \mathbf{B}=\begin{bmatrix} 5 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix}, \text{ then } \mathbf{A}+\mathbf{B}=\begin{bmatrix} 1 & 5 & 3 \\ 3 & 2 & 2 \end{bmatrix}.$$

$$\text{If} \quad \mathbf{a}=\begin{bmatrix} 5 & 7 & 2 \end{bmatrix} \text{ and } \mathbf{b}=\begin{bmatrix} -6 & 2 & 0 \end{bmatrix}, \text{ then } \mathbf{a}+\mathbf{b}=\begin{bmatrix} -1 & 9 & 2 \end{bmatrix}.$$

Matrix Addition and Scalar Multiplication

Scalar Multiplication (Multiplication by a Number)

The product of any m x n matrix $A=[a_{jk}]$ and any scalar c (number c) is written cA and is the m x n matrix $cA=[ca_{jk}]$ obtained by multiplying each entry of A by c.

Ex5) Scalar Multiplication

$$\text{If} \quad \mathbf{A} = \begin{bmatrix} 2.7 & -1.8 \\ 0 & 0.9 \\ 9.0 & -4.5 \end{bmatrix}, \quad \text{then} \quad \mathbf{-A} = \begin{bmatrix} -2.7 & 1.8 \\ 0 & -0.9 \\ -9.0 & 4.5 \end{bmatrix}$$

zero matrix

$$\frac{10}{9}\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 0 & 1 \\ 10 & -5 \end{bmatrix}, \qquad 0 \cdot \mathbf{A} = \boxed{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}$$

0 : zero matrix (of size m x n matrix)

Rules for Matrix Addition and Scalar Multiplication

$$(3)\quad \begin{cases} (a) & \mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A} \\ (b) & (\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C}) \quad (\text{written } \mathbf{A}+\mathbf{B}+\mathbf{C}) \\ (c) & \mathbf{A}+0=\mathbf{A} \\ (d) & \mathbf{A}+(-\mathbf{A})=0 \end{cases}$$

$$(4)\quad \begin{cases} (a) & c(\mathbf{A}+\mathbf{B})=c\mathbf{A}+c\mathbf{B} \\ (b) & (c+k)\mathbf{A}=c\mathbf{A}+k\mathbf{A} \\ (c) & c(k\mathbf{A})=(ck)\mathbf{A} \quad (\text{written } ck\mathbf{A}) \\ (d) & 1\mathbf{A}=\mathbf{A} \end{cases}$$

7.2 MATRIX MULTIPLICATION

Multiplication of a Matrix by a Matrix

$$\underset{[m\times n]}{\mathbf{A}}\ \underset{[n\times r]}{\mathbf{B}} = \underset{[m\times r]}{\mathbf{C}}$$

$$\boxed{c_{jk} = \sum_{l=1}^{n} a_{jl}b_{lk} = a_{j1}b_{1k} + a_{j2}b_{2k} + \cdots + a_{jn}b_{nk} \cdots (1)}$$

$$j = 1, \cdots, m$$
$$k = 1, \cdots, p.$$

$$m=4\left\{\overbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}}^{n=3}\right. \left.\overbrace{\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}}^{p=2}\right\}(r=3) = \left.\overbrace{\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \\ c_{41} & c_{42} \end{bmatrix}}^{p=2}\right\} m=4$$

$$c_{21} = a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}$$

$$\underset{[m\times n]}{\mathbf{A}}\ \underset{[n\times r]}{\mathbf{B}} = \underset{[m\times r]}{\mathbf{C}}$$

Multiplication of a Matrix by a Matrix

Ex 1) Matrix Multiplication

$$\mathbf{AB} = \begin{bmatrix} 3 & 5 & -1 \\ 4 & 0 & 2 \\ -6 & -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 5 & 0 & 7 & 8 \\ 9 & -4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 22 & -2 & 43 & 42 \\ 26 & -16 & 14 & 6 \\ -9 & 4 & -37 & -28 \end{bmatrix}$$

$$c_{11} = 3\cdot 2 + 5\cdot 5 + (-1)\cdot 9 = 22$$

$$c_{23} = 4\cdot 3 + 0\cdot 7 + 2\cdot 1 = 14$$

The product $\mathbf{BA}$ is not defined.

Multiplication of a Matrix by a Matrix

$$\underset{[m\times n]}{\mathbf{A}}\ \underset{[n\times r]}{\mathbf{B}} = \underset{[m\times r]}{\mathbf{C}}$$

Ex 1) Matrix Multiplication

$$\begin{bmatrix} 4 & 2 \\ 1 & 8 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 4\cdot 3 + 2\cdot 5 \\ 1\cdot 3 + 8\cdot 5 \end{bmatrix} = \begin{bmatrix} 22 \\ 43 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 5 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 1 & 8 \end{bmatrix}$$ **is not defined.**

Ex3) Products of Row and Column Vectors

$$\begin{bmatrix} 3 & 6 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 19 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}\begin{bmatrix} 3 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 1 \\ 6 & 12 & 2 \\ 12 & 24 & 4 \end{bmatrix}.$$

Multiplication of a Matrix by a Matrix

$$\underset{[m\times n]}{\mathbf{A}}\ \underset{[n\times r]}{\mathbf{B}} \overset{=}{=} \underset{[m\times r]}{\mathbf{C}}$$

Ex 4) CAUTION! Matrix Multiplication is Not Commutative (가환성의), AB≠BA in General

Commutative rule (교환법칙)

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{BA} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix} = \begin{bmatrix} 99 & 99 \\ -99 & -99 \end{bmatrix}$$

$$\therefore \mathbf{AB} \neq \mathbf{BA}$$

It is interesting that this also shows that AB=0 does not necessarily imply BA=0 or A=0 or B=0.

Properties of Matrix Multiplication

$$\boxed{(2)\quad \begin{cases} (a) & (k\mathbf{A})\mathbf{B} = k(\mathbf{AB}) = \mathbf{A}(k\mathbf{B}) \\ (b) & \mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C} \\ (c) & (\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC}+\mathbf{BC} \\ (d) & \mathbf{C}(\mathbf{A}+\mathbf{B}) = \mathbf{CA}+\mathbf{CB} \end{cases}}$$

$\left(\text{written } k\mathbf{AB} \text{ } or \text{ } \mathbf{A}k\mathbf{B}\right)$

$\left(\text{written } \mathbf{ABC}\right)$

(2*b*) is called the associative law (결합법칙)

(2*c*) and (2*d*) is called the distributive law (분배법칙)

Properties of Matrix Multiplication

$$\underset{[m\times n]}{\mathbf{A}}\ \underset{[n\times r]}{\mathbf{B}} = \underset{[m\times r]}{\mathbf{C}}$$

$$c_{jk} = \sum_{l=1}^{n} a_{jl}b_{lk} = a_{j1}b_{1k} + a_{j2}b_{2k} + \cdots + a_{jn}b_{nk} \cdots (1)$$

Since matrix multiplication is a multiplication of rows into columns, we can write the defining formula (1) more compactly as

$$\boxed{c_{jk} = \mathbf{a}_j\mathbf{b}_k \cdots (3)} \qquad j = 1, \cdots, m; \quad k = 1, \cdots, p.$$

where $\mathbf{a}_j$: the jth row vector of $\mathbf{A}$

$\mathbf{b}_k$: the kth column vector of $\mathbf{B}$, so that in agreement with (1),

$$\mathbf{a}_j\mathbf{b}_k = \begin{bmatrix} a_{j1} & a_{j2} & \cdots & a_{jn} \end{bmatrix} \begin{bmatrix} b_{1k} \\ \vdots \\ b_{nk} \end{bmatrix} = a_{j1}b_{1k} + a_{j2}b_{2k} + \cdots + a_{jn}b_{nk}$$

Properties of Matrix Multiplication

Ex 5) Product in Terms of Row and Column Vectors

If $A=[a_{jk}]$ is of size $3x3$ and $B=[b_{jk}]$ is of size $3x4$, then

$$\mathbf{AB}=\begin{bmatrix}\mathbf{a}_1\mathbf{b}_1 & \mathbf{a}_1\mathbf{b}_2 & \mathbf{a}_1\mathbf{b}_3 & \mathbf{a}_1\mathbf{b}_4\\ \mathbf{a}_2\mathbf{b}_1 & \mathbf{a}_2\mathbf{b}_2 & \mathbf{a}_2\mathbf{b}_3 & \mathbf{a}_2\mathbf{b}_4\\ \mathbf{a}_3\mathbf{b}_1 & \mathbf{a}_3\mathbf{b}_2 & \mathbf{a}_3\mathbf{b}_3 & \mathbf{a}_3\mathbf{b}_4\end{bmatrix}\cdots(4).$$

$$\boxed{\mathbf{A}=\begin{bmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\end{bmatrix},\quad \mathbf{B}=\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4\end{bmatrix}}$$

Parallel processing of products on the computer is facilitated by a variant of (3) for computing $C=AB$, which is used by standard algorithms.

In this method, A is used as given, B is taken in terms of its column vectors, and the product is computed columnwise; thus,

$$\mathbf{AB}=\mathbf{A}\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p\end{bmatrix}=\begin{bmatrix}\mathbf{Ab}_1 & \mathbf{Ab}_2 & \cdots & \mathbf{Ab}_p\end{bmatrix}\cdots(5).$$

Properties of Matrix Multiplication

Ex 5) Product in Terms of Row and Column Vectors

$$\mathbf{AB} = \begin{bmatrix} \mathbf{a}_1\mathbf{b}_1 & \mathbf{a}_1\mathbf{b}_2 & \mathbf{a}_1\mathbf{b}_3 & \mathbf{a}_1\mathbf{b}_4 \\ \mathbf{a}_2\mathbf{b}_1 & \mathbf{a}_2\mathbf{b}_2 & \mathbf{a}_2\mathbf{b}_3 & \mathbf{a}_2\mathbf{b}_4 \\ \mathbf{a}_3\mathbf{b}_1 & \mathbf{a}_3\mathbf{b}_2 & \mathbf{a}_3\mathbf{b}_3 & \mathbf{a}_3\mathbf{b}_4 \end{bmatrix} \cdots (4).$$

$$\mathbf{AB} = \mathbf{A}\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix} = \begin{bmatrix} \mathbf{Ab}_1 & \mathbf{Ab}_2 & \cdots & \mathbf{Ab}_p \end{bmatrix} \cdots (5).$$

Columns of $\mathbf{B}$ are then assigned to different processors (individually or several to each processor), which simultaneously compute the columns of the product matrix $\mathbf{Ab}_1$, $\mathbf{Ab}_2$, etc.

Properties of Matrix Multiplication

Ex 6) Computing Products Columnwise by (5)

$$\mathbf{AB} = \mathbf{A}\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix} = \begin{bmatrix} \mathbf{Ab}_1 & \mathbf{Ab}_2 & \cdots & \mathbf{Ab}_p \end{bmatrix} \cdots (5).$$

$$\mathbf{AB} = \begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 & 7 \\ -1 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 11 & 4 & 34 \\ -17 & 8 & -23 \end{bmatrix}$$

To obtain $\mathbf{AB}$ from (5), calculate the columns

$$\begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 11 \\ -17 \end{bmatrix}, \begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ 6 \end{bmatrix} = \begin{bmatrix} 34 \\ -23 \end{bmatrix}$$

of $\mathbf{AB}$ and then write them as a single matrix, as shown in the second equation on the right.

Applications of Matrix Multiplication

Ex 11) Computer Production. Matrix Times Matrix

Supercomp Ltd produces two computer models PC1086 and PC1186.

Matrix $\mathbf{A}$: the cost per computer (in thousands of dollars)
Matrix $\mathbf{B}$: the production figures for the year 2010 (in multiples of 10,000 units)

Find a matrix $\mathbf{C}$ that shows the shareholders the cost per quarter (in millions of dollars) for raw material, labor, and miscellaneous.

$$\begin{array}{cc} & \begin{array}{cc} \text{PC1086} & \text{PC1186} \end{array} \\ \mathbf{A} = & \begin{bmatrix} 1.2 & 1.6 \\ 0.3 & 0.4 \\ 0.5 & 0.6 \end{bmatrix} \begin{array}{l} \text{Raw Components} \\ \text{Labor} \\ \text{Miscellaneous} \end{array} \end{array}$$

$$\begin{array}{cc} & \text{Quarter} \\ & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \mathbf{B} = & \begin{bmatrix} 3 & 8 & 6 & 9 \\ 6 & 2 & 4 & 3 \end{bmatrix} \begin{array}{l} \text{PC1086} \\ \text{PC1186} \end{array} \end{array}$$

Applications of Matrix Multiplication

Ex 11) Computer Production. Matrix Times Matrix

$$\mathbf{A} = \begin{array}{c} \begin{array}{cc} \textbf{PC1086} & \textbf{PC1186} \end{array} \\ \begin{bmatrix} 1.2 & 1.6 \\ 0.3 & 0.4 \\ 0.5 & 0.6 \end{bmatrix} \end{array} \begin{array}{l} \\ \textbf{Raw Components} \\ \textbf{Labor} \\ \textbf{Miscellaneous} \end{array}$$

$$\mathbf{B} = \begin{array}{c} \textbf{Quarter} \\ \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 3 & 8 & 6 & 9 \\ 6 & 2 & 4 & 3 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \textbf{PC1086} \\ \textbf{PC1186} \end{array}$$

Q ?

$$\mathbf{C} = \mathbf{AB} = \begin{array}{c} \textbf{Quarter} \\ \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 13.2 & 12.8 & 13.6 & 15.6 \\ 3.3 & 3.2 & 3.4 & 3.9 \\ 5.1 & 5.2 & 5.4 & 6.3 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \textbf{Raw Components} \\ \textbf{Labor} \\ \textbf{Miscellaneous} \end{array}$$

Applications of Matrix Multiplication

Ex 12) Weight Watching. Matrix Times Vector

A weight-watching program

A person of 185lb burns 350 cal/hr in walking (3 mph)
500 cal/hr in bicycling (13 mph)
950 cal/hr in jogging (5.5 mph)

Bill, weighing 185 lb, plans to exercise according to the matrix shown.

Verify the calculations (W = Walking, B = Bicycling, J=Jogging).

$$\begin{array}{c} \\ \text{MON} \\ \text{WED} \\ \text{FRI} \\ \text{SAT} \end{array} \begin{array}{c} \begin{array}{ccc} \text{W} & \text{B} & \text{J} \end{array} \\ \begin{bmatrix} 1.0 & 0 & 0.5 \\ 1.0 & 1.0 & 0.5 \\ 1.5 & 0 & 0.5 \\ 2.0 & 1.5 & 1.0 \end{bmatrix} \end{array} \begin{bmatrix} 350 \\ 500 \\ 950 \end{bmatrix} = \begin{bmatrix} 825 \\ 1325 \\ 1000 \\ 2400 \end{bmatrix} \begin{array}{c} \text{MON} \\ \text{WED} \\ \text{FRI} \\ \text{SAT} \end{array}$$

* 1 lb (파운드) = 0.453592 kg

Applications of Matrix Multiplication

Ex 13) Markov Process. Powers of a Matrix. Stochastic Matrix

Suppose that the 2004 state of land use in a city of 60 mi^2 of built-up area is
C : Commercially Used 25 %,
I : Industrially Used 20 %,
R : Residentially Used 55 %.

Transition probabilities for 5-year intervals : A and remain practically the same over the time considered.

Find the states 2009, 2014, 2019

$$\mathbf{A}: \quad \begin{array}{c} \\ \text{To C} \\ \text{To I} \\ \text{To R} \end{array} \begin{array}{c} \begin{array}{ccc} \text{From C} & \text{From I} & \text{From R} \end{array} \\ \begin{bmatrix} 0.7 & 0.1 & 0 \\ 0.2 & 0.9 & 0.2 \\ 0.1 & 0 & 0.8 \end{bmatrix} \end{array}$$

* Markov Process: 미래 상태가 과거 상태와는 독립적으로 현재 상태에 의해서만 결정된다는 것

Applications of Matrix Multiplication

Ex 13) Markov Process. Powers of a Matrix. Stochastic Matrix

A : stochastic matrix (확률행렬) : a square matrix
all entries nonnegative
all column sums equal to 1

Markov process : the probability of entering a certain state depends only on the last state occupied (and the matrix A), not on any earlier state.

solution.

$$\mathbf{A}: \quad \begin{array}{c|ccc} & \text{From C} & \text{From I} & \text{From R} \\ \hline \text{To C} & 0.7 & 0.1 & 0 \\ \text{To I} & 0.2 & 0.9 & 0.2 \\ \text{To R} & 0.1 & 0 & 0.8 \end{array}$$

area

$$\begin{array}{c} \text{C} \\ \text{I} \\ \text{R} \end{array} \begin{bmatrix} 0.7\cdot 25+0.1\cdot 20+\quad 0\cdot 55 \\ 0.2\cdot 25+0.9\cdot 20+0.2\cdot 55 \\ 0.1\cdot 25+\quad 0\cdot 20+0.8\cdot 55 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.1 & 0 \\ 0.2 & 0.9 & 0.2 \\ 0.1 & 0 & 0.8 \end{bmatrix} \begin{bmatrix} 25 \\ 20 \\ 55 \end{bmatrix} = \begin{bmatrix} 19.5 \\ 34.0 \\ 46.5 \end{bmatrix}.$$

* Markov Process: 미래 상태가 과거 상태와는 독립적으로 현재 상태에 의해서만 결정된다는 것

Applications of Matrix Multiplication

$$\mathbf{A}: \begin{array}{c|ccc} & \text{From C} & \text{From I} & \text{From R} \\ \hline \text{To C} & 0.7 & 0.1 & 0 \\ \text{To I} & 0.2 & 0.9 & 0.2 \\ \text{To R} & 0.1 & 0 & 0.8 \end{array}$$

Ex 13) Markov Process. Powers of a Matrix. Stochastic Matrix

area

$$\begin{array}{c} \text{C} \\ \text{I} \\ \text{R} \end{array} \begin{bmatrix} 0.7\cdot 25+0.1\cdot 20+\quad 0\cdot 55 \\ 0.2\cdot 25+0.9\cdot 20+0.2\cdot 55 \\ 0.1\cdot 25+\quad 0\cdot 20+0.8\cdot 55 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.1 & 0 \\ 0.2 & 0.9 & 0.2 \\ 0.1 & 0 & 0.8 \end{bmatrix}\begin{bmatrix} 25 \\ 20 \\ 55 \end{bmatrix} = \begin{bmatrix} 19.5 \\ 34.0 \\ 46.5 \end{bmatrix}.$$

We see that the 2009 state vector ($\mathbf{y}$) is the column vector

$$\mathbf{y} = \begin{bmatrix} 19.5 \\ 34.0 \\ 46.5 \end{bmatrix} = \mathbf{Ax} = \mathbf{A}\begin{bmatrix} 25 \\ 20 \\ 55 \end{bmatrix}$$

The vector ($\mathbf{x}$) is the given 2004 state vector

The sum of the entries of $\mathbf{y}$ is 100%.

Similarly, you may verify that for 2014 and 2019 we get the state vectors. $\boxed{\textbf{Q ?}}$

$$\mathbf{z} = \mathbf{Ay} = \mathbf{A}(\mathbf{Ax}) = \mathbf{A}^2\mathbf{x} = \begin{bmatrix} 17.05 & 43.80 & 39.15 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{u} = \mathbf{Az} = \mathbf{A}(\mathbf{Ay}) = \mathbf{A}^3\mathbf{x} = \begin{bmatrix} 16.315 & 50.660 & 33.025 \end{bmatrix}^{\mathrm{T}}$$

Transposition (전치)

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Transposition of Matrices and Vectors

The transposition of an m x n matrix $\mathbf{A}=[a_{jk}]$ is the n x m matrix $\mathbf{A}^T$ (read $\mathbf{A}$ transpose)

: the first row of $\mathbf{A}$ as its first column

the second row of $\mathbf{A}$ as its second column, and so on.

Thus the transpose of $\mathbf{A}$ in (2) is $\mathbf{A}^T=[a_{kj}]$, written out

As a special case, transposition converts row vectors to column vectors and conversely

$$\mathbf{A}^T = [a_{kj}] = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \cdot & \cdot & \cdots & \cdot \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} \cdots (9)$$

Transposition

Ex 7) Transposition of Matrices and Vectors

$$\text{If}\quad \mathbf{A} = \begin{bmatrix} 5 & -8 & 1 \\ 4 & 0 & 0 \end{bmatrix},\quad \text{then}\quad \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 5 & 4 \\ -8 & 0 \\ 1 & 0 \end{bmatrix}$$

Comparing this A little more compactly, we can write

$$\begin{bmatrix} 5 & -8 & 1 \\ 4 & 0 & 0 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 5 & 4 \\ -8 & 0 \\ 1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 3 & 8 \\ 0 & -1 \end{bmatrix},$$

$$\begin{bmatrix} 6 & 2 & 3 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}, \qquad \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 6 & 2 & 3 \end{bmatrix}$$

Rules for transposition

$$\begin{cases} (a) \quad (\mathbf{A}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{A} \\ (b) \quad (\mathbf{A}+\mathbf{B})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}} \\ (c) \quad (c\mathbf{A}^{\mathrm{T}}) = c\mathbf{A}^{\mathrm{T}} \\ (d) \quad (\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} \end{cases}$$

Caution! Note that in (d) the transposed matrices are in reversed order.

Special Matrices

Symmetric matrix (대칭행렬)

$$\mathbf{A}^{\mathrm{T}} = \mathbf{A}, \quad (\text{thus } a_{kj} = a_{jk})$$

Skew-symmetric matrix (반대칭행렬)

$$\mathbf{A}^{\mathrm{T}} = -\mathbf{A}, \quad (\text{thus } a_{kj} = -a_{jk}, \text{hence } a_{jj} = 0).$$

Ex 8) Symmetric and Skew-symmetric Matrices

$$\mathbf{A} = \begin{bmatrix} 20 & 120 & 200 \\ 120 & 10 & 150 \\ 200 & 150 & 30 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}$$

Symmetric matrix

Skew-symmetric matrix

Triangular Matrices

Ex 9) Upper and Lower Triangular Matrices

$$\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 6 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 & 0 \\ -8 & -1 & 0 \\ 7 & 6 & 8 \end{bmatrix}, \quad \begin{bmatrix} 3 & 0 & 0 & 0 \\ 9 & -3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 9 & 3 & 6 \end{bmatrix}.$$

Upper Triangular Matrices　　　　Lower Triangular Matrices

Upper triangular matrices (위삼각행렬)

- square matrices
- nonzero entries only on and above the main diagonal
- any entry below the diagonal must be zero.

Lower triangular matrices (아래삼각행렬)

- nonzero entries only on and below the main diagonal

- Any entry on the main diagonal of a triangular matrix may be zero or not.

Diagonal Matrices

Ex 10) Diagonal Matrix D. Scalar Matrix S. Unit Matrix I.

$$\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{S} = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}, \qquad \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Diagonal Matrices (대각행렬)

- **square matrices**
- **nonzero entries only on the main diagonal**
- **any entry above or below the main diagonal must be zero.**

Diagonal Matrices

$$\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}, \quad \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

S : scalar matrix (스칼라 행렬)

$$\mathbf{AS} = \mathbf{SA} = c\mathbf{A} \cdots (12).$$

I : unit matrix (or identity matrix) (단위행렬)
a scalar matrix whose entries on the main diagonal are all 1

$$\mathbf{AI} = \mathbf{IA} = \mathbf{A} \cdots (13).$$

Reference: Vectors and Linear Equations*

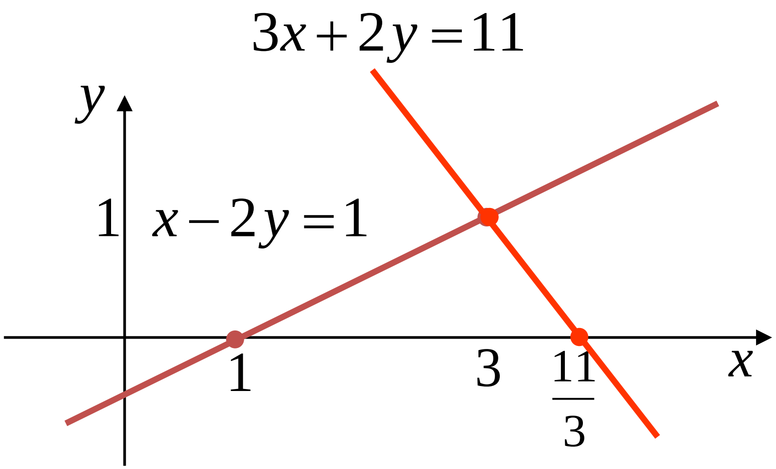
*Strang G., Introduction to Linear Algebra, Third edition, Wellesley-Cambridge Press, 2003, Ch.2.1, p21

Vectors and Linear Equations

The import problem of linear algebra is to solve a system of equations.

First example)

$$x - 2y = 1$$
$$3x + 2y = 11$$



Slopes are important in calculus and this is linear algebra.

The point $x = 3, y = 1$ lies on both lines.

This is the solution to our system of linear equations.

- solution of first equation
- solution of second equation

The left figure shows two lines meeting at a single point.

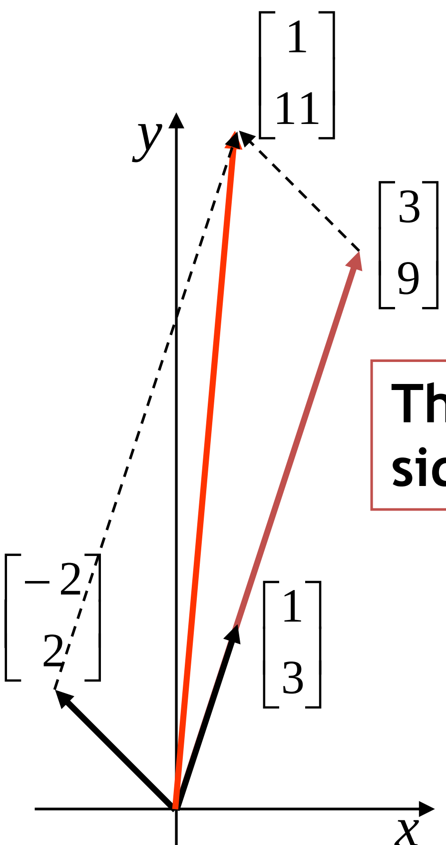
Vectors and Linear Equations

Linear system (선형연립방정식) can be a "vector equation". If you separate the original system into its columns instead of its rows, you get

$$x\begin{bmatrix}1\\3\end{bmatrix}+y\begin{bmatrix}-2\\2\end{bmatrix}=\begin{bmatrix}1\\11\end{bmatrix}=\mathbf{b}$$

This has two column vectors on the left side. The problem is to find the combination of those vectors that equals the vector on the right.

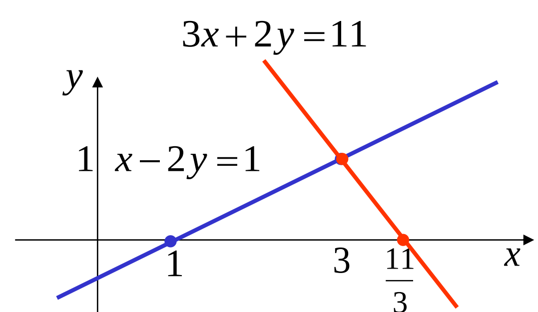
We are multiplying the first column by *x* and the second column by *y*, and adding. With the right choices *x* = 3, *y* = 1, this produces 3 (column 1) + 1 (column 2) = b.



A combination of columns produces the right side (1, 11).

The left figure combines the column vectors on the left side to produce the vector b on the right side.

비교



$$x-2y=1$$
$$3x+2y=11$$

Vectors and Linear Equations

$$x - 2y = 1$$
$$3x + 2y = 11$$

$$x\begin{bmatrix} 1 \\ 3 \end{bmatrix} + y\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix} = \mathbf{b} \qquad \therefore x = 3,\ y = 1$$

The left side of the vector equation is a linear combination of the columns.

The problem is to find the right coefficients $x = 3$ and $y = 1$.

We are combining scalar multiplication and vector addition into one step. :

$$\textbf{Linear combination} \quad 3\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 1\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

Vectors and Linear Equations
- Three Equations in Three Unknowns

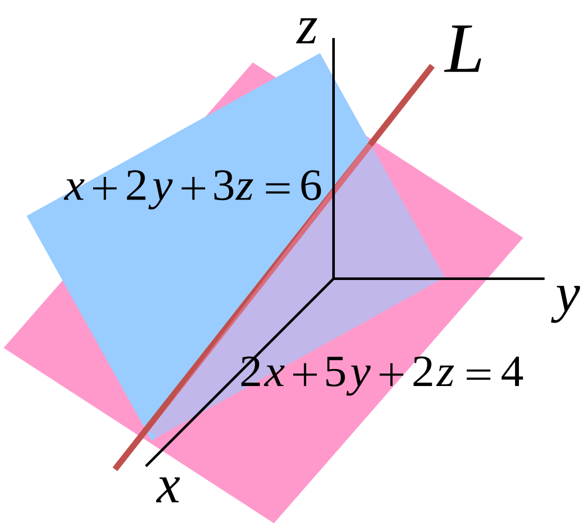
The three unknowns *x*, *y*, *z*. The linear equations Ax = b are

$$x + 2y + 3z = 6$$

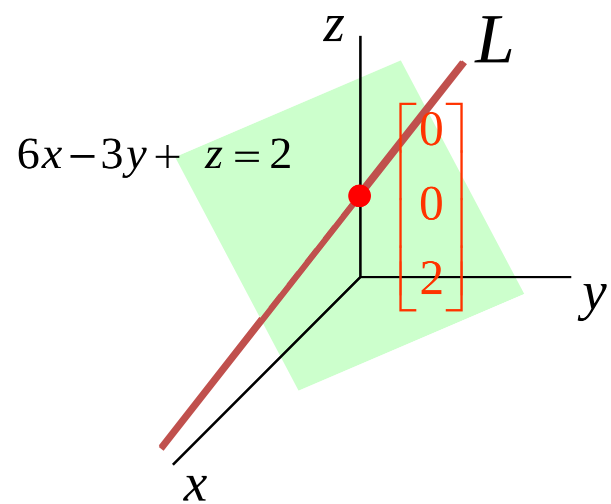
$$2x + 5y + 2z = 4$$

$$6x - 3y + z = 2$$

The right figure shows three planes meeting at a single point.



The usual result of two equations in three unknowns is a intersect line *L* of solutions.



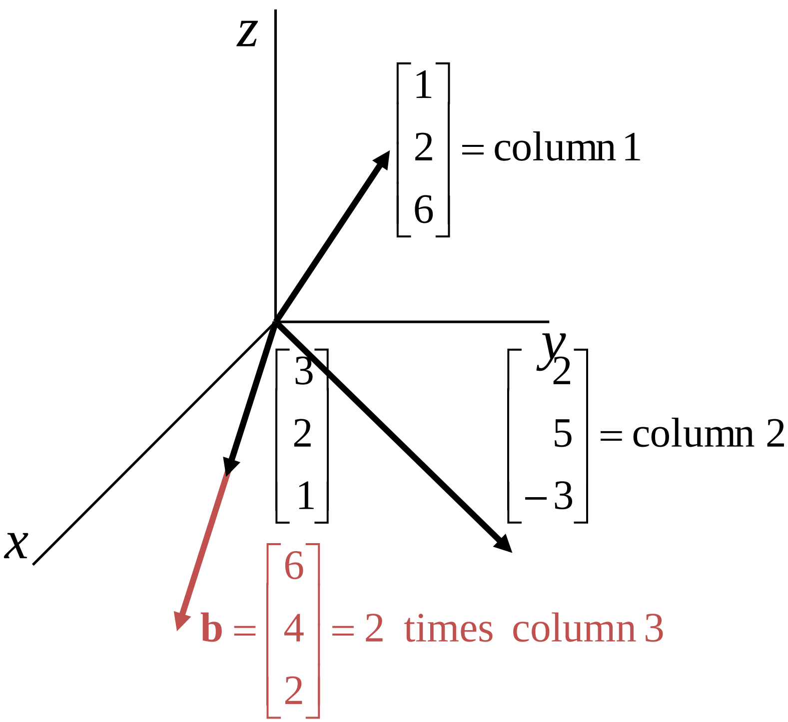
The third equation gives a third plane. It cuts the line L at a single point. That point lies on all three planes and it satisfies all three equations.

Vectors and Linear Equations
- Three Equations in Three Unknowns

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x + 5y + 2z &= 4 \\ 6x - 3y + z &= 2 \end{aligned}$$

The left figure starts with the vector form of the equations :

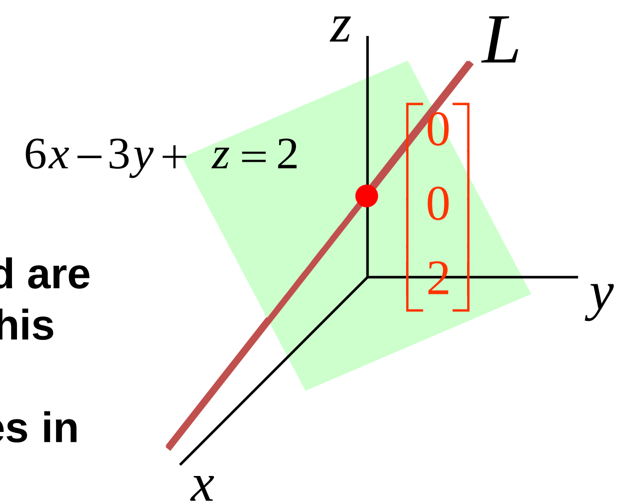
$$x\begin{bmatrix}1\\2\\6\end{bmatrix} + y\begin{bmatrix}2\\5\\-3\end{bmatrix} + z\begin{bmatrix}3\\2\\1\end{bmatrix} = \begin{bmatrix}6\\4\\2\end{bmatrix}$$



The left figure combines three columns to produce the vector (6,4,2)

(x, y, z) = (0, 0, 2) because 2(3, 2, 1) = (6, 4, 2) = b.

The coefficient we need are x = 0, y = 0 and z = 2. This is also the intersection point of the three planes in the right figure.



Vectors and Linear Equations
- The Matrix Form of the Equations

$$\begin{aligned} x+2y+3z&=6 \\ 2x+5y+2z&=4 \\ 6x-3y+\ z&=2 \end{aligned} \qquad (3)$$

Matrix equation :

$$\mathbf{Ax} = \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix}}_{\text{Coefficient matrix}} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\text{unknown vector}} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} = \mathbf{b}$$

We multiply the matrix A times the unknown vector x to get the right side b.

Multiplication by rows : Ax comes from dot products, each row time the column x :

$$\mathbf{Ax} = \begin{bmatrix} (\mathbf{row\,1})\bullet\mathbf{x} \\ (\mathbf{row\,2})\bullet\mathbf{x} \\ (\mathbf{row\,3})\bullet\mathbf{x} \end{bmatrix}.$$

Multiplication by columns : Ax is a combination of column vectors :

$$\mathbf{Ax} = x(\mathbf{column\ 1}) + y(\mathbf{column\ 2}) + z(\mathbf{column\ 3})$$

Vectors and Linear Equations
- The Matrix Form of the Equations

$$\mathbf{Ax} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} = \mathbf{b}$$

$$\mathbf{Ax} = x(\mathbf{column\ 1}) + y(\mathbf{column\ 2}) + z(\mathbf{column\ 3})$$

When we substitute the solution x = (0, 0, 2), the multiplication Ax produces b :

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = 2 \ \text{times column}\ 3 = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}.$$

The first dot product in row multiplication is (1, 2, 3) • (0, 0, 2) = 6. The other dot products are 4 and 2.

Multiplication by columns is simply 2 times column 3.

$\mathbf{Ax}$ is a combination of the columns of $\mathbf{A}$.

7.3 Linear Systems of Equations. Gauss Elimination

Linear System, Coefficient Matrix, Augmented Matrix

A linear system (선형연립방정식) of m equations in n unknowns $x_1, \cdots, x_n$ is a set of equations of the form

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
\quad \cdots (1)
\qquad
\begin{pmatrix} \text{given} \ : \\ a_{11}, \cdots a_{1n} \\ b_1, \cdots b_m \end{pmatrix}
$$

coefficients

If $b_1 = \cdots = b_m = 0$, homogeneous system (제차방정식), else nonhomogeneous system (비제차방정식).

The system is called linear because each variable x_j appears in the first power only, just as in the equation of a straight line.

Linear System, Coefficient Matrix, Augmented Matrix

$$\boxed{\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots&\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}} \quad \cdots(1)$$

A **solution** of (1) is a set of numbers $x_1, \cdots, x_n$ that satisfies all the m equations.

A **solution vector** of (1) : a vector $\mathbf{x}$ whose components form a solution of (1).
If the system (1) is homogeneous $\rightarrow$ the trivial solution (자명한 해) $x_1=0, \cdots, x_n=0$.

Matrix Form of the Linear system (1).

$$\boxed{\mathbf{Ax} = \mathbf{b} \cdots (2)}$$

Linear System, Coefficient Matrix, Augmented Matrix

$$\boxed{\mathbf{Ax}=\mathbf{b}\cdots(2)}$$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

the coefficient matrix A=[a_{jk}] : m x n matrix

$$\mathbf{A} = \begin{array}{c} \\ m \end{array}\left[\overbrace{\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{matrix}}^{n}\right], \quad \text{and } \mathbf{x} = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}, \text{ and } b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix},$$

- **A is not a zero matrix.**
- **x has n components, whereas b has m components.**

Linear System, Coefficient Matrix, Augmented Matrix

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array} \quad \cdots(1)$$

Augmented matrix (첨가행렬)

$$\tilde{\mathbf{A}} = \left[\begin{array}{cccc:c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

The dashed vertical line could be omitted.

The last column of $\tilde{\mathbf{A}}$ does not belong to $\mathbf{A}$.

The augmented matrix $\tilde{\mathbf{A}}$ determines the system (1) completely because it contains all the given numbers appearing in (1).

Linear System, Coefficient Matrix, Augmented Matrix

Ex 1) Geometric Interpretation. Existence and Uniqueness of Solutions

If $m = n = 2$, we have two equations in two unknowns x_1, x_2

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

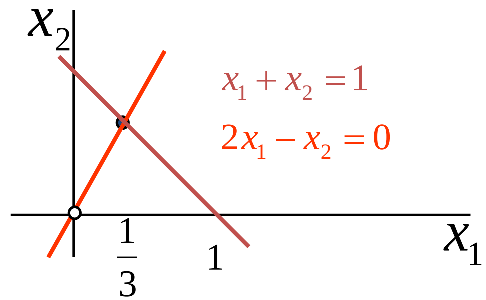
If we interpret x_1, x_2 as coordinate in the $x_1\, x_2$ -plane,

- **each of the two equations represents a straight line,**
- **(x_1, x_2) is a solution if and only if the point P with coordinates x_1, x_2 lies on both lines.**

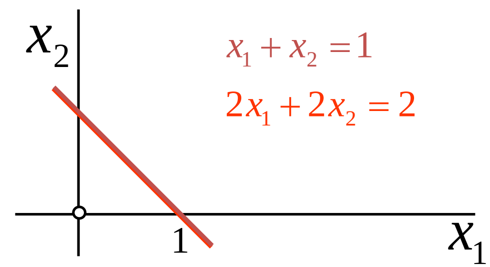
Hence there are three possible cases :

If the system is homogenous ($b_1 = b_2 = 0$), case (c) cannot happen

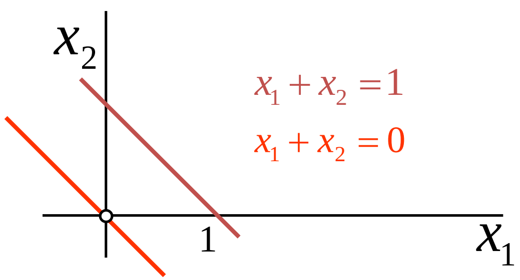
(a) precisely one solution if the lines intersect



(b) Infinitely many solutions if the lines coincide



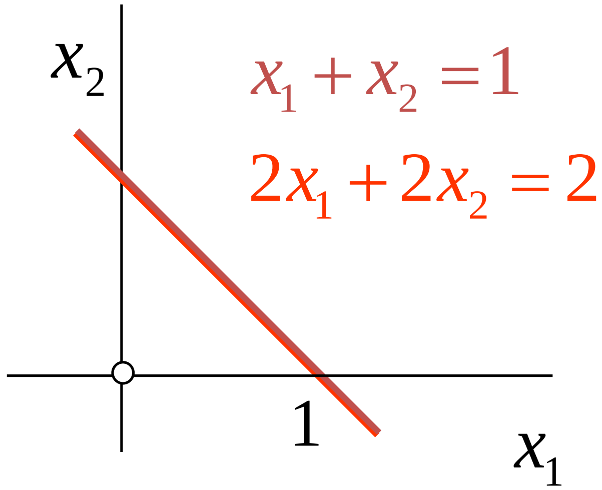
(c) No solution if the lines are parallel



Linear System, Coefficient Matrix, Augmented Matrix

Ex 1) Geometric Interpretation. Existence and Uniqueness of Solutions

(b) Infinitely many solutions if the lines coincide



Changing the form of the equation as

$$x_2 = 1 - x_1$$

if we choose a value of x_1, then the corresponding values of x_2 is uniquely determined.

x_1 : independent variable or design variable,
x_2 : dependent variable.

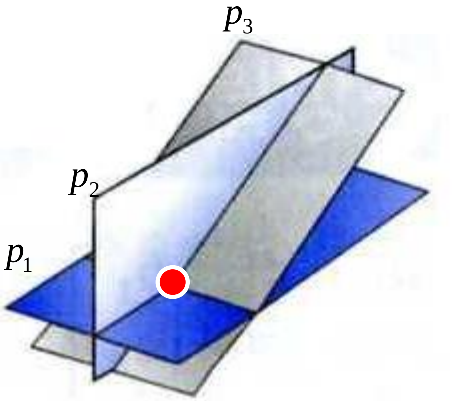
There would be many solutions, so we choose <u>the best solution to obtain proper result.</u>

Linear System, Coefficient Matrix, Augmented Matrix

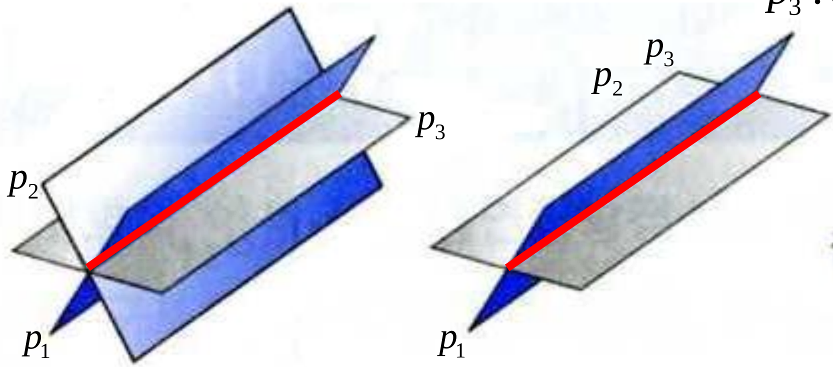
Ex 1) Geometric Interpretation. Existence and Uniqueness of Solutions

Three equations in three unknowns interpreted as planes in space

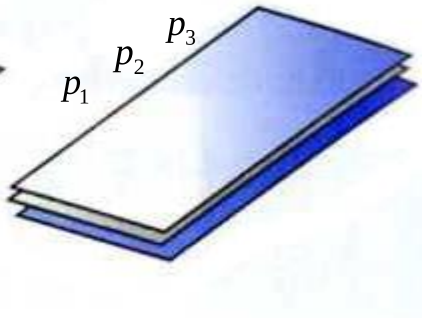
$$p_1 : \; a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$p_2 : a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$p_3 : a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$



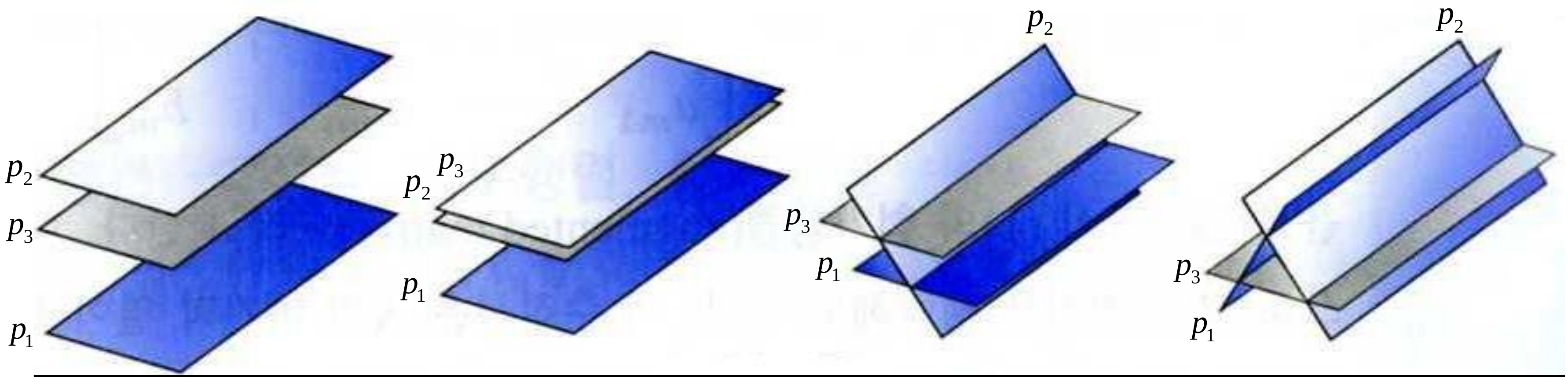
Unique solution



Infinitely many solutions



No solution



No solution

Gauss Elimination and Back Substitution

❖ **Gauss Elimination (가우스 소거법) and Back Substitution (후 치환)**

- **A standard elimination method for solving linear systems**
- **If a system is in "triangular form", we can solve it by "back substitution".**

Triangular Matrices

$$\begin{bmatrix} 1 & 4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 6 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ -8 & -1 & 0 \\ 7 & 6 & 8 \end{bmatrix}$$

Upper triangular matrix **Lower triangular matrix**

Gauss Elimination and Back Substitution

$$x_1 + 2x_2 + x_3 = 1$$
$$3x_1 - x_2 - x_3 = 2$$
$$2x_1 + 3x_2 - x_3 = -3$$

row2 + row1x(−3)

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

row2 + row1x(−3)

$$\begin{array}{rl} & 3x_1 - x_2 - x_3 = 2 \\ +) & \underline{-3x_1 - 6x_2 - 3x_3 = -3} \\ & -7x_2 - 4x_3 = -1 \end{array}$$

$$x_1 + 2x_2 + x_3 = 1$$
$$-7x_2 - 4x_3 = -1$$
$$2x_1 + 3x_2 - x_3 = -3$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -7 & -4 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$$

Gauss Elimination and Back Substitution

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 3x_1 - x_2 - x_3 &= 2 \\ 2x_1 + 3x_2 - x_3 &= -3 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ -7x_2 - 4x_3 &= -1 \\ 2x_1 + 3x_2 - x_3 &= -3 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -7 & -4 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$$

row3 + row1x(−2)

$$\begin{aligned} & 2x_1 + 3x_2 - x_3 = -3 \\ +)\ & -2x_1 - 4x_2 - 2x_3 = -2 \\ \hline & -x_2 - 3x_3 = -5 \end{aligned}$$

row3 + row1x(−2)

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ -7x_2 - 4x_3 &= -1 \\ -x_2 - 3x_3 &= -5 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -7 & -4 \\ 0 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -5 \end{bmatrix}$$

Gauss Elimination and Back Substitution

$$
\begin{aligned}
x_1 + 2x_2 + x_3 &= 1 \\
3x_1 - x_2 - x_3 &= 2 \\
2x_1 + 3x_2 - x_3 &= -3
\end{aligned}
\qquad
\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}
$$

$$
\begin{aligned}
x_1 + 2x_2 + x_3 &= 1 \\
-7x_2 - 4x_3 &= -1 \\
-x_2 - 3x_3 &= -5
\end{aligned}
\qquad
\begin{bmatrix} 1 & 2 & 1 \\ 0 & -7 & -4 \\ 0 & -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -5 \end{bmatrix}
$$

row 2 ↔ row 3

$$
\begin{aligned}
x_1 + 2x_2 + x_3 &= 1 \\
-x_2 - 3x_3 &= -5 \\
-7x_2 - 4x_3 &= -1
\end{aligned}
\qquad
\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -3 \\ 0 & -7 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ -1 \end{bmatrix}
$$

row 2x(−1)

$$
\begin{aligned}
x_1 + 2x_2 + x_3 &= 1 \\
x_2 + 3x_3 &= 5 \\
-7x_2 - 4x_3 &= -1
\end{aligned}
\qquad
\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -7 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}
$$

Gauss Elimination and Back Substitution

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 3x_1 - x_2 - x_3 &= 2 \\ 2x_1 + 3x_2 - x_3 &= -3 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ x_2 + 3x_3 &= 5 \\ -7x_2 - 4x_3 &= -1 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -7 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}$$

row 3+ row 2x7

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ x_2 + 3x_3 &= 5 \\ 17x_3 &= 34 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 17 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 34 \end{bmatrix}$$

The last equations and matrix are equal to given original equations.

Gauss Elimination and Back Substitution

$$\begin{aligned} \mathbf{x_1 + 2x_2 + x_3} &= \mathbf{1} \\ \mathbf{x_2 + 3x_3} &= \mathbf{5} \\ \mathbf{17x_3} &= \mathbf{34} \end{aligned}$$

x_1

x_2

x_3

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 34 \end{bmatrix}$$

"**Back substitution**"

$$x_3 = \frac{34}{17} = 2$$

$$x_2 + 3x_3 = x_2 + 3 \cdot 2 = 5$$

$$\therefore x_2 = -1$$

$$x_1 + 2x_2 + x_3 = x_1 + 2 \cdot (-1) + 2 = 1$$

$$\therefore x_1 = 1$$

Gauss Elimination and Back Substitution

Since a linear system is completely determined by its augmented matrix, Gauss elimination can be done by merely considering the matrices.

augmented matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc:c} 1 & 2 & 1 & 1 \\ 3 & -1 & -1 & 2 \\ 2 & 3 & -1 & -3 \end{array}\right]$$

$$\vdots \qquad\qquad \vdots$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 17 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 34 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc:c} 1 & 2 & 1 & 1 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 17 & 34 \end{array}\right]$$

Elementary Row Operations. Row-Equivalent (행 동치) System

Elementary *Row* Operations for Matrices :

1) *Interchange of two rows*

2) *Addition of a constant multiple (상수배) of one row to another row*

3) *Multiplication of a row by a nonzero constant* c

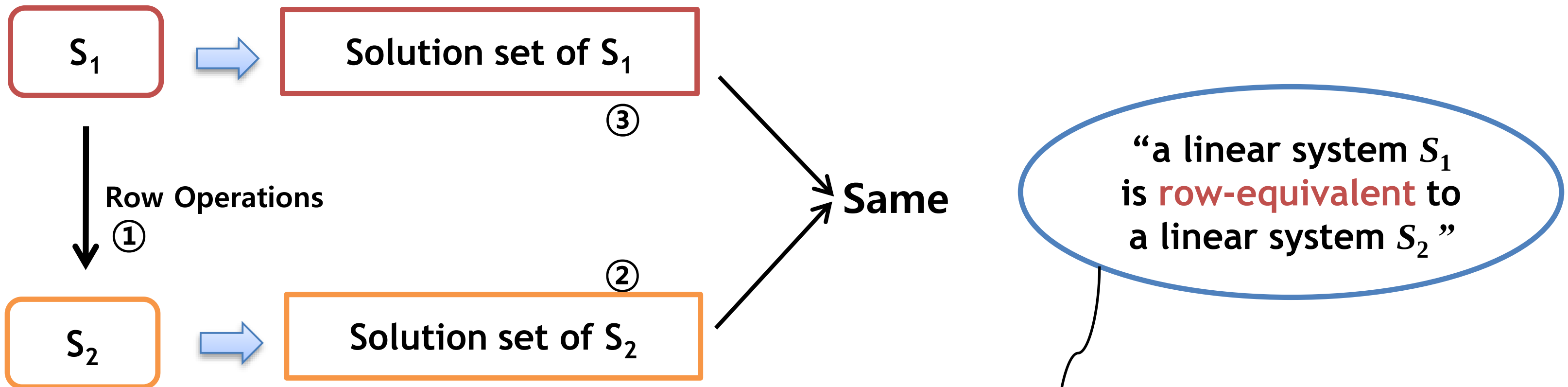
Elementary Operations for Equations :

1) *Interchange of two equations*

2) *Addition of a constant multiple of one equation to another equation*

3) *Multiplication of an equation by a nonzero constant* c

Elementary Row Operations. Row-Equivalent (행 동치) System



① : "elimination" → ② : Solution of S_2 found → ③ : $\boxed{\text{Solution of } S_1}$ found
goal

The interchange, addition, and multiplication of two equations does not alter the solution set. Because we can undo it by a corresponding subtraction.

A linear system S_1 is row-equivalent to a linear system S_2 if S_1 can be obtained from S_2 by row operations.

Because of this theorem, systems having the same solution sets are often called equivalent systems.

No column operations on the augmented matrix are permitted because they would generally alter the solution set.

Gauss Elimination : The Three Possible Cases of Systems

case 1 : Gauss Elimination if Infinitely Many Solutions Exist

three equations < four unknowns

$$\left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right]$$

$$\begin{aligned} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\ 0.6x_1 + 1.5x_2 + 1.5x_3 - 5.4x_4 &= 2.7 \\ 1.2x_1 - 0.3x_2 + 0.3x_3 + 2.4x_4 &= 2.1 \end{aligned}$$

$\Downarrow$

Row2-0.2*Row1

Row3-0.4*Row1

$$\left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & -1.1 & -1.1 & 4.4 & -1.1 \end{array}\right]$$

$$\begin{aligned} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\ 1.1x_2 + 1.1x_3 - 4.4x_4 &= 1.1 \\ -1.1x_2 - 1.1x_3 + 4.4x_4 &= -1.1 \end{aligned}$$

$\Downarrow$

Row3+Row2

$$\left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\ 1.1x_2 + 1.1x_3 - 4.4x_4 &= 1.1 \\ 0 &= 0 \end{aligned}$$

Gauss Elimination : The Three Possible Cases of Systems

case 1 : Gauss Elimination if Infinitely Many Solutions Exist

$$\left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\ 1.1x_2 + 1.1x_3 - 4.4x_4 &= 1.1 \\ 0 &= 0 \end{aligned}$$

Back substitution.

From the second equation : $x_2 = 1 - x_3 + 4x_4$

From the first equation : $x_1 = 2 - x_4$

Since x_3 and x_4 remain arbitrary, we have infinitely many solutions.

If we choose a value of x_3 and a value of x_4, then the corresponding values of x_1 and x_2 are uniquely determined.

Gauss Elimination : The Three Possible Cases of Systems

case 2 : Gauss Elimination if no Solution Exists

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right]$$

$$\begin{cases} 3x_1 + 2x_2 + x_3 = 3 \\ 2x_1 + x_2 + x_3 = 0 \\ 6x_1 + 2x_2 + 4x_3 = 6 \end{cases}$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & -2 & 2 & 0 \end{array}\right]$$

Row2-2/3*Row1

Row3-2*Row1

$$\begin{cases} 3x_1 + 2x_2 + x_3 = 3 \\ -1/3x_2 + 1/3x_3 = -2 \\ -2x_2 + 4x_3 = 0 \end{cases}$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & 0 & 0 & 12 \end{array}\right]$$

Row3-6*Row3

$$\begin{cases} 3x_1 + 2x_2 + x_3 = 3 \\ -1/3x_2 + 1/3x_3 = -2 \\ 0 = 12 \end{cases}$$

The false statement 0=12 show that the system has no solution.

Row Echelon Form and Information From It

Row Echelon Form

At the end of the Gauss elimination (before the back substitution) the row-echelon form (행 사다리꼴) of the augmented matrix will be

$$\left[\begin{array}{cccccc|c} a_{11} & a_{12} & \cdots\cdots & & & a_{1n} & b_1 \\ & c_{22} & \cdots\cdots & & & c_{2n} & \tilde{b}_2 \\ \text{Zero} & & \ddots & & & \vdots & \vdots \\ & & & k_{rr} & \cdots & k_{rn} & \tilde{b}_r \\ & & & & & & \tilde{b}_{r+1} \\ & & \text{Zero} & & & & \vdots \\ & & & & & & \tilde{b}_m \end{array}\right] \cdots (8)$$

(r : no. of equations, n : no. of unknowns, m : no. of rows)

Here, $r \le m$ and $a_{11} \neq 0,\ c_{22} \neq 0,\ \cdots,\ k_{rr} \neq 0$, and all the entries in the blue triangle as well as in the blue rectangle are zero.

Row Echelon Form and Information From It

Row Echelon Form (r : no. of equations, n : no. of unknowns, m : no. of rows)

(a) Exactly one solution

if $r = n$ and $\tilde{b}_{r+1},\cdots\tilde{b}_m$, if present, are zero,
solve the nth equation corresponding to (8) (which is $k_{nn}x_n=b_n$) for x_n, then the $(n\text{-}1)$st equation for $x_{n\text{-}1}$, and so on up the line (back substitution).

(b) Infinitely many solutions

if $r < n$ and $\tilde{b}_{r+1},\cdots\tilde{b}_m$ are zero,
to obtain any of these solutions, choose values of x_{r+1}, ···, x_n arbitrary.
Then solve the rth equation for x_r, then the $(r\text{-}1)$st equation for $x_{r\text{-}1}$, and so on up the line.

(c) No solution

if $r < m$ and one of the entries $\tilde{b}_{r+1},\cdots\tilde{b}_m$ is not zero, there is no solution.

$$n=3:\quad r=3\ \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right]\ m=4$$

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ 0\cdot x_1 + 10x_2 + 25x_3 = 90 \\ 0\cdot x_1 + 0\cdot x_2 - 95x_3 = -190 \\ 0\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 = 0 \end{cases}$$

$$n=4:\quad r=2\ \left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]\ m=3$$

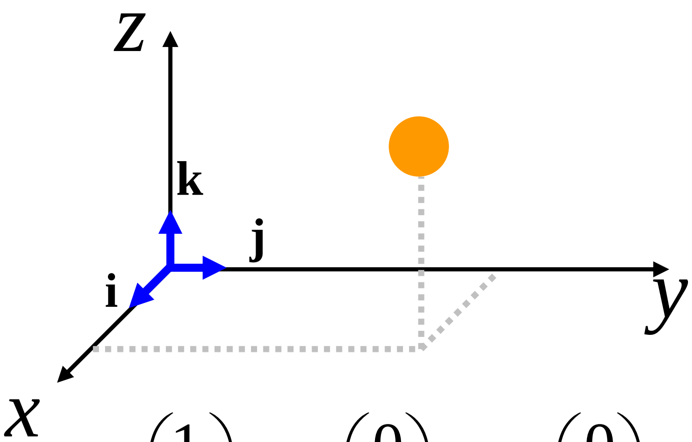
$$\begin{cases} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 = 8.0 \\ 0\cdot x_1 + 1.1x_2 + 1.1x_3 - 4.4x_4 = 1.1 \\ 0\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 + 0\cdot x_4 = 0.0 \end{cases}$$

$$n=3:\quad r=2\ \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & 0 & 0 & \boxed{12} \end{array}\right]\ m=3$$

$$\begin{cases} 3x_1 + 2x_2 + x_3 = 3 \\ -1/3x_2 + 1/3x_3 = -2 \\ 0 = 12 \end{cases}$$

7.4 LINEAR INDEPENDENCE. RANK OF MATRIX. VECTOR SPACE

Linearly independent vectors



$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

We can express the location of the point with i, j, k.

$$a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

If the point is at the origin, the equation becomes

$$a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The equation above is satisfied if and only if a=b=c=0.

Then, i, j, k are linearly independent.

Linear Independence and Dependence of Vectors

Given any set of m vectors $\mathbf{a}_{(1)}, \cdots, \mathbf{a}_{(m)}$ (with the same number of components), a linear combination of these vectors is an expression of the form

$$c_1\mathbf{a}_{(1)} + c_2\mathbf{a}_{(2)} + \cdots + c_m\mathbf{a}_{(m)}$$

$c_1, c_2, \cdots, c_m$ are any scalars. Now consider the equation.

$$\boxed{c_1\mathbf{a}_{(1)} + c_2\mathbf{a}_{(2)} + \cdots + c_m\mathbf{a}_{(m)} = \mathbf{0} \quad \cdots\cdots(1)}$$

When $c_1 = c_2 = \cdots = c_m = \mathbf{0}$

Vector $\mathbf{a}_{(1)}, \mathbf{a}_{(2)}, \cdots, \mathbf{a}_{(m)}$ **vectors linearly independent set or linearly independent.**

비교

Function

Definition 3.1 **Linear Dependence / Independence**

A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ is said to be 'linearly dependent' on an interval I if there exist constant $c_1, c_2, \ldots c_n$, not all zero such that $c_1 f_1(x) + c_2 f_2(x) + \cdots c_n f_n(x) = 0$
for every x in the interval.
If the set of functions is not linearly dependent on the interval, it is said to be 'linearly independent'

In other words, a set of functions is 'linearly independent' if the only constants for

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

are $c_1 = c_2 = \cdots = c_n = 0$

Linear Independence and Dependence of Vectors

$$\boxed{c_1\mathbf{a}_{(1)} + c_2\mathbf{a}_{(2)} + \cdots + c_m\mathbf{a}_{(m)} = \mathbf{0} \quad \cdots\cdots(1)}$$

If (1) also holds with scalars not all zero, we call these vectors linearly dependent, because then we can express (at least) one of them as a linear combination of the others. For instance, if (1) holds with, say, $c_1 \neq 0$, we can solve (1) for $\mathbf{a}_{(1)}$:

$$\mathbf{a}_{(1)} = k_2\mathbf{a}_{(2)} + \cdots + k_m\mathbf{a}_{(m)}, \qquad (\text{where } \; k_j = -c_j / c_1)$$

(Some k_j's may be zero. If $\mathbf{a}_{(1)}=0$, even all of them may be zero.)

Linear Independence and Dependence of Vectors

Ex 1) Linear Independence and Dependence

Vector

$$\mathbf{a}_{(1)} = [3,\ 0,\ 2,\ 2]$$
$$\mathbf{a}_{(2)} = [-6,\ 42,\ 24,\ 54]$$
$$\mathbf{a}_{(3)} = [21,\ -21,\ 0,\ -15]$$

$$6\mathbf{a}_{(1)} = [18,\ 0,\ 12,\ 12]$$
$$-\frac{1}{2}\mathbf{a}_{(2)} = [3, -21, -12, -27]$$
$$-\mathbf{a}_{(3)} = [-21,\ 21,\ 0,\ 15]$$

$$6\mathbf{a}_{(1)} - \frac{1}{2}\mathbf{a}_{(2)} - \mathbf{a}_{(3)} = [0,0,0,0]$$

Linear Systems

$$\begin{cases} ① & 3x_1 + 0\cdot x_2 + 2x_3 = 2 \\ ② & -6x_1 + 42x_2 + 24x_3 = 54 \\ ③ & 21x_1 - 21x_2 + 0\cdot x_3 = -15 \end{cases}$$

⇩ ①x2+②

$$\begin{cases} 3x_1 + 0\cdot x_2 + 2x_3 = 2 \\ 0\cdot x_1 + 42x_2 + 28x_3 = 58 \\ 21x_1 - 21x_2 + 0\cdot x_3 = -15 \end{cases}$$

⇩ ①x(-7)+③

$$\begin{cases} 3x_1 + 0\cdot x_2 + 2x_3 = 2 \\ \boxed{0\cdot x_1 + 42x_2 + 28x_3 = 58} \\ \boxed{0\cdot x_1 - 21x_2 - 14x_3 = -29} \end{cases}$$

⇩ ②x(0.5)+③

$$\begin{cases} 3x_1 + 0\cdot x_2 + 2x_3 = 2 \\ 0\cdot x_1 + 42x_2 + 28x_3 = 58 \\ \boxed{0\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 = 0} \end{cases}$$

Matrix

$$\left[\begin{array}{ccc|c} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{array}\right] \quad \begin{bmatrix} 3 & 0 & 2 \\ -6 & 42 & 24 \\ 21 & -21 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 21 & -21 & 0 & -15 \end{array}\right] \quad \begin{bmatrix} 3 & 0 & 2 \\ 0 & 42 & 28 \\ 21 & -21 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{array}\right] \quad \begin{bmatrix} 3 & 0 & 2 \\ 0 & 42 & 28 \\ 0 & -21 & -14 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{bmatrix} 3 & 0 & 2 \\ 0 & 42 & 28 \\ 0 & 0 & 0 \end{bmatrix}$$

The three vectors are linearly dependent

The three equations are linearly dependent

The three rows are linearly dependent

Rank of a Matrix

Rank (계수) of a Matrix

The rank of a matrix $\mathbf{A}$

: "<u>the maximum number of linearly independent row vectors" of</u> $\mathbf{A}$. rank $\mathbf{A}$.

Ex 2) Rank

The matrix

$$\mathbf{A}=\begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix} \cdots (2)$$

has rank 2, because Example 1 shows that the first two row vectors are linearly independent, whereas all three row vectors are linearly dependent.

Note further that rank $\mathbf{A}=0$ if and only if $\mathbf{A}=0$ (zero matrix).

Rank of a Matrix

☑ Example 1
Rank of 3 x 4 Matrix

Consider the 3 x 4 matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & -1 & 3 \\ 2 & -2 & 6 & 8 \\ 3 & 5 & -7 & 8 \end{pmatrix}.$$

With $\mathbf{u}_1$=(1 1 -1 3), $\mathbf{u}_2$=(2 -2 6 8), and $\mathbf{u}_3$=(3 5 -7 8) → $4\mathbf{u}_1 - 1/2\mathbf{u}_2 + \mathbf{u}_3 = 0$.

the set $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ is linearly dependent.

$\mathbf{u}_1 \neq c\,\mathbf{u}_2$ → $\mathbf{u}_1$, $\mathbf{u}_2$ is linearly independent.
Hence by Definition, rank($\mathbf{A}$) = 2.

$$\begin{pmatrix} 1 & 1 & -1 & 3 \\ 2 & -2 & 6 & 8 \\ 3 & 5 & -7 & 8 \end{pmatrix} \qquad \begin{cases} x_1 + x_2 - x_3 = 3 \\ 2x_1 - 2x_2 + 6x_3 = 8 \\ 3x_1 + 5x_2 - 7x_3 = 8 \end{cases}$$

⇩ ①x(-2)+②

$$\begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & -4 & 8 & 2 \\ 3 & 5 & -7 & 8 \end{pmatrix} \qquad \begin{cases} x_1 + x_2 - x_3 = 3 \\ 0 \cdot x_1 - 4x_2 + 8x_3 = 2 \\ 3x_1 + 5x_2 - 7x_3 = 8 \end{cases}$$

⇩ ①x(-3)+③

$$\begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & -4 & 8 & 2 \\ 0 & 2 & -4 & -1 \end{pmatrix} \qquad \begin{cases} x_1 + x_2 - x_3 = 3 \\ 0 \cdot x_1 - 4x_2 + 8x_3 = 2 \\ 0 \cdot x_1 + 2x_2 - 4x_3 = -1 \end{cases}$$

⇩ ②x(0.5)+③

$$\begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & -4 & 8 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{cases} x_1 + x_2 - x_3 = 3 \\ 0 \cdot x_1 - 4x_2 + 8x_3 = 2 \\ 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 0 \end{cases}$$

rank($\mathbf{A}$) = 2

Rank and Row-Equivalent Matrices

Theorem 1 : Row-Equivalent Matrices

Row-equivalent matrices have the same rank.

A_1 is row-equivalent to a matrix A_2

→ rank is invariant under elementary row operations.

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 0 & 10 & 25 \\ 20 & 10 & 0 \end{bmatrix} \Downarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 10 & 25 \\ 0 & 0 & -95 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3.0 & 2.0 & 2.0 & -5.0 \\ 0.6 & 1.5 & 1.5 & -5.4 \\ 1.2 & -0.3 & -0.3 & 2.4 \end{bmatrix} \Downarrow \begin{bmatrix} 3.0 & 2.0 & 2.0 & -5.0 \\ 0 & 1.1 & 1.1 & -4.4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 1 \\ 6 & 2 & 4 \end{bmatrix} \Downarrow \begin{bmatrix} 3 & 2 & 1 \\ 0 & -1/3 & 1/3 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank and Row-Equivalent Matrices

☑ Example 3
Linear Independence /Dependence

Determine whether the set of vectors

$$\mathbf{u}_1 = < 2,1,1 >$$
$$\mathbf{u}_2 = < 0,3,0 >$$
$$\mathbf{u}_3 = < 3,1,2 >$$

in R^3 in linearly dependent or linearly independent.

Solution) Q ?

- If we form a matrix $\mathbf{A}$ with the given vectors as rows,
- if we reduce $\mathbf{A}$ to a row-echelon form $\mathbf{B}$ with rank 3, then the set of vectors is linearly independent.
- If $\text{rank}(\mathbf{A}) < 3$, then the set of vectors is linearly dependent.

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 0 \\ 3 & 1 & 2 \end{pmatrix} \xrightarrow{\text{row operations}}$$

Rank in Terms of Column Vectors

Theorem 3 : Rank in Terms of Column Vectors

The rank r of a matrix $\mathbf{A}$ equals the maximum number of linearly independent column vectors of $\mathbf{A}$. Hence $\mathbf{A}$ and its transpose $\mathbf{A}^T$ have the same rank.

Proof) Let $\mathbf{A}$ be an m x n matrix of rank($\mathbf{A}$) = r

Then by definition of rank, $\mathbf{A}$ has r linearly independent rows which we denote by $\mathbf{v}_{(1)}, \cdots, \mathbf{v}_{(r)}$ and all the rows $\mathbf{a}_{(1)}, \cdots, \mathbf{a}_{(m)}$ of $\mathbf{A}$ are linear combinations of those.

Rank in Terms of Column Vectors - 3 by 3 matrix

$$\text{let rank } \mathbf{A} = 3$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix},$$

A의 rank가 3이므로 3개의 행벡터 (a_1, a_2, a_3)는 linearly independent 하다.

3 (= rank A) linearly independent rows (basis) :

$$\mathbf{v}_1 = [v_{11} \quad v_{12} \quad v_{13}]$$
$$\mathbf{v}_2 = [v_{21} \quad v_{22} \quad v_{23}]$$
$$\mathbf{v}_3 = [v_{31} \quad v_{32} \quad v_{33}]$$

그러므로 A의 행벡터가 이루는 공간은 3개의 basis를 갖는다. (ex : $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$)

즉 그 공간의 임의의 벡터(b)는 A의 행벡터로 표현될 수 있다. $\mathbf{b} = l\mathbf{a}_1 + m\mathbf{a}_2 + n\mathbf{a}_3 \quad (l, m, n : const)$

Ex) 3차원 공간상의 벡터 b를 basis $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ 의 linear combination으로 표현

$\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ 를 다른 basis (v_1, v_2, v_3)로 표현한다면 3개의 basis가 필요하다.

$$\begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} c_{11}\mathbf{v}_1 + c_{12}\mathbf{v}_2 + c_{13}\mathbf{v}_3 \\ \hline c_{21}\mathbf{v}_1 + c_{22}\mathbf{v}_2 + c_{23}\mathbf{v}_3 \\ \hline c_{31}\mathbf{v}_1 + c_{32}\mathbf{v}_2 + c_{33}\mathbf{v}_3 \end{bmatrix}$$

행벡터를 v_1, v_2, v_3의 일차결합으로 표현함

$$\begin{aligned} \mathbf{b} &= l\mathbf{a}_1 + m\mathbf{a}_2 + n\mathbf{a}_3 \\ &= l(c_{11}\mathbf{v}_1 + c_{12}\mathbf{v}_2 + c_{13}\mathbf{v}_3) + m(c_{21}\mathbf{v}_1 + c_{22}\mathbf{v}_2 + c_{23}\mathbf{v}_3) + n(c_{31}\mathbf{v}_1 + c_{32}\mathbf{v}_2 + c_{33}\mathbf{v}_3) \\ &= (lc_{11} + mc_{21} + nc_{31})\mathbf{v}_1 + (lc_{12} + mc_{22} + nc_{32})\mathbf{v}_2 + (lc_{13} + mc_{23} + nc_{33})\mathbf{v}_3 \end{aligned}$$
$$(l, m, n, c : const)$$

Ex) 3차원 공간상의 벡터 b를 basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ 의 linear combination으로 표현

$\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ 를 다른 2개의 basis로 표현한다면

$$\begin{aligned} \mathbf{b} &= l\mathbf{a}_1 + m\mathbf{a}_2 + n\mathbf{a}_3 \\ &= l(c_{11}\mathbf{v}_1 + c_{12}\mathbf{v}_2) + m(c_{21}\mathbf{v}_1 + c_{22}\mathbf{v}_2) + n(c_{31}\mathbf{v}_1 + c_{32}\mathbf{v}_2) \\ &= (lc_{11} + mc_{21} + nc_{31})\mathbf{v}_1 + (lc_{12} + mc_{22} + nc_{32})\mathbf{v}_2 \end{aligned}$$
$$(l, m, n, c : const)$$

Ex) $\mathbf{v}_1, \mathbf{v}_2$ 만으로는 3차원 공간상의 벡터 b를 표현할 수 없음

* basis (기저): 공간에 속한 1차독립인 벡터들의 최대집합

Rank in Terms of Column Vectors - 3 by 3 matrix

$$\mathbf{v}_1 = [v_{11} \quad v_{12} \quad v_{13}]$$
$$\mathbf{v}_2 = [v_{21} \quad v_{22} \quad v_{23}]$$
$$\mathbf{v}_3 = [v_{31} \quad v_{32} \quad v_{33}]$$

행 벡터들에 대한 방정식을 열 벡터들에 대한 식으로 변경

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} c_{11}\mathbf{v}_1 + c_{12}\mathbf{v}_2 + c_{13}\mathbf{v}_3 \\ c_{21}\mathbf{v}_1 + c_{22}\mathbf{v}_2 + c_{23}\mathbf{v}_3 \\ c_{31}\mathbf{v}_1 + c_{32}\mathbf{v}_2 + c_{33}\mathbf{v}_3 \end{bmatrix}$$

새로운 basis v의 성분으로 표현하면

$$[\mathbf{a}_1] = [a_{11}, a_{12}, a_{13}] = c_{11}\mathbf{v}_1 + c_{12}\mathbf{v}_2 + c_{13}\mathbf{v}_3$$
$$= c_{11}[v_{11}, v_{12}, v_{13}] + c_{12}[v_{21}, v_{22}, v_{23}] + c_{13}[v_{31}, v_{32}, v_{33}]$$
$$= [(c_{11}v_{11} + c_{12}v_{21} + c_{13}v_{31}), (c_{11}v_{12} + c_{12}v_{22} + c_{13}v_{32}), (c_{11}v_{13} + c_{12}v_{22} + c_{13}v_{32})]$$

$$\boxed{a_{11} = c_{11}v_{11} + c_{12}v_{21} + c_{13}v_{31}}$$
$$a_{21} = c_{21}v_{11} + c_{22}v_{21} + c_{23}v_{31}$$
$$a_{31} = c_{31}v_{11} + c_{32}v_{21} + c_{33}v_{31}$$

$$\boxed{a_{12} = c_{11}v_{12} + c_{12}v_{22} + c_{13}v_{32}}$$
$$a_{22} = c_{21}v_{12} + c_{22}v_{22} + c_{23}v_{32}$$
$$a_{32} = c_{31}v_{12} + c_{32}v_{22} + c_{33}v_{32}$$

$$\boxed{a_{13} = c_{11}v_{13} + c_{12}v_{23} + c_{13}v_{33}}$$
$$a_{23} = c_{21}v_{13} + c_{22}v_{23} + c_{23}v_{33}$$
$$a_{33} = c_{31}v_{13} + c_{32}v_{23} + c_{33}v_{33}$$

Column Vector of A

linearly independent basis

$$\begin{bmatrix} a_{1k} \\ a_{2k} \\ a_{3k} \end{bmatrix} = v_{1k}\begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix} + v_{2k}\begin{bmatrix} c_{12} \\ c_{22} \\ c_{32} \end{bmatrix} + v_{3k}\begin{bmatrix} c_{13} \\ c_{23} \\ c_{33} \end{bmatrix}$$

열벡터도 3개의 basis (c의 성분의 수는 v의 수와 동일함) 존재. 따라서 rank A^T = rank A

Rank in Terms of Column Vectors - 3 by 3 matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \hline a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} c_{11}\mathbf{v}_1 + c_{12}\mathbf{v}_2 + c_{13}\mathbf{v}_3 \\ \hline c_{21}\mathbf{v}_1 + c_{22}\mathbf{v}_2 + c_{23}\mathbf{v}_3 \\ \hline c_{31}\mathbf{v}_1 + c_{32}\mathbf{v}_2 + c_{33}\mathbf{v}_3 \end{bmatrix}$$

이 열벡터들이 linearly dependent 하다면?

$$\begin{bmatrix} a_{1k} \\ a_{2k} \\ a_{3k} \end{bmatrix} = v_{1k} \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix} + v_{2k} \begin{bmatrix} c_{12} \\ c_{22} \\ c_{32} \end{bmatrix} + v_{3k} u \begin{bmatrix} c_{12} \\ c_{22} \\ c_{23} \end{bmatrix}$$

$$\begin{bmatrix} a_{1k} \\ a_{2k} \\ a_{3k} \end{bmatrix} = v_{1k} \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix} + (v_{2k} + v_{3k} u) \begin{bmatrix} c_{12} \\ c_{22} \\ c_{32} \end{bmatrix}$$

A의 열벡터

$$\begin{bmatrix} a_{1k} \\ a_{2k} \\ a_{3k} \end{bmatrix} = v_{1k} \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix} + v_{2k} \begin{bmatrix} c_{12} \\ c_{22} \\ c_{32} \end{bmatrix} + v_{3k} \begin{bmatrix} c_{13} \\ c_{23} \\ c_{33} \end{bmatrix}$$

∴ 이 열벡터들은 linearly independent 한 basis

열벡터도 3개의 basis (c의 성분의 수는 v의 수와 동일함) 존재.
따라서 rank A^T = rank A

A의 행백터 관점에서 basis의 개수가 줄어들게 되어 모순이 됨 $\mathbf{v}_1, \mathbf{v}_{new}$

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} c_{11}\mathbf{v}_1 + c_{12}\mathbf{v}_2 + uc_{12}\mathbf{v}_3 \\ \hline c_{21}\mathbf{v}_1 + c_{22}\mathbf{v}_2 + uc_{22}\mathbf{v}_3 \\ \hline c_{31}\mathbf{v}_1 + c_{32}\mathbf{v}_2 + uc_{33}\mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} c_{11}\mathbf{v}_1 + c_{12}(1+u)(\mathbf{v}_2 + \mathbf{v}_3) \\ \hline c_{21}\mathbf{v}_1 + c_{22}(1+u)(\mathbf{v}_2 + \mathbf{v}_3) \\ \hline c_{31}\mathbf{v}_1 + c_{32}(1+u)(\mathbf{v}_2 + \mathbf{v}_3) \end{bmatrix} = \begin{bmatrix} c_{11}\mathbf{v}_1 + c_{12}(1+u)\mathbf{v}_{new} \\ \hline c_{21}\mathbf{v}_1 + c_{22}(1+u)\mathbf{v}_{new} \\ \hline c_{31}\mathbf{v}_1 + c_{32}(1+u)\mathbf{v}_{new} \end{bmatrix}$$

7.5 Solutions of Linear Systems : Existence, Uniqueness

Solutions of Linear Systems : Existence, Uniqueness

Theorem 1

(a) Existence (존재성): A linear systems of m equations in n unknowns $x_1, \cdots, x_n$

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m
\end{aligned}
$$

is consistent (모순이 없는), that is, has solutions (해를 갖는), if and only if the coefficient matrix A and the augmented matrix Â have same rank. Here

$$
\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad
\tilde{\mathbf{A}} = \left[\begin{array}{cccc:c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]
$$

Solutions of Linear Systems : Existence, Uniqueness

(b) Uniqueness (유일성)

The linear system has precisely one solution if and only if this common rank *r* of A and Â equals *n*.

(c) Infinitely Many Solutions

If this common rank *r* is less than *n*, the system has infinitely many solutions.

(d) Gauss Elimination

If solutions exist, they can all be obtained by the Gauss elimination.

Solutions of Linear Systems : Existence, Uniqueness

Theorem

Rank of a Matrix by Row Reduction

If a matrix A is row-equivalent to a row-echelon form B, then
i) the row space of A = the row space of B
ii) the nonzero rows of B form a basis for the row space of A, and
iii) <u>rank(A) = the number of nonzero rows in B</u>

$$\left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right] \Rightarrow \begin{array}{c} ① \\ ② \\ ③ \\ \\ \end{array}\left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

rank : 3

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ -x_1 + 11x_2 + 24x_3 = 90 \\ 3x_1 - 3x_2 - 92x_3 = -190 \\ 2x_1 - 2x_2 + 2x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 - x_2 + x_3 = 0 \\ 0 \cdot x_1 + 10x_2 + 25x_3 = 90 \\ 0 \cdot x_1 + 0 \cdot x_2 - 95x_3 = -190 \\ 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 0 \end{cases}$$

$$\left[\begin{array}{cccc:c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right] \Rightarrow \begin{array}{c} ① \\ ② \\ \\ \end{array}\left[\begin{array}{cccc:c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

rank : 2

$$\begin{cases} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 = 8.0 \\ -x_1 - 0.3x_2 - 0.3x_3 + 0.2x_4 = -2.3 \\ 1.5x_1 + 1.0x_2 + 1.0x_3 - 2.5x_4 = 4.0 \end{cases} \Rightarrow \begin{cases} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 = 8.0 \\ 0 \cdot x_1 + 1.1x_2 + 1.1x_3 - 4.4x_4 = 1.1 \\ 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 0.0 \end{cases}$$

$$\left[\begin{array}{ccc:c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right] \Rightarrow \begin{array}{c} ① \\ ② \\ ③ \end{array}\left[\begin{array}{ccc:c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & 0 & 0 & 12 \end{array}\right]$$

rank : 3

$$\begin{cases} 3x_1 + 2x_2 + x_3 = 3 \\ 9x_1 + 7x_2 + 2x_3 = 15 \\ 3x_1 + 2x_2 + x_3 = -9 \end{cases} \Rightarrow \begin{cases} 3x_1 + 2x_2 + x_3 = 3 \\ -1/3x_2 + 1/3x_3 = -2 \\ 0 = 12 \end{cases}$$

Solutions of Linear Systems : Existence, Uniqueness

$$\begin{aligned} x_1 + x_2 &= 1 \\ 4x_1 - x_2 &= -6 \\ 2x_1 - 3x_2 &= 8 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} 1 & 1 \\ 4 & -1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ 8 \end{bmatrix} \quad \Rightarrow \quad \mathbf{Ax} = \mathbf{B}$$

$$[\mathbf{A} \,|\, \mathbf{B}] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 4 & -1 & -6 \\ 2 & -3 & 8 \end{array}\right] \xrightarrow{\text{row operation}} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 16 \end{array}\right] \xrightarrow{\text{row operation}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

rank(A|B)=3

different

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 4 & -1 \\ 2 & -3 \end{bmatrix} \xrightarrow{\text{row operation}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

rank(A)=2

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

No solution case

$$\begin{cases} x_1 + 0 \cdot x_2 = 1 \\ 0 \cdot x_1 + x_2 = 0 \\ \boxed{0 \cdot x_1 + 0 \cdot x_2 = 1} \end{cases}$$

rank(A)≠rank(A|B) **False statement**

Solutions of Linear Systems : Existence, Uniqueness

Theorem

Consistency of AX=B

A linear system of equations AX=B is consistent if and only if the rank of the coefficient matrix A is the same as the rank of the augmented matrix of the system

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right]$$

$$\begin{array}{c} ① \\ ② \\ ③ \\ \\ \end{array}\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

rank (A|B): 3
rank (A): 3

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ -x_1 + 11x_2 + 24x_3 = 90 \\ 3x_1 - 3x_2 - 92x_3 = -190 \\ 2x_1 - 2x_2 + 2x_3 = 0 \end{cases}$$

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ 0 \cdot x_1 + 10x_2 + 25x_3 = 90 \\ 0 \cdot x_1 + 0 \cdot x_2 - 95x_3 = -190 \\ 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 0 \end{cases}$$

Solution
- One solution
(r = n)

$$\left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right]$$

$$\begin{array}{c} ① \\ ② \\ \\ \end{array}\left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

rank (A|B) : 2
rank (A): 2

$$\begin{cases} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 = 8.0 \\ -x_1 - 0.3x_2 - 0.3x_3 + 0.2x_4 = -2.3 \\ 1.5x_1 + 1.0x_2 + 1.0x_3 - 2.5x_4 = 4.0 \end{cases}$$

$$\begin{cases} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 = 8.0 \\ 0 \cdot x_1 + 1.1x_2 + 1.1x_3 - 4.4x_4 = 1.1 \\ 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 0.0 \end{cases}$$

Solution
- Many solutions
(r < n)

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right]$$

$$\begin{array}{c} ① \\ ② \\ ③ \end{array}\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & 0 & 0 & \boxed{12} \end{array}\right]$$

rank (A|B) : 3
rank (A): 2

$$\begin{cases} 3x_1 + 2x_2 + x_3 = 3 \\ 9x_1 + 7x_2 + 2x_3 = 15 \\ 3x_1 + 2x_2 + x_3 = -9 \end{cases}$$

No Solution

$$\begin{cases} 3x_1 + 2x_2 + x_3 = 3 \\ -1/3x_2 + 1/3x_3 = -2 \\ 0 = 12 \end{cases}$$

Solutions of Linear Systems : Existence, Uniqueness

Theorem 2

A homogeneous linear system (제차선형연립방정식)

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0
\end{aligned}
$$

- **A homogeneous linear system of *m* equations in *n* unknowns always has the trivial solution (자명한 해).**
- **Nontrivial solutions (자명하지 않은 해) exist if and only if rank $\mathbf{A} = r < n$.**
- **If rank $\mathbf{A} = r < n$, these solutions, together with $\mathbf{x} = \mathbf{0}$, form a vector space of dimension $n - r$, called the solution space (해공간).**
- **Linear combination of two solution vectors of the homogeneous linear system, $\mathbf{x} = c_1\mathbf{x}_{(1)} + c_2\mathbf{x}_{(2)}$ with any scalars c_1 and c_2, is a solution vector.**

Solutions of Linear Systems : Existence, Uniqueness

☑ Example

$$\left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 0 \\ 6 & 15 & 15 & -54 & 0 \\ 12 & -3 & -3 & 24 & 0 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 0 \\ 0 & 11 & 11 & -44 & 0 \\ 0 & -11 & -11 & 44 & 0 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 0 \\ 0 & 1 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(a) rank (A|B) : 2
rank (A): 2 ⇨ **Nontrivial solution exist! (r < n)**

(b) Solution space

$$3x_1 + 2x_2 + 2x_3 - 5x_4 = 0$$
$$x_2 + x_3 - 4x_4 = 0$$

$$\text{if } \ x_3 = 1, x_4 = 0 \Rightarrow x_2 = -1, x_1 = 0$$
$$[x_1, x_2, x_3, x_4] = [0, -1, 1, 0]$$
$$\text{if } \ x_3 = 0, x_4 = 1 \Rightarrow x_2 = 4, x_1 = -1$$
$$[x_1, x_2, x_3, x_4] = [-1, 4, 0, 1]$$

$$\therefore \text{dimension of solution vectors} = (n - r) = (4 - 2)$$
$$\mathbf{x}_{(1)} = [x_1, x_2, x_3, x_4] = [0, -1, 1, 0]$$
$$\mathbf{x}_{(2)} = [x_1, x_2, x_3, x_4] = [-1, 4, 0, 1]$$

(b) Linear combination of solution vectors

$$\mathbf{x} = c_1\mathbf{x}_{(1)} + c_2\mathbf{x}_{(2)} = [-c_2, -c_1 + 4c_2, c_1, c_2]$$

$$\begin{bmatrix} 3 & 2 & 2 & -5 \\ 6 & 15 & 15 & -54 \\ 12 & -3 & -3 & 24 \end{bmatrix} \begin{bmatrix} -c_2 \\ -c_1 + 4c_2 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\because \mathbf{A}(c_1\mathbf{x}_{(1)} + c_2\mathbf{x}_{(2)}) = c_1\mathbf{A}\mathbf{x}_{(1)} + c_2\mathbf{A}\mathbf{x}_{(2)} = 0$$

Solutions of Linear Systems : Existence, Uniqueness

Theorem 3 : Homogeneous Linear System with Fewer Equations Than Unknowns

A homogeneous linear system with fewer equations than unknowns has always nontrivial solutions.

Theorem 4 :Nonhomogeneous Linear System (비제차 선형연립방정식)

If a nonhomogeneous linear system (1) is consistent, then all of its solutions are obtained as

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_h$$

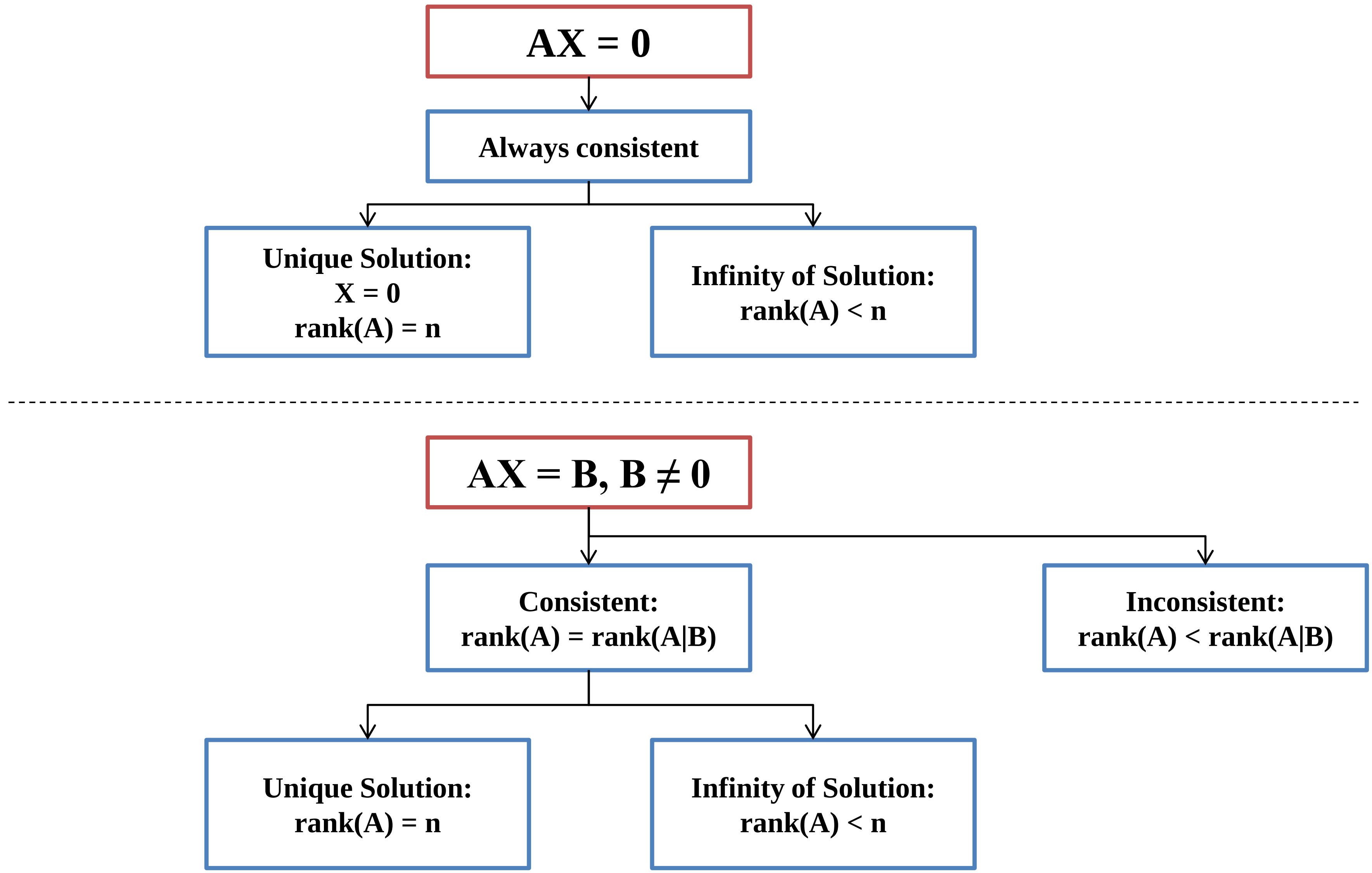
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots&\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \quad (1)$$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \cdots\cdots\cdots&\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned} \quad (2)$$

where $\mathbf{x}_0$ is any solution of the nonhomogeneous linear system (1) and $\mathbf{x}_h$ runs through all the solutions of the corresponding homogeneous system (2).

Proof) $\mathbf{x}_h = \mathbf{x} - \mathbf{x}_0 \quad A\mathbf{x}_h = A(\mathbf{x} - \mathbf{x}_0) = \mathbf{b} - \mathbf{b} = \mathbf{0}$

Connection between the concept of rank of a matrix and the solution of linear system



7.6 Second- and Third-Order Determinants

Determinant (행렬식) of second- and third order

Determinant of second order

$$D = \det A = \det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Determinant of third order

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= + a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

7.7 DETERMINANTS. CRAMER'S RULE

Determinant of Order n

Terms

In D we have n^2 entries a_{jk}, also n rows and n columns, and a main diagonal on which a_{11}, a_{12}, …, a_{nn} stand.

M_{jk} is called the minor (소행렬식) of a_{jk} in D, and C_{jk} the cofactor (여인수) of a_{jk} in D

For later use we note that D may also be written in terms of minors

$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \quad (j = 1, 2, \cdots, n)$$

$$D = \sum_{j=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \quad (k = 1, 2, \cdots, n)$$

$$C_{jk} = (-1)^{j+k} M_{jk}$$

Determinant of Order n

A determinant of order n is a scalar associated with an n x n matrix $\mathbf{A}=[a_{jk}]$, which is written

$$D = \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

and is defined for n=1 by

$$D = a_{11}$$

Determinant of Order n

For $n \geq 2$ by

$$D = a_{j1}C_{j1} + a_{j2}C_{j2} + \cdots + a_{jn}C_{jn} \quad (j = 1, 2, \cdots, n)$$

or

$$D = a_{1k}C_{1k} + a_{2k}C_{2k} + \cdots + a_{nk}C_{nk} \quad (k = 1, 2, \cdots, n)$$

Here,

$$C_{jk} = (-1)^{j+k} M_{jk}$$

M_{jk} is a determinant of order n-1, namely, the determinant of the submatrix of $\mathbf{A}$ obtained from $\mathbf{A}$ by omitting the row and column of the entry a_{jk}, that is, the jth row and the kth column.

Determinant of Order *n*

1) *n*=1

$$\mathbf{A} = \left[a_{11}\right] \qquad \therefore \det \mathbf{A} = a_{11}$$

2) *n*=2

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\det \mathbf{A} = a_{11}\begin{vmatrix} & \\ & a_{22} \end{vmatrix} - a_{12}\begin{vmatrix} & \\ a_{21} & \end{vmatrix}$$

$$= a_{11}a_{22} - a_{12}a_{21}$$

Determinant of Order n

3) n**=3**

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det \mathbf{A} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) - a_{12}\left(a_{21}a_{33} - a_{23}a_{31}\right) + a_{13}\left(a_{21}a_{32} - a_{22}a_{31}\right)$$

Determinant :

$$C_{jk} = (-1)^{j+k} M_{jk}$$

(Minors and Cofactors of a Third-Order Determinant)

Find minors and cofactors.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

1) 1st row

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$C_{11} = (-1)^{1+1} M_{11} = M_{11}$$

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{12} = (-1)^{1+2} M_{12} = -M_{12}$$

$$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$C_{13} = (-1)^{1+3} M_{13} = M_{13}$$

Determinant : (Minors and Cofactors of a Third-Order Determinant)

$$C_{jk} = (-1)^{j+k} M_{jk}$$

Find minors and cofactors.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

2) 2nd row

$$M_{21} = \begin{vmatrix} & a_{12} & a_{13} \\ & & \\ & a_{32} & a_{33} \end{vmatrix}$$

$$C_{21} = (-1)^{2+1} M_{21} = -M_{21}$$

$$M_{22} = \begin{vmatrix} a_{11} & & a_{13} \\ & & \\ a_{31} & & a_{33} \end{vmatrix}$$

$$C_{22} = (-1)^{2+2} M_{22} = M_{22}$$

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} & \\ & & \\ a_{31} & a_{32} & \end{vmatrix}$$

$$C_{23} = (-1)^{2+3} M_{23} = -M_{23}$$

Determinant :
(Minors and Cofactors of a Third-Order Determinant)

$$C_{jk} = (-1)^{j+k} M_{jk}$$

Find minors and cofactors.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

$$C_{32} = (-1)^{3+2} M_{32} = -M_{32}$$

3) 3rd row

$$M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$C_{31} = (-1)^{3+1} M_{31} = M_{31}$$

$$M_{13} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$C_{33} = (-1)^{3+3} M_{33} = M_{33}$$

Determinant :

$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \quad (j = 1, 2, \cdots, n)$$

(Minors and Cofactors of a Third-Order Determinant)

(Expansions of a Third-Order Determinant)

Find determinant

$$\det \mathbf{A} = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix}$$

1) 1st rows

$$= 1 \begin{vmatrix} 6 & 4 \\ 0 & 2 \end{vmatrix} - 3 \begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} + 0 \begin{vmatrix} 2 & 6 \\ -1 & 0 \end{vmatrix}$$

$$= 1(12 - 0) - 3(4 + 4) + 0(0 + 6) = -12$$

Determinant :

$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \quad (j = 1,\ 2,\ \cdots,\ n)$$

(Minors and Cofactors of a Third-Order Determinant)

(Expansions of a Third-Order Determinant)

Find determinant.

$$\det \mathbf{A} = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix}$$

2) 2nd rows

$$= -2 \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} + 6 \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} - 4 \begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix}$$

$$= -2(6-0) + 6(2+0) - 4(0+3) = -12$$

Determinant : (Minors and Cofactors of a Third-Order Determinant)

$$D=\sum_{k=1}^{n}(-1)^{j+k}a_{jk}M_{jk} \quad (j=1,\,2,\,\cdots,\,n)$$

(Expansions of a Third-Order Determinant)

Find determinant.

$$\det \mathbf{A}=\begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix}$$

2) 3rd rows

$$=-1\begin{vmatrix} 3 & 0 \\ 6 & 4 \end{vmatrix}+0\begin{vmatrix} 1 & 0 \\ 2 & 4 \end{vmatrix}+2\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix}$$

$$=-1(12-0)+0(4-0)+2(0-0)=-12$$

Determinant : (Minors and Cofactors of a Third-Order Determinant)

$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \quad (j = 1, 2, \cdots, n)$$

(Expansions of a Third-Order Determinant)

Find determinant.

$$\det \mathbf{A} = \begin{vmatrix} -3 & 0 & 0 \\ 6 & 4 & 0 \\ -1 & 2 & 5 \end{vmatrix}$$

$$= -3 \begin{vmatrix} 4 & 0 \\ 2 & 5 \end{vmatrix} - 0 \begin{vmatrix} 6 & 0 \\ -1 & 5 \end{vmatrix} + 0 \begin{vmatrix} 6 & 4 \\ -1 & 2 \end{vmatrix}$$

$$= -3 \begin{vmatrix} 4 & 0 \\ 2 & 5 \end{vmatrix} = -3 \cdot 4 \cdot 5 = -60$$

Behavior of an nth-Order Determinant under Elementary Row Operations

Theorem 1. Behavior of an nth-Order Determinant under Elementary Row Operations (기본행연산)

(a) Interchange of two rows multiplies the value of the determinant by -1.

(b) Addition of a multiple of a row to another row does not alter the value of the determinant.

(c) Multiplication of a row by a nonzero constant c multiplies the value of the determinant by c.

Behavior of an nth-Order Determinant under Elementary Row Operations

Proof. ***(a) Interchange of two rows multiplies the value of the determinant by -1*** **by induction.**

The statement holds for n=2 because

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \qquad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad$$

(a) holds for determinants of order n-1(≥2) and show that it then holds determinants of order n.
Let D be of order n. Let E be one of those interchanged. Expand D and E by a row that is not one of those interchanged, call it the *j*th row.

$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk}, \qquad E = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} N_{jk}$$

Behavior of an nth-Order Determinant under Elementary Row Operations

$$D=\sum_{k=1}^{n}(-1)^{j+k}a_{jk}M_{jk}, \qquad E=\sum_{k=1}^{n}(-1)^{j+k}a_{jk}N_{jk}$$

N_{jk} is obtained from the minor M_{jk} of a_{jk} in D by interchange of those two rows which have been interchanged in D.
Now these minors are of order n-1.

$$Ex)\quad D=\begin{array}{c} \begin{array}{ccc} ① & & ② \end{array} \\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \end{array} \quad E=\begin{array}{c} \begin{array}{ccc} ① & & ② \end{array} \\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{vmatrix} \end{array}$$

➔ **The induction hypothesis (귀납법의 가정) applies**

$$N_{jk}=-M_{jk}$$ **(n-1차의 행렬식에 대해 본 정리가 성립한다는 가정)**

$$D=\sum_{k=1}^{n}(-1)^{j+k}a_{jk}M_{jk}=\sum_{k=1}^{n}(-1)^{j+k}a_{jk}\left(-N_{jk}\right)=-E$$

Behavior of an nth-Order Determinant under Elementary Row Operations

Proof. ***(b) Addition of a multiple of a row to another row does not alter the value of the determinant.***

Add c times Row i to Row j.

Let $\tilde{D}$ be the new determinant. Its entries in Row j are $a_{jk}+ca_{ik}$.

$$D=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \Rightarrow \tilde{D}=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{j1}+ca_{i1} & a_{j2}+ca_{i2} & \cdots & a_{jn}+ca_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Behavior of an nth-Order Determinant under Elementary Row Operations

We can write $\tilde{D}$ by the jth row.

$$\tilde{D} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{j1}+ca_{i1} & a_{j2}+ca_{i2} & \cdots & a_{jn}+ca_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{k=1}^{n} (-1)^{j+k} (a_{jk} + ca_{ik}) M_{jk}$$

$$= \underbrace{\sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk}}_{D_1} + c \underbrace{\sum_{k=1}^{n} (-1)^{j+k} a_{ik} M_{jk}}_{D_2} = D_1 + cD_2$$

Behavior of an nth-Order Determinant under Elementary Row Operations

$$D_1 = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = D$$

Behavior of an nth-Order Determinant under Elementary Row Operations

It has a_{ik} in both Row i and Row j.
Interchanging these two rows gives D_2 back,

$$D_2 = \sum_{k=1}^{n} (-1)^{j+k} a_{ik} M_{jk} = \begin{array}{c} \\ \\ ⓘ \\ \\ ⓙ \\ \\ \\ \end{array} \underbrace{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}}_{D_2} \xrightarrow[\text{Same}]{\text{Interchanging ⓘ,ⓙ}} \begin{array}{c} \\ \\ ⓙ \\ \\ ⓘ \\ \\ \\ \end{array} \underbrace{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}}_{D_2}$$

but on the other hand interchanging these two rows gives $-D_2$ by Theorem (a).

$$D_2 \xrightarrow{\text{Interchanging ⓘ,ⓙ}} -D_2$$

Together $D_2 = -D_2$ ➡ $D_2 = 0$

Behavior of an nth-Order Determinant under Elementary Row Operations

$$D_1 = D$$
$$D_2 = 0$$

$$\tilde{D} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{j1} + ca_{i1} & a_{j2} + ca_{i2} & \cdots & a_{jn} + ca_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + c \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= D_1 + cD_2$$

$$= D + c \cdot 0$$

$$= D$$

Behavior of an nth-Order Determinant under Elementary Row Operations

Proof. ***(c) Multiplication of a row by a nonzero constant*** c ***multiplies the value of the determinant by*** c

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \Rightarrow \tilde{D} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ ca_{j1} & ca_{j2} & \cdots & ca_{jn} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Expand the determinant by the *j*th row.

$$\tilde{D} = \sum_{k=1}^{n} (-1)^{j+k} ca_{jk} M_{jk} = c \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} = cD$$

Further Properties of nth-Order Determinants

Theorem 2. Further Properties of nth-Order Determinants

(d) Transposition leaves the value of a determinant unaltered.

(e) A zero row or column renders the value of a determinant zero.

(f) Proportional rows or columns render the value of a determinant zero. In particular, a determinant with two identical rows or columns has the value zero.

Further Properties of *n*th-Order Determinants

Proof.

(d) Transposition leaves the value of a determinant unaltered.

Transposition is defined as for matrices, that is, the *j*th row becomes the *j*th column of the transpose.

Proof.

(e) A zero row or column renders the value of a determinant zero.

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk}$$

$$= \sum_{k=1}^{n} (-1)^{j+k} 0 \cdot M_{jk} = 0$$

Further Properties of nth-Order Determinants

Proof.

(f) Proportional rows or columns render the value of a determinant zero. In particular, a determinant with two identical rows or columns has the value zero.

$$D=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ ca_{i1} & ca_{i2} & \cdots & ca_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = c\times \begin{array}{c} \\ \\ \text{ⓘ} \\ \\ \text{ⓙ} \\ \\ \\ \end{array} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \;\text{ⓘx(-1)+ⓙ}\; = c \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = c\times 0 = 0$$

Theorem (1.c) Multiplication of a row by a nonzero constant c multiplies the value of the determinant by c

Theorem. ***(1.b) Addition of a multiple of a row to another row does not alter the the value of the determinant.***

Determinant of a Triangular Matrix

$$D = \det \mathbf{A} = \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \Rightarrow M_{11}$$

➔ It is also a determinant of a triangular matrix.

$$= a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

$$= a_{11}C_{11} \qquad (\because a_{12} = a_{13} = \cdots = a_{1n} = 0)$$

$$= a_{11}M_{11}$$

$$= a_{11} \times a_{22} \times \cdots \times a_{nn}$$

Rank in Terms of Determinants

Theorem 3. Rank in Terms of Determinants

Consider an m x n matrix $A=[a_{jk}]$

1) A has rank $r \geq 1$ if and only if A has an r x r submatrix with nonzero determinant.
2) The determinant of any square submatrix with more than r rows, contained in A (if such a matrix exists!) has a value equal to zero.
3) In particular, if A is square, n x n, it has rank n if and only if $\det D \neq 0$

(1) Example

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 0 & 10 & 25 \\ 20 & 10 & 0 \end{bmatrix} \Downarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 10 & 25 \\ 0 & 0 & -95 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank=3

(3) Example

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 1 \\ 6 & 2 & 4 \end{bmatrix} \Downarrow \begin{bmatrix} 3 & 2 & 1 \\ 0 & -1/3 & 1/3 \\ 0 & 0 & 0 \end{bmatrix}$$

det D=0
Rank=2

Solving linear systems of two equations

Solve the linear systems of two equations

$$a_{11}x_1 + a_{12}x_2 = b_1 \quad \cdots①$$

$$a_{21}x_1 + a_{22}x_2 = b_2 \quad \cdots②$$

1. General Solution

$$①\times a_{22} - ②\times a_{12} :$$

$$(a_{11}a_{22} - a_{12}a_{21})x_1$$

$$= b_1a_{22} - a_{12}b_2$$

$$\therefore x_1 = \frac{b_1a_{22} - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}$$

$$(a_{11}a_{22} - a_{12}a_{21} \neq 0)$$

$$①\times(-a_{21}) + ②\times a_{11} :$$

$$(a_{11}a_{22} - a_{12}a_{21})x_1$$

$$= a_{11}b_2 - b_1a_{21}$$

$$\therefore x_2 = \frac{a_{11}b_2 - b_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}}$$

$$(a_{11}a_{22} - a_{12}a_{21} \neq 0)$$

Solving linear systems of two equations

Solve the linear systems of two equations

$$x_1 = \frac{b_1 a_{22} - a_{12} b_2}{a_{11}a_{22} - a_{12}a_{21}}$$

$$x_2 = \frac{a_{11} b_2 - b_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}}$$

$$D = \det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

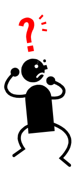
$$a_{11}x_1 + a_{12}x_2 = b_1 \qquad \ldots①$$

$$a_{21}x_1 + a_{22}x_2 = b_2 \qquad \ldots②$$

2. Use Cramer's rule

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{D} = \frac{b_1 a_{22} - a_{12} b_2}{D} \qquad (D \neq 0)$$

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{D} = \frac{a_{11} b_2 - b_1 a_{21}}{D} \qquad (D \neq 0)$$



$(D = 0)$

Solving linear systems of three equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$x_1 = \frac{D_1}{D}, \qquad x_2 = \frac{D_2}{D}; \qquad x_3 = \frac{D_3}{D}$$

$$D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}, \quad D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \quad D_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

Solving linear systems of three equations

Note that D_1, D_2, D_3 are obtained by replacing Columns 1, 2, 3.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \Rightarrow \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{\mathbf{b}}$$

$$\mathbf{A} = \begin{bmatrix} \boxed{\begin{matrix} b_1 \\ b_2 \\ b_3 \end{matrix}} & \begin{matrix} a_{12} \\ a_{22} \\ a_{32} \end{matrix} & \begin{matrix} a_{13} \\ a_{23} \\ a_{33} \end{matrix} \end{bmatrix}, \quad \mathbf{b} = \boxed{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}} \Rightarrow D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

Solving linear systems of three equations

Note that D_1, D_2, D_3 are obtained by replacing Columns 1, 2, 3.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$\Rightarrow \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{\mathbf{b}}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & \boxed{b_1} & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Rightarrow D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$$

Solving linear systems of three equations

Note that D_1, D_2, D_3 are obtained by replacing Columns 1, 2, 3.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3
\end{aligned}
\quad \Rightarrow \quad
\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{\mathbf{A}}
\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}}
=
\underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{\mathbf{b}}
$$

$$
\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \boxed{b_1} \\ a_{21} & a_{22} & \boxed{b_2} \\ a_{31} & a_{32} & \boxed{b_3} \end{bmatrix}, \quad
\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}
\quad \Rightarrow \quad
D_3 = \begin{vmatrix} a_{11} & a_{21} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{23} & b_3 \end{vmatrix}
$$

Cramer's Rule

Cramer's Theorem (Solution of Linear Systems by Determinants)

(a) If a linear system of n equations in the same number of unknowns $x_1, \ldots, x_n$

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

has a nonzero coefficient determinant D=det($\mathbf{A}$), the system has precisely one solution. This solution is given by the formulas

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \quad , \quad \cdots \quad , \quad x_n = \frac{D_n}{D}$$

Where D_k is the determinant obtained from D by replacing in D the kth column by the column with the entries $b_1, \ldots, b_n$.

Cramer's Rule

Cramer's Theorem (Solution of Linear Systems by Determinants)

$$D_k = \begin{vmatrix} a_{11} & \cdots & \boxed{b_1} & \cdots & a_{1n} \\ a_{21} & \cdots & b_2 & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & \cdots & b_n & \cdots & a_{nn} \end{vmatrix} \qquad \begin{matrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_n \end{matrix}$$

replace

$$D_k = b_1 C_{1k} + b_2 C_{2k} + \cdots + b_n C_{nk}$$

(b) Hence if the system is homogeneous and D≠0, it has only the trivial solution $x_1=0$, … , $x_n=0$. If D=0, the homogeneous system also has nontrivial solutions.

Example

Q ? Solve by Cramer's rule.

$$
\begin{aligned}
3y - 4z &= 16 \\
2x - 5y + 7z &= -27 \\
-x - 9z &= 9
\end{aligned}
$$

Solving linear systems of two equations

Solve the linear systems of two equations

$$(D=0) \quad D=\det A=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=a_{11}a_{22}-a_{12}a_{21}=0$$

$$\Downarrow$$

$$a_{11}=\frac{a_{12}a_{21}}{a_{22}}$$

$$\begin{cases} ① \; a_{11}x_1+a_{12}x_2=b_1 \\ ② \; a_{21}x_1+a_{22}x_2=b_2 \end{cases}$$

$$\frac{a_{12}a_{21}}{a_{22}}x_1+a_{12}x_2=b_1$$

$$\begin{cases} a_{21}x_1+a_{22}x_2=\dfrac{a_{22}}{a_{12}}b_1 \\ a_{21}x_1+a_{22}x_2=b_2 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} a_{21}x_1+a_{22}x_2=\dfrac{a_{22}}{a_{12}}b_1 \\ 0\cdot x_1+0\cdot x_2=\dfrac{a_{22}}{a_{12}}b_1-b_2 \end{cases}$$

when $\dfrac{a_{22}}{a_{12}}b_1-b_2=0$

$$\begin{cases} a_{21}x_1+a_{22}x_2=\dfrac{a_{22}}{a_{12}}b_1 \\ 0\cdot x_1+0\cdot x_2=0 \end{cases}$$

Linearly independent equation : 1
Variables : 2

$$\left[\begin{array}{cc:c} a_{21} & a_{22} & \dfrac{a_{22}}{a_{12}}b_1 \\ 0 & 0 & 0 \end{array}\right]$$

rank(A)=1=rank(A|B)
rank(A)=1<2 unknowns
(Many solutions)

$$\frac{a_{22}}{a_{12}}b_1-b_2\neq 0=b'$$

$$\begin{cases} a_{21}x_1+a_{22}x_2=\dfrac{a_{22}}{a_{12}}b_1 \\ 0\cdot x_1+0\cdot x_2=b' \end{cases}$$

False statement

$$\left[\begin{array}{cc:c} a_{21} & a_{22} & \dfrac{a_{22}}{a_{12}}b_1 \\ 0 & 0 & b' \end{array}\right]$$

rank(A)=1≠2=rank(A|B)
(No solution)

Homogeneous linear systems $\begin{cases} a_{11}x_1+a_{12}x_2=0 \\ a_{21}x_1+a_{22}x_2=0 \end{cases}$

$$A\mathbf{x}=\mathbf{0}$$

$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Trivial Solution $\mathbf{x}=\mathbf{0}$

$$\det A=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=0 \quad \Longrightarrow$$

$$A\mathbf{x}=\mathbf{0}$$

$$A=\begin{bmatrix} a_{11} & a_{12} \\ 0 & 0 \end{bmatrix}$$

Nontrivial many solutions

Cramer's Rule - Proof

Proof

(a) The augmented matrix **Ã** is of size nx(n+1). Hence its rank can be at most n.

If D = det **A** ≠ 0, then rank **A** = n. Thus rank **Ã** = rank **A.** → The system has a unique solution.

(b)

If *kth* column is replaced by *lth* column

$$D = \begin{vmatrix} a_{11} & a_{1k} & \cdots & a_{1l} & a_{1n} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ a_{j1} & a_{jk} & \cdots & a_{jl} & a_{jn} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ a_{n1} & a_{nk} & \cdots & a_{nl} & a_{jn} \end{vmatrix}$$

$$D = \begin{vmatrix} a_{11} & a_{1l} & \cdots & a_{1l} & a_{1n} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ a_{j1} & a_{jl} & \cdots & a_{jl} & a_{jn} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ a_{n1} & a_{nl} & \cdots & a_{nl} & a_{jn} \end{vmatrix}$$

$$D = a_{1k}C_{1k} + a_{2k}C_{2k} + \cdots + a_{nk}C_{nk}$$

$$\boxed{\hat{D} = a_{1l}C_{1k} + a_{2l}C_{2k} + \cdots + a_{nl}C_{nk} = 0}$$

Cramer's Rule - Proof

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

We now multiply the first equation in the linear system by C_{1k} on both sides, the second by C_{2k} ,the last by C_{nk} and add the resulting equations. This gives

$$C_{1k}(a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n) + \cdots + C_{nk}(a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n)$$

$$= b_1C_{1k} + \cdots + b_nC_{nk}$$

Collecting terms with the same x_j, we can write the left side as

$$x_1(a_{11}C_{1k} + a_{21}C_{2k} + \cdots + a_{n1}C_{nk}) + \cdots + x_n(a_{1n}C_{1k} + a_{2n}C_{2k} + \cdots + a_{nn}C_{nk})$$

Only one term led by x_j remains.

$$x_k(a_{1k}C_{1k} + a_{2k}C_{2k} + \cdots + a_{nk}C_{nk}) = x_kD$$

$$\boxed{\hat{D} = a_{1l}C_{1k} + a_{2l}C_{2k} + \cdots + a_{nl}C_{nk} = 0}$$

$$x_1(a_{11}C_{1k} + a_{21}C_{2k} + \cdots + a_{n1}C_{nk}) = 0 \ \ since\ 1 \neq k$$

$$x_l(a_{1l}C_{1k} + a_{2l}C_{2k} + \cdots + a_{nl}C_{nk}) = 0 \ \ since\ l \neq k$$

$$D_k = \begin{vmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & \cdots & b_2 & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & \cdots & b_n & \cdots & a_{nn} \end{vmatrix}$$

Therefore,

$$\boxed{x_kD = b_1C_{1k} + \cdots + b_nC_{nk} = D_k}$$

$$\boxed{\therefore x_k = \frac{D_k}{D}}$$

7.8 Inverse of a Matrix. Gauss-Jordan Elimination

Notation of Inverse Matrix

In this inverse section, only *square matrices* are considered exclusively.

Notation of inverse of an $n \times n$ matrix $A=[a_{jk}]$: A^{-1}

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$, where I is the $n \times n$ unit matrix.

Nonsingular matrix (정칙행렬): A matrix that has an inverse.

(If a matrix has an inverse, the inverse is unique)

Singular matrix (특이행렬): A matrix that has no inverse.

Proof of uniqueness of inverse matrix

If B and C are inverses of A $(\mathbf{AB} = \mathbf{I} \ \& \ \mathbf{CA} = \mathbf{I})$,

We obtain $\mathbf{B} = \mathbf{IB} = (\mathbf{CA})\mathbf{B} = \mathbf{C}(\mathbf{AB}) = \mathbf{CI} = \mathbf{C}$

(the uniqueness of inverse)

Theorem1. Existence of the Inverse

Theorem 1. Existence of the Inverse

The inverse A^{-1} of an n x n matrix A exists if and only if rank $A = n$, thus if and only if $\det A \neq 0$.

Hence A is nonsingular if rank $A = n$,
and is singular if rank $A < n$.

Inverse by the Gauss-Jordan Method

For Practical determination of the inverse $\mathbf{A}^{-1}$ of a nonsingular n x n matrix $\mathbf{A}$, Gauss elimination can be used.
: This method is called Gauss-Jordan elimination

Step 1. Make augmented matrix. $\tilde{\mathbf{A}} = [\mathbf{A} \;\; \mathbf{I}]$

Step 2. Make Multiplication of $\mathbf{AX=I}$ by $\mathbf{A}^{-1}$
(by applying Gauss elimination to $\tilde{\mathbf{A}} = [\mathbf{A} \;\; \mathbf{I}]$)

$\longrightarrow$ **This gives a matrix of the form $[\mathbf{U} \;\; \mathbf{H}]$** $\quad$ **U : upper triangular**

Step 3. Reduce U by further elementary row operations to diagonal form.
(Eliminate the entries of U above the main diagonal and making the diagonal entries all 1 by multiplication. See the example next page.)

Inverse of a Matrix. Gauss-Jordan elimination.

Determine the inverse $\mathbf{A}^{-1}$ of

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

Step 1. Make augmented matrix.

$$\left[\mathbf{A}\ \mathbf{I}\right] = \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

Step 2. Make Multiplication of $\mathbf{AX=I}$ by $\mathbf{A}^{-1}$ by applying Gauss elimination to

Inverse of a Matrix. Gauss-Jordan elimination

$$\mathbf{AX} = \mathbf{I} \Rightarrow \mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{I} \Rightarrow \mathbf{IX} = \mathbf{A}^{-1}$$

$$[\mathbf{A} \,|\, \mathbf{I}] \Rightarrow$$ **Gauss-Jordan Elimination** $$\Rightarrow [\mathbf{I} \,|\, \mathbf{A}^{-1}]$$

$$[\mathbf{A} \;\; \mathbf{I}] = \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

Row2 + 3Row1

Row3 − Row1

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right]$$

Row3 − Row2

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right]$$

Row1 +0.4Row3

Row2 +1.4Row3

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 0 & -0.6 & -0.4 & 0.4 \\ 0 & 2 & 0 & -2.6 & -0.4 & 1.4 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right]$$

Row1 -0.5Row2

$$\left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 0.7 & -0.2 & -0.3 \\ 0 & 2 & 0 & -2.6 & -0.4 & 1.4 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right]$$

diagonal matrix

-Row1

0.5Row2

-0.2Row3

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right]$$

Inverse of a Matrix. Gauss-Jordan elimination

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right]$$

$$\mathbf{A}^{-1}$$

Check the result.

Let
AA^{-1} = B

$$= \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$$

$$b_{11} = (-1)\times(-0.7) + 1\times(-1.3) + 2\times 0.8 = 1$$
$$b_{12} = (-1)\times(0.2) + 1\times(-0.2) + 2\times 0.2 = 0$$
$$b_{13} = (-1)\times(0.3) + 1\times(0.7) + 2\times(-0.2) = 0$$
$$b_{21} = (3)\times(-0.7) + (-1)\times(-1.3) + 1\times(0.8) = 0$$
$$b_{22} = (3)\times(0.2) + (-1)\times(-0.2) + 1\times(0.2) = 1$$
$$b_{23} = (3)\times(0.3) + (-1)\times(0.7) + 1\times(-0.2) = 0$$
$$b_{31} = (-1)\times(-0.7) + (3)\times(-1.3) + 4\times(0.8) = 0$$
$$b_{32} = (-1)\times(0.2) + (3)\times(-0.2) + 4\times(0.2) = 0$$
$$b_{33} = (-1)\times(0.3) + (3)\times(0.7) + 4\times(-0.2) = 1$$

$$\therefore \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

cofactor : $C_{jk} = (-1)^{j+k} M_{jk}$

M_{jk} : a determinant of order $n-1$

Theorem 2. Inverse of a Matrix by determinant

The inverse of a nonsingular $n \times n$ matrix $\mathbf{A}=[a_{jk}]$ is given by

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}[\mathbf{C}_{jk}]^{\mathsf{T}} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ . & . & . & . \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \cdots (4)$$

Where C_{jk} is *the cofactor of a_{jk}* in $\det \mathbf{A}$

CAUTION! Note well that in $\mathbf{A}^{-1}$, the cofactor C_{jk} occupies the same place as $\mathbf{a}_{kj}$ (not $\mathbf{a}_{jk}$) does in $\mathbf{A}$.)

Proof)

Let $$\mathbf{B} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ . & . & . & . \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$ **and show that $\mathbf{BA}=\mathbf{I}$**

Useful formulas for Inverses
Theorem 2. Inverse of a Matrix

Sec 7.7

$$(k = l),\ \ D = a_{1k}C_{1k} + a_{2k}C_{2k} + \cdots + a_{nk}C_{nk} \cdots (9)$$

$$(k \neq l),\ \ a_{1l}C_{1k} + a_{2l}C_{2k} + \cdots + a_{nl}C_{nk} = 0 \cdots (10)$$

Let

$$\mathbf{BA} = \mathbf{G} = [g_{kl}] \ldots (5)$$

and then show that G = I.

By definition of matrix multiplication and because of the form of B in (4)

$$g_{kl} = \sum_{s=1}^{n} \frac{C_{sk}}{\det \mathbf{A}} a_{sl}$$

$$= \frac{1}{\det \mathbf{A}} \left(a_{1l}C_{1k} + \cdots + a_{nl}C_{nk} \right) \cdots (6)$$

In Sec 7.7 (9) and (10),

If $l = k$, $a_{1l}C_{1k} + \cdots + a_{nl}C_{nk} = \det \mathbf{A}$

$l \neq k$, $a_{1l}C_{1k} + \cdots + a_{nl}C_{nk} = 0$

Hence,

$$g_{kk} = \frac{1}{\det \mathbf{A}} \det \mathbf{A} = 1$$

$$g_{lk} = 0$$

g_{kk} are the entries of main diagonal of matrix G. and that means only entries of main diagonal is 1.

$\therefore$ **G = I**

$\therefore$ **B = A^{-1}**

$$\mathbf{BA} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ . & . & . & . \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & . & . \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Inverse of 2 x 2 Matrix

$$C_{jk} = (-1)^{j+k} M_{jk}$$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \cdot & \cdot & \cdot & \cdot \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

Ex 2)

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

$$\det \mathbf{A} = 3 \cdot 4 - 1 \cdot 2 = 10$$

$$C_{11} = 4, \qquad C_{21} = -1,$$

$$C_{12} = -2, \qquad C_{22} = 3,$$

$$\therefore \mathbf{A}^{-1} = \frac{1}{10} \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4 & -0.1 \\ -0.2 & 0.3 \end{bmatrix}$$

Inverse of 3 x 3 Matrix

$$C_{jk} = (-1)^{j+k} M_{jk}$$

Q ?

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}, \qquad \mathbf{A}^{-1} \ ?$$

Inverse of Diagonal Matrices

$\mathbf{A}=[a_{jk}]$, $a_{jk}=0$ when $j \neq k$, have an inverse if and only if all $a_{jj} \neq 0$. Then $\mathbf{A}^{-1}$ is diagonal, too, with entries $1/a_{11}, \cdots, 1/a_{nn}$.

Proof) For a diagonal matrix we have in (4)

$$\frac{C_{11}}{D} = \frac{a_{22} \cdots a_{nn}}{a_{11}a_{22} \cdots a_{nn}} = \frac{1}{a_{11}}, \quad etc.$$

Ex 4) Inverse of Diagonal Matrix

$$\mathbf{A} = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \therefore \mathbf{A}^{-1} = \begin{bmatrix} -\frac{1}{0.5} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{1} \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Inverse of Products

Products can be inverted by taking the inverse of each factor and multiplying these inverses in reverse order,

Hence for more than two factors, $(\mathbf{AC})^{-1} = \mathbf{C}^{-1}\mathbf{A}^{-1}$

$$(\mathbf{AC}\cdots\mathbf{PQ})^{-1} = \mathbf{Q}^{-1}\mathbf{P}^{-1}\cdots\mathbf{C}^{-1}\mathbf{A}^{-1}$$

$$(\mathbf{AC})(\mathbf{C}^{-1}\mathbf{A}^{-1}) = \mathbf{ACC}^{-1}\mathbf{A}^{-1} = \mathbf{AIA}^{-1} = \mathbf{AA}^{-1} = \mathbf{I}$$

$$(\mathbf{C}^{-1}\mathbf{A}^{-1})(\mathbf{AC}) = \mathbf{C}^{-1}\mathbf{A}^{-1}\mathbf{AC} = \mathbf{C}^{-1}\mathbf{IC} = \mathbf{C}^{-1}\mathbf{C} = \mathbf{I}$$

Cancellation Laws

Theorem 3. Cancellation Laws (약분법)

Let A, B, C be n x n matrices. Then

(a) If rank A = n and AB = AC, then B = C

(b) If rank A = n, then AB = 0 implies B = 0. Hence if AB = 0, but A ≠ 0 as well as B ≠ 0, then rank A < n and rank B < n.

(c) If A is singular, so are BA and AB

Determinants of Matrix Products

Theorem 4. Determinant of a Product of Matrices

For any n x n matrices $\mathbf{A}$ and $\mathbf{B}$,

$$\det(\mathbf{AB}) = \det(\mathbf{BA}) = \det \mathbf{A} \cdot \det \mathbf{B}$$

If $\mathbf{A}$ or $\mathbf{B}$ is singular, so are $\mathbf{AB}$ and $\mathbf{BA}$

Now $\mathbf{A}$ and $\mathbf{B}$ be nonsingular. Then we can reduce $\mathbf{A}$ to a diagonal matrix $\hat{\mathbf{A}}$ = $[a_{jk}]$ by Gauss-Jordan steps.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \hat{\mathbf{A}} = \begin{bmatrix} \hat{a}_{11} & 0 & \cdots & 0 \\ 0 & \hat{a}_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \hat{a}_{nn} \end{bmatrix}.$$

$\mathbf{A}$ and $\hat{\mathbf{A}}$ are row-equivalent matrices.

Determinants of Matrix Products

Proof)

$\boxed{\text{A and } \hat{\text{A}} \text{ are row-equivalent matrices.}}$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \hat{\mathbf{A}} = \begin{bmatrix} \hat{a}_{11} & 0 & \cdots & 0 \\ 0 & \hat{a}_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \hat{a}_{nn} \end{bmatrix}.$$

$$\mathbf{Ax} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ b_n \end{bmatrix} = \mathbf{b}$$

$$\hat{\mathbf{A}}\mathbf{x} = \begin{bmatrix} \hat{a}_{11} & 0 & \cdots & 0 \\ 0 & \hat{a}_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \hat{a}_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \cdot \\ \hat{b}_n \end{bmatrix} = \hat{\mathbf{b}}$$

$$\boxed{\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \hat{\mathbf{A}}^{-1}\hat{\mathbf{b}}}$$

Determinants of Matrix Products

A **and** $\hat{\mathrm{A}}$ **are row-equivalent matrices.**

Addition of a multiple of a row to another row does not alter the value of the determinant.

$$\therefore \det(\mathbf{AB}) = \det(\hat{\mathbf{A}}\mathbf{B})$$

$$\hat{\mathbf{A}}\mathbf{B} = \begin{bmatrix} \hat{a}_{11} & 0 & \cdots & 0 \\ 0 & \hat{a}_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \hat{a}_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ & & \vdots & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} \hat{a}_{11}b_{11} & \hat{a}_{11}b_{12} & \cdots & \hat{a}_{11}b_{1n} \\ \hat{a}_{22}b_{21} & \hat{a}_{22}b_{22} & \cdots & \hat{a}_{22}b_{2n} \\ & & \vdots & \\ \hat{a}_{nn}b_{n1} & \hat{a}_{nn}b_{n2} & \cdots & \hat{a}_{nn}b_{nn} \end{bmatrix}$$

$$\det(\hat{\mathbf{A}}\mathbf{B}) = \begin{vmatrix} \hat{a}_{11}b_{11} & \hat{a}_{11}b_{12} & \cdots & \hat{a}_{11}b_{1n} \\ \hat{a}_{22}b_{21} & \hat{a}_{22}b_{22} & \cdots & \hat{a}_{22}b_{2n} \\ & & \vdots & \\ \hat{a}_{nn}b_{n1} & \hat{a}_{nn}b_{n2} & \cdots & \hat{a}_{nn}b_{nn} \end{vmatrix} = \hat{a}_{11}\hat{a}_{22}\cdots\hat{a}_{nn} \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ & & \vdots & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{vmatrix}$$

Determinants of Matrix Products

$$\det(\mathbf{AB}) = \det(\mathbf{BA}) = \det\mathbf{A}\cdot\det\mathbf{B}$$

$$\tilde{D} = \sum_{k=1}^{n}(-1)^{j+k}\, c a_{jk} M_{jk} = cD$$

A and $\hat{\mathbf{A}}$ are row-equivalent matrices.

$$\therefore \det(\mathbf{AB}) = \det(\hat{\mathbf{A}}\mathbf{B})$$

$$\det(\hat{\mathbf{A}}\mathbf{B}) = \begin{vmatrix} \hat{a}_{11}b_{11} & \hat{a}_{11}b_{12} & \cdots & \hat{a}_{11}b_{1n} \\ \hat{a}_{22}b_{21} & \hat{a}_{22}b_{22} & \cdots & \hat{a}_{22}b_{2n} \\ & & \vdots & \\ \hat{a}_{nn}b_{n1} & \hat{a}_{nn}b_{n2} & \cdots & \hat{a}_{nn}b_{nn} \end{vmatrix} = \hat{a}_{11}\hat{a}_{22}\cdots\hat{a}_{nn} \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ & & \vdots & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{vmatrix}$$

$$= \hat{a}_{11}\hat{a}_{22}\cdots\hat{a}_{nn}\cdot\det(\mathbf{B})$$

$$= \det(\hat{\mathbf{A}})\cdot\det(\mathbf{B}) = \det(\mathbf{A})\cdot\det(\mathbf{B})$$

$$\therefore \det(\mathbf{AB}) = \det(\mathbf{A})\cdot\det(\mathbf{B})$$