CHAPTER 8. LINEAR ALGEBRA : MATRIX EIGENVALUE PROBLEMS

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서유택

※ 본 강의 자료는 이규열, 장범선, 노명일 교수님께서 만드신 자료를 바탕으로 일부 편집한 것입니다.

8.1 The Matrix Eigenvalue Problem. Determining Eigenvalues and Eigenvectors

Eigenvalues, Eigenvectors ?

$$\mathbf{A} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \quad \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \mathbf{A}\mathbf{x}_1 = ? \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \mathbf{A}\mathbf{x}_2 = ? \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

$$\mathbf{x}_3 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \mathbf{A}\mathbf{x}_3 = ? \begin{bmatrix} -12 \\ 6 \end{bmatrix}$$

$$\mathbf{x}_p = \begin{bmatrix} 5 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \mathbf{A}^{10}\mathbf{x}_p = ?$$

Eigenvalues (고유값), Eigenvectors (고유벡터)

Let $\mathbf{A}=[a_{jk}]$ be a given matrix n x n matrix and consider the vector equation

$$\mathbf{Ax} = \lambda \mathbf{x}$$

Our task is to determine $\mathbf{x}$'s and λ's that satisfy this equation. Clearly, the zero vector $\mathbf{x} = \mathbf{0}$ is a solution of this equation for any value λ, because $\mathbf{A0}=\mathbf{0}$. This is of no interest.

(a) Eigenvalue (고유값): A value of λ for which this equation has a solution $\mathbf{x} \neq \mathbf{0}$ (= characteristic value (특성값), latent root (잠정근))

(b) Eigenvector (고유벡터): The corresponding solutions $\mathbf{x} \neq \mathbf{0}$ of this equation

Eigenvalues, Eigenvectors

Ex) Determination of Eigenvalues and Eigenvectors

Determine eigenvalues and eigenvectors

$$\mathbf{A} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

(a) Eigenvalues

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$-5x_1 + 2x_2 = \lambda x_1$$
$$2x_1 - 2x_2 = \lambda x_2$$

$$(-5-\lambda)x_1 + 2x_2 = 0$$
$$2x_1 + (-2-\lambda)x_2 = 0$$

This can be written in matrix notation.

$$\mathbf{Ax} - \lambda \mathbf{x} = 0$$
$$\mathbf{Ax} - \lambda \mathbf{Ix} = 0$$
$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0$$

Eigenvalues, Eigenvectors

$D(\lambda)$: characteristic determinant

$D(\lambda)=0$: characteristic equation

Ex) Determination of Eigenvalues and Eigenvectors

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

→ homogeneous linear system

By Cramer's rule it has a nontrivial solution $\mathbf{x}\neq\mathbf{0}$ if and only if its coefficient determinant is zero.

$$D(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix}=0$$

$$(-5-\lambda)(-2-\lambda)-4=0$$

$$\lambda^2+7\lambda+6=0$$

$$\lambda_1=-1$$

$$\lambda_2=-6$$

$$(-5-\lambda)x_1+2x_2=0\cdots(a)$$

$$2x_1+(-2-\lambda)x_2=0\cdots(b)$$

$x_1=0, x_2=0$: Trivial solution

To have non-trivial solution

No. of Linearly independent equation < Unknowns

(rank $r<n$)

$(a)\times n=(b)$ Linearly dependent, $n\neq 0$

$$(-5-\lambda)n=2\cdots(c)$$

$$2n=(-2-\lambda)\cdots(d)$$

$(c)/(d)$

$$\frac{(-5-\lambda)}{2}=\frac{2}{(-2-\lambda)}$$

$$\therefore(-5-\lambda)(-2-\lambda)-4=0$$

Eigenvalues, Eigenvectors

Ex) Determination of Eigenvalues and Eigenvectors

$$(-5-\lambda)x_1 + 2x_2 = 0 \qquad \lambda_1 = -1$$

$$2x_1 + (-2-\lambda)x_2 = 0 \qquad \lambda_2 = -6$$

-Eigenvector of $\mathbf{A}$ corresponding to λ_1

$$\lambda_1 = -1$$

The equation is

$$-4x_1 + 2x_2 = 0$$

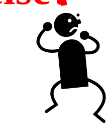
$$2x_1 - x_2 = 0$$

$$\therefore x_2 = 2x_1$$

- This determines eigenvector corresponding to λ_1 up to a scalar multiple.
- If we choose $x_1=1$, we obtain

$$\mathbf{x}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

What happen if choose x_1=something else?



$$(x_1 = 1): \mathbf{x}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

$$(x_1 = 2): \mathbf{x}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2\boxed{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

$$(x_1 = 3): \mathbf{x}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\boxed{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

$\vdots$

'basis (기저)'

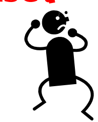
Eigenvalues, Eigenvectors

Ex) Determination of Eigenvalues and Eigenvectors

$$(-5-\lambda)x_1 + 2x_2 = 0 \qquad \lambda_1 = -1$$

$$2x_1 + (-2-\lambda)x_2 = 0 \qquad \lambda_2 = -6$$

What happen if choose x_2=something else?



-Eigenvector of $\mathbf{A}$ corresponding to λ_1

$$\lambda_2 = -6$$

The equation is

$$x_1 + 2x_2 = 0$$

$$2x_1 + 4x_2 = 0$$

$$\therefore\ x_2 = -x_1/2$$

- This determines eigenvector corresponding to λ_2 up to a scalar multiple.
- If we choose x_1=2, we obtain

$$\mathbf{x}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1/2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$(x_1 = 2):\mathbf{x}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1/2 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}$$

$$(x_1 = 4):\mathbf{x}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1/2 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix} = 2\boxed{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}$$

$$(x_1 = 6):\mathbf{x}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1/2 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix} = 3\boxed{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}$$

$$\vdots$$

'basis'

General case of Eigenvectors, Eigenvalues

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$$

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= \lambda x_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= \lambda x_2 \\ &\vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n &= \lambda x_n \end{aligned}$$

$$\begin{aligned} (a_{11} - \lambda)x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + (a_{22} - \lambda)x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + (a_{nn} - \lambda)x_n &= 0 \end{aligned}$$

In matrix notation,

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0$$

General case of Eigenvectors, Eigenvalues

$$D(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

(a) Characteristic matrix (특성행렬): $\mathbf{A}-\lambda\mathbf{I}$

(b) Characteristic determinant (특성행렬식): $D(\lambda)$

(c) Characteristic polynomial (특성다항식): a polynomial of nth degree in λ by developing $D(\lambda)$

Eigenvalues, Eigenvectors

Theorem 8.1 Eigenvalues

The eigenvalues of a square matrix A are the roots of the characteristic equation (특성방정식) of A.

$$D(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

Hence an n x n matrix has at least one eigenvalue and at most n numerically different eigenvalues.

Eigenvalues, Eigenvectors

Ex) Multiple Eigenvalues

Find the eigenvalues and eigenvector of

$$\mathbf{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Sol) For our matrix, the characteristic determinant gives the characteristic equation

$$-\lambda^3 - \lambda^2 + 21\lambda + 45 = 0$$

The roots (eigenvalues of $\mathbf{A}$) are

$$\lambda_1 = 5,\ \lambda_2 = \lambda_3 = -3$$

Eigenvalues, Eigenvectors

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

Ex) Multiple Eigenvalues

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}$$

$$=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

$(1)\ \lambda=5$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-5\mathbf{I}$$

$$=\begin{bmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{bmatrix}$$

row reduction

$$\begin{bmatrix}-7 & 2 & -3\\ 0 & -24/7 & -48/7\\ 0 & 0 & 0\end{bmatrix}$$

Choosing

$$x_3=-1$$

we have

$$-\frac{24}{7}x_2-\frac{48}{7}x_3=0$$

$$\therefore x_2=2$$

$$-7x_1+2x_2-3x_3=0$$

$$\therefore x_1=1$$

$$\mathbf{x}_1=\boxed{\begin{bmatrix}1 & 2 & -1\end{bmatrix}^{\mathbf{T}}}$$

if choose

$$x_3=1$$

we have

$$-\frac{24}{7}x_2-\frac{48}{7}x_3=0$$

$$\therefore x_2=-2$$

$$-7x_1+2x_2-3x_3=0$$

$$\therefore x_1=-1$$

$$\mathbf{x}_1=\begin{bmatrix}-1 & -2 & 1\end{bmatrix}^{\mathbf{T}}$$

$$\mathbf{x}_1=(-1)\times\boxed{\begin{bmatrix}1 & 2 & -1\end{bmatrix}^{\mathbf{T}}}$$

'basis'

Eigenvalues, Eigenvectors

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}$$

$$=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

Ex) Multiple Eigenvalues

$(2)\ \lambda=-3$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-3\mathbf{I}$$

$$=\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}$$

row reduction:

$$\begin{bmatrix}1 & 2 & -3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$x_1+2x_2-3x_3=0$$

$$x_1=-2x_2+3x_3$$

Choosing

$x_2=1,\ x_3=0$

we have

$x_1=-2$

$$\mathbf{x}_2=\begin{bmatrix}-2 & 1 & 0\end{bmatrix}^{\mathbf{T}}$$

Choosing

$x_2=0,\ x_3=1$

we have

$x_1=3$

$$\mathbf{x}_3=\begin{bmatrix}3 & 0 & 1\end{bmatrix}^{\mathbf{T}}$$

$$\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix} \Rightarrow \begin{cases}x_1+2x_2-3x_3=0\\ 2x_1+4x_2-6x_3=0\\ -x_1-2x_2+3x_3=0\end{cases}$$

$$\Downarrow$$

$$x_1+2x_2-3x_3=0$$

we have only one equation

and three variables

two free variables (x_2, x_3)

Eigenvalues, Eigenvectors

$$\mathbf{Ax} = \lambda \mathbf{x}$$

Ex) Real Matrices with Complex Eigenvalues and Eigenvectors

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Characteristic Equation:

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 1 = 0$$

$$\therefore \lambda = \pm i$$

$(1)\ \lambda = i$

$$\mathbf{A} - \lambda \mathbf{I} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}$$

$$-ix_1 + x_2 = 0$$

Choosing $x_1 = 1$

we have $x_2 = i$

$$\mathbf{x}_1 = \begin{bmatrix} 1 & i \end{bmatrix}^{\mathrm{T}}$$

Ex) Real Matrices with Complex Eigenvalues and Eigenvectors

$(2)\ \lambda = -i$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}$$

$$ix_1 + x_2 = 0$$

Choosing $x_1 = 1$

we have $x_2 = -i$

$$\mathbf{x}_2 = \begin{bmatrix} 1 & -i \end{bmatrix}^{\mathbf{T}}$$

Eigenvalues of the Transpose

Theorem 8.3 Eigenvalues of the Transpose

The transpose $\mathbf{A}^T$ of a square matrix $\mathbf{A}$ has the same eigenvalues as $\mathbf{A}$.

<u>Proof</u>

Transposition does not change the value of the characteristic determinant.

Theorem 2. Further Properties of *n*th-Order Determinants

(d) Transposition leaves the value of a determinant unaltered.

(e) A zero row or column renders the value of a determinant zero.

(f) Proportional rows or columns render the value of a determinant zero. In particular, a determinant with two identical rows or columns has the value zero.

8.2 Some Applications of Eigenvalue Problems

Sum and Multiplication of Eigenvalues

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$D(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix}$$

$$= (a_{11} - \lambda)(a_{22} - \lambda) - a_{12} \cdot a_{21}$$

$$= \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12} \cdot a_{21}$$

$$= \lambda^2 - (a_{11} + a_{22})\lambda + \det \mathbf{A}$$

$$= (\lambda - \lambda_1)(\lambda - \lambda_2)$$

$$= \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = 0$$

$$\therefore a_{11} + a_{22} = \lambda_1 + \lambda_2$$

$$\therefore \det \mathbf{A} = \lambda_1 \cdot \lambda_2$$

The sum of these *n* eigenvalues equals the sum of the entries on the main diagonal of A, called trace of A;

$$\therefore \text{trace } \mathbf{A} = \sum_{j=1}^{n} a_{jj} = \sum_{k=1}^{n} \lambda_k$$

The product of the eigenvalues equals the determinant of A,

$$\det \mathbf{A} = \lambda_1 \lambda_2 \cdots \lambda_n$$

Markov Matrix

We need the hundredth power $\mathbf{A}^{100}$.

$$\underset{\mathbf{A}}{\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}} \quad \underset{\mathbf{A}^2}{\begin{bmatrix} 0.70 & 0.45 \\ 0.30 & 0.55 \end{bmatrix}} \quad \underset{\mathbf{A}^3}{\begin{bmatrix} 0.650 & 0.525 \\ 0.350 & 0.475 \end{bmatrix}} \quad \cdots \quad \underset{\mathbf{A}^{100}}{\begin{bmatrix} 0.6000 & 0.6000 \\ 0.4000 & 0.4000 \end{bmatrix}}$$

$\mathbf{A}^{100}$ was found by using the eigenvalues of A, not by multiplying 100 matrices.

Strang G., Introduction to Linear Algebra, Third edition, Wellesley-Cambridge Press, 2003, Ch.6.1, p274 need the hundredth power $\mathbf{A}^{100}$

Markov Matrix

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

Almost all vectors change direction, when they are multiplied by $\mathbf{A}$.

Certain exceptional vectors $\mathbf{x}$ are in the same direction as $\mathbf{Ax}$ → "eigenvectors"

Multiply an eigenvector by $\mathbf{A}$, and the vector $\mathbf{Ax}$ is a number λ times the original $\mathbf{x}$.
The basic equation is $\mathbf{Ax} = \lambda\mathbf{x}$.

$$\mathbf{x}_1 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} \quad \text{and} \quad \mathbf{Ax}_1 = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}\begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \boxed{}\mathbf{x}_1 \qquad \therefore \lambda_1 = 1$$

A vector　Multiply by A　Same direction　Eigenvector　Eigenvalue

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{Ax}_2 = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix} = \boxed{\frac{1}{2}}\mathbf{x}_2 \qquad \therefore \lambda_2 = \frac{1}{2}$$

A vector　Multiply by A　Same direction　Eigenvector　Eigenvalue

need the hundredth power A^{100}

Markov Matrix

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

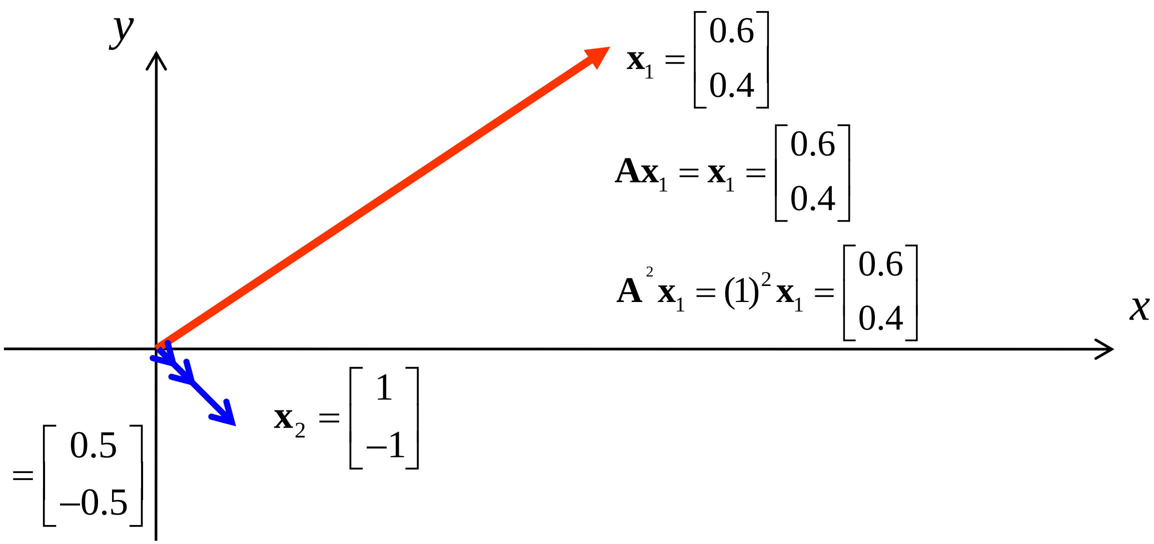
$$\mathbf{A}\mathbf{x}_1 = \mathbf{x}_1,$$

$$\mathbf{A}^2\mathbf{x}_1 = \mathbf{A}(\mathbf{A}\mathbf{x}_1) = \mathbf{A}\mathbf{x}_1 = \mathbf{x}_1,$$

$$\mathbf{A}\mathbf{x}_2 = \frac{1}{2}\mathbf{x}_2$$

$$\mathbf{A}^2\mathbf{x}_2 = \mathbf{A}(\mathbf{A}\mathbf{x}_2) = \mathbf{A}\left(\frac{1}{2}\mathbf{x}_2\right) = \frac{1}{2}\mathbf{A}\mathbf{x}_2 = \frac{1}{2}\frac{1}{2}\mathbf{x}_2 = \frac{1}{2}^2\mathbf{x}_2$$

The eigenvalues of A^{100} are $1^{100} = 1$ and $½^{100}$ = very small number.



$$\mathbf{x}_1 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$$

$$\mathbf{A}\mathbf{x}_1 = \mathbf{x}_1 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$$

$$\mathbf{A}^2\mathbf{x}_1 = (1)^2\mathbf{x}_1 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$$

$$\mathbf{A}^2\mathbf{x}_2 = (\frac{1}{2})^2\mathbf{x}_2 = \begin{bmatrix} 0.25 \\ 0.25 \end{bmatrix}$$

$$\mathbf{A}\mathbf{x}_2 = \lambda_2\mathbf{x}_2 = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}$$

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

need the hundredth power A^{100}

Markov Matrix

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}, \quad \mathbf{x}_1 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{A}^n \mathbf{x}_1 = \mathbf{x}_1, \quad \mathbf{A}^n \mathbf{x}_2 = \tfrac{1}{2}^n \mathbf{x}_2$$

- Other vectors do change direction. But all other vectors are combinations of the two eigenvectors.
- The first column of $\mathbf{A}$ is the combination $\mathbf{x}_1 + 0.2\mathbf{x}_2$:

$$\begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = \mathbf{x}_1 + 0.2\mathbf{x}_2 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} + \begin{bmatrix} 0.2 \\ -0.2 \end{bmatrix}$$

Multiplying by $\mathbf{A}$ gives the first column of $\mathbf{A}^2$. Do it separately for $\mathbf{x}_1$ and $0.2\mathbf{x}_2$.

$$\mathbf{A}\begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = \mathbf{A}(\mathbf{x}_1 + 0.2\mathbf{x}_2) = \mathbf{A}\mathbf{x}_1 + 0.2\mathbf{A}\mathbf{x}_2 = \boxed{}\mathbf{x}_1 + 0.2 \cdot \boxed{\tfrac{1}{2}}\mathbf{x}_2 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} + \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix}$$

Eigenvalue

$$\lambda_1 = 1$$

Eigenvalue

$$\lambda_2 = \frac{1}{2}$$

$$\mathbf{A}^{99}\begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = \mathbf{x}_1 + 0.2 \cdot \tfrac{1}{2}^{99}\mathbf{x}_2 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} + \begin{bmatrix} very \\ small \\ vector \end{bmatrix}$$

need the hundredth power A^{100}

Markov Matrix

$$\mathbf{A} = \begin{bmatrix} 0.8 & \boxed{0.3} \\ 0.2 & \boxed{0.7} \end{bmatrix}, \quad \mathbf{x}_1 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \mathbf{A}^n \mathbf{x}_1 = \mathbf{x}_1, \quad \mathbf{A}^n \mathbf{x}_2 = \tfrac{1}{2}^n \mathbf{x}_2$$

The second column of A is the combination $\mathbf{x}_1 - 0.3\mathbf{x}_2$:

$$\begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} = \mathbf{x}_1 - 0.3\mathbf{x}_2 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} + \begin{bmatrix} 0.3 \\ -0.3 \end{bmatrix}$$

Multiplying by A gives the first column of A^2 . Do it separately for $\mathbf{x}_1$ and $0.3\mathbf{x}_2$.

$$\mathbf{A}\begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} = \mathbf{A}(\mathbf{x}_1 - 0.3\mathbf{x}_2) = \mathbf{A}\mathbf{x}_1 - 0.3\mathbf{A}\mathbf{x}_2 = \mathbf{x}_1 - 0.3 \cdot \tfrac{1}{2}\mathbf{x}_2 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} + \begin{bmatrix} 0.15 \\ -0.15 \end{bmatrix}$$

$$\mathbf{A}^{99}\begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} = \mathbf{x}_1 - 0.3 \cdot \tfrac{1}{2}^{99} \mathbf{x}_2 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} + \begin{bmatrix} very \\ small \\ vector \end{bmatrix}$$

need the hundredth power $\mathbf{A}^{100}$

Markov Matrix

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}, \quad \mathbf{x}_1 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \mathbf{A}^n\mathbf{x}_1 = \mathbf{x}_1, \ \mathbf{A}^n\mathbf{x}_2 = \tfrac{1}{2}^n\mathbf{x}_2$$

$$\mathbf{A}^{99}\begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = \mathbf{x}_1 + 0.2 \cdot \tfrac{1}{2}^{99}\mathbf{x}_2 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} + \begin{bmatrix} \textit{very} \\ \textit{small} \\ \textit{vector} \end{bmatrix}$$

$$\mathbf{A}^{99}\begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} = \mathbf{x}_1 - 0.3 \cdot \tfrac{1}{2}^{99}\mathbf{x}_2 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} - \begin{bmatrix} \textit{very} \\ \textit{small} \\ \textit{vector} \end{bmatrix}$$

$$\Rightarrow \quad \boxed{\therefore \mathbf{A}^{99}\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}}$$

$$\therefore \mathbf{A}^{100} = \mathbf{A}^{99}\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}$$

The eigenvector $\mathbf{x}_1$ is a "steady state" that doesn't change (because $\lambda_1 = 1$).

The eigenvector $\mathbf{x}_2$ is a "decaying mode" that virtually disappears (because $\lambda_2 = 0.5$).

The higher power of $\mathbf{A}$, the closer its columns approach the steady state.

Markov matrix

- Its entries are positive and every column adds to 1 ⇨ largest eigenvalue is $\lambda = 1$.
- Its eigenvector $\mathbf{x}_1 = (0.6, 0.4)$ is the steady state - which all columns of $\mathbf{A}^k$ will approach.

Markov Matrix

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}, \quad \mathbf{x}_1 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}, \; \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{A}^n \mathbf{x}_1 = \mathbf{x}_1, \quad \mathbf{A}^n \mathbf{x}_2 = \frac{1}{2}^n \mathbf{x}_2$$

Q ?

$$\text{find} : \mathbf{A}^{100} \begin{bmatrix} 3.8 \\ -0.8 \end{bmatrix}$$

Markov Matrix

always $\lambda_1 = 1,\ |\lambda_2| < 1$

Given : Markov matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} a & b \\ 1-a & 1-b \end{bmatrix}$$

$$0 \le a,\ b < 1$$

Find : The eigenvalues of $\mathbf{A}$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} a-\lambda & b \\ 1-a & 1-b-\lambda \end{vmatrix}$$

$$= (a-\lambda)(1-b-\lambda) - b(1-a)$$

$$= \lambda^2 - (1+a-b)\lambda + a - ab - b + ab$$

$$= \lambda^2 - (1+a-b)\lambda + a - b$$

$$= \lambda^2 - (1+a-b)\lambda + 1\cdot(a-b)$$

$$= (\lambda-1)(\lambda-(a-b))$$

$$(\lambda-1)(\lambda-(a-b)) = 0$$

$$\therefore \lambda_1 = 1,\ \lambda_2 = a-b$$

$$0 < a < 1$$

$$0 < b < 1$$

$$0 - 1 = -1 < \lambda_2 = a - b < 1 = 1 - 0$$

$$\therefore \lambda_1 = 1,\ |\lambda_2| < 1$$

Matrix as a Transformation

Solving linear systems

$$x_1 + 2x_2 + x_3 = 1$$
$$0 \cdot x_1 + x_2 + 3x_3 = 5$$
$$0 \cdot x_1 + 2x_2 + 6x_3 = 10$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix} \qquad \mathbf{Ax} = \mathbf{B}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 3 & 5 \\ 0 & 2 & 6 & 10 \end{array}\right] \qquad [\mathbf{A} \mid \mathbf{B}]$$

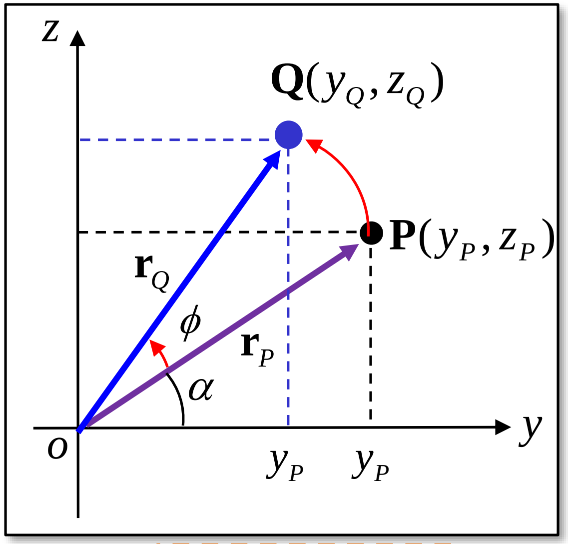
$$\mathbf{y} = \mathbf{Az}$$

$\mathbf{z}$: input

$\mathbf{y}$: output

$\mathbf{A}$: Transformation

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$



$$\begin{bmatrix} y_Q \\ z_Q \end{bmatrix} = \boxed{\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}} \begin{bmatrix} y_P \\ z_P \end{bmatrix}$$

점의 회전 변환

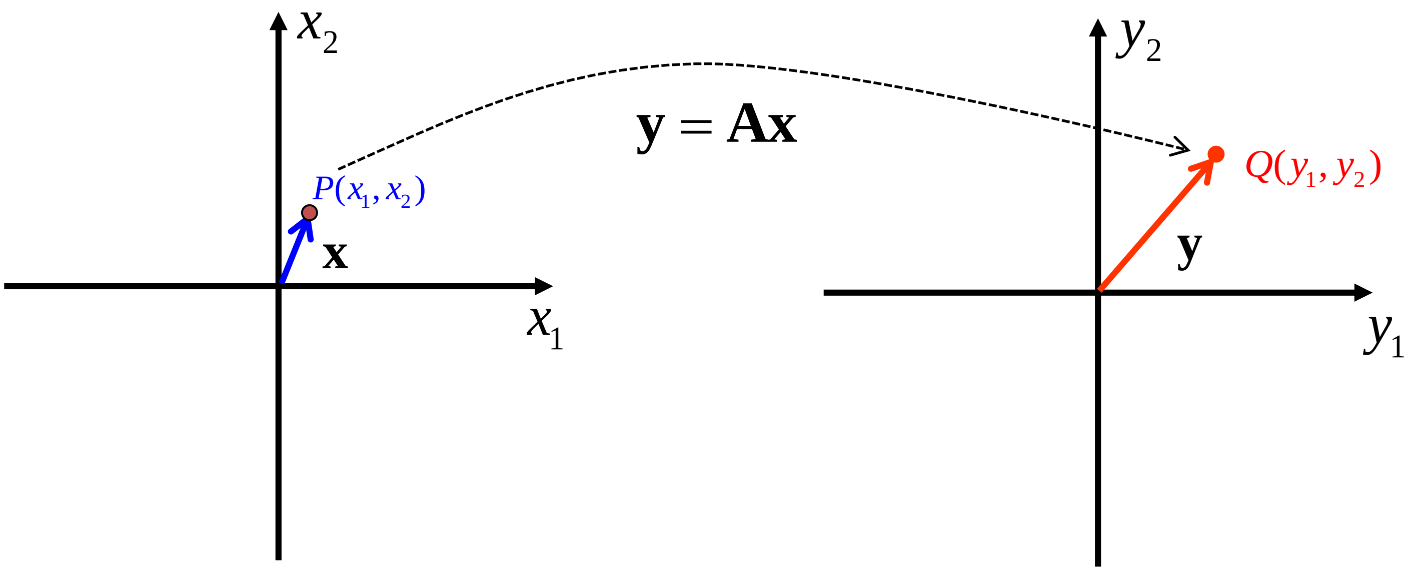
$$\mathbf{Ax} = \lambda \mathbf{x}$$

To see transformation properties

Stretching of an Elastic Membrane (탄성막의 팽창)

An elastic membrane on the x_1x_2-plane with boundary circle $x_1^2+x_2^2=1$ is stretched so that a point $P(x_1, x_2)$ goes over into the point $Q(y_1, y_2)$ given by

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Ax} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



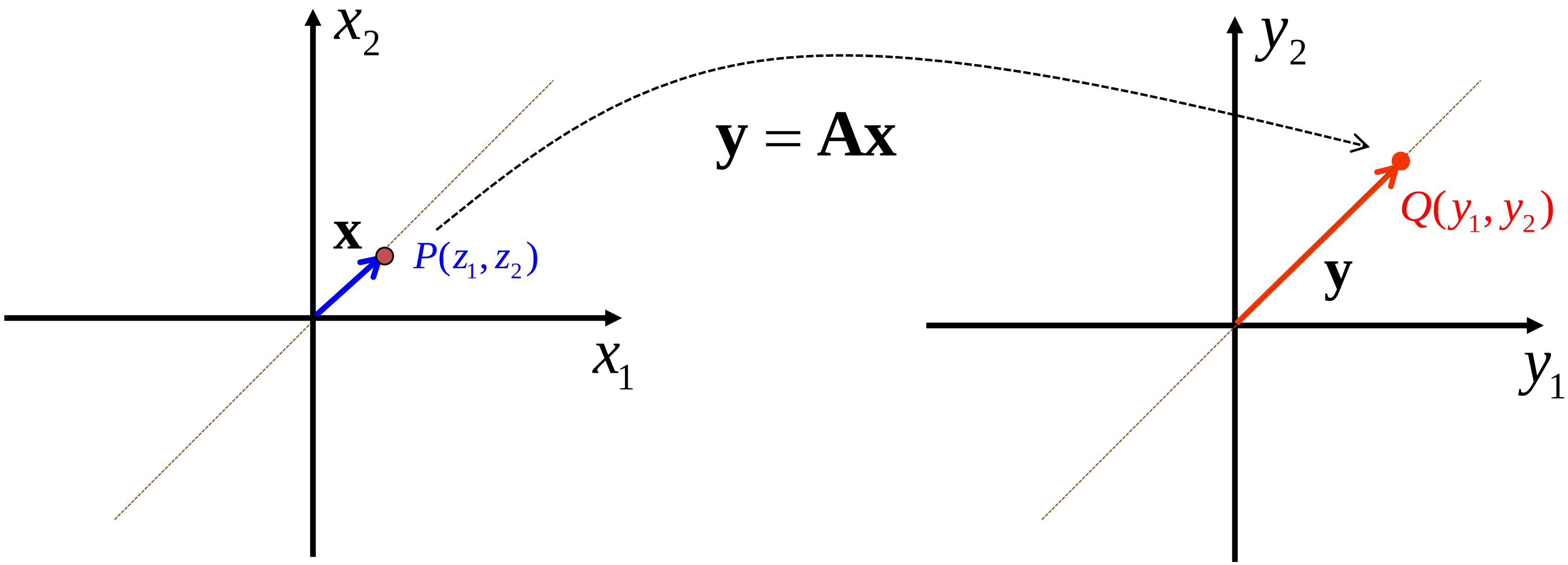
continues…

Stretching of an Elastic Membrane

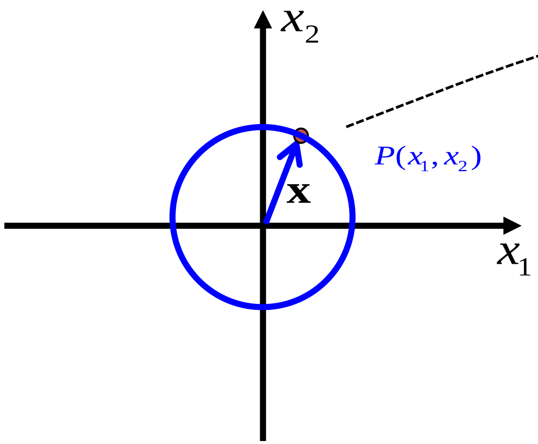
Find the principle directions (주방향):

the directions of the position vector $\mathbf{x}$ *of* P *for which the directions of the position vector* $\mathbf{y}$ *of* Q *is the same or exactly opposite.*

What shape does the boundary circle take under this deformation?



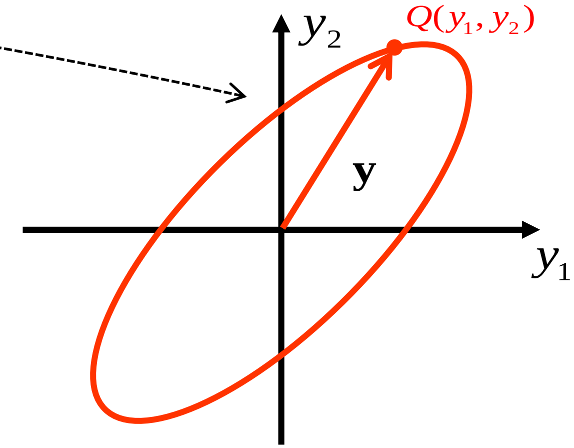
Stretching of an Elastic Membrane



Given:

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Ax} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



Transformed:

$$x_1^2 + x_2^2 = 1$$

$$\begin{cases} y_1 = 5x_1 + 3x_2 \\ y_2 = 3x_1 + 5x_2 \end{cases}$$

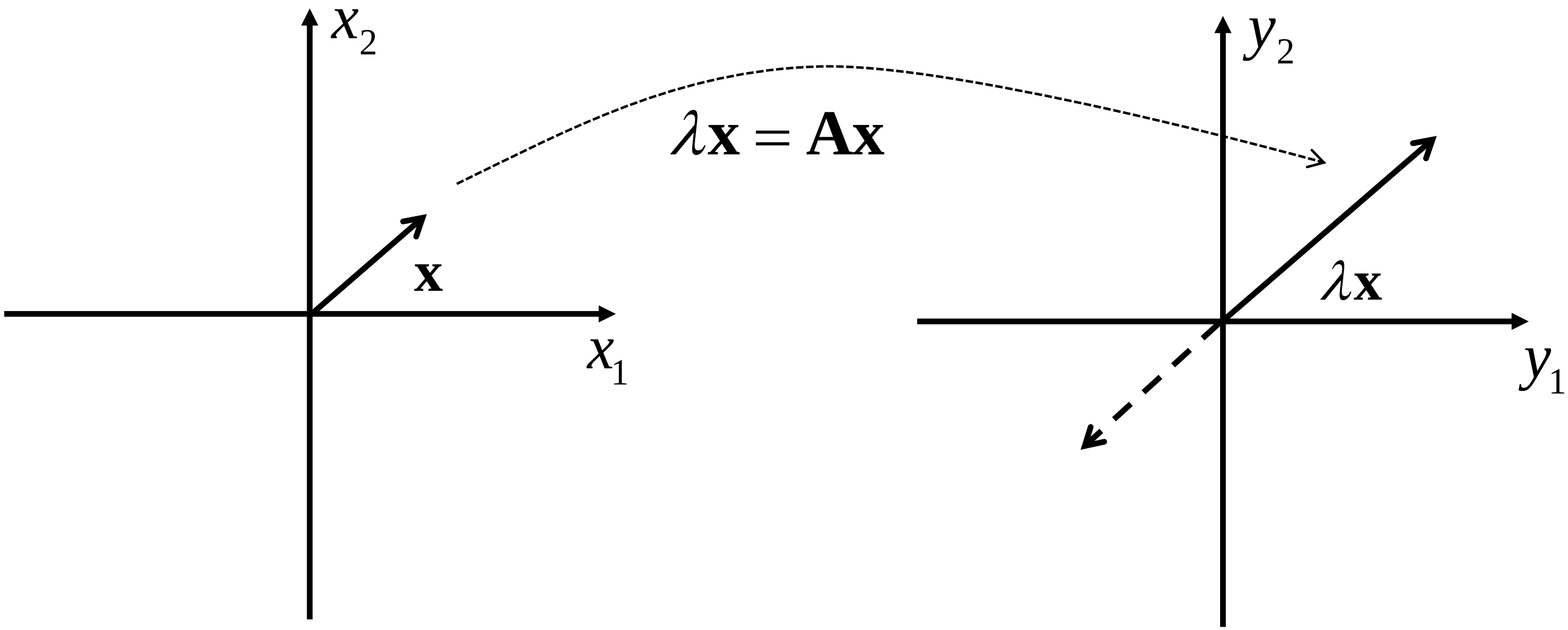
$$\begin{cases} x_1 = \dfrac{1}{16}(5y_1 - 3y_2) \\ x_2 = \dfrac{1}{16}(-3y_1 + 5y_2) \end{cases}$$

$$\frac{1}{16^2}(5y_1 - 3y_2)^2 + \frac{1}{16^2}(-3y_1 + 5y_2)^2 = 1$$

$$34y_1^2 - 60y_1y_2 + 34y_2^2 = 256$$

Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector



Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

The direction of $\mathbf{x}$ *after the stretch*

= *the same direction before the stretch*.

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

Characteristic Equation:

$$\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix}=0$$

$$\lambda^2-10\lambda+25-9=\lambda^2-10\lambda+16$$

$$=(\lambda-8)(\lambda-2)=0$$

$$\therefore \lambda_1=8,\ \lambda_2=2$$

(1) $\lambda=\lambda_1=8$

$$-3x_1+3x_2=0$$

$$3x_1-3x_2=0$$

$$\therefore x_1=x_2$$

For instance, $x_1=x_2=1$

$$\mathbf{x}_1=[1\quad 1]^{\mathbf{T}}$$

$$\mathbf{Ax}_1=\begin{bmatrix}5 & 3\\ 3 & 5\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}8\\8\end{bmatrix}$$

$$=8\begin{bmatrix}1\\1\end{bmatrix}=\lambda_1\mathbf{x}_1$$

Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

The direction of $\mathbf{x}$ *after the stretch*

= *the same direction before the stretch*.

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

Characteristic Equation:

$$\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix}=0$$

$$\lambda^2-10\lambda+25-9=\lambda^2-10\lambda+16$$

$$=(\lambda-8)(\lambda-2)=0$$

$$\therefore \lambda_1=8,\ \lambda_2=2$$

(2) $\lambda=\lambda_2=2$

$$3x_1+3x_2=0$$

$$3x_1+3x_2=0$$

$$\therefore x_1=-x_2$$

For instance, $x_1=1,\ x_2=-1$

$$\mathbf{x}_2=\begin{bmatrix}1 & -1\end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{Ax}_2=\begin{bmatrix}5 & 3\\ 3 & 5\end{bmatrix}\begin{bmatrix}1\\ -1\end{bmatrix}=\begin{bmatrix}2\\ -2\end{bmatrix}$$

$$=2\begin{bmatrix}1\\ -1\end{bmatrix}=\lambda_2\mathbf{x}_2$$

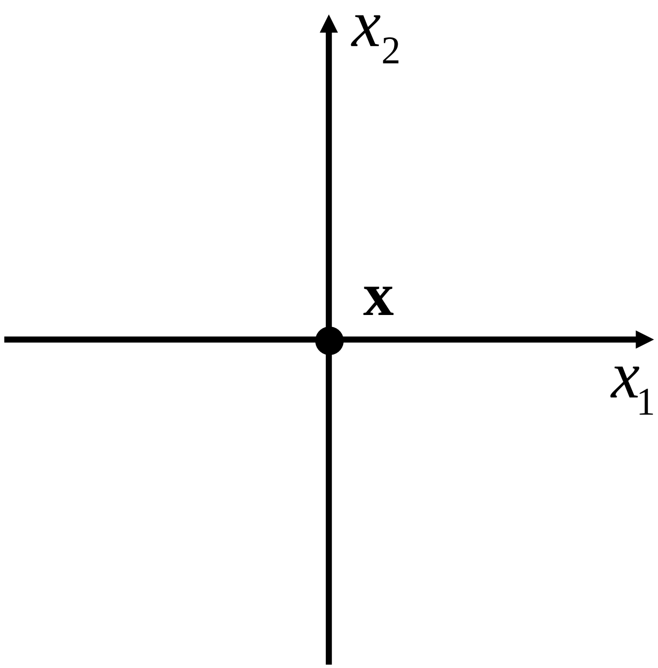
Transformation and Trivial Solution

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{A}\mathbf{x} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Find the <u>principle directions</u>:

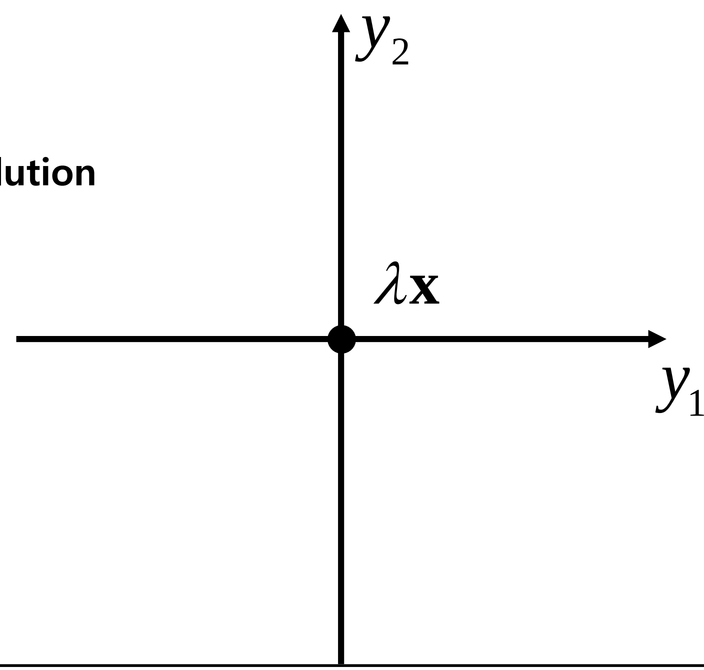
the directions of the position vector $\mathbf{x}$ of P for which the directions of the position vector $\mathbf{y}$ of Q is the same or exactly opposite.

What shape does the boundary circle take under this deformation?



$$\lambda \mathbf{x} = \mathbf{A}\mathbf{x}$$

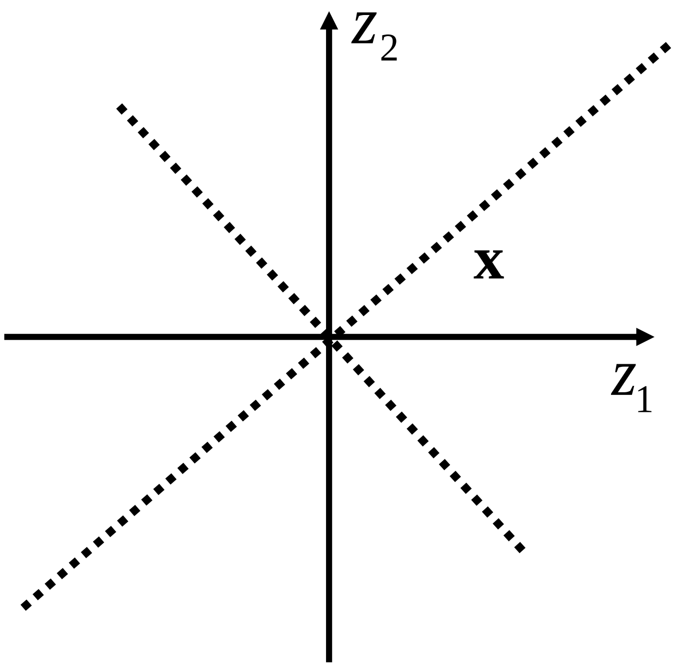
$\mathbf{x} = \mathbf{0}$:Trivial solution



Transformation and Trivial Solution

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{A}\mathbf{x} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Find the principle directions:

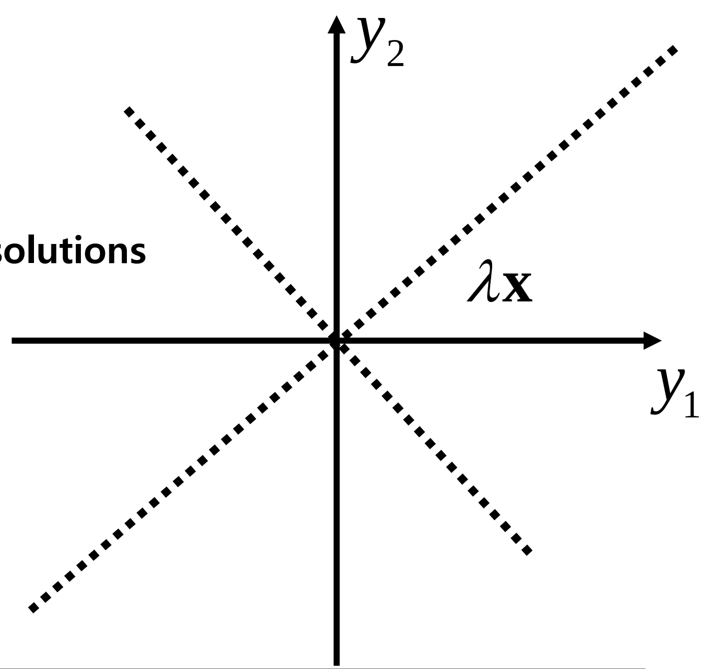


$$\lambda \mathbf{x} = \mathbf{A}\mathbf{x}$$

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0$$

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0$$

$\mathbf{x}$ **:Nontrivial many solutions**



Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

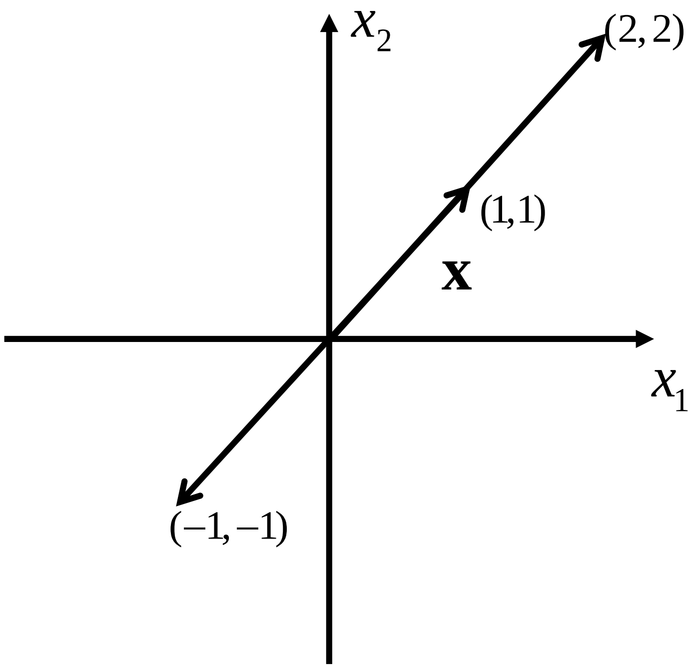
When $\mathbf{Ax}=\lambda\mathbf{x}$.

The direction of $\mathbf{x}$ *after the stretch*

= *the same direction before the stretch*.

1) $\lambda_1 = 8$

$$\mathbf{Ax}_1 = \lambda_1 \mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

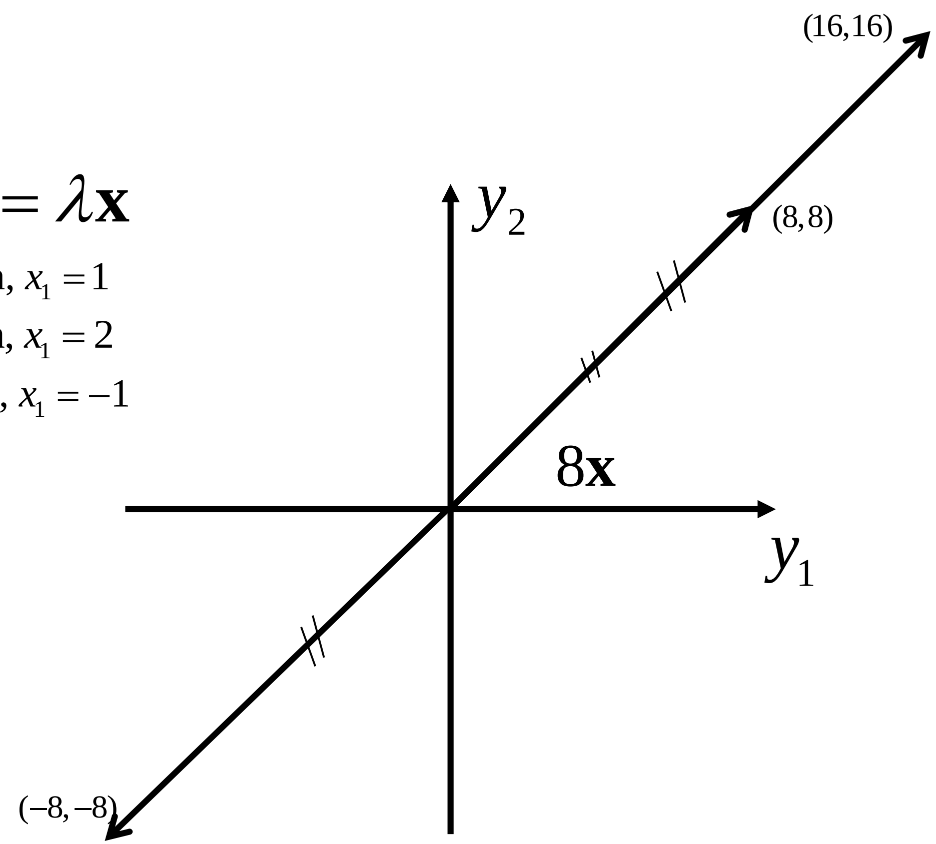


$$\mathbf{Ax} = \lambda\mathbf{x}$$

when, $x_1 = 1$

when, $x_1 = 2$

when, $x_1 = -1$



Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

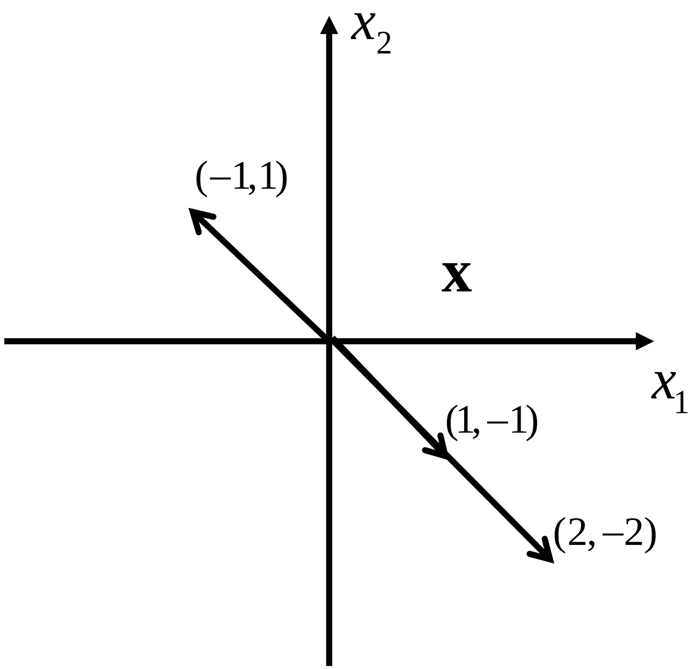
When $\mathbf{Ax}=\lambda\mathbf{x}$.

The direction of $\mathbf{x}$ after the stretch

= *the same direction before the stretch*.

1) $\lambda_1 = 2$

$$\mathbf{Ax}_2 = \lambda_2\mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

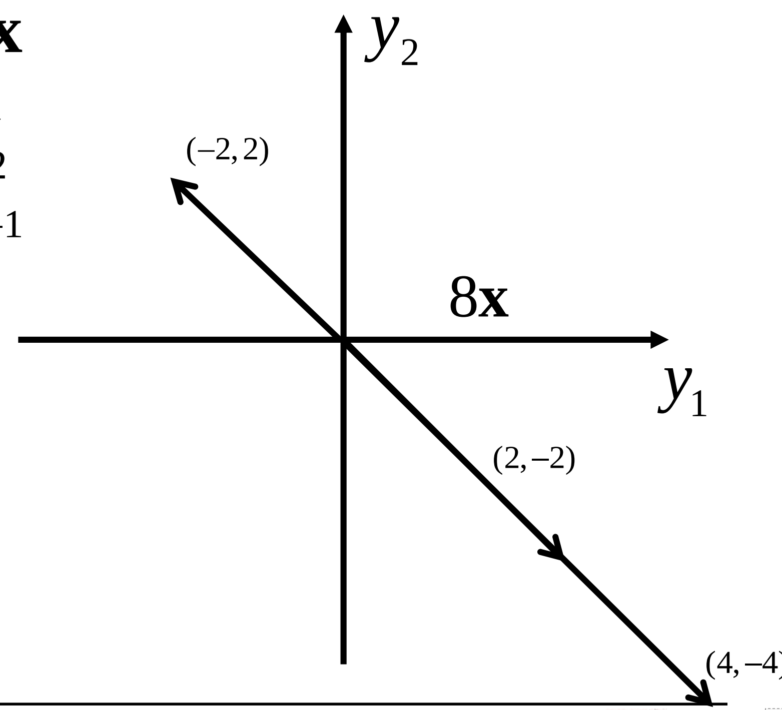


$$\mathbf{Ax} = \lambda\mathbf{x}$$

when, $x_1 = 1$

when, $x_1 = 2$

when, $x_1 = -1$



Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

$$\mathbf{A}\mathbf{x}_1 = \lambda_1\mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

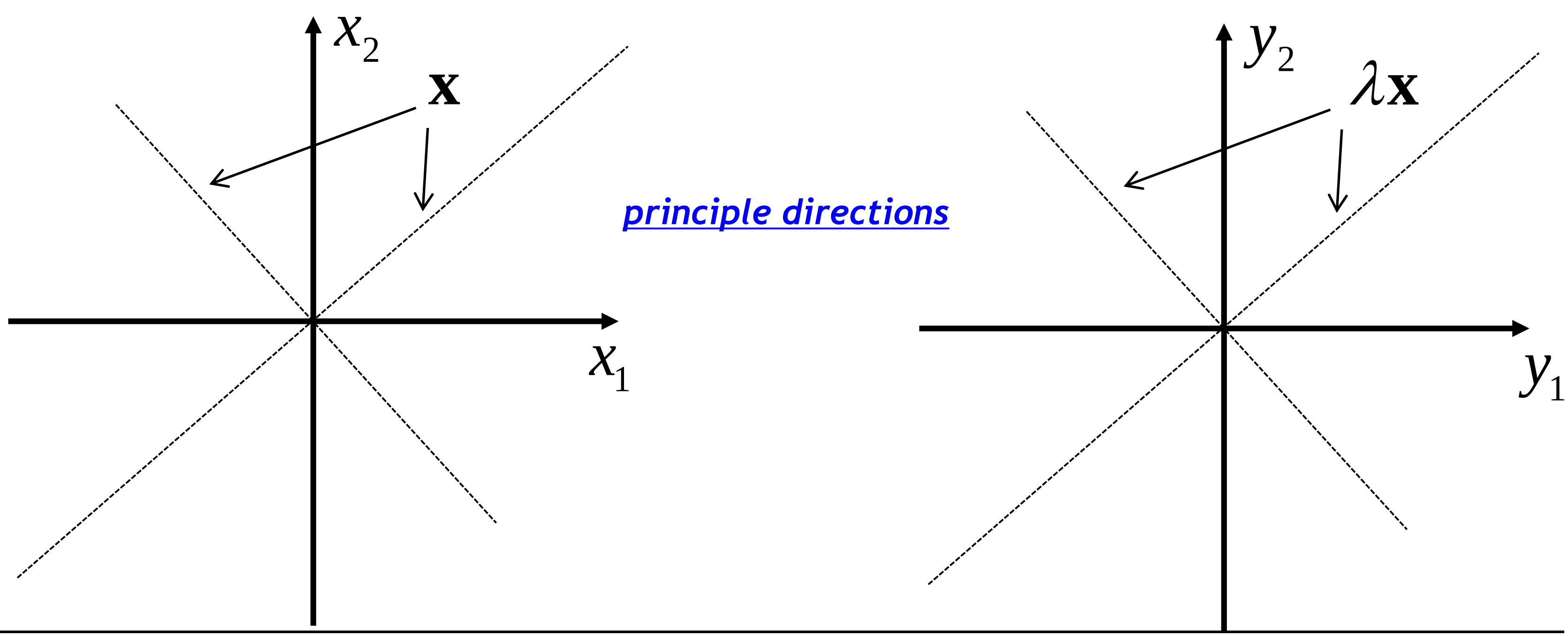
$$\mathbf{A}\mathbf{x}_2 = \lambda_2\mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

When $\mathbf{Ax}=\lambda\mathbf{x}$.

The direction of $\mathbf{x}$ after the stretch

= *the same direction before the stretch*.

$$\lambda\mathbf{x} = \mathbf{A}\mathbf{x}$$



Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

$$\mathbf{A}\mathbf{x}_1 = \lambda_1 \mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

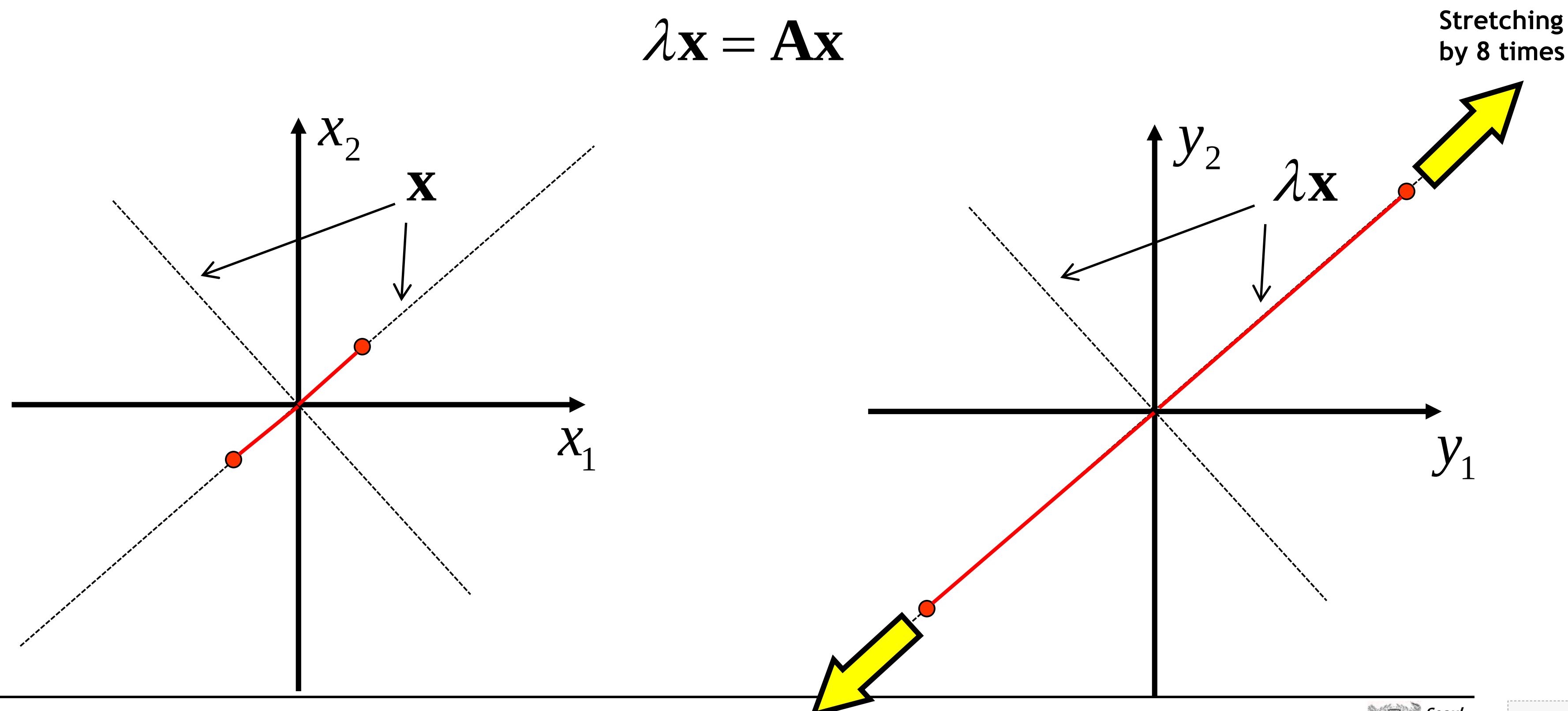
$$\mathbf{A}\mathbf{x}_2 = \lambda_2 \mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

When $\mathbf{Ax}=\lambda\mathbf{x}$.

The direction of $\mathbf{x}$ *after the stretch*

= *the same direction before the stretch*.

$$\lambda\mathbf{x} = \mathbf{A}\mathbf{x}$$



Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

$$\mathbf{A}\mathbf{x}_1 = \lambda_1 \mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

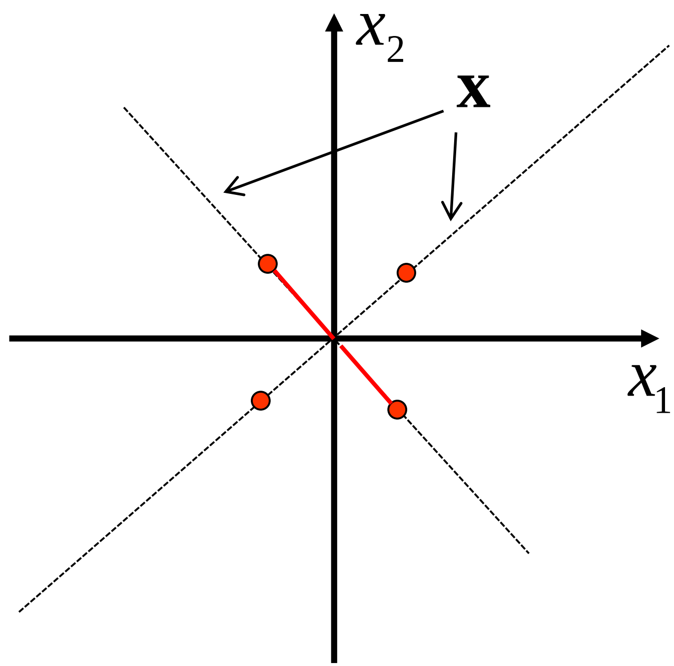
$$\mathbf{A}\mathbf{x}_2 = \lambda_2 \mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

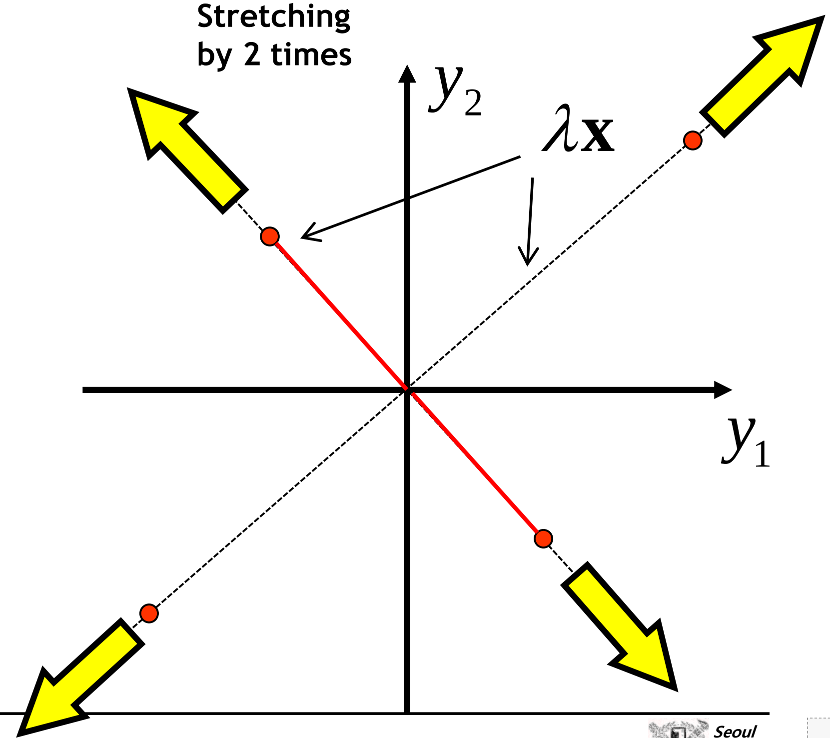
When $\mathbf{Ax}=\lambda\mathbf{x}$.

The direction of $\mathbf{x}$ *after the stretch*

= *the same direction before the stretch*.

$$\lambda\mathbf{x} = \mathbf{A}\mathbf{x}$$





Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

$$\mathbf{A}\mathbf{x}_1 = \lambda_1 \mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

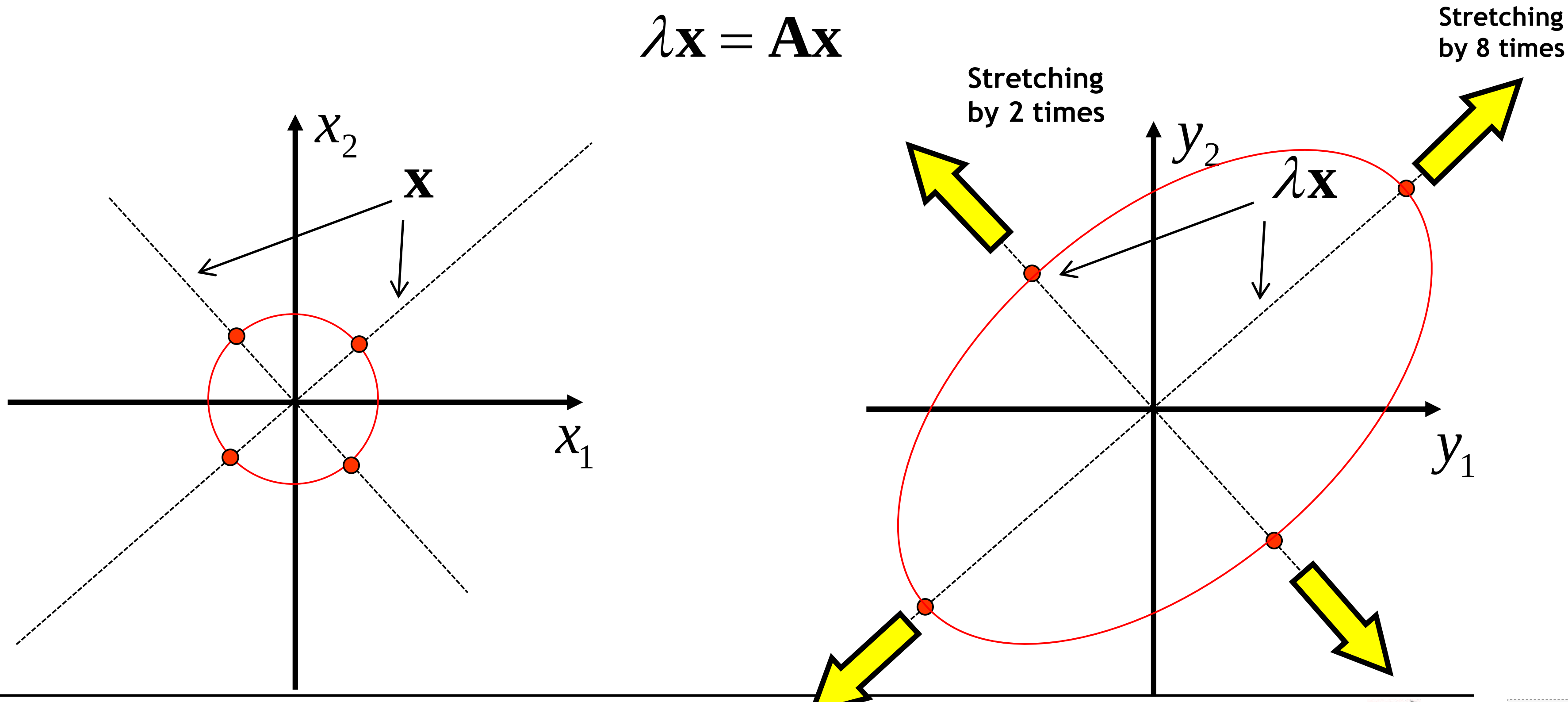
$$\mathbf{A}\mathbf{x}_2 = \lambda_2 \mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

When $\mathbf{Ax}=\lambda\mathbf{x}$.

The direction of $\mathbf{x}$ *after the stretch*

= *the same direction before the stretch*.

$$\lambda \mathbf{x} = \mathbf{A}\mathbf{x}$$



Stretching of an Elastic Membrane

So

$$\lambda_1 = 8, \ \mathbf{x}_1 = [1 \quad 1]^{\mathrm{T}}$$

➔ stretching by $8(=\lambda_1)$ *times* to the direction of $\mathbf{x}_1 = [1 \ \ 1]^{\mathrm{T}}$.

$$\lambda_2 = 2, \ \mathbf{x}_2 = [1 \quad -1]^{\mathrm{T}}$$

➔ stretching by $2(=\lambda_2)$ *times* to the direction of $\mathbf{x}_2 = [1 \ \ -1]^{\mathrm{T}}$.

Principal directions.

This vector make 45° and 135° angles with the positive x_1-direction.

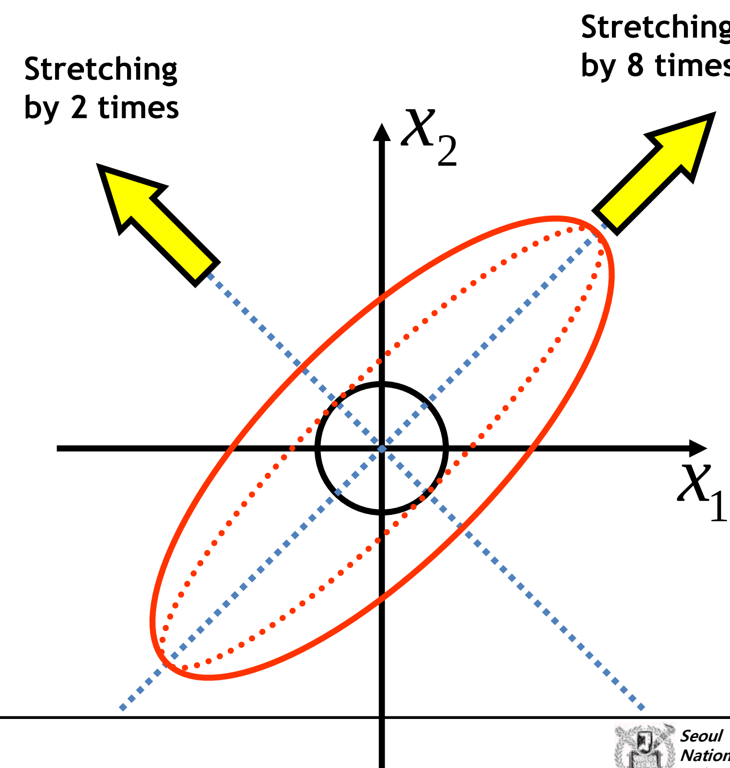
$$\mathbf{x}_1^{\mathrm{T}}\mathbf{x}_2 = [1 \quad 1]\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

these eigenvectors of a symmetric matrix corresponding to different eigenvalues are orthogonal

A real square matrix is orthogonal

if and only if column vectors $a_1, \ldots, a_n$ form an orthonormal system,

$$\mathbf{a}_j \cdot \mathbf{a}_k = \mathbf{a}_j^{\mathrm{T}}\mathbf{a}_k = \begin{cases} 0 & \text{if} \quad j \neq k \\ 1 & \text{if} \quad j = k \end{cases}$$



Stretching of an Elastic Membrane

$$\mathbf{Ax} = \lambda \mathbf{x}$$

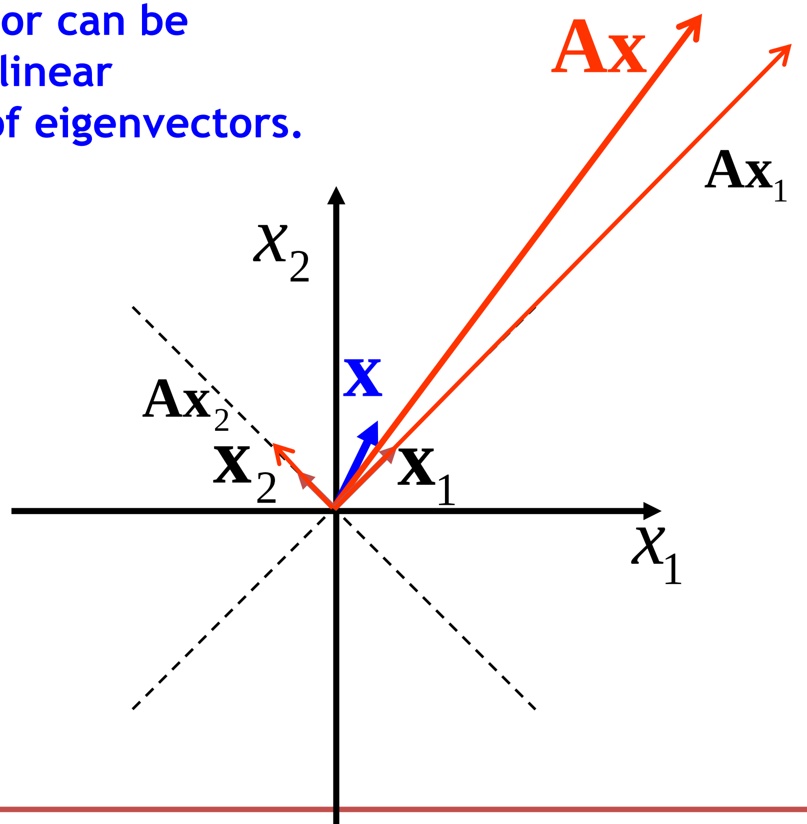
$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Ax} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\lambda_1 = 8, \ \mathbf{x}_1 = [1 \quad 1]^{\mathbf{T}}$$

$$\lambda_2 = 2, \ \mathbf{x}_2 = [1 \quad -1]^{\mathbf{T}}$$

$$\boxed{\mathbf{x} = \alpha\, \mathbf{x}_1 + \beta\, \mathbf{x}_2}$$

Arbitrary vector can be expressed by linear combination of eigenvectors.



Denoting the corresponding eigenvalues of the matrix A by $\lambda_1, \ldots, \lambda_n$, we have $\mathbf{A}\mathbf{x}_j = \lambda_j \mathbf{x}_j$, so that we simply obtain

$$\begin{aligned} \mathbf{y} = \mathbf{Ax} &= \mathbf{A}\left(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n\right) \\ &= c_1\mathbf{A}\mathbf{x}_1 + c_2\mathbf{A}\mathbf{x}_2 + \cdots + c_n\mathbf{A}\mathbf{x}_n \\ &= c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 + \cdots + c_n\lambda_n\mathbf{x}_n \end{aligned}$$

Stretching of an Elastic Membrane

$\mathbf{Ax} = \lambda \mathbf{x}$

$$\boxed{\mathbf{x} = \alpha\,\mathbf{x}_1 + \beta\,\mathbf{x}_2}$$

$$\mathbf{Ax} = \mathbf{A}(\alpha\,\mathbf{x}_1 + \beta\,\mathbf{x}_2)$$

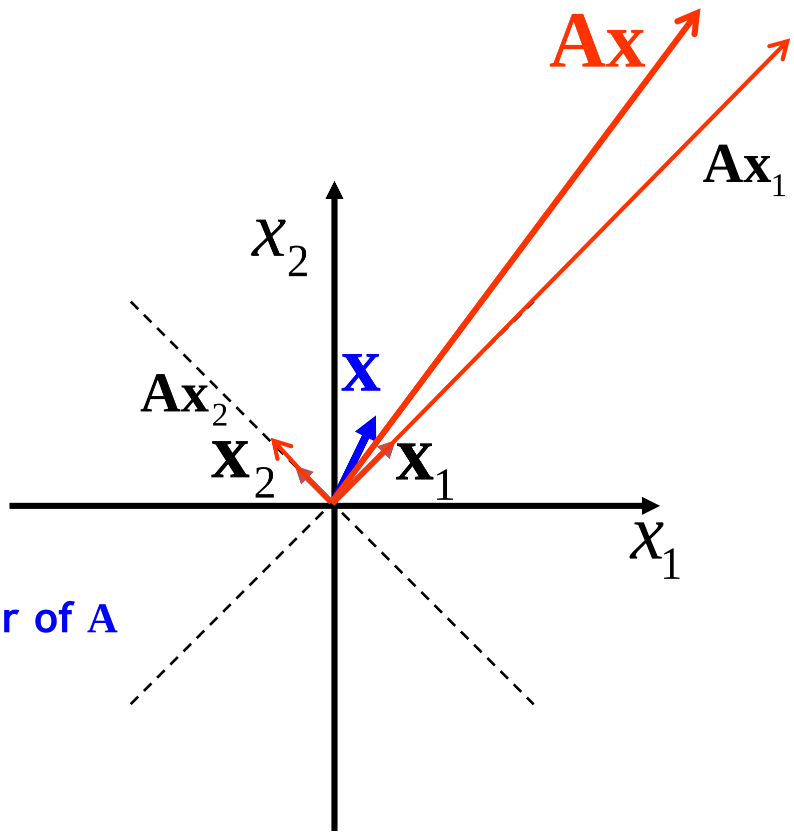
$$= \alpha\mathbf{A}\,\mathbf{x}_1 + \beta\,\mathbf{A}\mathbf{x}_2$$

$\mathbf{x}_1$, $\mathbf{x}_2$ is eigenvector of $\mathbf{A}$

$$= \alpha\lambda_1\,\mathbf{x}_1 + \beta\lambda_2\,\mathbf{x}_2$$

$$= 8\alpha\,\mathbf{x}_1 + 2\beta\,\mathbf{x}_2$$

$$\boxed{\therefore\ \mathbf{Ax} = 8\alpha\,\mathbf{x}_1 + 2\beta\,\mathbf{x}_2}$$



$\mathbf{Ax}$ — complicated action of $\mathbf{A}$ on an arbitrary vector $\mathbf{x}$

decompose ⇒

$8\alpha\,\mathbf{x}_1 + 2\beta\,\mathbf{x}_2$ — Sum of simple actions (multiplication by scalars) on the eigenvectors of $\mathbf{A}$

Stretching of an Elastic Membrane

$\mathbf{Ax} = \lambda \mathbf{x}$

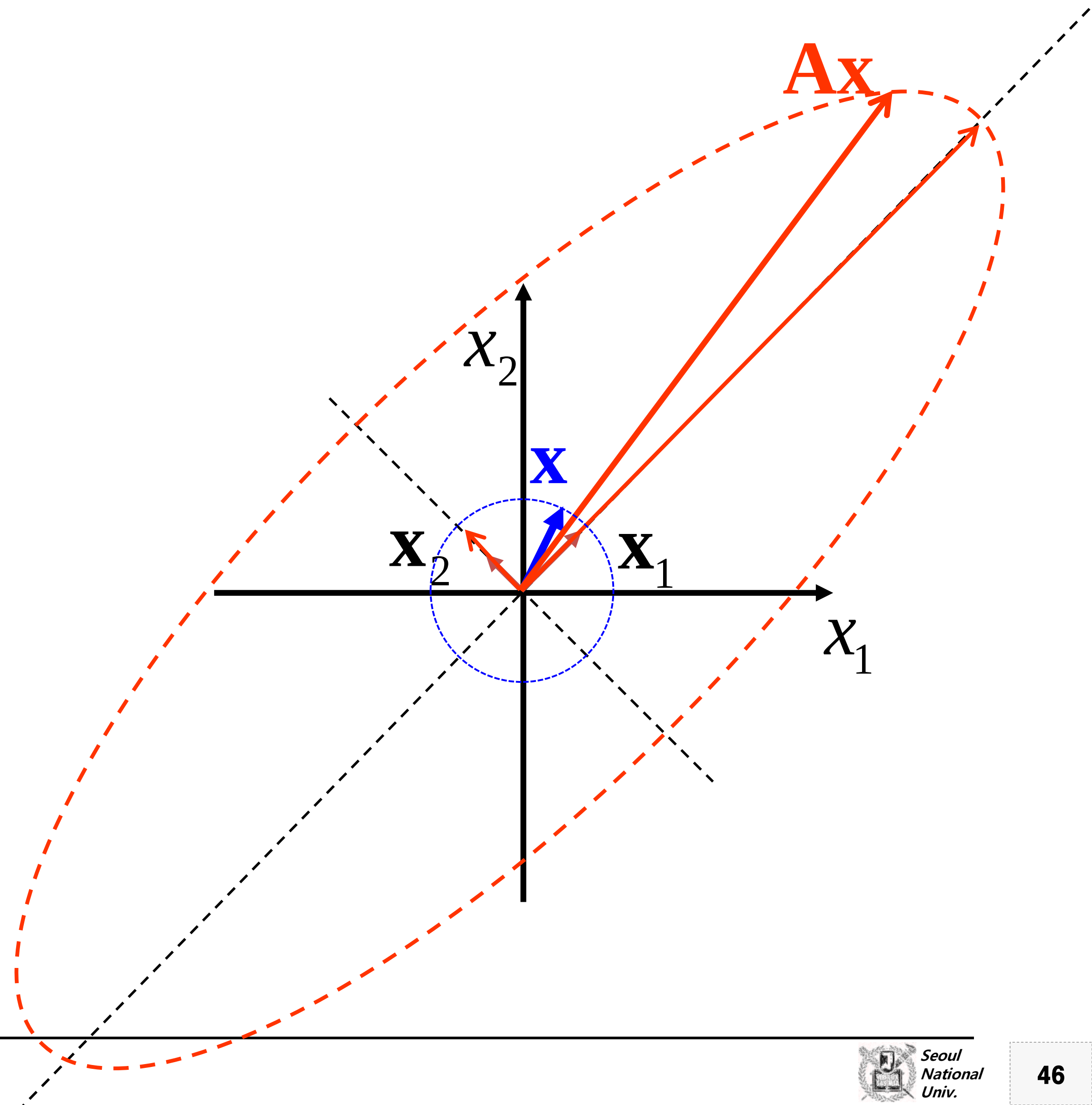
$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Ax} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\lambda_1 = 8, \ \mathbf{x}_1 = [1 \quad 1]^{\mathbf{T}}$$

$$\lambda_2 = 2, \ \mathbf{x}_2 = [1 \quad -1]^{\mathbf{T}}$$

$$\boxed{\mathbf{x} = \alpha \, \mathbf{x}_1 + \beta \, \mathbf{x}_2}$$

$$\boxed{\therefore \mathbf{Ax} = 8\alpha \, \mathbf{x}_1 + 2\beta \, \mathbf{x}_2}$$



Stretching of an Elastic Membrane

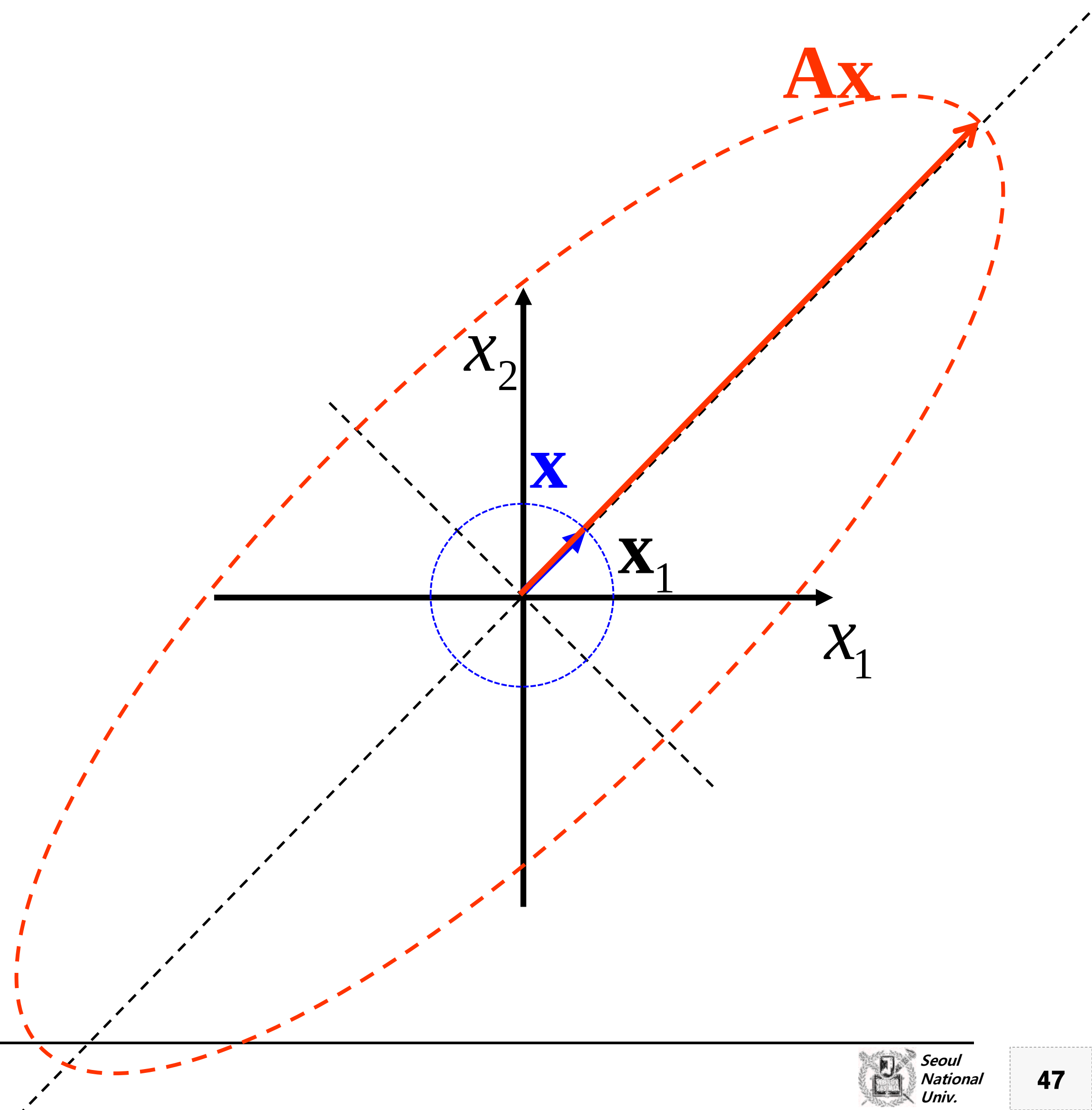
$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{A}\mathbf{x} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\lambda_1 = 8, \ \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathbf{T}}$$

$$\lambda_2 = 2, \ \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathbf{T}}$$

$$\boxed{\mathbf{x} = \alpha\,\mathbf{x}_1 + \beta\,\mathbf{x}_2}$$

$$\boxed{\therefore \mathbf{A}\mathbf{x} = 8\alpha\,\mathbf{x}_1 + 2\beta\,\mathbf{x}_2}$$



Stretching of an Elastic Membrane

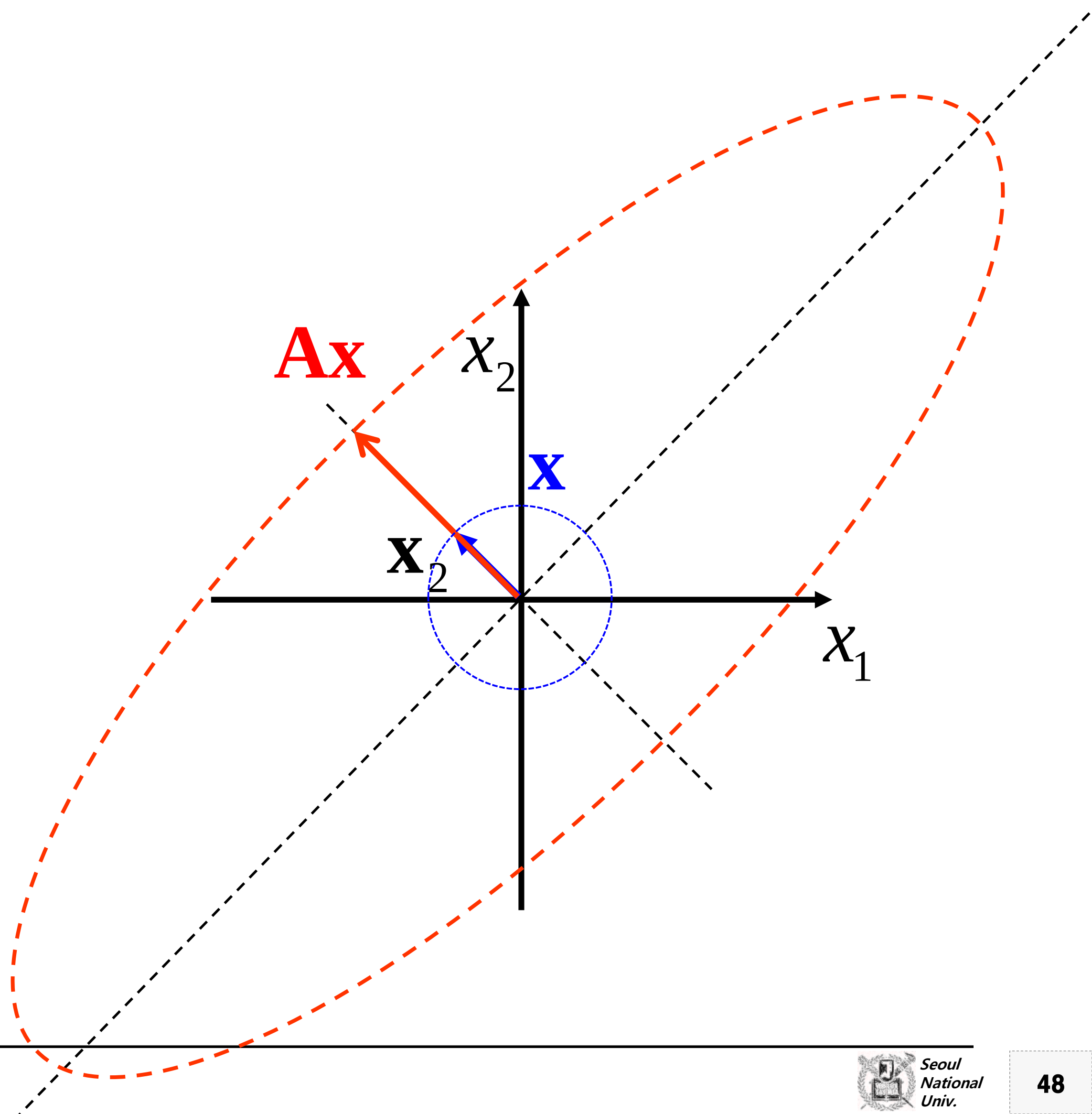
$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Ax} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\lambda_1 = 8, \ \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \ \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$\boxed{\mathbf{x} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2}$$

$$\boxed{\therefore \mathbf{Ax} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2}$$



8-3. Symmetric, Skew-Symmetric, and Orthogonal Matrices

Symmetric, Skew-Symmetric, and Orthogonal Matrices

A *real* square matrix $\mathbf{A} = [a_{jk}]$ is called

symmetric if transposition leaves it unchanged
(대칭)

$$\mathbf{A}^{\mathrm{T}} = \mathbf{A}, \qquad \text{thus} \quad a_{kj} = a_{jk}$$

skew-symmetric if transposition gives the negative of $\mathbf{A}$
(반대칭)

$$\mathbf{A}^{\mathrm{T}} = -\mathbf{A}, \qquad \text{thus} \quad a_{kj} = -a_{jk}$$

orthogonal if transposition gives the inverse of $\mathbf{A}$
(직교의)

$$\mathbf{A}^{\mathrm{T}} = \mathbf{A}^{-1}$$

Example 8.3-1 (1)
(Symmetric, Skew-Symmetric, and Orthogonal Matrices)

The matrices

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}, \quad \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}, \quad \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

[symmetric] [skew-symmetric] [orthogonal]

Every skew-symmetric matrix has all main diagonal entries zero.

Example 8.3-1 (2)

(Symmetric, Skew-Symmetric, and Orthogonal Matrices)

Symmetric: $A^T=A$
Skew-symmetric: $A^T=-A$
Orthogonal: $A^T=A^{-1}$

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}^T = \begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

➔ **symmetric**

$$\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & -9 & 12 \\ 9 & 0 & -20 \\ -12 & 20 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$$

➔ **skew-symmetric**

Example 8.3-1 (3)
(Symmetric, Skew-Symmetric, and Orthogonal Matrices)

Symmetric: $\mathbf{A}^T=\mathbf{A}$
Skew-symmetric: $\mathbf{A}^T=-\mathbf{A}$
Orthogonal: $\mathbf{A}^T=\mathbf{A}^{-1}$

$$\mathbf{A}=\begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}, \quad \mathbf{A}^{\mathbf{T}}=\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}^{\mathbf{T}}=\begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

$$\mathbf{A}\cdot\mathbf{A}^{\mathbf{T}}=\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}\begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore \mathbf{A}^{\mathbf{T}}=\mathbf{A}^{-1}$ ➔ **orthogonal**

Example 8.3-1 (4)
(Symmetric, Skew-Symmetric, and Orthogonal Matrices)

Symmetric: $\mathbf{A}^T=\mathbf{A}$
Skew-symmetric: $\mathbf{A}^T=-\mathbf{A}$
Orthogonal: $\mathbf{A}^T=\mathbf{A}^{-1}$

From definitions skew-symmetric matrix is

$$\mathbf{A}^{\mathrm{T}} = -\mathbf{A}, \qquad \text{thus} \quad a_{kj} = -a_{jk}$$

$\boxed{\text{Q? Prove all main diagonal entries zero.}}$

Example 8.3-2 (1)

Any real square matrix $\mathbf{A}$ may be written as the sum of a symmetric matrix $\mathbf{R}$ and a skew-symmetric matrix $\mathbf{S}$, where

$$\mathbf{R} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^{\mathrm{T}}\right), \qquad \mathbf{S} = \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^{\mathrm{T}}\right)$$

$$\mathbf{A} = \begin{bmatrix} 9 & 5 & 2 \\ 2 & 3 & -8 \\ 5 & 4 & 3 \end{bmatrix} \qquad \mathbf{R} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^{\mathrm{T}}\right) = \begin{bmatrix} 9.0 & 3.5 & 3.5 \\ 3.5 & 3.0 & -2.0 \\ 3.5 & -2.0 & 3.0 \end{bmatrix}$$

$$\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 9 & 2 & 5 \\ 5 & 3 & 4 \\ 2 & -8 & 3 \end{bmatrix} \qquad \mathbf{S} = \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^{\mathrm{T}}\right) = \begin{bmatrix} 0 & 1.5 & -1.5 \\ -1.5 & 0 & -6.0 \\ 1.5 & 6.0 & 0 \end{bmatrix}$$

Example 8.3-2 (2)

$$\mathbf{R} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^{\mathbf{T}}\right), \qquad \mathbf{S} = \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^{\mathbf{T}}\right)$$

$$\mathbf{A} = \begin{bmatrix} 9 & 5 & 2 \\ 2 & 3 & -8 \\ 5 & 4 & 3 \end{bmatrix} = \mathbf{R} + \mathbf{S}$$

$$= \begin{bmatrix} 9.0 & 3.5 & 3.5 \\ 3.5 & 3.0 & -2.0 \\ 3.5 & -2.0 & 3.0 \end{bmatrix} + \begin{bmatrix} 0 & 1.5 & -1.5 \\ -1.5 & 0 & -6.0 \\ 1.5 & 6.0 & 0 \end{bmatrix}$$

Eigenvalues of Symmetric and Skew-Symmetric Matrices

Theorem 8.1 Eigenvalues of Symmetric and Skew-Symmetric Matrices

(a) The eigenvalues of a symmetric matrix are *real*

(b) The eigenvalues of a skew-symmetric matrix are *pure imaginary or zero.*

Example 8.3-3
(Eigenvalues of Symmetric and Skew-Symmetric Matrices)

From example 8.2-1

$$\mathbf{A} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

➔ **symmetric**

Characteristic Equation:

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)^2 - 3 = 0$$

$$\therefore \lambda = 2,\ \ 8$$ ➔ **real**

From example 8.3-1

$$\mathbf{A} = \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$$

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -\lambda & 9 & -12 \\ -9 & -\lambda & 20 \\ 12 & -20 & -\lambda \end{vmatrix} = 0$$

$$-\lambda(\lambda^2 + 400) - 9(9\lambda - 240) - 12(180 + 12\lambda) = 0$$

$$\lambda^3 + 625\lambda = 0$$

$$\therefore \lambda = 0,\ \ \pm 25i$$ ➔ **imaginary**

Orthogonal Transformations and Orthogonal Matrices

Orthogonal Transformations (직교 변환) are transformations

$$\mathbf{y} = \mathbf{A}\mathbf{x}$$ where $\mathbf{A}$ is an orthogonal matrix (직교 행렬).

With each vector $\mathbf{x}$ in R^n such a transformation assigns a vector $\mathbf{y}$ in R^n.
For instance, the plane rotation through an angle θ.

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is an orthogonal transformation.

Any orthogonal transformation in the plane or in three-dimensional space is a *rotation*.

Orthogonal Transformations and Orthogonal Matrices

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\mathbf{A}\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \mathbf{A}^{\mathrm{T}} = \mathbf{A}^{-1}$$

Invariance of Inner Product (내적의 불변)

Theorem 8.3.2 Invariance of Inner Product

An orthogonal transformation preserves the value of the inner product of vectors $\mathbf{a}$ and $\mathbf{b}$ in R^n, defined by

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{a}^{\mathrm{T}}\mathbf{b} = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

That is, for any $\mathbf{a}$ and $\mathbf{b}$ in R^n, orthogonal n x n matrix A, and $\mathbf{u} = \mathbf{A}\mathbf{a}$, $\mathbf{v} = \mathbf{A}\mathbf{b}$ we have $\mathbf{u}\cdot\mathbf{v}=\mathbf{a}\cdot\mathbf{b}$.

Hence the transformation also preserves the *length* or *norm* of any vector $\mathbf{a}$ in R^n given by

$$\|\mathbf{a}\| = \sqrt{\mathbf{a}\cdot\mathbf{a}} = \sqrt{\mathbf{a}^{\mathrm{T}}\mathbf{a}}$$

Invariance of Inner Product (Proof)

Let $\mathbf{A}$ be orthogonal. Let $\mathbf{u} = \mathbf{Aa}$ and $\mathbf{v} = \mathbf{Ab}$. We must show that $\mathbf{u} \cdot \mathbf{v} = \mathbf{a} \cdot \mathbf{b}$.

$$(\mathbf{Aa})^{\mathrm{T}} = \mathbf{a}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$$ ➔ by (10d) in Sec. 7.2

$$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$ ➔ $\mathbf{A}$ is orthogonal.

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} = \mathbf{u}^{\mathrm{T}}\mathbf{v} = (\mathbf{Aa})^{\mathrm{T}}\mathbf{Ab} &= \mathbf{a}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{Ab} \\ &= \mathbf{a}^{\mathrm{T}}\mathbf{Ib} \\ &= \mathbf{a}^{\mathrm{T}}\mathbf{b} \\ &= \mathbf{a} \cdot \mathbf{b} \end{aligned}$$

Orthonormality (정규직교성) of Column and Row Vectors

Theorem 8.3.3 Orthonormality of Column and Row Vectors

A real square matrix is orthogonal if and only if column vectors $a_1, \ldots, a_n$ (and also its row vectors) form an orthonormal system, that is,

orthogonal (orthonomal) vectors

$$\mathbf{a}_j \cdot \mathbf{a}_k = \mathbf{a}_j^{\mathbf{T}} \mathbf{a}_k = \begin{cases} 0 & if \quad j \neq k \\ 1 & if \quad j = k \end{cases}$$

$$\mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathbf{T}}$$

$$\mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathbf{T}}$$

$$\mathbf{x}_1^{\mathbf{T}} \mathbf{x}_2 = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

Let A be orthogonal. Then $A^{-1}A = A^{T}A = I$, in terms of column vector $\mathbf{a}_1, \ldots, \mathbf{a}_n$,

orthogonal matrix

$$\mathbf{I} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{A}^{\mathbf{T}}\mathbf{A} = \begin{bmatrix} \mathbf{a}_1^{\mathbf{T}} \\ \vdots \\ \mathbf{a}_n^{\mathbf{T}} \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^{\mathbf{T}}\mathbf{a}_1 & \mathbf{a}_1^{\mathbf{T}}\mathbf{a}_2 & \cdots & \mathbf{a}_1^{\mathbf{T}}\mathbf{a}_n \\ \cdot & \cdot & \cdots & \cdot \\ \mathbf{a}_n^{\mathbf{T}}\mathbf{a}_1 & \mathbf{a}_n^{\mathbf{T}}\mathbf{a}_2 & \cdots & \mathbf{a}_n^{\mathbf{T}}\mathbf{a}_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Orthonormality of Column and Row Vectors (Proof)

(a) Let $\mathbf{A}$ be orthogonal. Then $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{I}$, in terms of column vector $\mathbf{a}_1$, … , $\mathbf{a}_n$,

$$\mathbf{I} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{bmatrix} \mathbf{a}_1{}^{\mathrm{T}} \\ \vdots \\ \mathbf{a}_n{}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{a}_1{}^{\mathrm{T}}\mathbf{a}_1 & \mathbf{a}_1{}^{\mathrm{T}}\mathbf{a}_2 & \cdots & \mathbf{a}_1{}^{\mathrm{T}}\mathbf{a}_n \\ \cdot & \cdot & \cdots & \cdot \\ \mathbf{a}_n{}^{\mathrm{T}}\mathbf{a}_1 & \mathbf{a}_n{}^{\mathrm{T}}\mathbf{a}_2 & \cdots & \mathbf{a}_n{}^{\mathrm{T}}\mathbf{a}_n \end{bmatrix}$$

c.f.) Expression of a matrix-transpose in terms of column vectors

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$$

$$\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1{}^{\mathrm{T}} \\ \mathbf{a}_2{}^{\mathrm{T}} \\ \mathbf{a}_3{}^{\mathrm{T}} \end{bmatrix}$$

여기서 행렬 A는 열벡터(column vector)의 곱이 아닌 배열이므로 행렬을 transpose할 때 그 순서가 바뀌지 않음

Orthonormality of Column and Row Vectors

Q: Prove the Orthonormality of Column and Row Vector A.

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$$

Theorem 8.3.3 orthonormal system

$$\mathbf{a}_j \cdot \mathbf{a}_k = \mathbf{a}_j^{\mathbf{T}} \mathbf{a}_k = \begin{cases} 0 & if \quad j \neq k \\ 1 & if \quad j = k \end{cases}$$

$$\mathbf{a}_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$\mathbf{a}_1^{\mathbf{T}} \mathbf{a}_1 = \begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = 1, \qquad \mathbf{a}_2^{\mathbf{T}} \mathbf{a}_2 = \begin{bmatrix} -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = 1,$$

$$\mathbf{a}_1^{\mathbf{T}} \mathbf{a}_2 = \begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = 0,$$

Determinant of an Orthogonal Matrix

Theorem 8.3.4 Determinant of an Orthogonal Matrix
The determinant of an orthogonal matrix has the value +1 or -1.

Proof

From $\det \mathbf{AB} = \det \mathbf{A} \cdot \det \mathbf{B}$ (Sec. 7.8, Theorem 4) and $\det \mathbf{A}^{\mathrm{T}} = \det \mathbf{A}$ (Sec. 7.7, Theorem 2d), we get for an orthogonal matrix

$$1 = \det \mathbf{I} = \det\left(\mathbf{A}\mathbf{A}^{-1}\right) = \det\left(\mathbf{A}\mathbf{A}^{\mathrm{T}}\right)$$

$$= \det \mathbf{A} \cdot \det \mathbf{A}^{\mathrm{T}}$$

$$= \det \mathbf{A} \cdot \det \mathbf{A}$$

$$= \left(\det \mathbf{A}\right)^2$$

Example 8.3-4

From example 1

$$\mathbf{A} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

➔ **orthogonal**

$$\det \mathbf{A} = \frac{2}{3}\left(-\frac{4}{9}-\frac{2}{9}\right) - \frac{1}{3}\left(\frac{4}{9}-\frac{1}{9}\right) + \frac{2}{3}\left(-\frac{4}{9}-\frac{2}{9}\right)$$
$$= -1$$

From example 3

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

➔ **orthogonal**

$$\det \mathbf{A} = \cos^2\theta + \sin^2\theta$$
$$= 1$$

Eigenvalues of an Orthogonal Matrix

Theorem 8.3.5 Eigenvalues of an Orthogonal Matrix
The eigenvalues of an orthogonal matrix $\mathbf{A}$ are *real or complex* conjugated in pairs and *have absolute value 1*.

<u>Proof</u>

The first part of the statement holds for any real matrix A because *its characteristic polynomial has real coefficients*.
$|\lambda|=1$ ➔ proved in Sec. 8.5.

Example 8.3.5 (1)

From example 1

$$\mathbf{A} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \qquad \det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} \frac{2}{3}-\lambda & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3}-\lambda & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3}-\lambda \end{vmatrix} = 0$$

➔ **orthogonal**

$$\left(\frac{2}{3}-\lambda\right)\left[\left(\frac{2}{3}-\lambda\right)\left(-\frac{2}{3}-\lambda\right)-\frac{1}{3}\cdot\frac{2}{3}\right]-\frac{1}{3}\left[-\frac{2}{3}\left(-\frac{2}{3}-\lambda\right)-\frac{1}{3}\cdot\frac{1}{3}\right]$$

$$+\frac{2}{3}\left[-\frac{2}{3}\cdot\frac{2}{3}-\left(\frac{2}{3}-\lambda\right)\frac{1}{3}\right]=0$$

$$-\lambda^3+\frac{2}{3}\lambda^2+\frac{2}{3}\lambda-1=0$$

Example 8.3.5 (2)

$$-\lambda^3+\frac{2}{3}\lambda^2+\frac{2}{3}\lambda-1=0$$

$$(\lambda+1)\left(\lambda^2-\frac{5}{3}\lambda+1\right)=0$$

$$\therefore\ \lambda=-1,\ \ \frac{5\pm i\sqrt{11}}{6}$$

Problem Set 8.3-16 (Orthogonality)

Symmetric: $A^T=A$
Skew-symmetric: $A^T=-A$
Orthogonal: $A^T=A^{-1}$

Prove that eigenvectors of a symmetric matrix corresponding to different eigenvalues are orthogonal. Give an example.

$$\text{Let} \quad \mathbf{Ax} = \lambda \mathbf{x}, \; \mathbf{Ay} = \mu \mathbf{y}$$

where $\mathbf{x} \neq \mathbf{0}, \; \mathbf{y} \neq \mathbf{0}$ **We need to prove** $\mathbf{x}^{\mathrm{T}}\mathbf{y} = 0$

$$\textit{Thus} \quad \lambda \mathbf{x}^T = (\mathbf{Ax})^T = \mathbf{x}^T \mathbf{A}^T = \mathbf{x}^T \mathbf{A} \; (\because \mathbf{A}^{\mathrm{T}} = \mathbf{A})$$

$$\lambda \mathbf{x}^{\mathrm{T}}\mathbf{y} = \mathbf{x}^{\mathrm{T}}\mathbf{Ay} = \mathbf{x}^{\mathrm{T}}\mu\mathbf{y} = \mu\mathbf{x}^{\mathrm{T}}\mathbf{y}.$$

$$\mathbf{x}^{\mathrm{T}}\mathbf{y} = 0, \; (\because \lambda \neq \mu)$$

It proves orthogonality (직교성).

Problem Set 8.3-16 (Orthogonality)

Symmetric: $A^T=A$
Skew-symmetric: $A^T=-A$
Orthogonal: $A^T=A^{-1}$

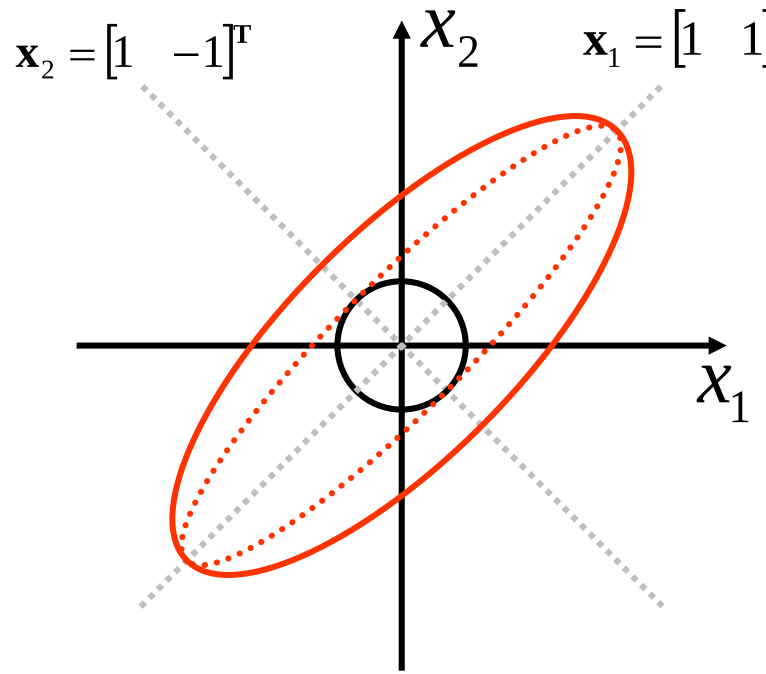
Prove that eigenvectors of a symmetric matrix corresponding to different eigenvalues are orthogonal. Give an example.

example 8.2-1

$$\mathbf{A}=\begin{bmatrix}5 & 3\\ 3 & 5\end{bmatrix},$$

$$\lambda_1=8,\ \mathbf{x}_1=\begin{bmatrix}1 & 1\end{bmatrix}^{\mathbf{T}}$$

$$\lambda_2=2,\ \mathbf{x}_2=\begin{bmatrix}1 & -1\end{bmatrix}^{\mathbf{T}}$$



$$\mathbf{x}_1\cdot\mathbf{x}_2=\mathbf{x}_1{}^{\mathbf{T}}\mathbf{x}_2=\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\ -1\end{bmatrix}=0$$

Problem Set 8.3-17 (Skew-Symmetric matrix)

Symmetric: $A^T=A$
Skew-symmetric: $A^T=-A$
Orthogonal: $A^T=A^{-1}$

Show that the inverse of a skew-symmetric matrix is skew-symmetric matrix.

Let A is a skew-symmetric matrix, and $B = A^{-1}$ then,

Q?

8.4 Eigenbases. Diagonalization. Quadratic Forms

EIGENBASES

Eigenbasis (고유벡터의 기저, 고유기저)

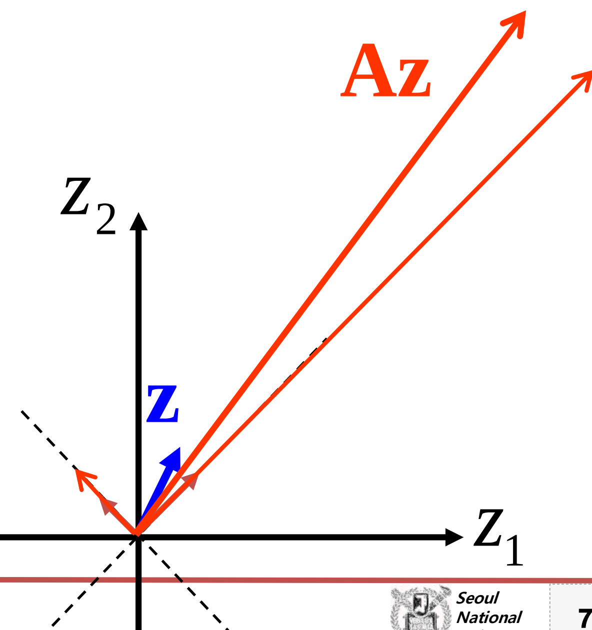
$\mathbf{y} = \mathbf{Az}$: a transformation

- If we are interested in a transformation $\mathbf{y} = \mathbf{Az}$,
- "*eigenbasis*" (basis of eigenvectors) is of great advantage because any $\mathbf{z}$ in R^n uniquely is represented as *a linear combination of the eigenvectors* $\mathbf{x}_1, \ldots, \mathbf{x}_n$, say,

$$\mathbf{z} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n$$

And denoting the corresponding eigenvalues of the matrix $\mathbf{A}$ by $\lambda_1, \ldots, \lambda_n$, we have $\mathbf{Ax}_j = \lambda_j\mathbf{x}_j$, so that we simply obtain

$$\begin{aligned}\mathbf{y} = \mathbf{Az} &= \mathbf{A}\left(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n\right) \\ &= c_1\mathbf{Ax}_1 + c_2\mathbf{Ax}_2 + \cdots + c_n\mathbf{Ax}_n \\ &= c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 + \cdots + c_n\lambda_n\mathbf{x}_n\end{aligned}$$



Ex.)

$$\mathbf{y} = \mathbf{Az} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad \begin{matrix} \lambda_1 = 8, \ \mathbf{x}_1 = [1 \quad 1]^{\mathrm{T}} \\ \lambda_2 = 2, \ \mathbf{x}_2 = [1 \quad -1]^{\mathrm{T}} \end{matrix}$$

Basis of Eigenvectors

Theorem 8.4.1 Basis of Eigenvectors

If an n x n matrix $\mathbf{A}$ has n distinct eigenvalues, then $\mathbf{A}$ has a basis of eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ for R^n.

Proof

All we have to show is that $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent.

Suppose they are not linearly independent.

1 to r : independent
r is the largest integer that is a linearly independent (r<n)

$\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$: independent

$\{\mathbf{x}_1, \cdots, \mathbf{x}_{r+1}\}$: dependent

$$c_1\mathbf{x}_1 + \cdots + c_{r+1}\mathbf{x}_{r+1} = 0$$

(not all scalars are zero)

Basis of Eigenvectors (Proof)

$$c_1\mathbf{x}_1+\cdots+c_{r+1}\mathbf{x}_{r+1}=0 \quad \rightarrow ①$$

(not all scalars are zero)

Multiply both sides by A

$$c_1\mathbf{A}\mathbf{x}_1+\cdots+c_{r+1}\mathbf{A}\mathbf{x}_{r+1}=0$$

Use $\mathbf{A}\mathbf{x}_j = \lambda_j\mathbf{x}_j$

$$c_1\lambda_1\mathbf{x}_1+\cdots+c_{r+1}\lambda_{r+1}\mathbf{x}_{r+1}=0 \rightarrow ②$$

② - λ_{r+1} x ①:

$$c_1(\lambda_1-\lambda_{r+1})\mathbf{x}_1+\cdots+c_r(\lambda_r-\lambda_{r+1})\mathbf{x}_r=0$$

$\mathbf{x}_1, \ldots, \mathbf{x}_r$ is linearly independent.

$$c_1(\lambda_1-\lambda_{r+1})=\cdots=c_r(\lambda_r-\lambda_{r+1})=0$$

All the eigenvalues are distinct.

$$\lambda_1-\lambda_{r+1}\neq 0$$

$$\vdots$$

$$\lambda_r-\lambda_{r+1}\neq 0$$

$$\therefore c_1=\cdots=c_r=0$$

With this, ① reduces to

$$c_{r+1}\mathbf{x}_{r+1}=0$$

$$\therefore c_{r+1}=0$$

This contradicts the fact that not all scalars in ① are zero.

Basis of Eigenvectors

Example **(Eigenbasis. Nondistinct Eigenvalues. Nonexistence)**

$$\mathbf{A} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

Characteristic Equation:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 9 = 0$$

$$\lambda_1 = 8, \ \lambda_2 = 2$$

(1) $\lambda = \lambda_1 = 8$

$$-3x_1 + 3x_2 = 0$$
$$3x_1 - 3x_2 = 0$$
$$\therefore x_1 = x_2$$
$$x_1 = x_2 = 1$$
$$\mathbf{x}_1 = [1 \quad 1]^{\mathrm{T}}$$

(2) $\lambda = \lambda_2 = 2$

$$3x_1 + 3x_2 = 0$$
$$3x_1 + 3x_2 = 0$$
$$\therefore x_1 = -x_2$$
$$x_1 = 1, \ x_2 = -1$$
$$\mathbf{x}_2 = [1 \quad -1]^{\mathrm{T}}$$

➔ eigenbasis for R^n

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Characteristic Equation:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = (-\lambda)^2 = 0$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0 \quad , \lambda = 0$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$0 \cdot x_1 + 1 \cdot x_2 = 0 \qquad \therefore \mathbf{x} = \begin{bmatrix} k \\ 0 \end{bmatrix}$$

Matrix A may *not have enough linearly independent eigenvectors* to make up a basis.

Symmetric Matrices

Theorem 8.4.2 Symmetric Matrices
A symmetric matrix has an orthonormal basis of eigenvectors for R^n.

From example 8.4.1

$$\mathbf{A} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

$$\lambda_1 = 8, \quad \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

normalize $\rightarrow \mathbf{x}_1 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^{\mathrm{T}}$

$$\lambda_2 = 2, \quad \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

normalize $\rightarrow \mathbf{x}_2 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}^{\mathrm{T}}$

$$\begin{aligned} \mathbf{x}_1 \cdot \mathbf{x}_2 &= \mathbf{x}_1^{\mathrm{T}} \mathbf{x}_2 \\ &= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \\ &= 0 \end{aligned}$$

So $\mathbf{x}_1$, $\mathbf{x}_2$ is an orthonormal basis of eigenvectors.

Similar Matrices

Similar Matrices. Similarity Transformation (상사변환)

Definition. Similar Matrices (유사행렬). Similarity Transformation (유사변환)

An n x n matrix $\hat{\mathbf{A}}$ is called similar to an n x n matrix $\mathbf{A}$ if

$$\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

for some (nonsingular!) n x n matrix $\mathbf{P}$. This transformation, which gives $\hat{\mathbf{A}}$ from $\mathbf{A}$, is called *a similarity transformation.*

* Nonsingular matrix (정칙행렬): A matrix that has an inverse.

Eigenvalues and Eigenvectors of Similar Matrices

$\mathbf{Ax} = \lambda\mathbf{x}$

Theorem 8.4.3 Eigenvalues and Eigenvectors of Similar Matrices
If $\hat{\mathrm{A}}$ is similar to A, then $\hat{\mathrm{A}}$ has <u>the same eigenvalues</u> as A. Furthermore, if x is an eigenvector of A, then $\mathrm{y} = \mathrm{P}^{-1}\mathrm{x}$ is an eigenvector of $\hat{\mathrm{A}}$ corresponding to the same eigenvalue.

<u>Proof</u>

$$\mathbf{P}^{-1}\mathbf{Ax} = \mathbf{P}^{-1}(\lambda\mathbf{x}) = \lambda\mathbf{P}^{-1}\mathbf{x} \quad \Rightarrow ①$$

Use $\mathrm{I} = \mathrm{PP}^{-1}$

$$\mathbf{P}^{-1}\mathbf{Ax} = \mathbf{P}^{-1}\mathbf{AIx} = \underbrace{\mathbf{P}^{-1}\mathbf{A}\mathbf{P}}_{\hat{\mathbf{A}}}\mathbf{P}^{-1}\mathbf{x}$$

$$= \hat{\mathbf{A}}\mathbf{P}^{-1}\mathbf{x} \quad \Rightarrow ②$$

From ①, ②

$$\hat{\mathbf{A}}\mathbf{P}^{-1}\mathbf{x} = \lambda\mathbf{P}^{-1}\mathbf{x}$$

If $\mathrm{y} = \mathrm{P}^{-1}\mathrm{x}$, then

$$\hat{\mathbf{A}}\mathbf{y} = \lambda\mathbf{y}$$

➔ y is an eigenvector of $\hat{\mathrm{A}}$

Indeed, $\mathrm{P}^{-1}\mathrm{x} = 0$ would give

$$\mathbf{x} = \mathbf{Ix} = \mathbf{PP}^{-1}\mathbf{x} = \mathbf{P}\cdot 0 = 0$$

➔ contradicting $\mathrm{x} \neq 0$

$$\therefore \mathbf{y} = \mathbf{P}^{-1}\mathbf{x} \neq 0$$

Example 8.4-3 (1)
(Eigenvalues and Vectors of Similar Matrices)

$$\mathbf{A} = \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$$

$$\mathbf{P}^{-1} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$$

Get similar matrix $\hat{\mathbf{A}}$.

$$\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

$$= \begin{bmatrix} 4 & -3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 6-\lambda & -3 \\ 4 & -1-\lambda \end{vmatrix} = 0$$

$$(6-\lambda)(-1-\lambda) + 12 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0 \quad \Rightarrow \quad \lambda_1 = 3, \quad \lambda_2 = 2$$

$$\det(\hat{\mathbf{A}} - \lambda\mathbf{I}) = \begin{vmatrix} 3-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(2-\lambda) = 0 \quad \Rightarrow \quad \lambda_1 = 3, \quad \lambda_2 = 2$$

So $\mathbf{A}$ and $\hat{\mathbf{A}}$ has the same eigenvalue.

Example 8.4-3 (2)
(Eigenvalues and Vectors of Similar Matrices)

$$\mathbf{P}^{-1} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$$

$$\hat{\mathbf{A}} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \qquad \lambda_1 = 3, \quad \lambda_2 = 2$$

$(1)\ \lambda = \lambda_1 = 3$

$$\mathbf{A} - \lambda_1 \mathbf{I} = \begin{bmatrix} 3 & -3 \\ 4 & -4 \end{bmatrix}$$

$$3x_1 - 3x_2 = 0$$

Let x_1 = 1, $\therefore \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathbf{T}}$

$(2)\ \lambda = \lambda_2 = 2$

$$\mathbf{A} - \lambda_2 \mathbf{I} = \begin{bmatrix} 4 & -3 \\ 4 & -3 \end{bmatrix}$$

$$4x_1 - 3x_2 = 0$$

Let x_1 = 3, $\therefore \mathbf{x}_2 = \begin{bmatrix} 3 & 4 \end{bmatrix}^{\mathbf{T}}$

Eigenvectors of $\hat{\mathbf{A}}$ is

$$\mathbf{y}_1 = \mathbf{P}^{-1}\mathbf{x}_1 = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathbf{y}_2 = \mathbf{P}^{-1}\mathbf{x}_2 = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

DIAGONALIZATION

Diagonalization of a Matrix (행렬의 대각화)

Theorem 8.4.4 Diagonalization of Matrix

If an n x n matrix $\mathbf{A}$ has a basis of eigenvectors, then

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X}$$

is *diagonal matrix* (대각행렬), with the eigenvalues of $\mathbf{A}$ as the entries on the main diagonal.

Here $\mathbf{X}$ is the matrix with these eigenvectors as column vectors.

Also $\mathbf{D}^m = \mathbf{X}^{-1}\mathbf{A}^m\mathbf{X}$

Diagonalization of a Matrix (Proof) (1)

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ constitute a basis of eigenvectors of $\mathbf{A}$ for R^n.

Let the corresponding eigenvalues of A be $\lambda_1, \ldots, \lambda_n$.

$$\mathbf{A}\mathbf{x}_1 = \lambda_1 \mathbf{x}_1$$
$$\vdots$$
$$\mathbf{A}\mathbf{x}_n = \lambda_n \mathbf{x}_n$$

Then $\mathbf{X} = [\,\mathbf{x}_1 \;\ldots\; \mathbf{x}_n\,]$ has rank n, by Theorem 3 in Sec. 7.4. Hence $\mathbf{X}^{-1}$ exists.

$$\mathbf{AX} = \mathbf{A}\begin{bmatrix}\mathbf{x}_1 & \cdots & \mathbf{x}_n\end{bmatrix}$$
$$= \begin{bmatrix}\mathbf{A}\mathbf{x}_1 & \cdots & \mathbf{A}\mathbf{x}_n\end{bmatrix}$$
$$= \begin{bmatrix}\lambda_1\mathbf{x}_1 & \cdots & \lambda_n\mathbf{x}_n\end{bmatrix}$$
$$= \begin{bmatrix}\mathbf{x}_1 & \cdots & \mathbf{x}_n\end{bmatrix}\underbrace{\begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_n\end{bmatrix}}_{\mathbf{D}}$$
$$= \mathbf{XD}$$

$$\mathbf{AX} = \mathbf{XD}$$
$$\therefore \mathbf{D} = \mathbf{X}^{-1}\mathbf{AX}$$

$$\mathbf{D}^2 = \mathbf{DD} = \mathbf{X}^{-1}\mathbf{AXX}^{-1}\mathbf{AX}$$
$$= \mathbf{X}^{-1}\mathbf{AIAX} = \mathbf{X}^{-1}\mathbf{A}^2\mathbf{X}$$

Diagonalization of a Matrix (Proof) (2)

$$\mathbf{AX} = \mathbf{A}[\mathbf{x}_1 \quad \cdots \quad \mathbf{x}_n] = [\mathbf{Ax}_1 \quad \cdots \quad \mathbf{Ax}_n]$$

Let n = 2.

$$\mathbf{x}_1 = [x_{11} \quad x_{12}]^{\mathbf{T}}, \ \mathbf{x}_2 = [x_{21} \quad x_{22}]^{\mathbf{T}}$$

$$\begin{aligned}
\mathbf{AX} &= \mathbf{A}[\mathbf{x}_1 \quad \mathbf{x}_2] \\
&= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{bmatrix} \\
&= \begin{bmatrix} a_{11}x_{11} + a_{12}x_{21} & a_{11}x_{12} + a_{12}x_{22} \\ a_{21}x_{11} + a_{22}x_{21} & a_{21}x_{12} + a_{22}x_{22} \end{bmatrix} \\
&= [\mathbf{Ax}_1 \quad \mathbf{Ax}_2]
\end{aligned}$$

Diagonalization of a Matrix (Proof) (3)

$\mathbf{X} = [\mathbf{x}_1 \quad \cdots \quad \mathbf{x}_n]$

$$\mathbf{A}\mathbf{x}_1 = \lambda_1 \mathbf{x}_1 \qquad \mathbf{A}\mathbf{x}_1 = \mathbf{x}_1 \cdot \lambda_1 + \mathbf{x}_2 \cdot 0 + \mathbf{x}_3 \cdot 0$$

$$\mathbf{A}\mathbf{x}_2 = \lambda_2 \mathbf{x}_2 \quad \Rightarrow \quad \mathbf{A}\mathbf{x}_2 = \mathbf{x}_1 \cdot 0 + \mathbf{x}_2 \cdot \lambda_2 + \mathbf{x}_3 \cdot 0$$

$$\mathbf{A}\mathbf{x}_3 = \lambda_3 \mathbf{x}_3 \qquad \mathbf{A}\mathbf{x}_3 = \mathbf{x}_1 \cdot 0 + \mathbf{x}_2 \cdot 0 + \mathbf{x}_3 \cdot \lambda_3$$

$$\mathbf{A}[\mathbf{x}_1 \quad \mathbf{x}_2 \quad \mathbf{x}_3]$$

$$= [\mathbf{x}_1 \cdot \lambda_1 + \mathbf{x}_2 \cdot 0 + \mathbf{x}_3 \cdot 0 \mid \mathbf{x}_1 \cdot 0 + \mathbf{x}_2 \cdot \lambda_2 + \mathbf{x}_3 \cdot 0 \mid \mathbf{x}_1 \cdot 0 + \mathbf{x}_2 \cdot 0 + \mathbf{x}_3 \cdot \lambda_3]$$

$$= [\mathbf{x}_1 \quad \mathbf{x}_2 \quad \mathbf{x}_3] \underbrace{\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}}_{\mathbf{D}} = \mathbf{XD}$$

$$\therefore \mathbf{AX} = \mathbf{XD}$$

$$\therefore \mathbf{D} = \mathbf{X}^{-1}\mathbf{AX}$$

$$\mathbf{A} = \mathbf{XDX}^{-1}$$

Example 8.4-4 (1)
(Diagonalization)

Diagonalize

$$\mathbf{A} = \begin{bmatrix} 7.3 & 0.2 & -3.7 \\ -11.5 & 1.0 & 5.5 \\ 17.7 & 1.8 & -9.3 \end{bmatrix}$$

Characteristic Equation:

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0$$

$$\begin{vmatrix} 7.3-\lambda & 0.2 & -3.7 \\ -11.5 & 1.0-\lambda & 5.5 \\ 17.7 & 1.8 & -9.3-\lambda \end{vmatrix} = 0$$

$$(7.3-\lambda)[(1.0-\lambda)(-9.3-\lambda)-5.5\cdot 1.8]$$
$$-0.2[-11.5(-9.3-\lambda)-5.5\cdot 17.7]$$
$$-3.7[-11.5\cdot 1.8-(1.0-\lambda)17.7]=0$$

$$-\lambda^3 - \lambda^2 + 12\lambda = 0$$

$$\lambda_1 = 3 \qquad \mathbf{x}_1 = [-1 \quad 3 \quad -1]^{\mathrm{T}}$$
$$\lambda_2 = -4 \quad \Rightarrow \quad \mathbf{x}_2 = [1 \quad -1 \quad 3]^{\mathrm{T}}$$
$$\lambda_3 = 0 \qquad \mathbf{x}_3 = [2 \quad 1 \quad 4]^{\mathrm{T}}$$

$$\therefore \mathbf{X} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$
$$\qquad\quad \mathbf{x}_1 \quad \mathbf{x}_2 \quad \mathbf{x}_3$$

$$\mathbf{X}^{-1} = \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$$

Example 8.4-4 (2)
(Diagonalization)

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X}$$

$$= \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix} \begin{bmatrix} 7.3 & 0.2 & -3.7 \\ -11.5 & 1.0 & 5.5 \\ 17.7 & 1.8 & -9.3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$\mathbf{D}$ has <u>the same eigenvalues</u> as $\mathbf{A}$ because $\mathbf{D}$ is a kind of a similar matrix of $\mathbf{A}$.

Example

Q: Find an eigenbasis and diagonalize

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 1/5 & 2/5 \\ -2/5 & 1/5 \end{bmatrix} \mathbf{A} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$$

QUADRATIC FORMS

Quadratic Forms

Definition: a quadratic form Q in the components $x_1, \dots, x_n$ of a vector $\mathbf{x}$ is a sum of n^2 terms, namely,

$$Q = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}x_j x_k$$

$$\begin{aligned}
&= a_{11}x_1^2 + a_{12}x_1x_2 + \cdots + a_{1n}x_1x_n \\
&\quad + a_{21}x_2x_1 + a_{22}x_2^2 + \cdots + a_{2n}x_2x_n \\
&\quad + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + \\
&\quad + a_{n1}x_nx_1 + a_{n2}x_nx_2 + \cdots + a_{nn}x_n^2
\end{aligned}$$

$\mathbf{A} = [a_{jk}]$ is called of the *coefficient matrix* of the form.

We may assume that $\mathbf{A}$ is *symmetric*, because we can take off-diagonal terms together in pairs and write the result as a sum of two equal terms.

Example 8.4-5
(Quadratic Form. Symmetric Coefficient Matrix)

다음 2차형식을 $\mathbf{x}^T\mathbf{C}\mathbf{x}$의 형태로 나타내시오.

$$Q = 3x_1^2 + \boxed{4x_1x_2 + 6x_2x_1} + 2x_2^2$$

$$= x_1(3x_1 + 4x_2) + x_2(6x_1 + 2x_2)$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3x_1 + 4x_2 \\ 6x_1 + 2x_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{x}^T\mathbf{C}\mathbf{x}$$

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \mathbf{x}^T\mathbf{C}\mathbf{x}$$

Symmetric matrix

$$Q = 3x_1^2 + \boxed{5x_1x_2 + 5x_2x_1} + 2x_2^2 \quad = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Principal Axes Theorem (주축정리)

$$Q = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} \longleftarrow \mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X}$$
$$= \mathbf{x}^{\mathrm{T}}\mathbf{X}\mathbf{D}\mathbf{X}^{\mathrm{T}}\mathbf{x} \longleftarrow \mathbf{y} = \mathbf{X}^{-1}\mathbf{x} = \mathbf{X}^{\mathrm{T}}\mathbf{x}$$
$$= \mathbf{y}^{\mathrm{T}}\mathbf{D}\mathbf{y} \qquad \mathbf{y}^{\mathrm{T}} = (\mathbf{X}^{\mathrm{T}}\mathbf{x})^{\mathrm{T}} = \mathbf{x}^{\mathrm{T}}\mathbf{X}$$

Theorem 8.4.5 Principal Axes Theorem

The substitution $\mathbf{x} = \mathbf{X}\mathbf{y}$ transforms a quadratic form

$$Q = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}x_j x_k \qquad (a_{kj} = a_{jk})$$

to the *principal axes form* (주축형식) or *canonical form* (표준형)

$$Q = \mathbf{y}^{\mathrm{T}}\mathbf{D}\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

where $\lambda_1, \ldots, \lambda_n$ are the (not necessarily distinct) eigenvalues of the *symmetric* matrix $\mathbf{A}$, and $\mathbf{X}$ is an *orthogonal* matrix with corresponding eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$, respectively, as column vectors.

Principal Axes Theorem

Theorem 8.4.2 Symmetric Matrices

A symmetric matrix has an orthonormal basis of eigenvectors for R^n.

(Quadratic Forms)
(2차 형식)

$$Q = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk} x_j x_k$$

By the Theorem 8.4.2 the *symmetric* coefficient matrix $\mathbf{A}$ has an *orthonormal basis of eigenvectors* $\mathbf{x}_1, \ldots, \mathbf{x}_n$. Let $\mathbf{X}$ be

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{bmatrix}$$

$\mathbf{X}$ is *orthogonal*, so that $\mathbf{X}^{-1}=\mathbf{X}^{\mathrm{T}}$, we obtain

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X}$$

$$\therefore \mathbf{A} = \mathbf{X}\mathbf{D}\mathbf{X}^{-1} = \mathbf{X}\mathbf{D}\mathbf{X}^{\mathrm{T}}$$

$$Q = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} = \mathbf{x}^{\mathrm{T}}(\mathbf{X}\mathbf{D}\mathbf{X}^{\mathrm{T}})\mathbf{x}$$

Symmetric: $\mathbf{A}^{\mathrm{T}}=\mathbf{A}$
Skew-symmetric: $\mathbf{A}^{\mathrm{T}}=-\mathbf{A}$
Orthogonal: $\mathbf{A}^{\mathrm{T}}=\mathbf{A}^{-1}$

Principal Axes Theorem

$$Q = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} = \mathbf{x}^{\mathrm{T}}\mathbf{X}\mathbf{D}\mathbf{X}^{\mathrm{T}}\mathbf{x}$$

If we set $\mathbf{X}^{\mathrm{T}}\mathbf{x} = \mathbf{y}$, then, since $\mathrm{X}^{\mathrm{T}} = \mathrm{X}^{-1}$, we get

$$\mathbf{x} = \left(\mathbf{X}^{\mathrm{T}}\right)^{-1}\mathbf{y} = \mathbf{X}\mathbf{y}$$

Furthermore, we have

$$\mathbf{x}^{\mathrm{T}}\mathbf{X} = \left(\mathbf{X}^{\mathrm{T}}\mathbf{x}\right)^{\mathrm{T}} = \mathbf{y}^{\mathrm{T}}$$

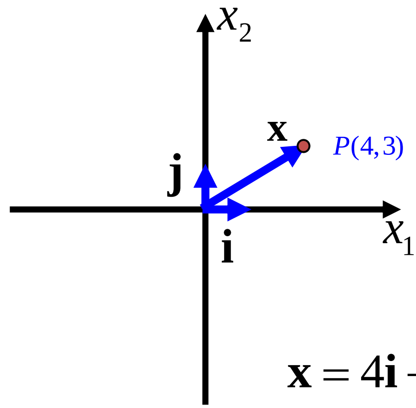
So Q becomes simply

$$Q = \mathbf{x}^{\mathrm{T}}\mathbf{X}\mathbf{D}\mathbf{X}^{\mathrm{T}}\mathbf{x} = \mathbf{y}^{\mathrm{T}}\mathbf{D}\mathbf{y}$$

$$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

Principal Axes Theorem

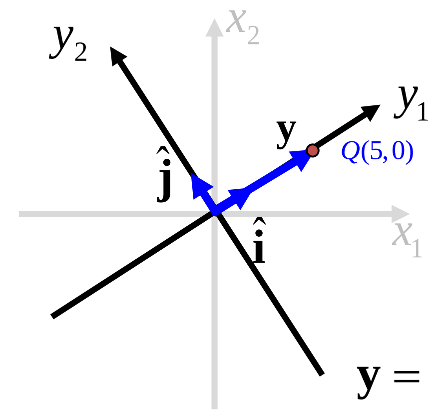
Transformation of Axis



$$\mathbf{x} = 4\mathbf{i} + 3\mathbf{j}$$

Choose proper axis

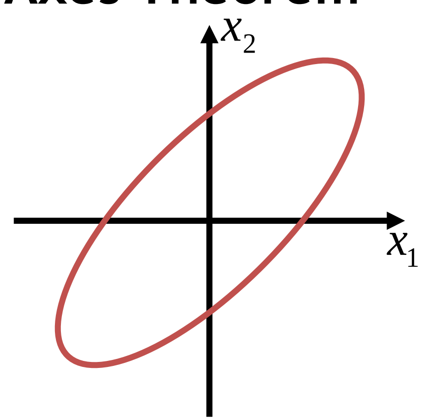
(Coordinate Transformation)



$$\mathbf{y} = 5\hat{\mathbf{i}}$$

Simple expression.
Easily recognition of magnitude of vector

Principal Axes Theorem

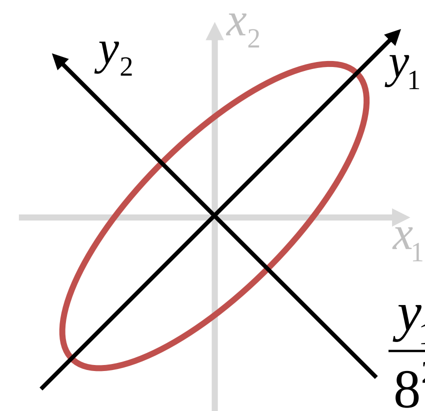


$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$$

Choose proper axis

(Coordinate Transformation)

$$\mathbf{x} = \mathbf{X}\mathbf{y}$$



$$\frac{y_1^2}{8^2} + \frac{y_2^2}{2^2} = 1$$

Simple expression.
Easily recognition of magnitude of principal axis of ellipse.

Principal Axes Theorem

Ex) Transformation to Principal Axes. Conic Sections

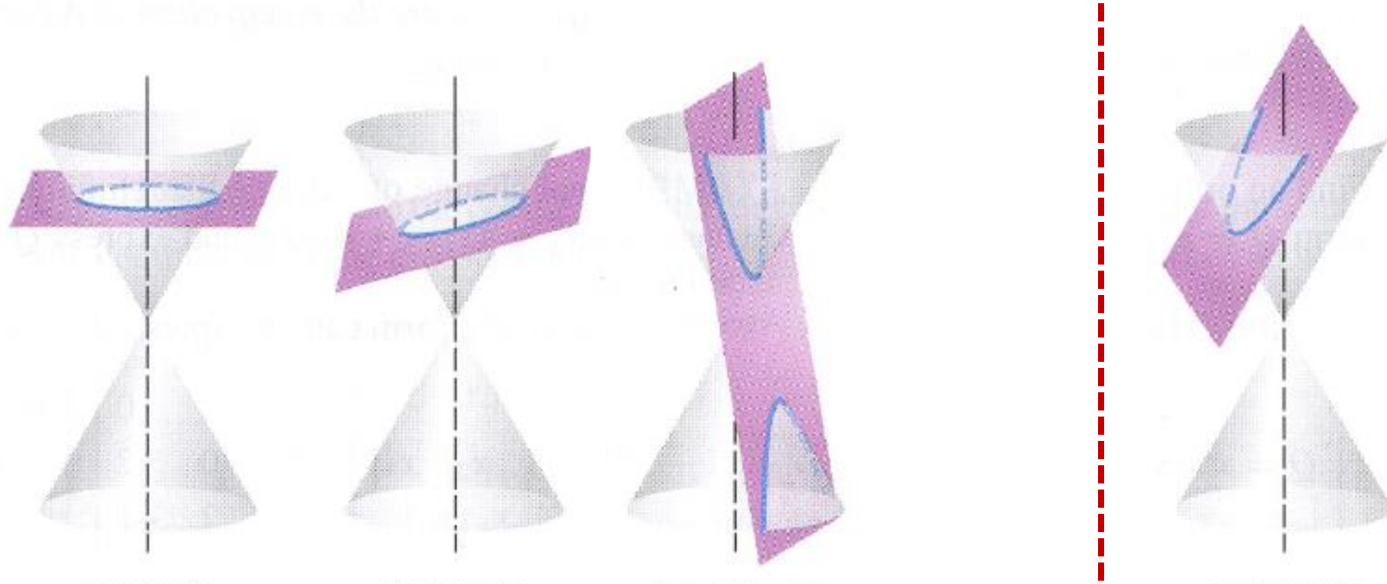
Find out what type of conic section the following quadratic form represents and transform it to principal axes. (Ex 8.2-1)

$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$$

Conic Section ?

Principal Axes Theorem
(Reference: Conic section)

Conic Section (Conic: 원뿔) : curve that results by cutting a double-napped cone with a plane[1)]



Standard form : $ax_1^2 + cx_2^2 + f = 0$,($b=d=e=0$)

Central conic : $ax_1^2 + 2bx_1x_2 + cx_2^2 + f = 0$,($d=e=0$)

: rotated conic in standard position about the origin

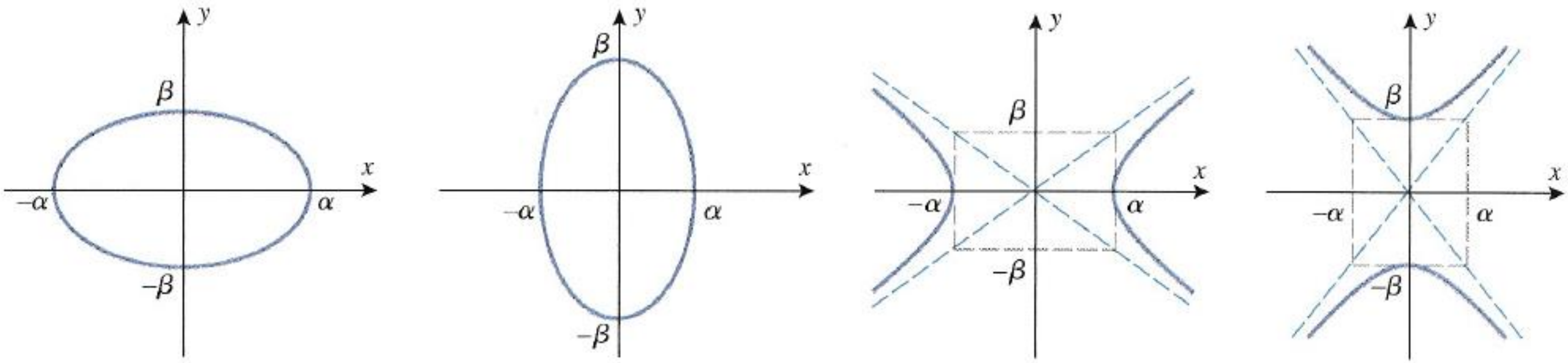
Conic section : $ax_1^2 + 2bx_1x_2 + cx_2^2 + dx_1 + ex_2 + f = 0$,(a~f : constants)

1)Anton, Busby, Contemporary Linear Algebra, Wiley, 2003, Ch.8.4, p.502

Principal Axes Theorem
(Reference: Conic section)

Standard forms of the central conics (represent a conic in standard position)[1]

$$ax_1^{\ 2} + cx_2^{\ 2} + f = 0$$



$\dfrac{x^2}{\alpha^2} + \dfrac{y^2}{\beta^2} = 1$ $(\alpha \geq \beta > 0)$	$\dfrac{x^2}{\alpha^2} + \dfrac{y^2}{\beta^2} = 1$ $(\beta \geq \alpha > 0)$	$\dfrac{x^2}{\alpha^2} - \dfrac{y^2}{\beta^2} = 1$ $(\alpha > 0, \beta > 0)$	$\dfrac{y^2}{\beta^2} - \dfrac{x^2}{\alpha^2} = 1$ $(\alpha > 0, \beta > 0)$

[1] Anton, Busby, Contemporary Linear Algebra, Wiley, 2003, Ch.8.4, p.502

Principal Axes Theorem

Ex) Transformation to Principal Axes. Conic Sections

Find out what type of conic section the following quadratic form represents and transform it to principal axes. (Ex 8.2-1)

$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$$

$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}$$

Characteristic Equation:

$$\begin{vmatrix} 17-\lambda & -15 \\ -15 & 17-\lambda \end{vmatrix} = 0$$

$$(17-\lambda)^2 - 15^2 = 0 \qquad \therefore \lambda_1 = 2,\ \lambda_2 = 32$$

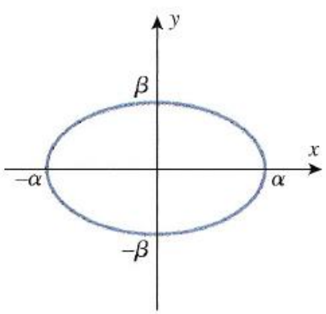
Eigenvectors $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and $\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$$Q = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} = \mathbf{x}^{\mathrm{T}}\mathbf{X}\mathbf{D}\mathbf{X}^{\mathrm{T}}\mathbf{x} = \mathbf{y}^{\mathrm{T}}\mathbf{D}\mathbf{y}$$

$$= [y_1, y_2] \begin{bmatrix} 2 & 0 \\ 0 & 32 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= 2y_1^2 + 32y_2^2 = 128$$

$$\therefore \frac{y_1^2}{8^2} + \frac{y_2^2}{2^2} = 1$$



$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$$
$$(\alpha \ge \beta > 0)$$

* **D** is diagonal matrix, with the eigenvalues of **A** as the entries on the main diagonal

Principal Axes Theorem

Ex) Transformation to Principal Axes. Conic Sections

$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$$

$$Q = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$$

$$\lambda_1 = 2,\ \lambda_2 = 32$$

Principal Axes

$$\mathbf{A} = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}$$

1) $\lambda = \lambda_1 = 2$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix}$$

$$15x_1 - 15x_2 = 0$$

From this we get normalized eigenvector $\mathbf{x}_1$.

$$\mathbf{x}_1 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^{\mathrm{T}}$$

2) $\lambda = \lambda_2 = 32$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} -15 & -15 \\ -15 & -15 \end{bmatrix}$$

$$-15x_1 - 15x_2 = 0$$

From this we get normalized eigenvector $\mathbf{x}_2$.

$$\mathbf{x}_2 = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^{\mathrm{T}}$$

Principal Axes Theorem

Ex) Transformation to Principal Axes. Conic Sections

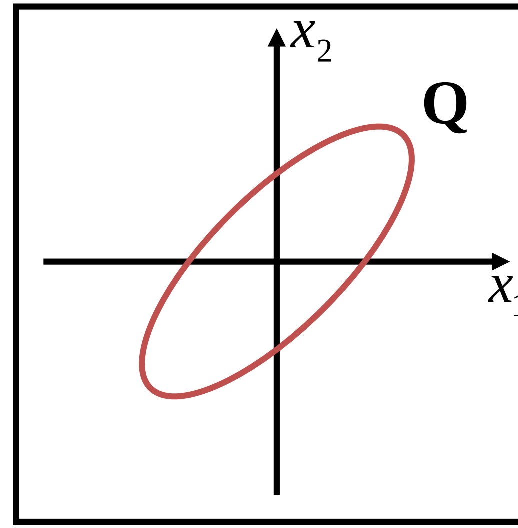
$$\mathbf{x} = \mathbf{X}\mathbf{y}$$

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix}$$: X is orthogonal matrix ($\mathrm{X}^{\mathrm{T}}=\mathrm{X}^{-1}$) with these eigenvectors as column vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{x}_2 = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^{\mathrm{T}}$$

$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$$

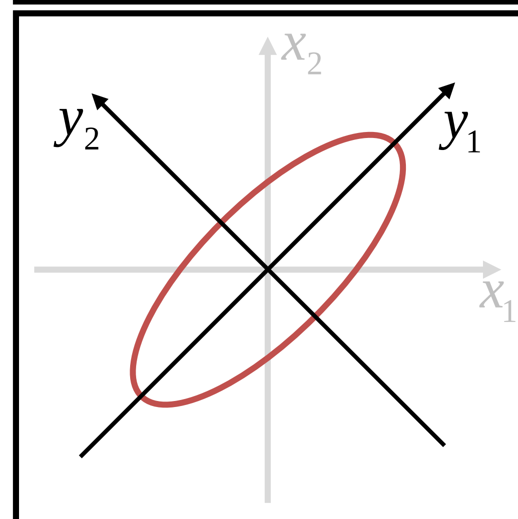


$$\mathbf{x} = \mathbf{X}\mathbf{y}$$

$$\therefore \mathbf{x} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\frac{y_1^2}{8^2} + \frac{y_2^2}{2^2} = 1$$



➔This means a 45° rotation (of principal axes)

(Ref.) Rotational Transformation

$$\frac{z_1'^2}{8^2}+\frac{z_2'^2}{2^2}=1$$

$$\frac{1}{2}\left(\frac{(z_1+z_2)^2}{8^2}+\frac{(-z_1+z_2)^2}{2^2}\right)=1$$

$$\frac{(z_1+z_2)^2}{8^2}+\frac{(-z_1+z_2)^2}{2^2}=2$$

$$4(z_1^2+2z_1z_2+z_2^2)+64(z_1^2-2z_1z_2+z_2^2)=2$$

$$68z_1^2-120z_1z_2+68z_2^2=512$$

$$17z_1^2-30z_1z_2+17z_2^2=128$$

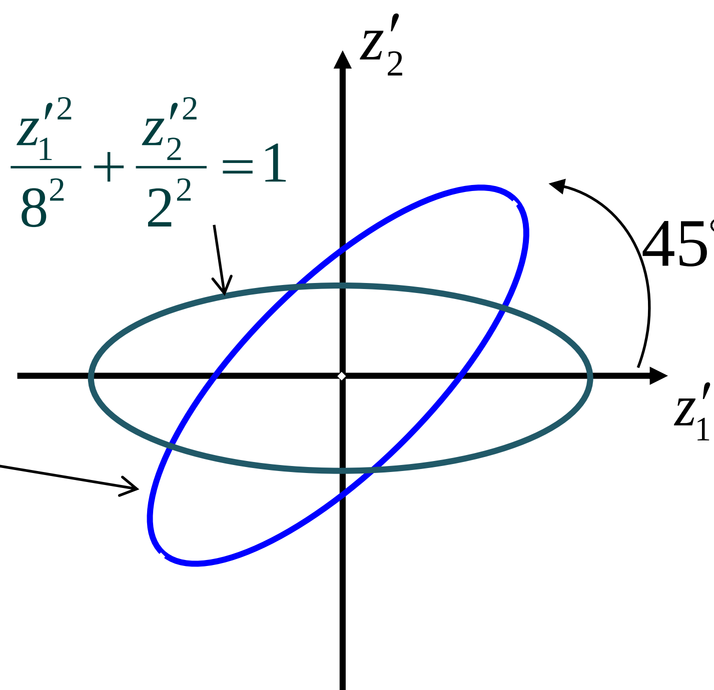
$$z_1'=\frac{1}{\sqrt{2}}(z_1+z_2)$$

$$z_2'=\frac{1}{\sqrt{2}}(-z_1+z_2)$$

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}=\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} z_1' \\ z_2' \end{bmatrix}$$

$$\begin{bmatrix} z_1' \\ z_2' \end{bmatrix}=\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\begin{bmatrix} z_1' \\ z_2' \end{bmatrix}=\begin{bmatrix} \cos 45^\circ & \sin 45^\circ \\ -\sin 45^\circ & \cos 45^\circ \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$



Principal Axes Theorem

Ex) Transformation to Principal Axes. Conic Sections

$$\mathbf{x} = \mathbf{X}\mathbf{y}$$

$$\mathbf{X} = [\mathbf{x}_1 \quad \mathbf{x}_2]$$

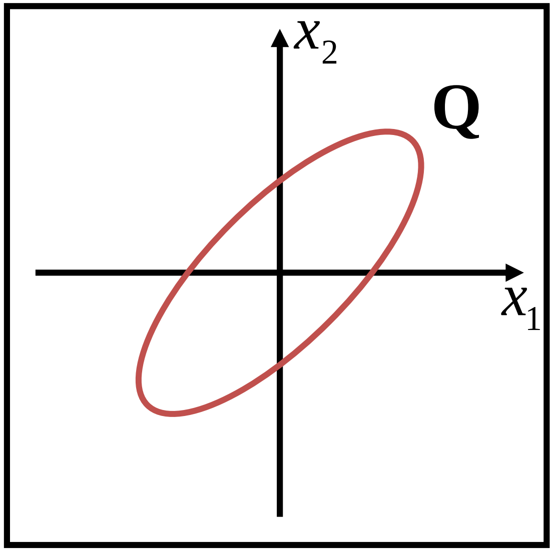
$$\mathbf{x}_1 = \left[1/\sqrt{2} \quad 1/\sqrt{2}\right]^{\mathrm{T}}$$

$$\mathbf{x}_2 = \left[-1/\sqrt{2} \quad 1/\sqrt{2}\right]^{\mathrm{T}}$$

$$\therefore \mathbf{x} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

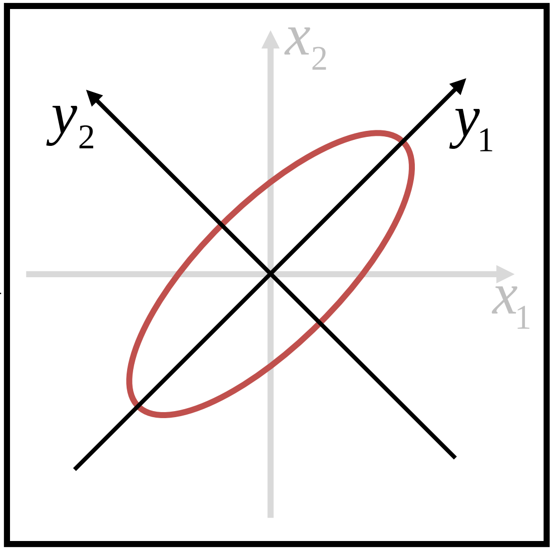
$$= \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$$



$$\mathbf{x} = \mathbf{X}\mathbf{y}$$

$$\frac{y_1^2}{8^2} + \frac{y_2^2}{2^2} = 1$$



(Review) Stretching of Elastic Membrane

Object / Transform Matrix	Conic (Circle, Ellipse, Hyperbola, Parabola)	Arbitrary shape (Rectangle in this example)
Symmetric matrix (eigenvalues are orthogonal)	**Case I**	**Case II**
Non-symmetric matrix (eigenvalues are not orthogonal)	**Case III**	**Case IV**

➔This example of transformation of principal axes corresponds to Case I, and magnitude of transformation matrix is 1.

Principal Axes Theorem

Q: What kind of conic section is given by the quadratic form? Transform it to principal axes. Express **x** in terms of **y**.

$$7x_1^2 + 6x_1 x_2 + 7x_2^2 = 200$$

Quadratic form (Definiteness)

A quadratic form $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ and its (symmetric!) matrix A are called

(a) positive definite if $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq 0$,

(b) negative definite if $Q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq 0$,

(c) indefinite if Q(x) takes both positive and negative values.

Quadratic form (Definiteness: 부호성)

A quadratic form $Q(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ and its (symmetric!) matrix A are called

(a) positive definite (양의 정부호) if $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq 0$,

(b) negative definite (음의 정부호) if $Q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq 0$,

(c) indefinite (부정부호) if Q(x) takes both positive and negative values.

A necessary and sufficient condition for positive definiteness is that all the "principal minors (주 소행렬식)" are positive, that is,

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \cdots \quad , \quad \det \mathbf{A} > 0$$

Show that the form in Prob. 23 is positive definite, whereas that in Prob. 19 is indefinite.

Quadratic form (Definiteness)

A necessary and sufficient condition for positive definiteness is that all the "principal minors" are positive, that is,

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \cdots \quad , \quad \det \mathbf{A} > 0$$

$$\mathbf{A} = \begin{bmatrix} 4 & \sqrt{3} \\ \sqrt{3} & 2 \end{bmatrix}$$

$$a_{11} = 4 > 0$$

$$\begin{vmatrix} 4 & \sqrt{3} \\ \sqrt{3} & 2 \end{vmatrix} = 4 \cdot 2 - \left(\sqrt{3}\right)^2 = 5 > 0$$

➔ positive definite

$$\mathbf{A} = \begin{bmatrix} 1 & 12 \\ 12 & -6 \end{bmatrix}$$

$$a_{11} = 1 > 0$$

$$\begin{vmatrix} 1 & 12 \\ 12 & -6 \end{vmatrix} = -6 - 12^2 = -150 < 0$$

➔ indefinite

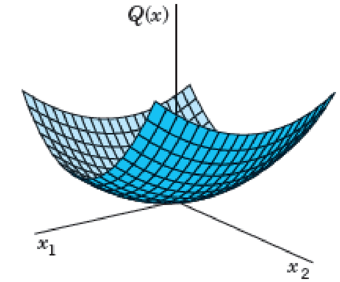
Quadratic form (Definiteness)

the eigenvalues of $\mathbf{A}$ are

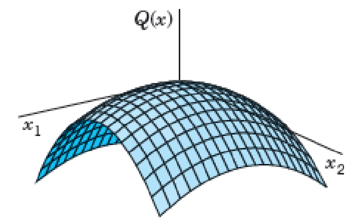
(a) positive definite: all positive

(b) negative definite: all negative

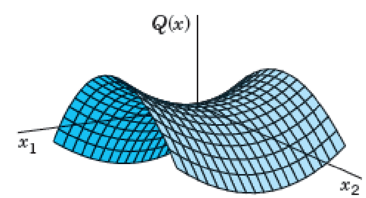
(c) indefinite: both positive and negative



(a) Positive definite form



(b) Negative definite form



(c) Indefinite form

$$Q = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}, \quad \mathbf{x} = \mathbf{X}\mathbf{y}$$

$$= \mathbf{y}^{\mathrm{T}}\mathbf{A}\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \quad (1)$$

Because $\mathbf{y} = \mathbf{X}^{-1}\mathbf{x}$, if $\mathbf{x} \neq 0$, then $\mathbf{y} \neq 0$.

From equation (1),

If all eigenvalues are positive, $Q(\mathbf{x})$ is positive.

If all eigenvalues are negative, $Q(\mathbf{x})$ is negative.

(참고) Taylor Series Expansion과 극소점 (1)

$$\text{Given} : f(x), \frac{df}{dx}, \frac{d^2 f}{dx^2}, \cdots \text{ at } x$$

$$\text{Find} : f(x+\Delta x)$$

Taylor series expansion

Higher Order Terms

$$f(x+\Delta x) = f(x) + \frac{df}{dx}\Delta x + \frac{1}{2}\frac{d^2 f}{dx^2}\Delta x^2 + H.O.T$$

$f(x)$가 극소값을 가질 조건은?

$f(x)$가 주위의 함수값 $f(x+\Delta x)$보다 항상 작아야 함

$$f(x+\Delta x) - f(x) = \frac{df}{dx}\Delta x + \frac{1}{2}\frac{d^2 f}{dx^2}\Delta x^2 + H.O.T > 0$$

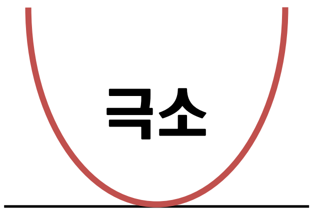
(참고) Taylor Series Expansion과 극소점 (2)

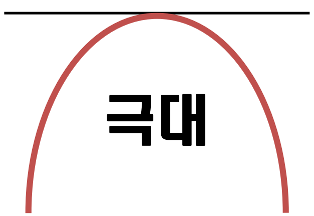
$$f(x+\Delta x) - f(x) = \frac{df}{dx}\underline{\Delta x} + \frac{1}{2}\frac{d^2 f}{dx^2}\underline{\Delta x^2} + H.O.T > 0$$

$$\begin{array}{llll} if \ \ \Delta x < 0, & \quad < 0 & \quad > 0 \\ if \ \ \Delta x > 0, & \quad > 0 & \quad > 0 \end{array}$$

$x + \Delta x$ 가 x보다 크거나 작은 것과 관계 없이 항상 $f(x+\Delta x)$가 $f(x)$ 보다 커야 하므로,

$$\boxed{\therefore \left.\frac{df}{dx}\right|_{x=x^*} = 0}$$





$f(x+\Delta x) - f(x)$ 의 다른 항 중에 가장 값이 큰 항이 두 번째 항(2계 미분계수)이므로,

$$\boxed{\therefore \left.\frac{d^2 f}{dx^2}\right|_{x=x^*} > 0}$$

(참고) Taylor Series Expansion과 극소점 (3)

$$\text{Given}: f(x_1, x_2), \frac{\partial f}{\partial x_1}, \frac{\partial^2 f}{\partial x_1 \partial x_2}, \frac{\partial^2 f}{\partial {x_1}^2}, \frac{\partial^2 f}{\partial {x_2}^2} \cdots \text{ at } (x_1, x_2)$$

$$\text{Find}: \quad f(x_1 + \Delta x_1, x_2 + \Delta x_2)$$

Taylor series expansion (단, 3차 이상의 고차항은 무시할 경우)

$$f(x_1 + \Delta x_1, x_2 + \Delta x_2) = f(x_1, x_2) + \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2$$

$$+ \frac{1}{2}\left(\frac{\partial^2 f}{\partial x_1^2}\Delta {x_1}^2 + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}\Delta x_1 \cdot \Delta x_2 + \frac{\partial^2 f}{\partial x_2^2}\Delta {x_2}^2 \right)$$

$f(x_1, x_2)$가 극소값을 가질 조건은?

$f(x_1, x_2)$가 주위의 함수값 $f(x_1 + \Delta x_1, x_2 + \Delta x_2)$보다 항상 작아야 함

(참고) Taylor Series Expansion과 극소점 (4)

$$f(x_1+\Delta x_1, x_2+\Delta x_2) - f(x_1, x_2)$$
$$= \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x_1^2}\Delta x_1{}^2 + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}\Delta x_1 \cdot \Delta x_2 + \frac{\partial^2 f}{\partial x_2^2}\Delta x_2{}^2\right) > 0$$

(x_1, x_2)가 $(x_1+\Delta x_1, x_2+\Delta x_2)$보다 크거나 작은 것과 관계 없이 항상 $f(x_1, x_2)$가 $f(x_1+\Delta x_1, x_2+\Delta x_2)$ 보다 커야 하므로,

$$\therefore \left.\frac{\partial f}{\partial x_1}\right|_{x_1,\, x_2} = \left.\frac{\partial f}{\partial x_2}\right|_{x_1,\, x_2} = 0.$$

$f(x_1+\Delta x_1, x_2+\Delta x_2) - f(x_1, x_2)$의 다른 항 중에 가장 값이 큰 항이 2계 미분계수와 관련된 항임

$$\therefore \frac{\partial^2 f}{\partial x_1^2}\Delta x_1{}^2 + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}\Delta x_1 \cdot \Delta x_2 + \frac{\partial^2 f}{\partial x_2^2}\Delta x_2{}^2 > 0$$

(참고) Taylor Series Expansion과 극소점 (5)

$f(x_1^*, x_2^*)$가 (x_1^*, x_2^*)주위의 함수값보다 항상 작아야 함

$$\frac{\partial^2 f}{\partial x_1^2}\Delta x_1{}^2 + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}\Delta x_1 \cdot \Delta x_2 + \frac{\partial^2 f}{\partial x_2^2}\Delta x_2{}^2 > 0$$

(x_1, x_2)가 (x_1^*, x_2^*)보다 크거나 작은 것과 관계 없이 항상 위 식이 성립하게 하는 $\frac{\partial^2 f}{\partial x_1^2}, \frac{\partial^2 f}{\partial x_1 \partial x_2}, \frac{\partial^2 f}{\partial x_2^2}$ **에 대한 조건을 알아야 함**

위 식을 행렬을 이용해서 표현하면 다음과 같음

$$\frac{\partial^2 f}{\partial x_1^2}\Delta x_1{}^2 + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}\Delta x_1 \cdot \Delta x_2 + \frac{\partial^2 f}{\partial x_2^2}\Delta x_2{}^2$$

$$= \begin{bmatrix} \Delta x_1 & \Delta x_2 \end{bmatrix} \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ x_2 - x_2^* \end{bmatrix}$$

(참고) Taylor Series Expansion과 극소점 (6)

$$\frac{\partial^2 f}{\partial x_1^2}\Delta x_1{}^2 + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}\Delta x_1 \cdot \Delta x_2 + \frac{\partial^2 f}{\partial x_2^2}\Delta x_2{}^2$$

$$= \underbrace{\begin{bmatrix} \Delta x_1 & \Delta x_2 \end{bmatrix}}_{\mathbf{x}^{\mathrm{T}}} \underbrace{\begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 f}{\partial x_1 \partial x_2} & \dfrac{\partial^2 f}{\partial x_2^2} \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}}_{\mathbf{x}} = \mathbf{x}^{\mathrm{T}}\mathbf{Hx} > 0$$

따라서 x에 무관하게 행렬 $\mathrm{x^T H x}$가 항상 양수가 되는 H에 대한 조건을 알면, 언제 $f(x_1, x_2)$가 극소값을 갖는지 알 수 있음

H의 모든 고유치가 양수이면, $\mathrm{x^T H x}$가 항상 양수임 (2차 형식 부분에서 설명)

Reference

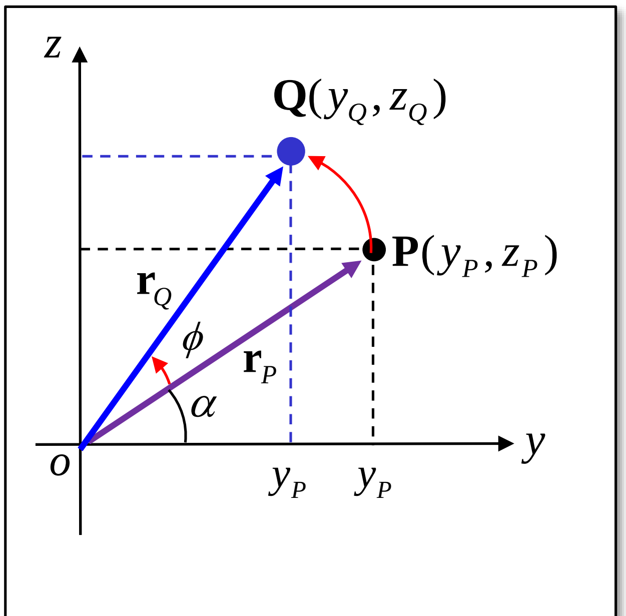
Point Transformation and Coordinate System Transformation

Point Transformation and Coordinate System Transformation (1)

✓ 고정된 좌표계에서 물체의 회전

Given: oyz 에서 정의된 점 P의 좌표값

Find : 점 P 을 oyz 에 대해 ϕ 만큼 회전시킨 점 Q구하기



$$\begin{bmatrix} y_Q \\ z_Q \end{bmatrix} = \boxed{\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}} \begin{bmatrix} y_P \\ z_P \end{bmatrix}$$

점의 회전 변환

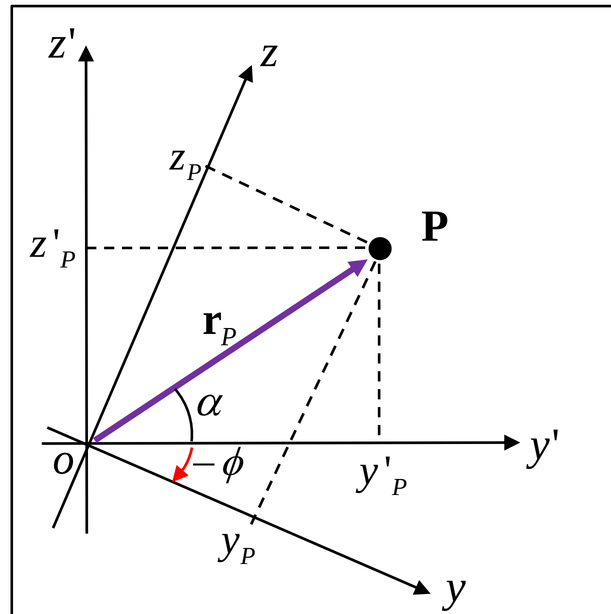
$oy'z'$: *Body fixed coordinate*

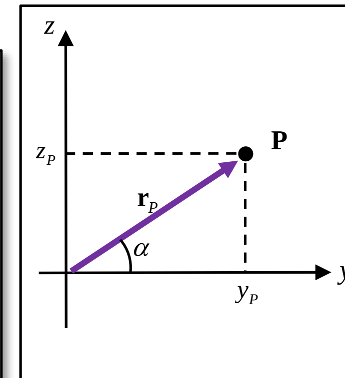
oyz : *Global coordinate*

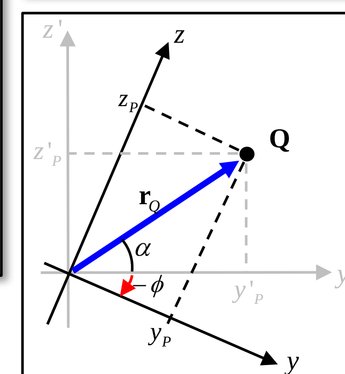
✓ 좌표계 회전

Given: $oy'z'$ 에서 정의된 점 P의 좌표값

Find: $oy'z'$ 에 대해 $-\phi$ 만큼 회전한 새로운 좌표계 oyz 에서의 P의 좌표값







$$\begin{bmatrix} y_P \\ z_P \end{bmatrix} = \boxed{\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}} \begin{bmatrix} y'_P \\ z'_P \end{bmatrix}$$

좌표계 회전 변환

점을 ϕ 만큼 회전시키는 변환 행렬과

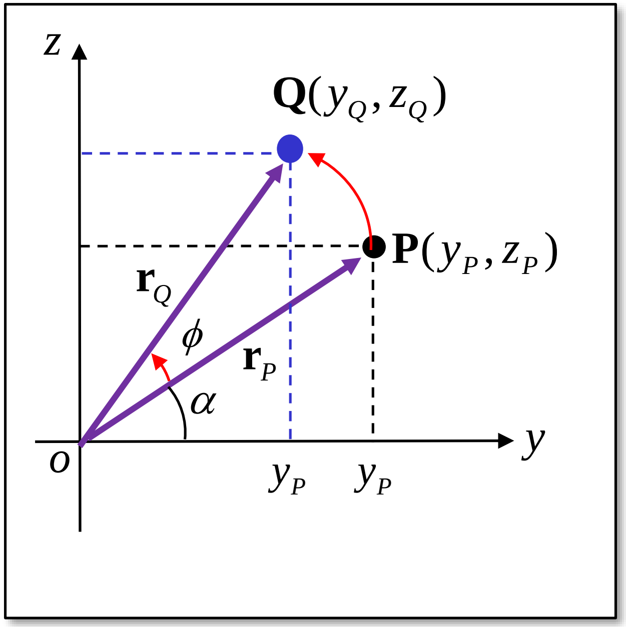
좌표계를 $-\phi$ 만큼 회전시키는 변환 행렬이 동일함

$\mathbf{r}_p = \mathbf{r}_Q$

Point Transformation and Coordinate System Transformation (2)

Given: oyz 에서 정의된 점 P의 좌표값

Find: 점 P 을 oyz 에 대해 ϕ 만큼 회전시킨 점 Q구하기



① 점 P,Q의 좌표를 각으로 표현하면,

$$y_P = |\mathbf{r}_P|\cos\alpha \qquad y_Q = |\mathbf{r}_Q|\cos(\alpha+\phi)$$

$$z_P = |\mathbf{r}_P|\sin\alpha \qquad z_Q = |\mathbf{r}_Q|\sin(\alpha+\phi)$$

② 삼각함수 합공식

$$\sin(\alpha+\phi) = \sin\alpha\cos\phi + \cos\alpha\sin\phi$$

$$\cos(\alpha+\phi) = \cos\alpha\cos\phi - \sin\alpha\sin\phi$$

③ 점 Q의 좌표를 삼각함수의 차공식으로 전개하면,

$$\begin{aligned} y_Q &= |\mathbf{r}_Q|\cos(\alpha+\phi) \\ &= |\mathbf{r}_Q|\cos\alpha\cos\phi - |\mathbf{r}_Q|\sin\alpha\sin\phi \\ &= (|\mathbf{r}_P|\cos\alpha)\cos\phi - (|\mathbf{r}_P|\sin\alpha)\sin\phi \quad ,(|\mathbf{r}_P| = |\mathbf{r}_Q|) \\ &= y_P\cos\phi - z_P\sin\phi \end{aligned}$$

$$\begin{aligned} z_Q &= |\mathbf{r}_Q|\sin(\alpha+\phi) \\ &= |\mathbf{r}_Q|\sin\alpha\cos\phi + |\mathbf{r}_Q|\cos\alpha\sin\phi \\ &= (|\mathbf{r}_P|\sin\alpha)\cos\phi + (|\mathbf{r}_P|\cos\alpha)\sin\phi \quad ,(|\mathbf{r}_P| = |\mathbf{r}_Q|) \\ &= z_P\cos\phi + y_P\sin\phi \end{aligned}$$

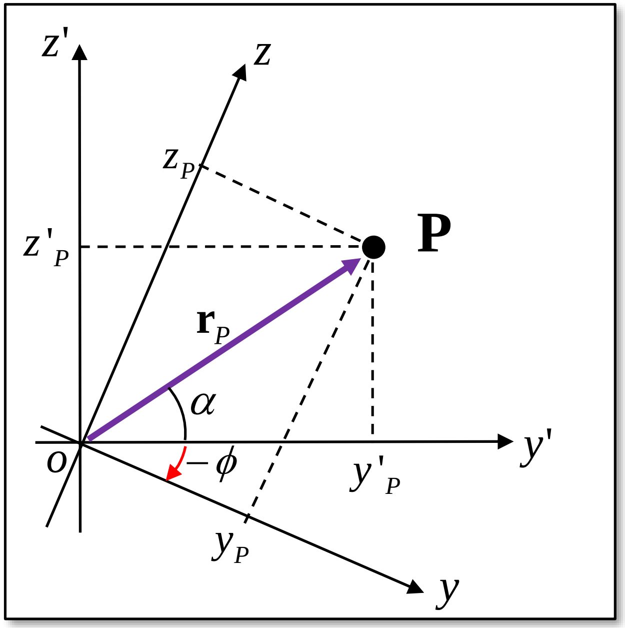
④ 행렬로 표현하면,

$$\boxed{\begin{bmatrix} y_Q \\ z_Q \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} y_P \\ z_P \end{bmatrix}}$$

Point Transformation and Coordinate System Transformation (3)

Given: $oy'z'$ 에서 정의된 점 P의 좌표값

Find: $oy'z'$ 에 대해 $-\phi$ 만큼 회전한 새로운 좌표계 oyz 에서의 P의 좌표값



① 점 P의 좌표를 각으로 표현하면,

$$y'_P = |\mathbf{r}_P|\cos\alpha \qquad y_P = |\mathbf{r}_P|\cos(\alpha+\phi)$$

$$z'_P = |\mathbf{r}_P|\sin\alpha \qquad z_P = |\mathbf{r}_P|\sin(\alpha+\phi)$$

② 삼각함수 합공식

$$\sin(\alpha+\phi) = \sin\alpha\cos\phi + \cos\alpha\sin\phi$$

$$\cos(\alpha+\phi) = \cos\alpha\cos\phi - \sin\alpha\sin\phi$$

③ 점 P의 좌표를 삼각함수의 차공식으로 전개하면,

$$\begin{aligned} y_P &= |\mathbf{r}_P|\cos(\alpha+\phi) \\ &= |\mathbf{r}_P|\cos\alpha\cos\phi - |\mathbf{r}_P|\sin\alpha\sin\phi \\ &= \left(|\mathbf{r}_P|\cos\alpha\right)\cos\phi - \left(|\mathbf{r}_P|\sin\alpha\right)\sin\phi \\ &= y'_P\cos\phi - z'_P\sin\phi \end{aligned}$$

$$\begin{aligned} z_P &= |\mathbf{r}_P|\sin(\alpha+\varphi) \\ &= |\mathbf{r}_P|\sin\alpha\cos\varphi + |\mathbf{r}_P|\cos\alpha\sin\varphi \\ &= \left(|\mathbf{r}_P|\sin\alpha\right)\cos\varphi + \left(|\mathbf{r}_P|\cos\alpha\right)\sin\varphi \\ &= z'_P\cos\varphi + y'_P\sin\varphi \end{aligned}$$

④ 행렬로 표현하면,

$$\boxed{\begin{bmatrix} y_P \\ z_P \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} y'_P \\ z'_P \end{bmatrix}}$$