CHAPTER 1. FIRST-ORDER ODE

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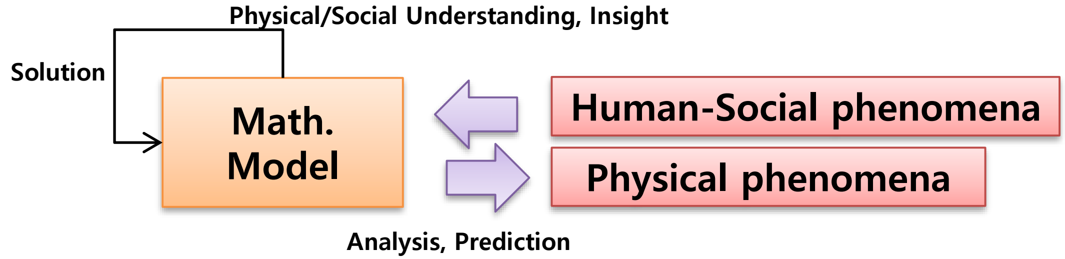
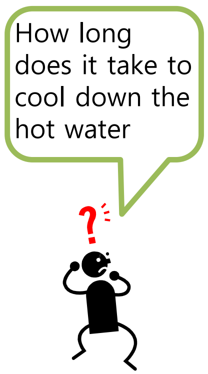
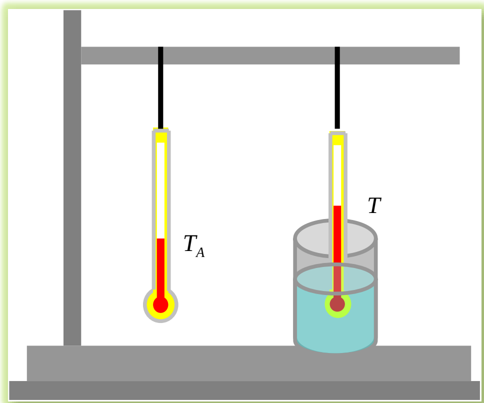
서유택

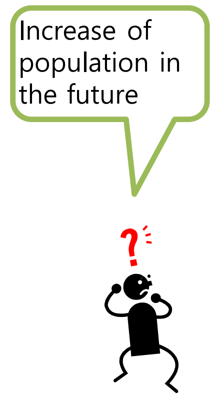
※ 본 강의 자료는 이규열, 장범선, 노명일 교수님께서 만드신 자료를 바탕으로 일부 편집한 것입니다.

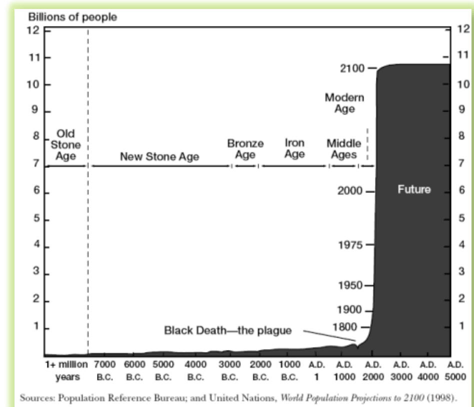
Why Mathematics?











Changing **rate** of water temperature	$\propto$	Temperature difference between water and outside

$$\frac{dT(t)}{dt} = k(T - T_A), k < 0$$

T : Water temperature

T_A :Outside temperature (constant)

<Newton's law of cooling>

Increasing **rate** of population	$\propto$	Present Population

$$\frac{dy(t)}{dt} = k \cdot y(t)$$

y : population

t : time

k : proportional constant

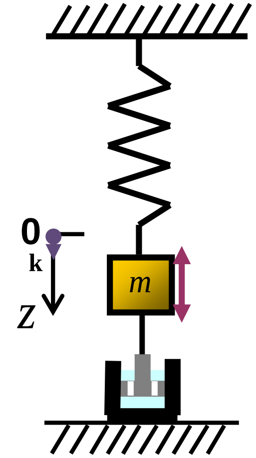
<Malthus's population dynamics>

Why do we need to study "Engineering Mathematics"?

Real World

Idealization or Simplification*

- Insight
- Constitutive Relations



Problem-Solving Abilities

Linearization

- Taylor Series

Explain or Predict

Mathematical Model

$$m\mathbf{z}'' + c\mathbf{z}' + k\mathbf{z} = \mathbf{F}_0 \cos \omega_0 t$$

Solution

$$z(t) = e^{-\alpha t}(A\cos \omega t + B\sin \omega t) + C\cos \omega_0 t$$

- Linearly Dependent / Independent
- Basis
- Orthogonality
- Linear Combination

* Keener, J.P., Principles of Applied Mathematics, Westview Press, 2000, p.xi : ...there is the goal to explain or predict the behavior of some physical situation. One begins by constructing a mathematical model which captures the essential features of the problem without ,asking its content with overwhelming detail

1.1 Basic Concepts. Modeling

☑ Modeling

❖ **The typical steps of modeling in detail**

Step 1. **The transition from the physical situation to its mathematical formulation**

Step 2. **The solution by a mathematical method**

Step 3. **The physical interpretation of differential equations and their applications**

1.1 Basic Concepts. Modeling

- **Differential Equation (미분방정식)**: An equation containing derivatives of an unknown function

Differential Equation
- **Ordinary Differential Equation (상미분 방정식)**
- **Partial Differential Equation (편미분 방정식)**

- **Ordinary Differential Equation**: An equation that contains one or several derivatives of an unknown function (y) of one independent variable (x)

ex) $y' = \cos x, \quad y'' + 9y = e^{-2x}, \quad y'y''' - \frac{3}{2}(y')^2 = 0$

- **Partial Differential Equation**: An equation involving partial derivatives of an unknown function (u) of two or more variables (x, y)

ex) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

1.1 Basic Concepts. Modeling

❖ **Order (계)**: The highest derivative of the unknown function

ex) (1) $y' = \cos x \quad \Rightarrow$ First order

(2) $y'' + 9y = e^{-2x} \quad \Rightarrow$ Second order

(3) $y'y''' - \frac{3}{2}(y')^2 = 0 \quad \Rightarrow$ Third order

❖ **First-order ODE**: Equations contain only the first derivative *y′* and may contain *y* and any given functions of *x*

- Explicit (**양함수**) Form: $y' = f(x, y)$
- Implicit (**음함수**) Form: $F(x, y, y') = 0$

1.1 Basic Concepts. Modeling

- **Solution**: Functions that make the equation hold true
  - General Solution (일반해)

    : a solution containing an arbitrary constant

  - Particular Solution (특수해)

    : a solution that we choose a specific constant

  - Singular Solution (Problem 16) (특이해)

    : an additional solution that cannot be obtained from the general solution

  - **Ex. (Problem 16) ODE :** $(y')^2 - xy' + y = 0$

    General solution : $y = cx - c^2$

    Particular solution : $y = 2x - 4$

    Singular solution : $y = x^2 / 4$

1.1 Basic Concepts. Modeling

❖ **Initial Value Problems (초기값 문제)**: An ordinary differential equation together with specified value of the unknown function at a given point in the domain of the solution

$$y' = f(x, y)\ , \quad y(x_0) = y_0$$

☑ Ex.4 Solve the initial value problem

$$y' = \frac{dy}{dx} = 3y, \quad y(0) = 5.7$$

Step 1 Find the general solution.

General solution: $y(x) = ce^{3x}$

Step 2 Apply the initial condition. $y(0) = ce^{0} = c = 5.7$

Particular solution: $y(x) = 5.7e^{3x}$

1.1 Basic Concepts. Modeling

☑ **Ex. 5 Given an amount of a radioactive substance, say 0.5 g (gram), find the amount present at any later time.**

Physical Information.

Experiments show that at each instant a radioactive substance decomposes at a rate proportional to the amount present.

Step 1 Setting up a mathematical model (a differential equation) of the physical process.

By the physical law : $\dfrac{dy}{dt} \propto -y \quad \Rightarrow \dfrac{dy}{dt} = -ky$

The initial condition : $y(0) = 0.5$

Step 2 Mathematical solution.

General solution: $y(t) = ce^{-kt}$

Particular solution: $y(0) = ce^{0} = c = 0.5 \quad \Rightarrow y(t) = 0.5e^{-kt}$

Always check your result: $\dfrac{dy}{dt} = -0.5ke^{-kt} = -ky, \;\; y(0) = 0.5e^{0} = 0.5$

Step 3 Interpretation of result. The limit of y as $t \to \infty$ is zero.

1.2 Geometric Meaning of $y'=f(x, y)$. Direction Fields, Euler's Method

1.2 Geometric Meaning of y′=f(x, y). Direction Fields, Euler′s Method

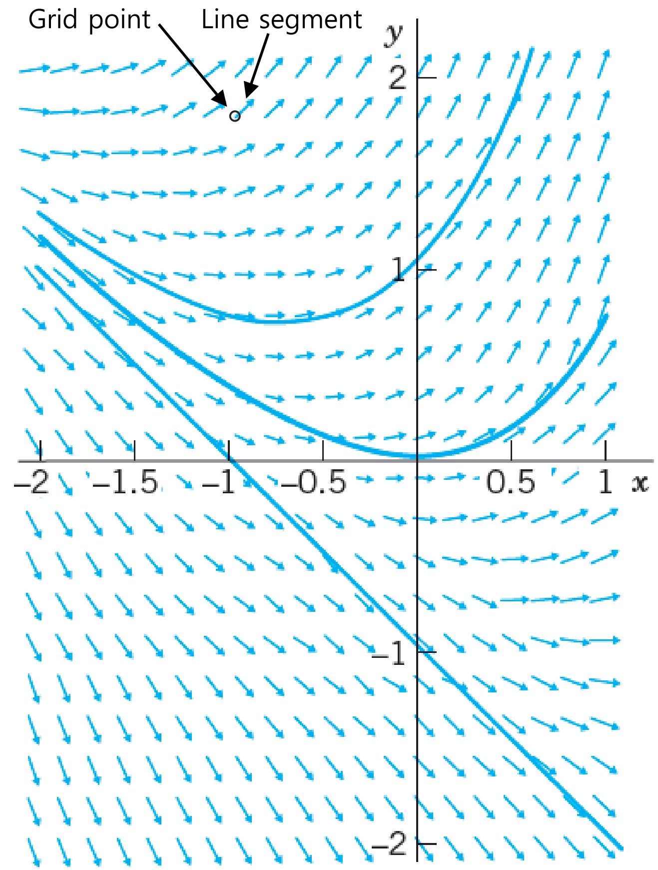
- **A first-order ODE** $y' = f(x, y)$

  : A solution curve (해 곡선) that passes through a point (x_0, y_0) must have, at that point, the slope $y'(x_0)$ equal to the value of f at that point

  $$y'(x_0) = f(x_0, y_0)$$

- **Direction Field (방향장)**
  - The graph includes pairs of grid points and line segments
  - The line segment at grid point coincides with the tangent line to the solution.

- **Reason of importance of the direction field**
  - You need not solve a ODE.
  - The method shows the whole family of solutions and their typical properties.



Direction field of with three approximate solution $y' = y + x$, curves passing through (0, 1), (0, 0), (0, -1), respectively

1.2 Geometric Meaning of y'=f(x, y). Direction Fields, Euler's Method

❖ Numeric Method by Euler

: yields approximate solution values at equidistant *x*-values with an initial value x_0.

$$y(x_0) = y_0$$

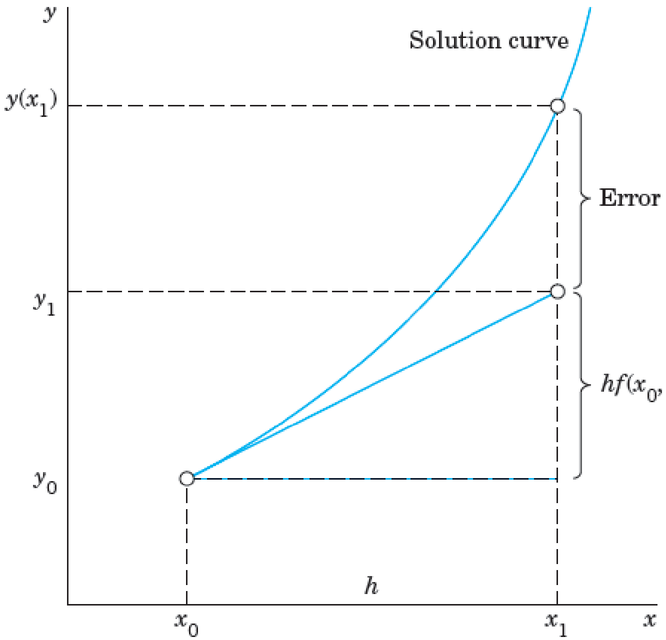
$$x_1 = x_0 + h, \quad y_1 = y_0 + hf(x_0, y_0)$$

$$x_2 = x_0 + 2h, \quad y_2 = y_1 + hf(x_1, y_1)$$

$$x_3 = x_0 + 3h, \quad y_3 = y_2 + hf(x_2, y_2)$$

$$\vdots \qquad\qquad \vdots$$

where the step h : a smaller value for greater accuracy e.g. 0.1 or 0.2



First Euler step, showing a solution curve, its tangent at (x_0, y_0), step h and increment $hf\ (x_0, y_0)$ in the formula for y_1

1.2 Geometric Meaning of y'=f(x, y). Direction Fields, Euler's Method

- *ODE*

$$y' = y+x,\ \ x=0,\ \ y(0)=0,\ \ h=0.2$$

- *Exact Solution*

$$y = e^x+x+1$$

Euler method for $y'=y+x$, $y(0)=0$ for $x=0,\dots,1.0$ with step $h=0.2$

n	x_n	y_n	$y(x_n)$	Error
0	0.0	0.000	0.000	0.000
1	0.2	0.000	0.021	0.021
2	0.4	0.04	0.092	0.052
3	0.6	0.128	0.222	0.094
4	0.8	0.274	0.426	0.152
5	1.0	0.488	0.718	0.230

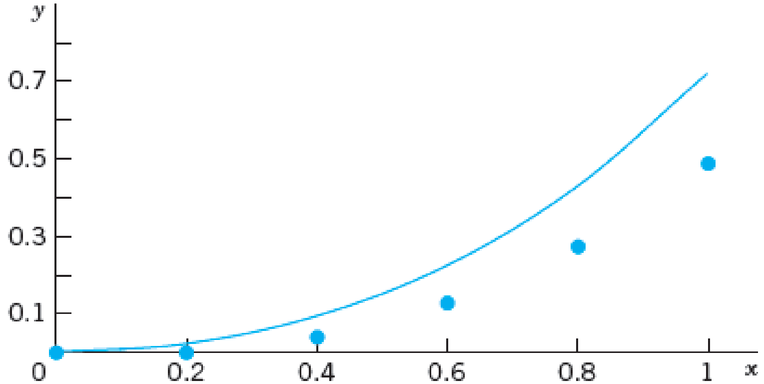


Fig. 9. Euler method: Approximate values in Table 1.1 and solution curve

$y' = f(x, y) = y+x$

$x_1 = x_0+h,\ \ y_1 = y_0+ hf(x_0, y_0) = y_0+h\,(y_0+x_0)$

$x_1 = 0 + 0.2 \ = \mathbf{0.2}$

$y_1 = 0 + 0.2\cdot 0 = \mathbf{0}$

$x_2=x_1+h,\ \ y_2 = y_1 + hf(x_1, y_1) = y_1+h\,(y_1+x_1)$

$x_2 = 0.2+0.2 \ = \mathbf{0.4}$

$y_2 = 0 + 0.2\cdot(0+0.2) = \mathbf{0.04}$

$x_3=x_2+h,\ \ y_3 = y_2 + hf(x_2, y_2) = y_2+h(y_2+x_2)$

$x_3 = 0.4 + 0.2 = \mathbf{0.6}$

$y_3 = 0.04 + 0.2\cdot(0.04+0.4) = \mathbf{0.128}$

$x_4=x_3+h,\ \ y_4 = y_3 + hf(x_3, y_3) = y_3+h(y_3+x_3)$

$x_4 = 0.6 + 0.2 = \mathbf{0.8}$

$y_4 = 0.128 + 0.2\cdot(0.128+0.6) = \mathbf{0.274}$

1.3 Separable ODEs. Modeling

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❖ **Separable Equation (변수분리형 방정식):**

A differential equation to be separable all the y's in the differential equation must be multiplied by the derivative and all the x's in the differential equation must be on the other side of the equal sign.

$$g(y)y' = f(x) \quad \Rightarrow \quad g(y)dy = f(x)dx \quad \left(\because y' = \frac{dy}{dx}\right)$$

❖ **Method of Separating Variables (변수분리법)**

$$g(y)y' = f(x) \quad \Rightarrow \quad \int g(y)\,dy = \int f(x)\,dx + c \quad \left(\because \frac{dy}{dx}dx = dy\right)$$

☑ **Ex. 1 Solve** $y' = 1 + y^2$

$$\frac{y'}{1+y^2} = 1 \quad \Rightarrow \quad \frac{dy/dx}{1+y^2} = 1 \quad \Rightarrow \quad \frac{dy}{1+y^2} = dx$$

$$\Rightarrow \quad \int \frac{1}{1+y^2}dy = \int dx + c \quad \Rightarrow \quad \arctan y = x + c \quad \Rightarrow \quad y = \tan(x+c)$$

$$\frac{d}{dx}(\arctan x) = \arctan' x = \frac{1}{1+x^2}$$

1.3 Separable ODEs. Modeling

☑ Example

Q?

Solve the IVP (Initial Value Problems).

$$\frac{dy}{dx} = -\frac{x}{y}, \quad y(4) = -3$$

1.3 Separable ODEs. Modeling

☑ Ex. 5: Find the amount of salt in the tank at any time t.

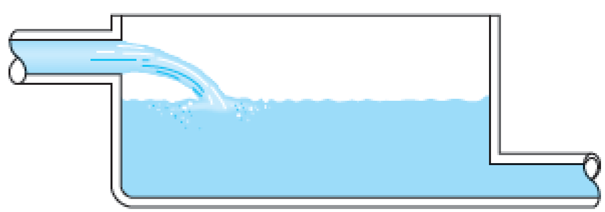
→ the amount of salt in the tank = y(t) in 1,000 gal

- The tank contains 1,000 *gal* of water in which initially 100*lb* of salt is dissolved. → Initial condition $y(0) = 100$ *lb*
- Brine (소금물) runs in at a rate of 10 *gal/min*, and each gallon contains 5*lb* of dissolved salt. → y_inflow = 10 gal/min × 5 lb/gal = 50 lb/min
- The mixture in **the tank is kept uniform** by stirring (휘저음).
- Brine runs out at 10 *gal/min*

  → y_outflow = 10 gal/min × y/1000 (lb/gal) = (y/100) lb/min
- Find the amount of salt in the tank at any time t.

the amount of salt = y *lb* in 1,000 *gal*

10 *gal/min* × 5 *lb/gal*
= 50 *lb/min*



10 *gal/min* × y/1000 *lb/gal*
= y/100 *lb/min*

1 *gal* = 3.78 *l*
1 *lb* = 0.454 *kg*

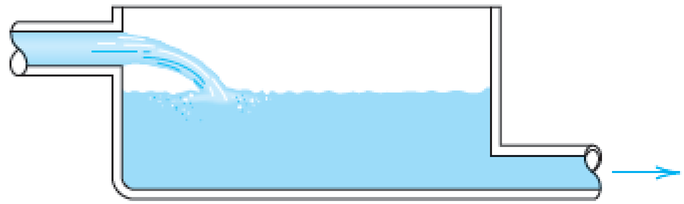
Tank

1.3 Separable ODEs. Modeling

☑ Ex. 5

the amount of salt = y *lb* in 1000 *gal*

10 *gal/min* × 5 *lb/gal*
= 50 *lb/min*



10 *gal/min* × y/1000 *lb/gal*
= y/100 *lb/min*

Tank

Step 1 Setting up a model.

▶ Salt's time rate of change = Salt inflow rate – Salt outflow rate "**Balance law**"

Salt inflow rate = 10 gal/min × 5 lb/gal = 50 lb/min

Salt outflow rate = 10 gal/min × y/1000 lb/gal = y/100 lb/min

$$\left(dy/dt = y'\right)$$

$$\Rightarrow \qquad y' = 50 - \frac{y}{100} = \frac{1}{100}\left(5{,}000 - y\right)$$

$1\ gal = 3.78\ l$
$1\ lb\ = 0.454\ kg$

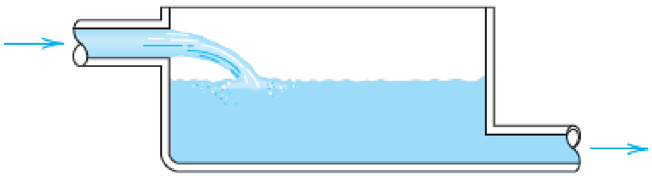
▶ The initial condition :

$$y(0) = 100$$

1.3 Separable ODEs. Modeling

Step 1 Setting up a model.

$$\Rightarrow \quad y' = 50 - \frac{y}{100} = \frac{1}{100}(5{,}000 - y) \qquad y(0) = 100$$



Tank

Step 2 Solution of the model.

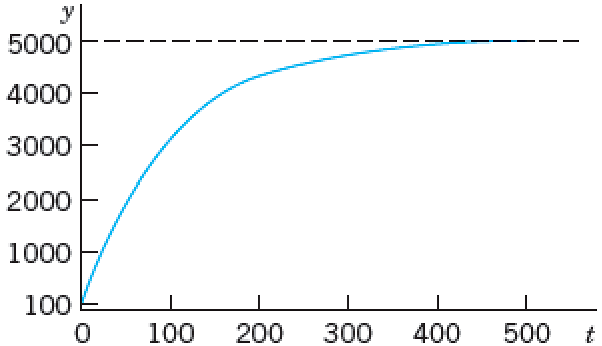
- General solution :

$$\frac{dy}{y - 5{,}000} = -\frac{1}{100}dt \quad \Rightarrow \quad \ln|y - 5{,}000| = -\frac{1}{100}t + c^* \quad \Rightarrow \quad y - 5{,}000 = ce^{-\frac{t}{100}}$$

- Particular solution :

$$y(0) = 5{,}000 + ce^0 = 5{,}000 + c = 100 \quad \Rightarrow \quad c = -4{,}900$$

$$y = 5{,}000 - 4{,}900 \; e^{-\frac{t}{100}}$$



Salt content $y(t)$

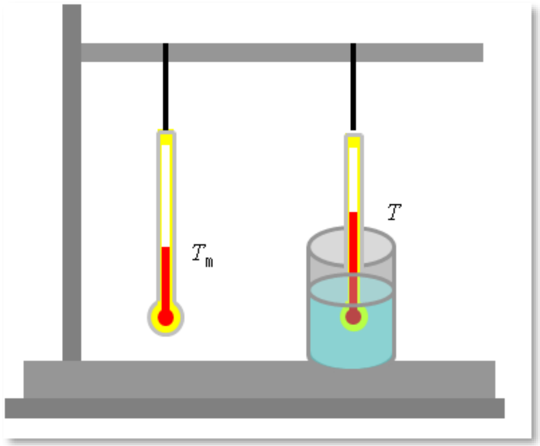
1.3 Separable ODEs. Modeling

☑ Newton's Law of Cooling

- The rate at which the temperature change of a body (물)
  is proportional to
  the difference between
  temperature of the body and
  the temperature of the surrounding medium.

$$\frac{dT}{dt} \propto (T - T_A) \Rightarrow \frac{dT}{dt} = k(T - T_A)$$

$$where,\ k < 0$$



T : Body temperature

T_A : Surronding medium temperature

1.3 Separable ODEs. Modeling

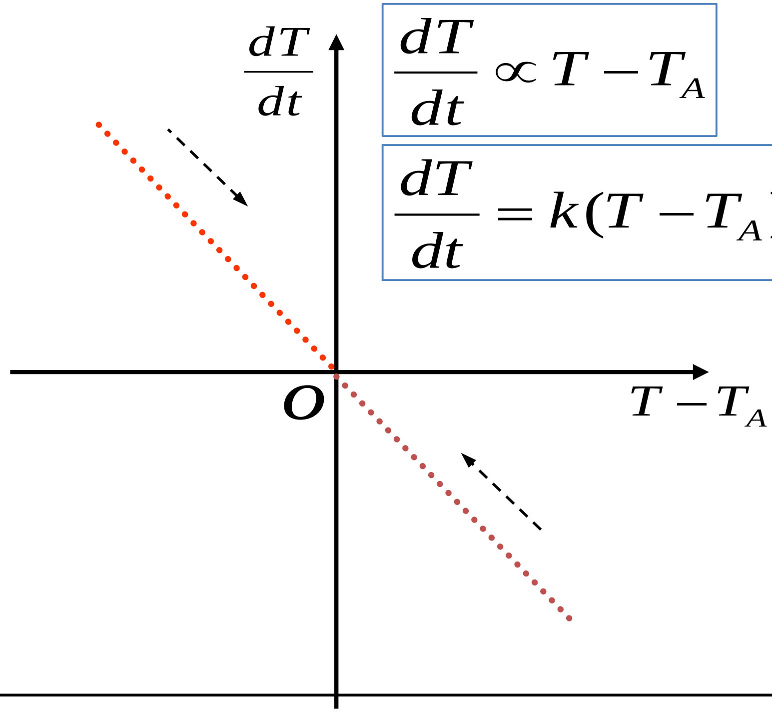
☑ Newton's Law of Cooling

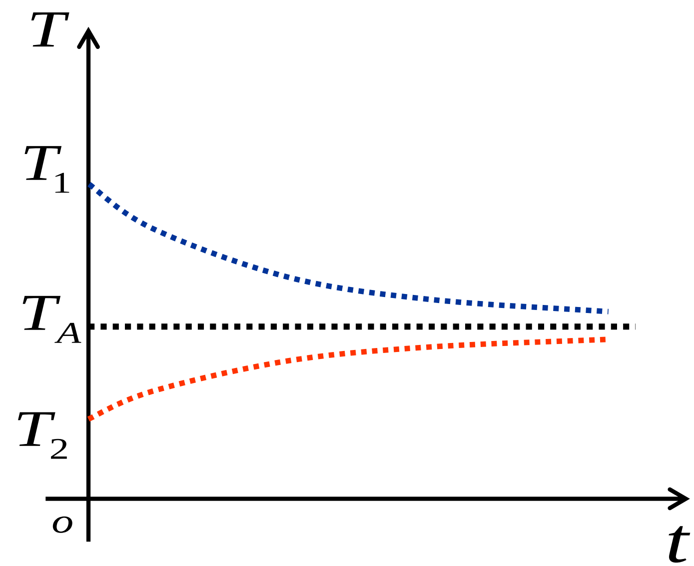
Great Idea !!

Relation between $\dfrac{dT(t)}{dt}$ and $T - T_A$.

$$\frac{dT}{dt} \propto T - T_A$$

$$\frac{dT}{dt} = k(T - T_A),\ k < 0$$





T : *body temperature*

T_A : *outside temperature (constant)*

T_1, T_2 : *Initial Body temperatures*

$$T_1 > T \quad , \quad T_2 < T$$

1.3 Separable ODEs. Modeling

☑ Newton's Law of Cooling

$$\boxed{\frac{dT}{dt} = k(T - T_A), \quad k < 0}$$

$$\frac{dT}{T - T_A} = k \cdot dt$$

$$Y = T - T_A$$

$$\frac{dY}{dT} = 1 \Rightarrow dY = dT$$

$$\frac{dT}{T - T_A} = k \cdot dt$$

$$\frac{dY}{Y} = k \cdot dt$$

$$\int \frac{dY}{Y} = \int k \cdot dt$$

$$\ln|Y| + c_L = kt + c_R$$

$$\ln|Y| = kt + c_R - c_L$$

$$= kt + c$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

1.3 Separable ODEs. Modeling

☑ Newton's Law of Cooling

①

$$\boxed{\frac{dT}{dt} = k(T - T_A), \quad k < 0}$$

$$\ln|Y| = kt + c$$

$$e^{\ln|Y|} = e^{kt+c}$$

$$|Y| = e^{kt+c}$$

②

$$|Y| = \tilde{c}e^{kt}$$

If $Y \geq 0,$

$$Y = \tilde{c}e^{kt}$$

If $Y < 0,$

$$-Y = \tilde{c}e^{kt}$$

$$Y = -\tilde{c}e^{kt}$$

③

$$Y = Ce^{kt}$$

$,(C = \mathrm{sgn}(Y = T - T_A) \cdot \tilde{c})$

$$T - T_A = Ce^{kt}$$

$$\boxed{T = Ce^{kt} + T_A}$$

1.3 Separable ODEs. Modeling

$$\frac{dT}{dt} = k(T - T_A)$$

☑ **Ex. 6** Suppose that in winter the daytime temperature in a certain office building is maintained at 70ºF → Initial condition

- The heating is shut off at 10 P.M. and turned on again at 6 A.M.
- On a certain day the temperature inside the building = 65ºF at 2 A.M.
- The outside temperature: 50ºF at 10 P.M. ~ 40ºF by 6 A.M.

What was the temperature inside the building (T) when the heat was turned on at 6 A.M.?

Step 1 Setting up a model

Temperature inside the building T(t), Outside temperature T_A

$$\frac{dT}{dt} = k(T - T_A)$$

Step 2 General Solution

T_A varied between 50ºF to 40ºF,

Golden Rule: If you cannot solve your problem, try to solve a simpler one.

T_A = 45ºF

$$\frac{dT}{(T-45)} = k\,dt \qquad T(t) = 45 + Ce^{kt}$$

Step 3 Particular solution Let 10 P.M to t=0. → $T(0)=70$

$$T(0) = 45 + Ce^0 = 70 \quad \Rightarrow C = 25, \quad T_p(t) = 45 + 25e^{kt}$$

1.3 Separable ODEs. Modeling

☑ **Ex. 6** Suppose that in winter the daytime temperature in a certain office building is maintained at 70°F.

- The heating is shut off at 10 P.M. and turned on again at 6 A.M.
- On a certain day the temperature inside the building = 65°F at 2 A.M.
- The outside temperature: 50°F at 10 P.M. ~ 40°F by 6 A.M.

What was the temperature inside the building (T) when the heat was turned on at 6 A.M.?

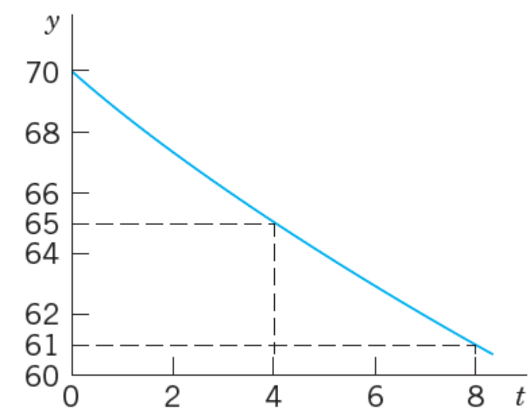
Step 4 Determination of k $T(4)=65$

$$T_p(4) = 45 + 25e^{4k} = 65 \qquad e^{4k} = 0.8 \quad k = \frac{1}{4}\ln 0.8 = -0.056$$

$$T_p(t) = 45 + 25e^{-0.056t}$$

Step 5 Answer and interpretation 6 A.M is t=8

$$T_p(8) = 45 + 25e^{-0.056t} = 61[^\circ \mathrm{F}]$$



Particular solution (temperature) in Example 6

1.3 Separable ODEs. Modeling

❖ **Extended Method (확장방법)** : Reduction to Separable Form. Certain first order equations that are not separable can be made separable by a simple change of variables.

- A homogeneous ODE $y' = f\left(\frac{y}{x}\right)$ can be reduced to separable form by the substitution of ***y=ux***

$$y' = f\left(\frac{y}{x}\right) \Rightarrow u'x + u = f(u) \Rightarrow \frac{du}{f(u)-u} = \frac{dx}{x} \quad \left(y = ux \Rightarrow u = \frac{y}{x} \ \& \ y' = (ux)' = u'x + u\right)$$

Q?

☑ **Ex. 8** Solve $2xyy' = y^2 - x^2$

1.4 Exact ODEs, Integrating Factors

1.4 Exact ODEs, Integrating Factors

- **Exact Differential Equation (완전미분 방정식)**:

  The ODE $M(x, y)dx + N(x, y)dy = 0$ whose the differential form $M(x, y)dx + N(x, y)dy$ is exact (완전미분), that is, this form is the differential $du = \underbrace{\frac{\partial u}{\partial x}}_{M(x,y)} dx + \underbrace{\frac{\partial u}{\partial y}}_{N(x,y)} dy$ of $u(x, y)$

- If ODE is an exact differential equation, then

$$M(x,y)dx + N(x,y)dy = 0 \quad \Rightarrow \quad du = 0 \quad \Rightarrow \quad u(x,y) = c$$

- Condition for exactness: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \left(\because \quad \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial N}{\partial x} \right)$

- Solve the exact differential equation.

$$M(x,y) = \frac{\partial u}{\partial x} \quad \Rightarrow \quad u(x,y) = \int M(x,y)dx + k(y) \quad \Rightarrow \quad \frac{\partial u}{\partial y} = N(x,y) \quad \Rightarrow \quad \frac{dk}{dy} \ \& \ k(y)$$

$$N(x,y) = \frac{\partial u}{\partial y} \quad \Rightarrow \quad u(x,y) = \int N(x,y)dy + l(x) \quad \Rightarrow \quad \frac{\partial u}{\partial x} = M(x,y) \quad \Rightarrow \quad \frac{dl}{dx} \ \& \ l(x)$$

1.4 Exact ODEs, Integrating Factors

☑ **Ex. 1 Solve** $\cos(x+y)dx + \left(3y^2 + 2y + \cos(x+y)\right)dy = 0$

Step 1 Test for exactness.

$$M(x,y) = \cos(x+y) \quad \Rightarrow \quad \frac{\partial M}{\partial y} = -\sin(x+y)$$

$$N(x,y) = 3y^2 + 2y + \cos(x+y) \quad \Rightarrow \quad \frac{\partial N}{\partial x} = -\sin(x+y)$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Step 2 Implicit general solution.

$$u(x,y) = \int M(x,y)dx + k(y) = \int \cos(x+y)dx + k(y) = \sin(x+y) + k(y)$$

$$\Rightarrow \quad \frac{\partial u}{\partial y} = \cos(x+y) + \frac{dk}{dy} = N(x,y) \quad \Rightarrow \quad \frac{dk}{dy} = 3y^2 + 2y \quad \Rightarrow \quad k = y^3 + y^2 + c^*$$

$$\therefore \quad u(x,y) = \sin(x+y) + y^3 + y^2 = c$$

Step 3 Checking an implicit solution. $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \cos(x+y)dx + (\cos(x+y) + 3y^2 + 2y)dy = 0$$

1.4 Exact ODEs, Integrating Factors

☑ Example

Q?

Solving an Exact DE

Solve $2xydx + (x^2 - 1)dy = 0$

1.4 Exact ODEs, Integrating Factors

$$du = \boxed{\frac{\partial u}{\partial x}} dx + \boxed{\frac{\partial u}{\partial y}} dy$$

$M(x, y)$ $N(x, y)$

❖ Reduction to Exact Form, Integrating Factors (적분 인자)

Some equations can be made exact by multiplication by some function, $F(x, y) \neq 0$, which is usually called the Integrating Factor.

☑ Ex. 3 Breakdown in the Case of Nonexactness

$$-ydx + xdy = 0$$

$\because \quad \frac{\partial}{\partial y}(-y) = -1, \quad \frac{\partial}{\partial x}(x) = 1 \quad \Rightarrow$ That equation is not exact.

If we multiply it by $1/x^2$, we get an exact equation

$$-\frac{y}{x^2}dx + \frac{1}{x}dy = 0 \left(\because \quad \frac{\partial}{\partial y}\left(-\frac{y}{x^2}\right) = -\frac{1}{x^2} = \frac{\partial}{\partial x}\left(\frac{1}{x}\right) \right)$$

General solution $y/x = c$

1.4 Exact ODEs, Integrating Factors

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

❖ How to Find Integrating Factors (F) ?

$$FPdx + FQdy = 0$$

The exactness condition : $\dfrac{\partial}{\partial y}(FP) = \dfrac{\partial}{\partial x}(FQ) \quad \Rightarrow \quad \dfrac{\partial F}{\partial y}P + F\dfrac{\partial P}{\partial y} = \dfrac{\partial F}{\partial x}Q + F\dfrac{\partial Q}{\partial x}$

Golden Rule : If you cannot solve your problem, try to solve a simpler one.

Hence we look for an integrating factor depending only on one variable.

Case 1)

$$F = F(x) \qquad \Rightarrow \qquad \frac{\partial F}{\partial x} = F', \quad \frac{\partial F}{\partial y} = 0$$

"$F=F(x,y)$ 가 일반적이지만, 단순히 $F(x)$로 가정"

$$FP_y = F'Q + FQ_x$$

$$F'Q = F(P_y - Q_x)$$

$$\frac{F'}{F} = \frac{1}{Q}(P_y - Q_x)$$

$$\frac{1}{F}\frac{dF}{dx} = R(x) \quad \text{where } R(x) = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$\therefore F(x) = \exp\left(\int R(x)\,dx\right)$$

Q?

Case 2)

1.4 Exact ODEs, Integrating Factors

☑ **Ex. Find an integrating factor and solve the initial value problem**

$$\left(e^{x+y} + ye^{y}\right)dx + \left(xe^{y} - 1\right)dy = 0, \quad y(0) = -1$$

$$R(x) = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$F(x) = \exp\left(\int R(x)dx\right)$$

$$R^*(y) = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$F^*(y) = \exp\left(\int R^*(y)dy\right)$$

Step 1 Nonexactness.

$$P(x,y) = e^{x+y} + ye^{y} \quad \Rightarrow \quad \frac{\partial P}{\partial y} = e^{x+y} + e^{y} + ye^{y}$$

$$Q(x,y) = xe^{y} - 1 \quad \Rightarrow \quad \frac{\partial Q}{\partial x} = e^{y}$$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$$

Step 2 Integrating factor. General solution.

$$R^*(y) = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \frac{1}{e^{x+y} + ye^{y}}\left(e^{y} - e^{x+y} - e^{y} - ye^{y}\right) = -1 \quad \Rightarrow \quad F^*(y) = e^{-y}$$

$$\therefore \quad \left(e^{x} + y\right)dx + \left(x - e^{-y}\right)dy = 0$$ is the exact equation.

Q? Why not R?

1.4 Exact ODEs, Integrating Factors

☑ **Ex. Find an integrating factor and solve the initial value problem**

$$\left(e^{x+y}+ye^{y}\right)dx+\left(xe^{y}-1\right)dy=0,\quad y(0)=-1$$

$$\boxed{M(x,y)=\frac{\partial u}{\partial x}}$$

$$\boxed{u(x,y)=\int M(x,y)\,dx+k(y)}$$

$$\boxed{\frac{\partial u}{\partial y}=N(x,y)}$$

Step 2 Integrating factor. General solution. $F^*(y)=e^{-y}$

$$\underbrace{\left(e^{x}+y\right)}_{M(x,\,y)}dx+\underbrace{\left(x-e^{-y}\right)}_{N(x,\,y)}dy=0$$

$$u=\int\left(e^{x}+y\right)dx=e^{x}+xy+k(y)$$

$$\Rightarrow\quad \frac{\partial u}{\partial y}=x+k'(y)=x-e^{-y}\quad\Rightarrow\quad k'(y)=-e^{-y},\quad k(y)=e^{-y}$$

The general solution is $u(x,y)=e^{x}+xy+e^{-y}=c$

Step 3 Particular solution

$$y(0)=-1\quad\Rightarrow\quad u(0,-1)=e^{0}+0+e=3.72$$

$$\therefore\ u(x,y)=e^{x}+xy+e^{-y}=3.72$$

1.4 Exact ODEs, Integrating Factors

☑ Example

Nonexact ODE

Solve

Q? Integration Factor

$$xydx + (2x^2 + 3y^2 - 20)dy = 0$$

1.5 Linear ODEs. Bernoulli Equation. Population Dynamics

1.5 Linear ODEs. Bernoulli Equation. Population Dynamics

ODEs
- Linear ODEs (선형 미분방정식)
  - Homogeneous Linear ODEs (제차 선형미분방정식)
  - Nonhomogeneous Linear ODEs (비제차 선형미분방정식)
- Nonlinear ODEs (비선형 미분방정식)

❖ **Linear ODEs**: ODEs which is linear in both the unknown function (y) and its derivative (y').

Ex. $y' + p(x)y = r(x)$: Linear differential equation

$y' + p(x)y = r(x)y^2$: Nonlinear differential equation

▪ Standard Form : $y' + p(x)y = r(x)$ ($r(x)$: Input, $y(x)$: Output)

❖ **Homogeneous, Nonhomogeneous Linear ODE**

$y' + p(x)y = 0$: Homogeneous Linear ODE

$y' + p(x)y = r(x) \neq 0$: Nonhomogeneous Linear ODE

1.5 Linear ODEs. Bernoulli Equation. Population Dynamics

- **Homogeneous Linear ODE** (Apply the method of separating variables)

$$y' + p(x)y = 0 \quad \Rightarrow \quad y = ce^{-\int p(x)dx} \qquad (1^*)$$

- **Nonhomogeneous Linear ODE** (Find integrating factor and solve)

$$y' + p(x)y = r(x) \quad \Rightarrow \quad (py - r)dx + dy = 0$$ is not exact

$$\left(\because \ \frac{\partial}{\partial y}(py - r) = p \neq 0 = \frac{\partial}{\partial x}(1) \right)$$

- Find integrating factor. We multiply $F(x)$.

$$Fy' + \boxed{pFy} = rF \quad \Rightarrow \quad \textbf{If } pFy = F'y \quad \Rightarrow \quad Fy' + \boxed{F'y} = rF$$

$$\Rightarrow \quad (Fy)' = rF \quad \Rightarrow \quad F(x)y = \left(\int r(x)F(x)dx + c\right)$$

$$y = \frac{1}{F(x)}\left(\int r(x)F(x)dx + c\right)$$

- 즉, ***pF=F'*** **이 되는** F**를 찾아서 양변에 곱하면** $\rightarrow$ ***Exact ODE***

❖ **Nonhomogeneous Linear ODE** (Find integrating factor and solve)

$$y' + p(x)y = r(x) \quad \Rightarrow \quad (py - r)dx + dy = 0$$ is not exact $\left(\because \frac{\partial}{\partial y}(py - r) = p \neq 0 = \frac{\partial}{\partial x}(1) \right)$

From exactness condition

$$Fy' + pFy = rF$$ ➡ $$Fdy + (pFy - rF)\,dx = 0$$

➡ $$(pFy - rF)dx + Fdy = 0$$

➡ $$\frac{\partial}{\partial y}(pFy - rF) = \frac{\partial F}{\partial x}$$

➡ $$pF = F'$$

1.5 Linear ODEs. Bernoulli Equation. Population Dynamics

- **Nonhomogeneous Linear ODE** (Find integrating factor and solve)

$$y' + p(x)y = r(x) \quad \Rightarrow \quad (py - r)dx + dy = 0$$

- Find integrating factor (F) from $pF = F'$

- By separating variables, $$p = \frac{F'}{F} \quad \Rightarrow p = \frac{dF}{dx}\frac{1}{F} \Rightarrow pdx = \frac{dF}{F}$$

- By integration, writing $h = \int p\,dx$,

$$\ln|F| = h = \int p\,dx, \;\Rightarrow\; F = e^h$$

- With $F = e^h$ and $h' = p$, Eq. $Fy' + pFy = rF$ becomes

$$e^h y' + h' e^h y = e^h y' + (e^h)' y = \boxed{(e^h y)' = re^h} \;\Rightarrow\; e^h y = \int e^h r dx + c$$

- By integration,

$$y(x) = e^{-h}\left(\int e^h r dx + c\right) = e^{-h}\int e^h r dx + ce^{-h}, \qquad h = \int p(x)dx$$

☑ **Ex. 1 Solve the linear ODE** $y' - y = e^{2x}$

$$\frac{dy}{dx} - (y + e^{2x}) = 0 \quad \Rightarrow (y + e^{2x})dx - dy = 0$$

$$(y + e^{2x})Fdx - Fdy = 0, \qquad F = F(x)$$

$$\text{Exactness :} \ \frac{\partial}{\partial y}(yF + e^{2x}F) = -\frac{\partial F}{\partial x} \quad \Rightarrow \quad F = -F'$$

($pF = F'$ ➡ $p = -1$)

$$\frac{dF}{F} = -dx \quad \Rightarrow \ln F = -x \ \Rightarrow F = e^{-x}$$

$$(y + e^{2x})e^{-x}dx - e^{-x}dy = 0$$ **($Fy' + pFy = rF$)**

$$where, \ \frac{\partial}{\partial y}(ye^{-x} + e^{x}) = e^{-x}, \frac{\partial}{\partial x}(-e^{-x}) = e^{-x}$$

$$e^{-x}y' - e^{-x}y = e^{2x}e^{-x} \qquad \Rightarrow \quad (e^{-x}y)' = e^{x}$$

$$(e^{-x}y)dy = e^{x}dx \quad \Rightarrow \quad e^{-x}y = e^{x} + c$$

$$\boxed{y = e^{2x} + ce^{x}}$$

1.5 Linear ODEs. Bernoulli Equation. Population Dynamics

☑ **Ex. 1 Solve the linear ODE** $y' - y = e^{2x}$

$$y' + p(x)y = r(x)$$

$$\boxed{\begin{aligned} y(x) &= e^{-h}\left(\int e^h r\,dx + c\right), \qquad h = \int p(x)\,dx \\ &= e^{-h}\int e^h r\,dx + ce^{-h} \end{aligned}}$$

$$p = -1,\ \ r = e^{2x},\ h = \int p\,dx = -x \quad \Rightarrow \quad \therefore\ \ y = e^{-h}\left[\int e^h r\,dx + c\right] = e^x\left[\int e^{-x}e^{2x}\,dx + c\right] = e^x\left[e^x + c\right] = e^{2x} + ce^x$$

1.5 Linear ODEs. Bernoulli Equation. Population Dynamics

❖ **Bernoulli Equation**: $y' + p(x)y = g(x)y^a \quad (a \neq 0 \ \& \ 1)$: Nonlinear ODE

We set $u(x) = [y(x)]^{1-a}$

$$\boxed{y' = g(x)y^a - p(x)y}$$

$$\Rightarrow u' = (1-a)y^{-a}\boxed{y'} = (1-a)y^{-a}(gy^a - py) = (1-a)(g - py^{1-a}) = (1-a)(g - pu)$$

$$\Rightarrow u' + (1-a)pu = (1-a)g$$

: Now transformed to Linear ODE

$$\boxed{y' + p(x)y = r(x)}$$

$$\boxed{y(x) = e^{-h}\left(\int e^h r dx + c\right), \qquad h = \int p(x)dx}$$

1.5 Linear ODEs. Bernoulli Equation. Population Dynamics

❖ **Bernoulli Equation**: $y' + p(x)y = g(x)y^a \quad (a \neq 0 \ \& \ 1)$

We set $u(x) = [y(x)]^{1-a}$

☑ Ex. 4 **Logistic Equation**

Solve the following Bernoulli equation, known as the logistic equation (or Verhulst equation) $y' = Ay - By^2$

$$y' = Ay - By^2 \quad \Rightarrow \quad y' - Ay = -By^2 \ \ \& \ \ a = 2 \left(u = y^{-1}\right)$$

$$\Rightarrow \quad u' = -y^{-2}y' = -y^{-2}\left(Ay - By^2\right) = -Ay^{-1} + B = -Au + B \quad \Rightarrow \quad u' + Au = B$$

$$p = A,\ r = B \quad \Rightarrow \quad h = \int p\,dx = Ax \quad \& \quad u = e^{-h}\left[\int e^h r\,dx + c\right] = e^{-Ax}\left[\frac{B}{A}e^{Ax} + c\right] = ce^{-Ax} + \frac{B}{A}$$

$$\boxed{y(x) = e^{-h}\left(\int e^h r\,dx + c\right), \qquad h = \int p(x)\,dx}$$

The general solution of the equation is $y = \dfrac{1}{u} = \dfrac{1}{\left(B/A + ce^{-Ax}\right)}$

1.6 Orthogonal Trajectories (직교 절선) - Skip

❖ Orthogonal Trajectory

: A family of curves in the plane that intersect a given family of curves at given angles.

❖ Find the orthogonal trajectories by using ODEs.

Step 1 Find an ODE $y' = f(x, y)$ for which the give family is a general solution.

Step 2 Write down the ODE $y' = -\dfrac{1}{f(x, y)}$ of the orthogonal trajectories.

Step 3 Solve it.

☑ Ex. A one-parameter family of quadratic parabolas is given by $y = cx^2$

Step 1 $\dfrac{y}{x^2} = c \quad \Rightarrow \quad \dfrac{y'x^2 - 2xy}{x^4} = 0 \quad \Rightarrow \quad y' = \dfrac{2y}{x}$

Step 2 $y' = -\dfrac{x}{2y}$

Step 3 $2yy' + x = 0 \quad \Rightarrow \quad y^2 + \dfrac{1}{2}x^2 = c^*$

1.7 Existence and Uniqueness of Solutions for Initial Value Problems

1.7 Existence and Uniqueness of Solutions for Initial Value Problems

- An initial value problem may have no solution, precisely one solution, or more than one solution.

- Ex. $|y'|+|y|=0, \quad y(0)=1$ $\Rightarrow$ No solution

  $y'=2x, \quad y(0)=1$ $\Rightarrow$ Precisely one solution $\Rightarrow$ $y=x^2+1$

  $xy'=y-1, \quad y(0)=1$ $\Rightarrow$ Infinitely many solutions $\Rightarrow$ $y=1+cx$

Problem of Existence (존재성)

Under what conditions does an initial value problem have at least one solution (hence one or several solutions)?

Problem of Uniqueness (유일성)

Under what conditions does that problem have at most one solution (hence excluding the case that has more than one solution)?

1.7 Existence and Uniqueness of Solutions for Initial Value Problems

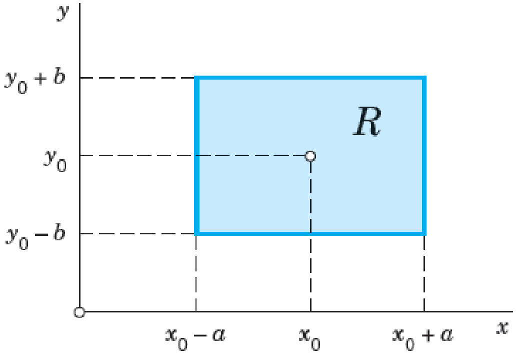
❖ **Theorem 1 Existence Theorem (존재 정리)**

Let the right side $f(x,y)$ of the ODE in the initial value problem.

(1) $$y' = f(x,y), \quad y(x_0) = y_0$$

be continuous at all points (x, y) in some rectangle

$$R: \ |x - x_0| < a, \quad |y - y_0| < b$$



and bounded in R ; that is, there is a number K such that

(2) $$|f(x,y)| \le K$$ for all (x, y) in R.

Then the initial value problem (1) has at least one solution $y(x)$. This solution exists at least for all x in the subinterval $|x - x_0| < \alpha$ of the interval $|x - x_0| < a$; here, α is the smaller of the two numbers a and b/K.

EX) $f(x, y) = x^2 + y^2$ *is bounded (with* $K = 2$) in the squre of $|x|<1$, $|y|<1$.
$f(x, y) = \tan(x+y)$ is not bounded for $|x+y|<\pi/2$

1.7 Existence and Uniqueness of Solutions for Initial Value Problems

❖ **Theorem 2 Uniqueness Theorem (유일성 정리)**

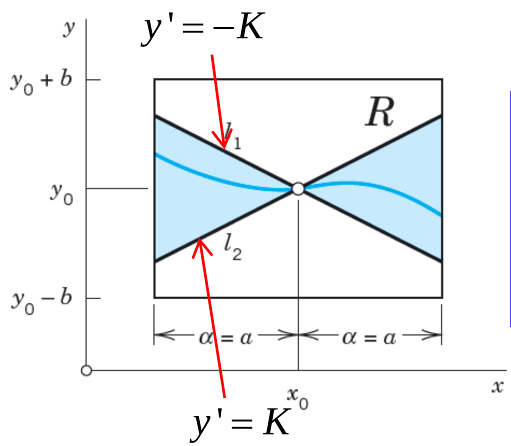
Let f and its partial derivative $f_y = \partial f / \partial y$ be continuous for all (x, y) in the rectangle R and bounded, say,

(3) (a) $\left|f(x,y)\right| \le K$ (b) $\left|f_y(x,y)\right| \le M$ for all (x,y) in R.

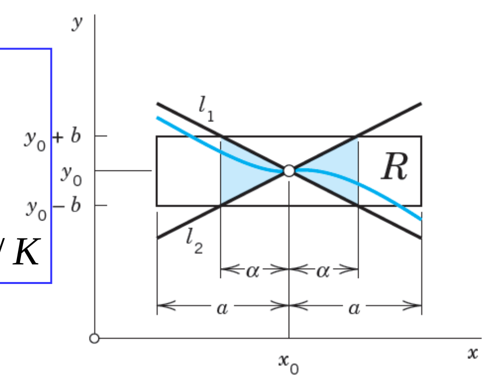
Then the initial value problem (1) has at most one solution $y(x)$. Thus, by the Existence Theorem, the problem has precisely one solution. This solution exists at least for all x in that subinterval $\left|x - x_0\right| < \alpha$.

$$y' = f(x, y), \text{the condition (2) implies that } |y'| \le K$$

Case of $b \ge aK$
$\Rightarrow \alpha = a$
Solution exists for
$x_0\text{-}a \le x \le x_0 + a$



Case of $b < aK$
$\Rightarrow \alpha = b / K < a$
Solution exists for
$x_0\text{-}b / K \le x \le x_0 + b / K$



1.7 Existence and Uniqueness of Solutions for Initial Value Problems

☑ **Ex. 1** Consider initial value problem

$$y' = 1 + y^2, \qquad y(0) = 0$$

$$R;\ |x| < 5,\ |y| < 3, \quad \text{then,} \ \ a = 5, b = 3 \ \mathit{and}$$

$$|f(x,y)| = |1 + y^2| \le K = 10$$

$$\left|\frac{\partial f}{\partial y}\right| = 2|y| \le M = 6 \ \Rightarrow \alpha = \frac{b}{K} = 0.3 < a$$

The solution of the problem $y = \tan x$. It is discontinuous at $\pm\pi/2$ and no continuous solution valid in the entire interval from which we started $|x|<5$.

$$\frac{y'}{1+y^2} = 1 \quad \Rightarrow \quad \frac{dy/dx}{1+y^2} = 1 \quad \Rightarrow \quad \frac{dy}{1+y^2} = dx$$

$$\Rightarrow \quad \int \frac{1}{1+y^2}\,dy = \int dx + c \quad \Rightarrow \quad \arctan y = x + c \quad \Rightarrow \quad y = \tan(x+c)$$

$$\Rightarrow \quad y = \tan(x+c) \quad (\because y(0) = 0)$$

[Reference] Natural Logarism (ln(x)) function

Rule name	Rule	Example
Product rule	$\ln(x \cdot y) = \ln(x) + \ln(y)$	$\ln(3 \cdot 7) = \ln(3) + \ln(7)$
Quotient rule	$\ln(x / y) = \ln(x) - \ln(y)$	$\ln(3 / 7) = \ln(3) - \ln(7)$
Power rule	$\ln(x^{y}) = y \cdot \ln(x)$	$\ln(2^8) = 8 \cdot \ln(2)$
ln derivative	$f(x) = \ln(x) \Rightarrow f'(x) = 1 / x$	
ln integral	$\int \ln(x)dx = x \cdot (\ln(x) - 1) + C$	
ln of negative number	$\ln(x)$ is undefined when $x \leq 0$	
ln of zero	$\ln(0)$ is undefined	
	$\lim_{x \to 0^+} \ln(x) = -\infty$	
ln of one	$\ln(1) = 0$	
ln of infinity	$\lim \ln(x) = \infty$,when $x \to \infty$	