# Manipulator Jacobians

Dongjun Lee (이동준)

Department of Mechanical & Aerospace Engineering
Seoul National University



## Spatial Manipulator Jacobian $J^s_{st}$

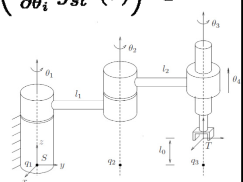
- Denote EF pose by $g_{st}(\theta) \in SE(3)$, $\theta = (\theta_1, \theta_2, .., \theta_n)$. Its differential relation between joint rate $\dot{\theta}$ and EF velocities $\Rightarrow$ **Jacobian.**

- First, write the spatial velocity of $g_{st}(\theta)$: using the chain rule,

$$\hat{V}^s_{st} = \dot{\bar{g}}_{st}(\theta)\bar{g}^{-1}_{st}(\theta) = \sum_{i=1}^{n} \left( \frac{\partial \bar{g}_{st}}{\partial \theta_i} \dot{\theta}_i \right) \bar{g}^{-1}_{st}(\theta) = \sum_{i=1}^{n} \left( \frac{\partial \bar{g}_{st}}{\partial \theta_i} \bar{g}^{-1}_{st}(\theta) \right) \dot{\theta}_i$$

where $\frac{\partial \bar{g}_{st}}{\partial \theta_i} = \begin{bmatrix} \frac{\partial R_{st}(\theta)}{\partial \theta_i} & \frac{\partial p_{st}(\theta)}{\partial \theta_i}; & 0 & 0 \end{bmatrix} \in \Re^{4\times 4}$ with $\left( \frac{\partial \bar{g}_{st}}{\partial \theta_i} \bar{g}^{-1}_{st}(\theta) \right) \in$ se(3) having the meaning of twist.

- We may then further write



$$\boxed{V^s_{st} = J^s_{st}(\theta)\dot{\theta} \in \Re^6}$$

where $J^s_{st}(\theta) \in \Re^{6\times n}$ is **spatial manipulator Jacobian:**

$$J^s_{st}(\theta) := \left[ \left( \frac{\partial \bar{g}_{st}}{\partial \theta_1} \bar{g}^{-1}_{st}(\theta) \right)^\vee, \ldots\ldots, \left( \frac{\partial \bar{g}_{st}}{\partial \theta_n} \bar{g}^{-1}_{st}(\theta) \right)^\vee \right]$$

which defines **linear** relation between $\dot{\theta}$ and $V^s_{st}$.

# Geometric Meaning of $J^s_{st}$

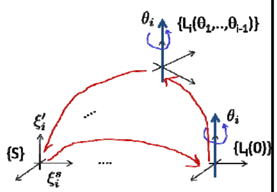
- Recall

$$g_{st}(\theta) = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}.....e^{\hat{\xi}_n\theta_n}g_{st}(0)$$

where $\xi^s_i$ is twist of joint $i$ at **reference** configuration expressed in $\{S\}$.

- The $i$-th column of $J^s_{st}(\theta)$ is then given by

$$\begin{aligned}\hat{\xi}'_i = \frac{\partial \bar{g}_{st}}{\partial \theta_i}\bar{g}_{st}^{-1} &= e^{\hat{\xi}_1\theta_1}.....e^{\hat{\xi}_{i-1}\theta_{i-1}}\left(\hat{\xi}_i\right)e^{\hat{\xi}_i\theta_i}e^{\hat{\xi}_{i+1}\theta_{i+1}}....e^{\hat{\xi}_n\theta_n}\bar{g}_{st}(0)\bar{g}_{st}^{-1} \\ &= e^{\hat{\xi}_1\theta_1}.....e^{\hat{\xi}_{i-1}\theta_{i-1}}\left(\hat{\xi}_i\right)e^{-\hat{\xi}_{i-1}\theta_{i-1}}....e^{-\hat{\xi}_1\theta_1} \\ &= \bar{g}^s_{l_i(0)l_i(\theta_1,\dots,\theta_{i-1})}\left(\hat{\xi}_i\right)\bar{g}^{s\,-1}_{l_i(0)l_i(\theta_1,\dots,\theta_{i-1})}\end{aligned}$$




- Thus, we can write spatial Jacobian $J^s_{st}(\theta)$ s.t.

$$\boxed{\begin{gathered}J^s_{st}(\theta) = \left[\ \xi_1 \quad \xi'_2 \quad \dots \quad \xi'_n\ \right] \in \Re^{6\times n} \\ \xi'_i = \mathrm{Ad}_{e^{\hat{\xi}_1\theta_1}.....e^{\hat{\xi}_{i-1}\theta_{i-1}}}\,\xi^s_i \in \Re^6\end{gathered}}$$

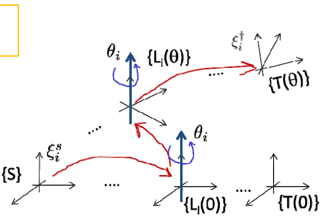
- Here, $\xi'_i = \mathrm{Ad}_{g_{sl_i(\theta_1,\dots,\theta_{i-1})}}\mathrm{Ad}_{g^{-1}_{sl_i(0)}}\xi^s_i$, i.e, denotes joint $i$ motion expressed in $\{S\}$ at the **current** configuration with $\theta_i$-axis moved by $\theta_1,\dots,\theta_{i-1}$ from the reference configuration (similar to the case of 2-DOF arm).



# Body Manipulator Jacobian $J^b_{st}$

- We also define **body manipulator Jacobian** $J^b_{st} \in \Re^{6\times n}$ s.t.

$$\boxed{V^b_{st} = \left(\bar{g}^{-1}_{st}(\theta)\dot{\bar{g}}_{st}(\theta)\right)^\vee = J^b_{st}(\theta)\dot{\theta}}$$




- Then, from $V^b_{st} = \mathrm{Ad}^{-1}_{g_{st}(\theta)}V^s_{st} = \mathrm{Ad}^{-1}_{g_{st}(\theta)}J^s_{st}(\theta)\dot{\theta}$,

$$J^b_{st}(\theta) = \mathrm{Ad}^{-1}_{g_{st}(\theta)}J^s_{st}$$

thus, we can obtain: given $\xi^s_i$,

$$\boxed{\begin{gathered}J^b_{st}(\theta) = \left[\ \xi^\dagger_1 \quad \xi^\dagger_2 \quad .... \quad \xi^\dagger_n\ \right] \in \Re^{6\times n} \\ \xi^\dagger_i = \mathrm{Ad}^{-1}_{e^{\hat{\xi}_i\theta_i}.....e^{\hat{\xi}_n\theta_n}\bar{g}_{st}(0)}\,\xi^s_i \in \Re^6\end{gathered}}$$

with $\xi^\dagger_i := \mathrm{Ad}^{-1}_{g_{st}(\theta)}\xi'_i = \mathrm{Ad}^{-1}_{g_{st}(\theta)}\mathrm{Ad}_{e^{\hat{\xi}_1\theta_1}.....e^{\hat{\xi}_{i-1}\theta_{i-1}}}\xi^s_i = \mathrm{Ad}^{-1}_{e^{\hat{\xi}_i\theta_i}.....e^{\hat{\xi}_n\theta_n}g_{st}(0)}\xi_i$.

- Here, $\xi^\dagger_i = \mathrm{Ad}^{-1}_{g_{st}(\theta)}\xi'_i = \mathrm{Ad}_{g^{-1}_{st(\theta)}}\mathrm{Ad}_{g_{sl_i(\theta)}}\mathrm{Ad}_{g^{-1}_{sl_i(0)}}\xi^s_i = \mathrm{Ad}_{g^{-1}_{l_i(\theta)t(\theta)}}\mathrm{Ad}_{g^{-1}_{sl_i(0)}}\xi^s_i$, i.e., $\theta_i$-motion expressed in $\{S\}$, mapped to $\{L_i(0)\}$, rotate to **current** configuration $\{L_i(\theta)\}$, then, mapped to $\{T(\theta)\}$ expressed in $\{T(\theta)\}$ (note: $\mathrm{Ad}_{g_{sl_i(\theta_{1:i-1})}}\xi^b_i = \mathrm{Ad}_{g_{sl_i(\theta_{1:i})}}\xi^b_i = \mathrm{Ad}_{g_{sl_i(\theta)}}\xi^b_i$ as they are the same motion expressed in $\{S\}$. cf. Lecture 4, slide p. 11)

# Example 3.8: SCARA

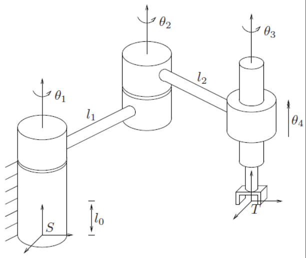
- Recall $\xi_i^s = [-w_i \times q_i; w_i]$ for revolute and $\xi_i^s = [v_i; 0]$ for prismatic.
- With $\xi_i^s$ defined at **reference** configuration in $\{S\}$, we can then compute $J_{st}^s$ with $\xi_i' = \mathrm{Ad}_{e^{\hat{\xi}_1\theta_1}\cdots e^{\hat{\xi}_{i-1}\theta_{i-1}}}\xi_i^s$.
- Or, we may compute $J_{st}^s$ at current configuration via observation:

$$J_{st}^s = \begin{bmatrix} \xi_1 & \xi_2' & \xi_3' & \xi_4' \end{bmatrix} = \begin{bmatrix} 0 & -w_2 \times q_2 & -w_3 \times q_3 & v_4 \\ w_1 & w_2 & w_3 & 0 \end{bmatrix}$$

where

$$q_1 = [0;0;0], \quad q_2 = [-l_1\, s_{\theta_1}; l_1\, c_{\theta_1}; 0]$$
$$q_3 = [-l_1\, s_{\theta_1} - l_2\, s_{\theta_1+\theta_2}; l_1\, c_{\theta_1} + l_2\, c_{\theta_1+\theta_1}; 0]$$

with $w_1 = w_2 = w_3 = [0;0;1]$ and $v_4 = [0;0;1]$.



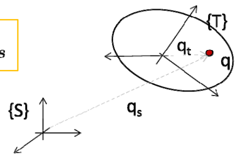

- We can also similarly compute $J_{st}^b$ via observation, i.e., joint twist written in $\{T\}$ at current configuration.



# Further on $J^s_{st}$ and $J^b_{st}$

- Now, rigidly attach a point $q$ on EF. Then, $\bar{q}_s = \bar{g}_{st}(\theta)\bar{q}_t$.
- Velocity of $q$ expressed in $\{S\}$ is then given by

$$\boxed{\bar{v}_q^s := \dot{\bar{q}}_s = \dot{\bar{g}}_{st}\bar{q}_t = \dot{\bar{g}}_{st}\bar{g}_{st}^{-1}\bar{q}_s = \hat{V}_{st}^s\bar{q}_s = \left(J_{st}^s(\theta)\dot{\theta}\right)^\wedge \bar{q}_s}$$




- The **same** velocity of $q$ is expressed in $\{T\}$ by

$$\boxed{\bar{v}_q^b := \bar{g}_{st}^{-1}\bar{v}_q^s = \bar{g}_{st}^{-1}\dot{\bar{g}}_{st}\bar{q}_t = \hat{V}_{st}^b\bar{q}_t = \left(J_{st}^b(\theta)\dot{\theta}\right)^\wedge \bar{q}_t}$$

- If $q$ is origin of $\{T\}$ (i.e., $q_t = 0$), with $g_{st}(\theta) = (R_{st}(\theta), q_s(\theta))$,

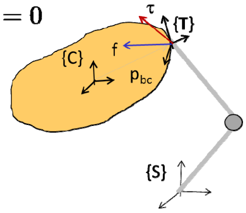
$$\bar{v}_q^s = \begin{pmatrix} \dot{q}_s \\ 0 \end{pmatrix} = \underbrace{\begin{bmatrix} \hat{w}_s & -\hat{w}_s q_s + \dot{q}_s \\ 0 & 0 \end{bmatrix}}_{\hat{V}_{st}^s} \begin{pmatrix} q_s \\ 1 \end{pmatrix} = \begin{pmatrix} R v_q^b \\ 0 \end{pmatrix} = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} \hat{w}_b & R^T \dot{q}_s \\ 0 & 0 \end{bmatrix}}_{\hat{V}_{st}^b} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- Note that $v_q^s = R v_q^b \neq v_{st}^s$ of the spatial velocity $V_{st}^s = (v_{st}^s, w_{st}^s)$, although $v_q^b = v_{st}^b$ of the body velocity $V_{st}^b = (v_{st}^b, w_{st}^b)$.

## Force Kinematics

- Suppose we apply joint torque $\tau \in \Re^n$ to resist body wrench $F_b$ acting through the origin of $\{T\}$.
- Then, with no friction, to maintain static equilibrium, **the principle of virtual work** should hold:

$$\delta W = \tau^T \delta\theta + F_b^T \delta g_b = \tau^T \delta\theta + F_b^T J_{st}^b \delta\theta = 0$$




implying that

$$\boxed{\tau = -(J_{st}^b)^T F_b}$$

- Also, using $F_b = \mathrm{Ad}_{g_{st}}^T F_s$ and $J_{st}^s = \mathrm{Ad}_{g_{st}} J_{st}^b$,

$$\boxed{\tau = -(J_{st}^s)^T F_s}$$

where $F_b, F_s$ are body and spatial wrenches applied at the orign of $\{T\}$.

- If $J_{st}$ is square/invertible (other cases later), this relation can be used for:
  - Given EF wrench $F$, what joint torque $\tau$ is required to resist this $F$?
  - Given joint torque $\tau$, what EF wrench will be generated?

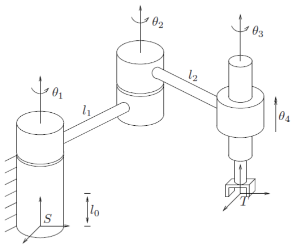


## Example 3.10: SCARA

- Force kinematics:

$$\tau = -(J_{st}^s)^T F_s$$

where $J_{st}^s \in \Re^{6\times 4}$ is tall $\Rightarrow$ **deficient** manipulator




- Some wrench $F$ resisted by mechanical structure.
- Basis vectors of null-space of $(J_{st}^s)^T$:

$$F_{n1} = [0;0;0;1;0;0]$$
$$F_{n2} = [0;0;0;0;1;0]$$

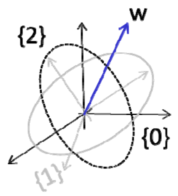
- Recall screw coordinates: $F = (f;\tau) = (w; -w \times q + hw)\theta$ or $F = (0;w)\theta$.
- $F_{n1}, F_{n2}$ are pure-torques along $x, y$-axes of $\{S\}$, which do not have any image on $\tau \Rightarrow$ balanced by robot's mechanical structure, not by joint torque $\tau$.

# Classical Jacobian: Angular Velocities

- Consider composition of rotations

$$R_2^0 = R_1^0 R_2^1$$

where $R_1^0$ represents rotation of $\{1\}$ relative to $\{0\}$ and $R_2^1$ that of $\{2\}$ relative to $\{1\}$. Combined angular velocity $w_{0,2}^0$ is given by:




$$\dot{R}_2^0 = S(w_{0,2}^0)R_2^0, \qquad S(w) := \hat{w}$$

- Yet, we also have: using $S(Rw) = RS(w)R^T$,

$$\begin{aligned}\dot{R}_2^0 &= \dot{R}_1^0 R_2^1 + R_1^0 \dot{R}_2^1 = S(w_{0,1}^0)R_1^0 R_2^1 + R_1^0 S(w_{1,2}^1)R_2^1 \\ &= S(w_{0,1}^0)R_2^0 + R_1^0 S(w_{1,2}^1)R_1^{0T} R_1^0 R_2^1 = S(w_{0,1}^0)R_2^0 + S(R_1^0 w_{1,2}^1)R_2^0 \\ &= S(w_{0,1}^0 + R_1^0 w_{1,2}^1)R_2^0 = S(w_{0,1}^0 + w_{1,2}^0)R_2^0\end{aligned}$$

- Thus, $w_{0,2}^0 = w_{0,1}^0 + w_{1,2}^0$. Or, more generally, for $R_n^0 = R_1^0 R_2^1 .... R_n^{n-1}$,

$$\boxed{w_{0,n}^0 = w_{0,1}^0 + w_{1,2}^0 + ... w_{n-1,n}^0, \qquad w_{i,i+1}^0 = R_i^0 w_{i,i+1}^i}$$

i.e., angular velocities can be simply added if expressed in same frame.

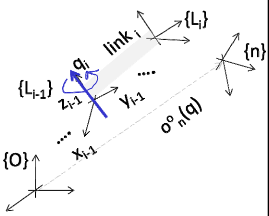


# Angular Velocity Manipulator Jacobian

- Denote pose of EF frame $\{n\}$ relative to base frame $\{0\}$ by

$$\bar{g}_{0,n}(q) = \begin{bmatrix} R_n^0(q) & o_n^0(q) \\ 0 & 1 \end{bmatrix}$$

where $q \in \Re^n$ joint variable, $o_n^0(q) \in \Re^3$ is of $\{n\}$.




- $z_{i-1}$ is along $q_i$-axis, while $\{L_i\}$ moves with link $i$ by $q_i$.
- Then, angular velocity of EF $\{n\}$-frame is given by

$$w_{0,n}^0 = w_{0,1}^0 + R_1^0 w_{1,2}^1 + ... + R_{n-1}^0 w_{n-1,n}^{n-1}$$

where, for $q_i$, $w_{i-1,i}^{i-1} = \dot{q}_i z_{i-1}^{i-1} = \dot{q}_i k$ if revolute, or $w_{i-1,i}^{i-1} = 0$ if prismatic.

- Thus, we can define **angular velocity Jacobian** $J_w(q) \in \Re^{3\times n}$ s.t.,

$$\boxed{\begin{aligned} w_{0,n}^0 &= \phi_1 z_0^0 \dot{q}_1 + \phi_2 R_1^0 z_1^1 \dot{q}_2 + ... + \phi_n R_{n-1}^0 z_{n-1}^{n-1}\dot{q}_n \\ &= \begin{bmatrix} \phi_1 z_0^0 & \phi_2 z_1^0 & ... & \phi_n z_{n-1}^0 \end{bmatrix} \dot{q} = J_w(q)\dot{q}\end{aligned}}$$

where $\phi_i = 1$ if $q_i$ is revolute; $\phi_i = 0$ if prismatic.

# Linear Velocity Manipulator Jacobian

- Linear velocity of origin of EF $\{n\}$-frame is given by

$$
\begin{aligned}
v^0_{0,n} := \dot{o}^0_n(q) &= \frac{\partial o^0_n}{\partial q_1}\dot{q}_1 + \frac{\partial o^0_n}{\partial q_2}\dot{q}_2 + \ldots + \frac{\partial o^0_n}{\partial q_n}\dot{q}_n \\
&= \left[ \begin{array}{cccc} \frac{\partial o^0_n}{\partial q_1} & \frac{\partial o^0_n}{\partial q_2} & \ldots. & \frac{\partial o^0_n}{\partial q_n} \end{array} \right] \dot{q} = J_v(q)\dot{q}
\end{aligned}
$$

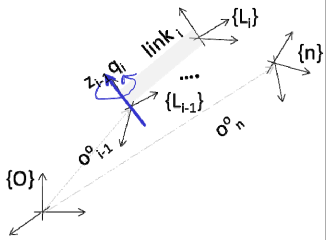
where $J_v(q) \in \Re^{3\times n}$ is **linear velocity Jacobian**.

- Can obtain $J_{vi} = \frac{\partial o^0_n}{\partial q_i}$ by seeing $v^0_{0,n}$ with only $\dot{q}_i$ and all other joints fixed.
- Then, we have, if $i$-th joint is revolute,

$$v^0_{0,n} = z^0_{i-1} \times (o^0_n - o^0_{i-1})\dot{q}_i$$

or if revolute

$$v^0_{0,n} = z^0_{i-1}\dot{q}_i$$




- Thus, linear velocity Jacobian can be obtained by

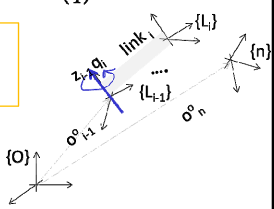
$$\boxed{J_{vi}(q) = \begin{cases} z^0_{i-1} \times (o^0_n - o^0_{i-1}) & \text{if } q_i \text{ is revolute} \\ z^0_{i-1} & \text{if } q_i \text{ is prismatic} \end{cases}}$$



# Manipulator Jacobian - Classical

- Combining this, we have **classical manipulator Jacobian** $J(q) \in \Re^{6\times n}$:

$$\boxed{\xi = \begin{pmatrix} v^0_{0,n} \\ w^0_{0,n} \end{pmatrix} = \begin{bmatrix} J_v(q) \\ J_w(q) \end{bmatrix} \dot{q} =: J(q)\dot{q}}$$




where

$$J_w(q) = \left[ \begin{array}{cccc} \phi_1 z^0_0 & \phi_2 z^0_1 & \ldots & \phi_n z^0_{n-1} \end{array} \right]$$

and

$$J_{vi}(q) = \begin{cases} z^0_{i-1} \times (o^0_n - o^0_{i-1}) & \text{if } q_i \text{ is revolute} \\ z^0_{i-1} & \text{if } q_i \text{ is prismatic} \end{cases}$$

- Note $\xi$ differs from spatial velocity $V^s_{st} = (v^s_{st}, w^s_{st})$ with $v^0_{0,n} \neq v^s_{st}$ although $w^0_{0,n} = w^s_{st}$.
- This $\xi$ is the same as body velocity $V^b_{st} = (v^b_{st}, w^b_{st})$ expressed in $\{0\}$ with $v^0_{0,n} = R_{st}v^b_{st}$ and $w^0_{0,n} = R_{st}w^b_{st}$.

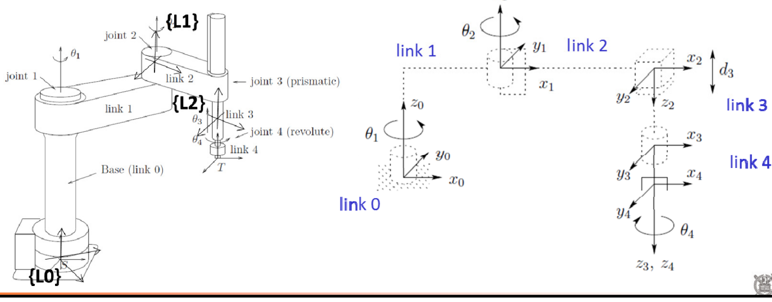
# Example: S4.8 SCARA

- Angular velocity Jacobian:

$$J_{w1} = [0;0;1], \quad J_{w2} = [0;0;1]$$
$$J_{w3} = [0;0;0], \quad J_{w4} = [0;0;-1]$$

- Linear velocity Jacobian:

$$J_{v1} = [0;0;1] \times o_4^0 = [-a_1\,\mathrm{s}_1 - a_2\,\mathrm{s}_{12}; a_1\,\mathrm{c}_1 + a_2\,\mathrm{c}_{12}; 0]$$
$$J_{v2} = [0;0;1] \times [a_2\,\mathrm{c}_{12}; a_2\,\mathrm{s}_{12}; -d_3 - d_4] = [-a_2\,\mathrm{s}_{12}; a_2\,\mathrm{c}_{12}; 0]$$
$$J_{v3} = [0;0;-1], \quad J_{v4} = [0;0;0]$$

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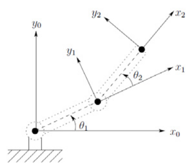
# Singularity



- Jacobian relations:

$$V_{st}^s = J_{st}^s(\theta)\dot{\theta}, \quad \tau = (J_{st}^s(\theta))^T F_s$$

- We say configuration $\theta$ is **singular** if its Jacobian $J_{st}^s(\theta)$ drops rank.
  - EF motion in certain direction not generatable (e.g., gimbal lock).
  - Very large $\dot{\theta}$ necessary to generate EF motion (internal instability).
  - EF force in certain direction resisted by mechanical structure.

- Recall $g_{st}(\theta)$ is global parameterization of SE(3). Instead, if we use local parameterization $f(\theta) = (x, y, z, p, r, y) \in \Re^6$, we have **analytical Jacobian** $J_a^s$:

$$J_a^s(\theta) = \frac{\partial f}{\partial \theta} \in \Re^{6\times n}$$

with

$$\boxed{\text{singularity}(J_a^s) = \text{singularity}(J_{st}^s) \bigcup \text{singularity}(f)}$$

i.e., encompassing manipulator Jacobian singularity (due to robot design) and local parameterization singularity (due to map $f$).

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## Manipulability

Manipulability measures how far a configuration is away from singularity

- For $\xi = J(\theta)\dot{\theta}$,

$$\mu_1(\theta) = \sigma_{\min}(J(\theta)) \geq 0$$

where

$$\sigma_{\min}(A) := \min_{||x||_2=1} ||Ax||_2 =: ||A||_2$$

is minimum singular value. This $\mu_1$ characterizes how "far" a configuration is from singularity; or minimum possible magnitude of $\xi$ given unit $\dot{q}$, i.e.,

$$\sigma^2_{\min}(J)||\dot{q}||^2 \leq \xi^T\xi = \dot{q}^T J^T J \dot{q} \leq \sigma^2_{\max}(J)||\dot{q}||^2$$

-
$$0 \leq \mu_2(\theta) = \frac{\sigma_{\min}(J(\theta))}{\sigma_{\max}(J(\theta))} \leq 1$$

This $\mu_2$ not only characterizes the closeness to the singularity, but also directionality as well. Ideally, we want $\mu_2(\theta) = 1$.

-
$$\mu_3(\theta) = |\det J(\theta)| = \sigma_1\sigma_2...\sigma_n$$

which characterizes the volume of the velocity ellipsoid.



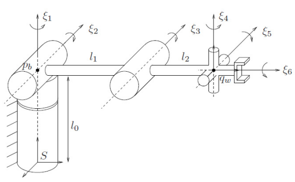
## Non-Square Jacobian

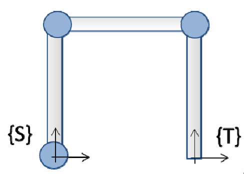
- In general, $\dim(\xi) \neq \dim(\dot{q})$, i.e., EF motion DOF of interest is not the same as joint variable DOF.
- If $\dim(\xi) < \dim(\dot{q}) \Rightarrow$ **redundant** manipulator.
- If $\dim(\xi) > \dim(\dot{q}) \Rightarrow$ **deficient** manipulator.
- Even if $\dim(\xi) \neq \dim(\dot{q})$, we still have

$$\boxed{\xi = J(q)\dot{q}, \quad J(q) \in \Re^{m\times n}}$$

i.e., all possible workspace velocity $\xi \in \Re^m$ is the one generatable (or permissible) by some joint motion $\dot{q} \in \Re^n$.

- Given $\dot{q} \Rightarrow \xi$ uniquely defined; given $\xi \Rightarrow \dot{q}$ may not be unique.



## Redundant Manipulator: Velocity Kinematics

- For redundant manipulator, we have

$$\boxed{\xi = J(q)\dot{q}, \quad \text{with } m < n \text{ and "fat" } J.}$$

- We can then decompose $\dot{q} \in \Re^n$ in its orthogonal components s.t.,

$$\dot{q} = \dot{q}_{\text{row}(J)} + \dot{q}_{\text{null}(J)} = J^T\alpha + \beta = \underbrace{J^T(JJ^T)^{-1}J\dot{q}}_{\in \text{ row}(J)} + \underbrace{[I - J^T(JJ^T)^{-1}J]\dot{q}}_{\in \text{ null}(J)}$$

  - $\dot{q}_{\text{row}(J)} \in \text{row}(J)$ is component of $\dot{q}$ producing **apparent velocity** $\xi$.
  - $\dot{q}_{\text{null}(J)} \in \text{null}(J) =\sim \text{col}(J^T)$ have no image to $\xi$: **internal motion**.

- This also shows that we can generate desired $\xi$ by

$$\boxed{\dot{q} = J^T(JJ^T)^{-1}\xi + [I - J^T(JJ^T)^{-1}J]b}$$

  - $b \in \Re^n$ can be **any** arbitrary vector.
  - $J^T(JJ^T)^{-1}\xi$ is **optimal** solution of $\dot{q}$ to produce $\xi$ (minimum $||\dot{q}||$).

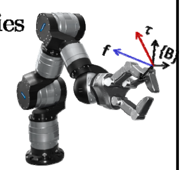


## Redundant Manipulator: Force Statics

- To maintain static equilibrium, **princple of virtual work** should hold:

$$\delta W = \tau^T\delta q + F^T\delta x = \tau^T\delta q + F^T J\delta q = 0$$

for **any** $\delta q \in \Re^n$ **given** $F$ with $\dot{x} = \xi = J(q)\dot{q}$, which then implies



$$\boxed{\tau = -J^T(q)F}$$

- We can also similarity decompose joint torque $\tau \in \Re^n$ s.t.

$$\begin{aligned}\tau = \tau_{\text{row}(J)} + \tau_{\text{null}(J)} &= \underbrace{J^T(JJ^T)^{-1}J\tau}_{\in\text{row}(J)} + \underbrace{[I - J^T(JJ^T)^{-1}J]\tau}_{\in\text{null}(J)} \\ &= -J^T F + [I - J^T(JJ^T)^{-1}J]b\end{aligned}$$

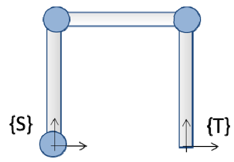
which will not maintain static equilibrium if $b \neq 0$ with $\delta W \neq 0 \ \forall \delta q \in \Re^n \Rightarrow$ not static problem any more, but dynamics/control problem.

- **Lagrange-D'Alembert principle** with generalized force:

$$\tau^T_{\text{generalized}}\delta q = \tau^T_{\text{joint}}\delta q + F^T_{\text{ext}}\delta x \quad \Rightarrow \quad \boxed{\tau_{\text{generalized}} = \tau_{\text{joint}} + J^T(q)F_{\text{ext}}}$$

## Example: Redundant Manipulator

- Only Cartesian velocity and force are considered.
- Jacobian relation given by




$$J = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

- Apparent motion (e.g., $\xi = [1;0]$); internal motion (i.e., $\dot{q} = [1;-1;1]$).

$$\dot{q} = \frac{1}{3}\begin{bmatrix} -1 & 2 \\ 1 & 1 \\ 2 & -1 \end{bmatrix}\xi + \frac{1}{3}\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} b$$

- Static balance torque (e.g., $F = [1;0]$ w/ $\tau = [0;1;1]$); motion-inducing torque in null($J$) (e.g., $\tau = [1;-1;1]$):

$$\tau = -J^T F + \frac{1}{3}\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} b$$

- Redundant manipulator force control $\Rightarrow$ hybrid force/position control or impedance/admittance control.



## Deficient Manipulator: Velocity Kinematics

- For deficient manipulator, we have

$$\boxed{\xi = J(q)\dot{q}, \qquad \text{with } m > n \text{ and "tall" } J.}$$

- $\xi$ must be in col($J$), i.e., $\xi$ should be **permissible** by some motion of $\dot{q}$.
- That means, in the orthogonal decomposition of $\xi$ s.t.

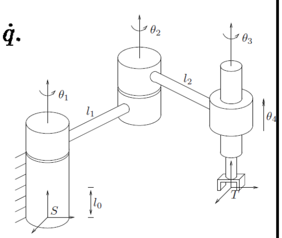
$$\xi = \xi_{\text{col}(J)} + \xi_{\sim\text{col}(J)} = \underbrace{J(J^TJ)^{-1}J^T\xi}_{\in\text{col}(J)} + \underbrace{[I - J(J^TJ)^{-1}J^T]\xi}_{\in\sim\text{col}(J)=\text{null}(J^T)}$$

  - $\xi_{\sim\text{col}(J)}$: **infeasible** velocity (e.g. velocity normal to the plane of planar robots), thus, should be zero.
  - $\xi_{\text{col}(J)}$: **feasible/permissible** by some motion of $\dot{q}$.

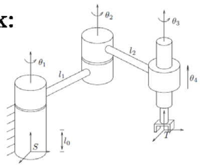
- By equating this with $\xi = J\dot{q}$, we have

$$\boxed{\dot{q} = (J^TJ)^{-1}J^T\xi.}$$

i.e., $\dot{q}$ to produce the desired $\xi \in \text{col}(J)$.



# Deficient Manipulator: Force Statics



- To maintain static equilibrium, from principle of virtual work:

$$\boxed{\tau = -J^T(q)F}$$

- We can decompose workspace force $F \in \Re^m$ s.t.

$$\boxed{F = F_{\text{col}(J)} + F_{\sim\text{col}(J)} = \underbrace{J(J^TJ)^{-1}J^TF}_{\in\text{col}(J)} + \underbrace{[I - J(J^TJ)^{-1}J^T]F}_{\in\sim\text{col}(J)}}$$

  - $F_{\text{col}(J)}$ needs to be sustained by $\tau = -J^TF = -J^TF_{\text{col}(J)}$
  - $F_{\sim\text{col}(J)}$ supported by **mechanical structure** of robot.
  - $\xi_{\sim\text{col}(J)} = 0$ (infeasible velocity), although $F_{\sim\text{col}(J)} \neq 0$ in general.

- If $F$ can resist any $\tau$ to maintain static posture of manipulator (i.e., $\ddot{q} = \dot{q} = 0$), the joint torque $\tau$ will generate external wrench

$$\boxed{F = -J(J^TJ)^{-1}\tau}$$

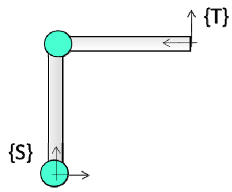
in the "actuated" direction col($J$); cannot generate workspace force in "un-actuated" direction $\sim$ col($J$) though.



# Example: Deficient Manipulator

- 2-DOF planar robot, yet, workspace of interest is 3D Cartesian motion.
- Jacobian relation given by

$$J = \begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$$




- Apparent motion (e.g., $\xi = [1;1;0]$) and infeasible motion (e.g., $\xi = [0;0;1]$)

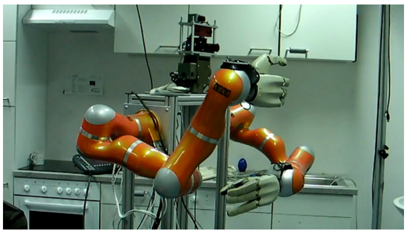
$$\dot{q} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}\xi$$

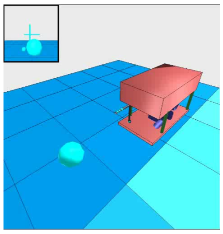
- Static balance torque (e.g., $F = [1;1;0]$ w/ $\tau = [0;-1]$); structurally-balanced force (e.g., $F = [0;0;1]$); and force generation in col($J$) (e.g., $\tau = [1;1]$ w/ $F = [0;1;0]$):

$$\tau = -J^TF, \quad F = -\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\tau$$

# Redundant Robots

- Consider a $n$-DOF robot with $q \in \mathcal{Q}$. Suppose we can define a task by $r = f(q) \in \Re^m$, where
$$f : \mathcal{Q}^n \to \mathcal{W}^m$$
- Robot is kinematically redundant if $n > m$.
- Redundancy can be utilized to:
  - Avoid collision with obstacle and self.
  - Avoid singularity and maintain manipulability.
  - Respect angle, velocity, acceleration limits.
  - Minimize energy consumption with minimum motion.







# Velocity Level Redundancy Resolution

- Consider a $n$-DOF robot with $q \in \mathcal{Q}$. Suppose we can define a task by $r = f(q) \in \Re^m$, where
$$f : \mathcal{Q}^n \to \mathcal{W}^m$$
- Typically, aim to achieve the **main task** $r(t) \to r_d(t)$, while optimizing cerain other requirements via **internal motion** (e.g., manipulability $\mu(q)$, collision distance $\varphi(q)$, etc.).
- Redundancy resolution in **configuration kinematics** level: given $r_d(t : t+T)$ and $H(q, \dot{q})$, find $q_d(t : t+T)$ s.t.
$$r_d(\tau) = f(q_d(\tau)) \quad \text{with} \quad H(q_d, \dot{q}_d) \geq \underline{H}, \quad \forall \tau \in [t, t+T]$$
- Redundancy resolution in **velocity kinematics** level: for Jacobian relation,
$$\dot{r} = \frac{\partial f}{\partial q}\dot{q} = J(q)\dot{q}, \quad J(q) \in \Re^{m \times n}$$
find $\dot{q}_d$ s.t.,
$$\dot{r}_d(t) = J(q(t))\dot{q}_d(t) \quad \text{with} \quad \dot{H}(t) \geq 0 \;\; \text{if} \;\; H(t^-) = \underline{H}$$

## Pseudo-Inverse Optimal Control

- Jacobian relation:

$$\dot{r} = \frac{\partial f}{\partial q}\dot{q} = J(q)\dot{q}, \quad J(q) \in \Re^{m \times n}$$

- Inverse Jacobian relation: for "fat" $J(q)$,

$$\dot{q} = J^T(JJ^T)^{-1}\dot{r} + [I - J^T(JJ^T)^{-1}J]b$$

where $b \in \Re^n$ can be arbitrary.

- The optimal solution is then

$$\boxed{\dot{q} = J^+\dot{r}, \quad J^+ := J^T(JJ^T)^{-1}}$$

where $J^+(q) \in \Re^{n \times m}$ is **Moore-Penrose pseudo-inverse** s.t.,

$$JJ^+J = J, \quad J^+JJ^+ = J^+, \quad (JJ^+)^T = JJ^+, \quad (J^+J)^T = J^+J$$

- Given any $J \in \Re^{m \times n}$, there always exists unique pseudo-inverse.



## Linear-Quadratic Optimization

- Linear-quadratic optimization formulation:

$$\boxed{\begin{array}{ll} \min\limits_{\dot{q} \in \Re^n} & ||\dot{q}||^2 := \frac{1}{2}\dot{q}^T\dot{q} \\ \text{subj.} & \dot{r} - J\dot{q} = 0 \end{array}}$$

  - Define Lagrangian $L(\dot{q}, \lambda) := \frac{1}{2}\dot{q}^T\dot{q} + \lambda^T(\dot{r} - J\dot{q})$, where $\lambda \in \Re^n$ is **Lagrangian multiplier**.
  - Necessary condition: $\frac{\partial L}{\partial \dot{q}} = \dot{q}^T - \lambda^T J = 0$ and $\frac{\partial L}{\partial \lambda} = (\dot{r} - J\dot{q})^T = 0$.
  - We can then obtain $\lambda = (JJ^T)^{-1}J\dot{q}$ and the optimal solution:

$$\dot{q}_{\text{optimal}} = J^T(JJ^T)^{-1}\dot{r} = J^+\dot{r}$$

- When $J$ drops rank, $\dot{q}$ will be unbounded. To maintain boundedness of $\dot{q}$ while crossing singularity,

$$\dot{q}_{\text{optimal}} = J^T(JJ^T + \alpha(t)I)^{-1}\dot{r} = J^+\dot{r}$$

i.e., **damped least square method** with variable damping $\alpha(t) > 0$, to ensure boundedness while compromising task (not exact inverse).

# Weighted Pseudo-Inverse Optimal Control

- Linear-quadratic optimization with positive-definite weight $W \in \Re^{n\times n}$:

$$\boxed{\begin{array}{ll} \min\limits_{\dot{q}\in\Re^n} & \|\dot{q}\|_W^2 := \frac{1}{2}\dot{q}^T W \dot{q} \\ \text{subj.} & \dot{r} - J\dot{q} = 0 \end{array}}$$

  - Define Lagrangian $L_W(\dot{q}, \lambda) := \frac{1}{2}\dot{q}^T W \dot{q} + \lambda^T(\dot{r} - J\dot{q})$.
  - Necessary condition: $\frac{\partial L}{\partial \dot{q}} = \dot{q}^T W - \lambda^T J = 0$ and $\frac{\partial L}{\partial \lambda} = (\dot{r} - J\dot{q})^T = 0$.
  - Weighted optimal solution

$$\dot{q}_{\text{optimal}} = W^{-1}J^T(JW^{-1}J^T)^{-1}\dot{r} = J_W^+\dot{r}$$

  with $\lambda = (JW^{-1}J^T)^{-1}J\dot{q}$
  - $J_W^+$ is a **generalized inverse** satisfying only some properties of the Moore-Penrose pseudo-inverse $J^+$.

- Choose gain $w_i$ of $W$ large along undesirable motion direction so that $\dot{q}_i$ will be small (e.g., small motion desired to avoid singularity/obstacle).



# Projected Gradient Control

- Inverse Jacobian relation of redundant manipulator:

$$\boxed{\dot{q} = J^+\dot{r} + [I - J^+J]\dot{q}_o = J^+\dot{r} + P\dot{q}_o}$$

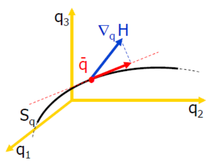
where $P := I - J^T(JJ^T)^{-1}J \in \Re^{n\times n}$ is null-space projection operator, and $\dot{q}_o$ defines internal motion.

- **Projected gradient method**: given cost function $H(q)$,

$$\boxed{\dot{q}_o := \left[\frac{\partial H}{\partial q}\right]^T =: \nabla_q H(q) \in \Re^n \quad \text{s.t.,} \quad \dot{q} = J^+\dot{r} + P\nabla_q H(q)}$$

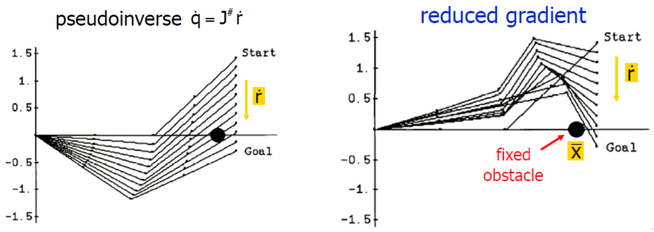
- Note that the gradient action $\nabla_q H(q)$ to avoid $H(q) < \underline{H}$ is **projected** into the null-space of $J$.

- Suppose $\dot{r} = 0$. Then,

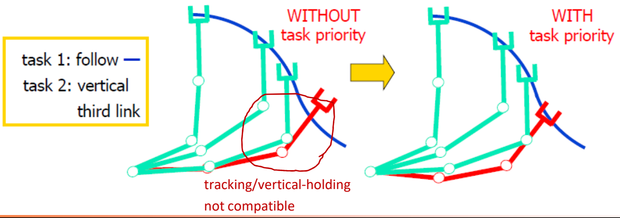



$$\frac{d}{dt}H(q) = \frac{\partial H}{\partial q}\dot{q} = \frac{\partial H}{\partial q}[J^+\dot{r} + P\nabla_q H(q)]$$
$$= \nabla_q^T H[I - J^T(JJ^T)^{-1}J]\nabla_q H \geq 0$$

# Simulation Example




- In this example, vertical motion and collision avoidance are compatible with each other, thus, can attain both of them at the same time.
- If collision is critical, yet, vertical motion not attainable while avoiding collision (i.e., not compatible), we want to put higher priority on collision while tolerating error in vertical motion $\Rightarrow$ **task priority control**




* exerpted from the lecture by Prof. A. De Luca, Spienza Universita Di Roma



# Task Priority Control

- Consider two tasks $r_1 = f_1(q)$ and $r_2 = f_2(q)$, with task $r_1$ having higher priority than task $r_2$.
- Want to guarantee task $r_1$ (e.g., collision) while trying to attain task $r_2$ (e.g., tracking) if permissible under task $r_1$.
- Highest priority task 1: for $\dot{r}_1^d = J_1\dot{q}$, the optimal solution $\dot{q}_1$ is

$$\boxed{\dot{q}_1 = J_1^+\dot{r}_1^d, \quad J_1^+ = J_1^T(J_1J_1^T)^{-1}}$$

- In the next level, we want to achieve task 2 under task 1 constraint, i.e., the solution $\dot{q}$ should have the form of

$$\dot{q}_2 = \dot{q}_1 + P_1v_1 = \underbrace{J_1^+\dot{r}_1^d}_{\in \text{row}(J_1)} + \underbrace{P_1v_1}_{\in \text{null}(J_1)}, \quad \text{where } v_1 \in \Re^n, \quad P_1 = [I - J_1^+J_1]$$

- Then, from $\dot{r}_2^d = J_2(\dot{q}_1 + P_1v_1)$, we have (optimal) solution $v_1 = (J_2P_1)^+[\dot{r}_2^d - J_2\dot{q}_1] = (J_2P_1)^+[\dot{r}_2^d - J_2J_1^+\dot{r}_1^d]$, thus, the combined (optimal) solution upto this level is

$$\boxed{\dot{q}_2 = \dot{q}_1 + (J_2P_1)^+[\dot{r}_2^d - J_2\dot{q}_1] = J_1^+\dot{r}_1^d + \underbrace{(J_2P_1)^+[\dot{r}_2^d - J_2J_1^+\dot{r}_1^d]}_{\in \text{row}(J_2) \text{ within } P_1 = \text{null}(J_1)}}$$

where we use $P_1(J_2P_1)^+ = (J_2P_1)^+$.

# Task Priority Control

- Highest priority task 1: $\dot{q}_1 = J_1^+ \dot{r}_1$, which can be also written by

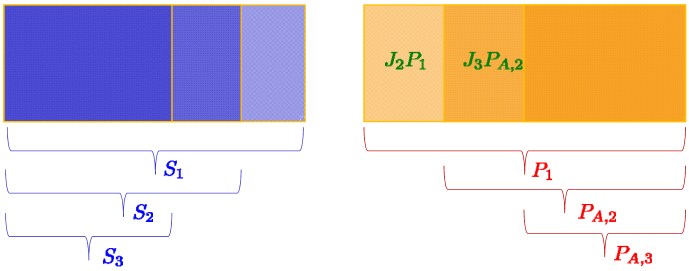
$$\dot{q}_1 = \mathrm{argmin}_{\dot{q} \in S_1} ||\dot{q}||^2, \quad S_1 =: \{\mathrm{argmin}_{\dot{q} \in S_o = \Re^n} ||\dot{r}_1^d - J_1 \dot{q}||^2\}$$

with $S_1 = \{\dot{q}_1 + \cancel{P_1 v_1}, v_1 \in \Re^{n-n_1}\}$, where $P_1 \approx \mathrm{null}(J_1)$.

- Next level task 2: $\dot{q}_2 = \dot{q}_1 + (J_2 P_1)^+ [\dot{r}_2^d - J_2 \dot{q}_1]$, which can be written as

$$\dot{q}_2 = \mathrm{argmin}_{\dot{q} \in S_2} ||\dot{q}||^2, \quad S_2 =: \{\mathrm{argmin}_{\dot{q} \in S_1} ||\dot{r}_2 - J_2 \dot{q}||^2\}$$

with $S_2 = \{\dot{q}_2 + \cancel{P_{A,2}} v_2, v_2 \in \Re^{n-n_1-n_2}\}$, where $P_{A,2} = P_1 - (J_2 P_1)^+ J_2 P_1$ is the subtraction of component of row$(J_2)$ from $P_1 = \mathrm{null}(J_1)$.






# Task Priority Control

- We can then obtain recursive formula with

$$\dot{q}_k = \mathrm{argmin}_{\dot{q} \in S_k} ||\dot{q}||^2, \quad S_k =: \{\mathrm{argmin}_{\dot{q} \in S_{k-1}} ||\dot{r}_k - J_k \dot{q}||^2\}$$

with $S_{k-1} = \{\dot{q}_{k-1} + P_{A,k-1} v_{k-1}, v_{k-1} \in \Re^n\}$ and the nested structure:

$$S_p \subset S_{p-1} \subset ... \subset S_1 \subset S_o = \Re^n$$

$$P_{A,k} = P_{A,k-1} - (J_k P_{k-1})^+ J_k P_{A,k-1}, P_{A,1} = P_1 = \mathrm{null}(J_1)$$