CHAPTER 4. SYSTEMS OF ODES. PHASE PLANE. QUALITATIVE METHODS

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서울대학교
조선해양공학과

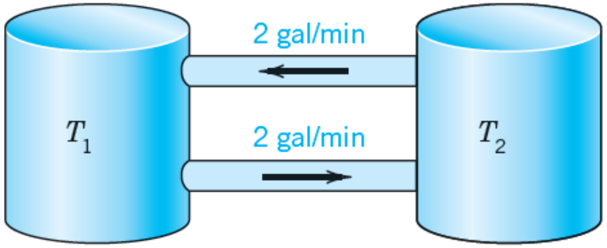
서 유 택

※ 본 강의 자료는 이규열, 장범선, 노명일 교수님께서 만드신 자료를 바탕으로 일부 편집한 것입니다.

4.1 Systems of ODEs as Models in Engineering Applications

☑ **Ex. 1 A mixing problem involving a single tank is modeled by a single ODE. The model will be a system of two first-order ODEs.**

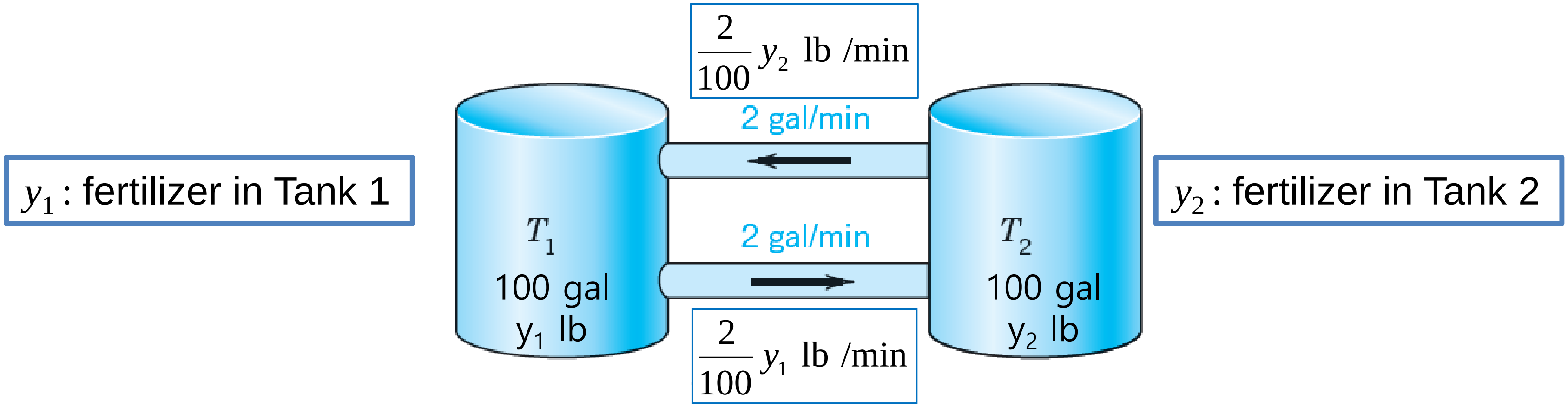
- Tank T_1 and T_2 contain initially 100 gal of water each.
- In T_1 the water is pure, whereas 150 lb of fertilizer (비료) are dissolved in T_2.



System of tanks

- By circulating liquid at a rate of 2 gal/min and stirring the amounts of fertilizer $y_1(t)$ in T_1 and $y_2(t)$ in T_2 change with time t.
- How long should we let the liquid circulate so that T_1 will contain at least half as much fertilizer as there will be left in T_2 ?

4.1 Systems of ODEs as Models in Engineering Applications



Step 1 Setting up the model

$$y_1' = \text{Inflow / min} - \text{Outflow / min} = \frac{2}{100} y_2 - \frac{2}{100} y_1 \qquad (\text{Tank } T_1) \qquad \Rightarrow \qquad y_1' = -0.02 y_1 + 0.02 y_2$$

$$y_2' = \text{Inflow / min} - \text{Outflow / min} = \frac{2}{100} y_1 - \frac{2}{100} y_2 \qquad (\text{Tank } T_2) \qquad \Rightarrow \qquad y_2' = 0.02 y_1 - 0.02 y_2$$

$$\therefore \quad \mathbf{y}' = \mathbf{A}\mathbf{y}, \qquad \mathbf{A} = \begin{bmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{bmatrix}$$

4.1 Systems of ODEs as Models in Engineering Applications

Step 2 General solution

Idea: We try an exponential function of t,

$$\boxed{\mathbf{y}' = \mathbf{A}\mathbf{y} = \mathbf{A}\mathbf{x}e^{\lambda t}}$$

$$\mathbf{y} = \mathbf{x}e^{\lambda t} \quad \Rightarrow \quad \boxed{\mathbf{y}'} = \lambda \mathbf{x}e^{\lambda t} = \mathbf{A}\mathbf{x}e^{\lambda t} \quad \Rightarrow \quad \mathbf{A}\mathbf{x} = \lambda \mathbf{x} \quad \Rightarrow$$

look for eigenvalues and eigenvectors of A

Characteristic equation:

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -0.02-\lambda & 0.02 \\ 0.02 & -0.02-\lambda \end{vmatrix} = (-0.02-\lambda)^2 - 0.02^2 = \lambda(\lambda + 0.04) = 0$$

$$\therefore \quad \lambda_1 = 0, \quad \mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad \lambda_2 = -0.04, \quad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Apply the superposition principle

$$\mathbf{y} = c_1 \mathbf{x}^{(1)} e^{\lambda_1 t} + c_2 \mathbf{x}^{(2)} e^{\lambda_2 t} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-0.04t} \qquad (c_1 \text{ and } c_1 \text{ are arbitrary constants})$$

4.1 Systems of ODEs as Models in Engineering Applications

Step 3 Use of initial conditions

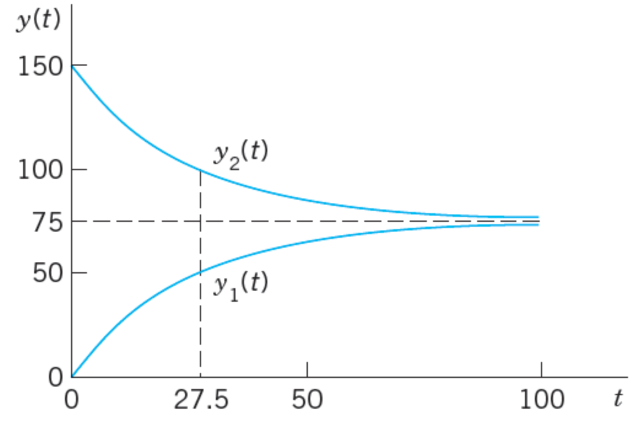
Initial conditions: $y_1(0)=0,\quad y_2(0)=150$

$$\mathbf{y}(0)=c_1\begin{bmatrix}1\\1\end{bmatrix}+c_2\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}c_1+c_2\\c_1-c_2\end{bmatrix}=\begin{bmatrix}0\\150\end{bmatrix}\quad\Rightarrow\quad c_1=75,\quad c_2=-75\quad\Rightarrow\quad \mathbf{y}=75\begin{bmatrix}1\\1\end{bmatrix}-75\begin{bmatrix}1\\-1\end{bmatrix}e^{-0.04t}$$

Step 4 Answer

T_1 contains half the fertilizer amount of T_2 if it contains 1/3 of the total amount, that is, 50lb.

$$y_1=75-75e^{-0.04t}=50\quad\Rightarrow\quad e^{-0.04t}=\frac{1}{3}\quad\Rightarrow\quad t=\ln 3/0.04=27.5\,(\text{about half an hour})$$



4.1 Systems of ODEs as Models in Engineering Applications

❖ **Conversion of an *n*th-Order ODE to a System**

(*n*계 미분방정식의 1계 연립상미분방정식으로의 변환)

▪ **Theorem 1 Conversion of an ODE**

An ***n*th-order ODE** $y^{(n)} = F\left(t, y, y', \cdots, y^{(n-1)}\right)$ can be converted to a system of

***n* first-order ODEs** by setting $y_1 = y,\ y_2 = y',\ y_3 = y'', \cdots, y_n = y^{(n-1)}$.

This system is of the form

$$
\begin{aligned}
y_1' &= y_2 \\
y_2' &= y_3 \\
&\vdots \\
y_{n-1}' &= y_n \\
y_n' &= F\left(t, y_1, y_2, \cdots, y_n\right)
\end{aligned}
$$

4.2 Basic Theory of Systems of ODEs. Wronskian

❖ First-order systems (1계 연립상미분방정식)

$$\begin{aligned} y_1' &= f_1(t, y_1, \cdots, y_n) \\ y_2' &= f_2(t, y_1, \cdots, y_n) \\ &\vdots \\ y_n' &= f_n(t, y_1, \cdots, y_n) \end{aligned} \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \longrightarrow \mathbf{y}' = \mathbf{f}(t, \mathbf{y})$$

For example, if $n = 1$, $y_1' = f_1(t, y_1)$ or $y' = f(t, y)$ (First ODE)

❖ Solution on some interval $a < t < b$

: A set of n differentiable functions $y_1 = h_1(t), \quad \cdots, \quad y_n = h_n(t)$.

In vector form, $\mathbf{y} = \mathbf{h}(t)$; where $\mathbf{h} = [h_1 \cdots h_n]^T$ is “solution vector” (column vector)

❖ Initial condition: $y_1(t_0) = K_1, \quad y_2(t_0) = K_2, \quad \cdots, \quad y_n(t_0) = K_n$

4.2 Basic Theory of Systems of ODEs. Wronskian

● **Theorem 1 Existence and Uniqueness Theorem**

Let $f_1, \cdots f_n$ be continuous functions having continuous partial derivatives $\partial f_1/\partial y_1$, $\cdots$, $\partial f_1/\partial y_n$, $\cdots$, $\partial f_n/\partial y_n$ in some domain R of $t\,y_1\,y_2\,y_n$-space containing the point $(t_0, K_{1,} \cdots_, K_n)$.

Then the first-order system has a solution on some interval $t_0 - \alpha < t < t_0 + \alpha$ satisfying the initial condition, and this solution is unique.

4.2 Basic Theory of Systems of ODEs. Wronskian

❖ **Linear Systems (선형연립상미분 방정식)**

- **Linear System**

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} g_1 \\ \vdots \\ g_2 \end{bmatrix}$$

$$\begin{aligned} y_1' &= a_{11}(t)y_1 + \cdots + a_{1n}(t)y_n + g_1(t) \\ &\vdots \\ y_n' &= a_{n1}(t)y_1 + \cdots + a_{nn}(t)y_n + g_n(t) \end{aligned} \longrightarrow \mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}$$

- **Homogeneous:** $\mathbf{y}' = \mathbf{A}\mathbf{y}$
- **Nonhomogeneous:** $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}, \quad \mathbf{g} \neq \mathbf{0}$

4.2 Basic Theory of Systems of ODEs. Wronskian

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} g_1 \\ \vdots \\ g_2 \end{bmatrix} \qquad \mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}$$

❖ **Theorem 2 Existence and Uniqueness in the Linear Case**

Let the a_{jk}'s and g_j's be continuous functions of t on an open interval $\alpha < t < \beta$ containing the point $t = t_0$.

Then the linear system has a solution $\mathbf{y}(t)$ on this interval satisfying initial condition, and this solution is unique.

❖ **Theorem 3 Superposition Principle (중첩의 원리) or Linearity Principle**

If $\mathbf{y}^{(1)}$ and $\mathbf{y}^{(2)}$ are solutions of the homogeneous linear system on some interval,

so is any linear combination $\mathbf{y} = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)}$.

4.2 Basic Theory of Systems of ODEs. Wronskian

❖ Basis (기저). General Solution. Wronskian

- **Basis (기저)** or **Fundamental System (기본계)** of solutions of the homogeneous system on some interval J: ⇨ A linearly independent set of n solutions $\mathbf{y}^{(1)} \cdots \mathbf{y}^{(n)}$ of the homogeneous system on J

- **General Solution** of the homogeneous system on J

  : A corresponding linear combination

$$\mathbf{y} = c_1\mathbf{y}^{(1)} + \cdots + c_n \mathbf{y}^{(\mathbf{n})} \quad \left(c_1, \ \cdots, \ c_n\text{: arbitrary constants}\right)$$

4.2 Basic Theory of Systems of ODEs. Wronskian

❖ **Wronskian** of $\mathbf{y}^{(1)}, \cdots, \mathbf{y}^{(n)}$: The determinant of **Y** $\mathbf{Y} = \left[\mathbf{y}^{(1)}, \ \cdots, \ \mathbf{y}^{(\mathbf{n})}\right]$

Indices for basis

$$W\left(\mathbf{y}^{(1)}, \ \cdots, \ \mathbf{y}^{(\mathbf{n})}\right) = \begin{vmatrix} y_1^{(1)} & y_1^{(2)} & \cdots & y_1^{(n)} \\ y_2^{(1)} & y_2^{(2)} & \cdots & y_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ y_n^{(1)} & y_n^{(2)} & \cdots & y_n^{(n)} \end{vmatrix}$$

❖ **Fundamental Matrix (기본행렬)**: An $n \times n$ matrix whose **columns are *n* solutions**

4.3 Constant-Coefficient Systems. Phase Plane Method

- $\mathbf{y}' = \mathbf{A}\mathbf{y}$: Homogeneous linear system under discussion has **constant coefficients**

  where $n \times n$ matrix $\mathbf{A} = \left[a_{jk}\right]$ has entries not depending on t

  Idea: Try $\mathbf{y} = \mathbf{x}e^{\lambda t}$

  $$\Rightarrow \quad \mathbf{y}' = \boxed{\lambda \mathbf{x}\cancel{e^{\lambda t}}} = \mathbf{A}\mathbf{y} = \boxed{\mathbf{A}\mathbf{x}\cancel{e^{\lambda t}}} \quad \Rightarrow \quad \mathbf{A}\mathbf{x} = \lambda \mathbf{x} \quad \text{(Eigenvalue Problem)}$$

Theorem 1 General Solution

- If the constant matrix **A** in the homogeneous linear system has a linearly independent set of n eigenvectors,
- then the corresponding solutions $\mathbf{y}^{(1)}, \cdots, \mathbf{y}^{(n)}$ form a basis of solutions, and
- the corresponding general solution is $\mathbf{y} = c_1\mathbf{x}^{(1)}e^{\lambda_1 t} + \cdots + c_n\mathbf{x}^{(n)}e^{\lambda_n t}$.

Proof

Let $\mathbf{y}^{(1)} = \mathbf{x}^{(1)} e^{\lambda_1 t}, \ldots, \ \mathbf{y}^{(n)} = \mathbf{x}^{(n)} e^{\lambda_n t}$

❖ **Wronskian** of $\mathbf{y}^{(1)}, \cdots, \mathbf{y}^{(n)}$: The determinant of Y $\quad \mathbf{Y} = \left[\mathbf{y}^{(1)}, \ \cdots, \ \mathbf{y}^{(\mathbf{n})}\right]$

$$\mathrm{W}\left(\mathbf{y}^{(1)}, \ \cdots, \ \mathbf{y}^{(\mathbf{n})}\right) = \begin{vmatrix} x_1^{(1)} e^{\lambda_1 t} & x_1^{(2)} e^{\lambda_2 t} & \cdots & x_1^{(n)} e^{\lambda_n t} \\ x_2^{(1)} e^{\lambda_1 t} & x_2^{(2)} e^{\lambda_2 t} & \cdots & x_2^{(n)} e^{\lambda_n t} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{(1)} e^{\lambda_1 t} & x_n^{(2)} e^{\lambda_2 t} & \cdots & x_n^{(n)} e^{\lambda_n t} \end{vmatrix}$$

$$= \underbrace{e^{\lambda_1 t + \lambda_2 t + \cdots + \lambda_n t}}_{\neq 0} \underbrace{\begin{vmatrix} x_1^{(1)} & x_1^{(2)} & \cdots & x_1^{(n)} \\ x_2^{(1)} & x_2^{(2)} & \cdots & x_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{(1)} & x_n^{(2)} & \cdots & x_n^{(n)} \end{vmatrix}}_{\neq 0 \text{ since its columns are the } n \text{ linearly independent eigenvectors}} \neq 0$$

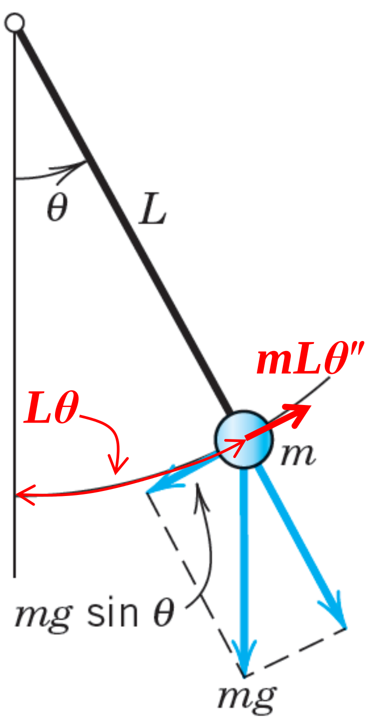
$\mathbf{y}^{(1)} = \mathbf{x}^{(1)} e^{\lambda_1 t}, \ldots, \ \mathbf{y}^{(n)} = \mathbf{x}^{(n)} e^{\lambda_n t}$
are the n linearly independent eigenvectors.

General solution
$$\Rightarrow \mathbf{y} = c_1 \mathbf{x}^{(1)} e^{\lambda_1 t} + \cdots + c_n \mathbf{x}^{(n)} e^{\lambda_n t}$$

4.3 Phase Plane Method (상평면법). Pendulum Example

☑ Ex Free Undamped Pendulum, Linearization

A pendulum consists of a body of mass *m* (the bob, 무게추) and a rod of length *L*. Assume that the mass of the rod and air resistance are negligible.



Pendulum

Step 1 Setting up the mathematical model

θ: the angular displacement

mg: the weight of the bob

A restoring force tangent to the curve of motion: ***mg*sinθ**

= the force of acceleration: ***mLθ''*** (***Lθ''***: acceleration)

$$\therefore\ mL\theta'' + mg\sin\theta = 0 \quad \rightarrow \quad \theta'' + k\sin\theta = 0 \quad \left(k = \frac{g}{L}\right)$$

Step 2 Linearization

$$\theta'' + k\sin\theta = 0$$
$$\theta'' + k\theta = 0$$

$$\xrightarrow[y_2 = \theta' \ :\ 각속도]{y_1 = \theta \ :\ 회전각}$$

$$y_1' = y_2$$
$$y_2' = \theta'' = -k\theta = -ky_1$$

* Taylor series expansion for sin(x)

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

* Original problem (Second ODE)

$$y_1'' + ky_1 = 0$$

4.3 Phase Plane Method. Pendulum Example

Step 3 Solve (k=g/L=4 for simplicity)

$$y_1' = y_2$$
$$y_2' = \theta'' = -4\theta = -4y_1$$

$$\Rightarrow \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \Rightarrow \mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \mathbf{y}$$

$$\mathbf{y} = \mathbf{x}e^{\lambda t}$$
$$\mathbf{y}' = \lambda \mathbf{x}e^{\lambda t}$$
$$\Rightarrow \mathbf{A}\mathbf{x}e^{\lambda t}$$

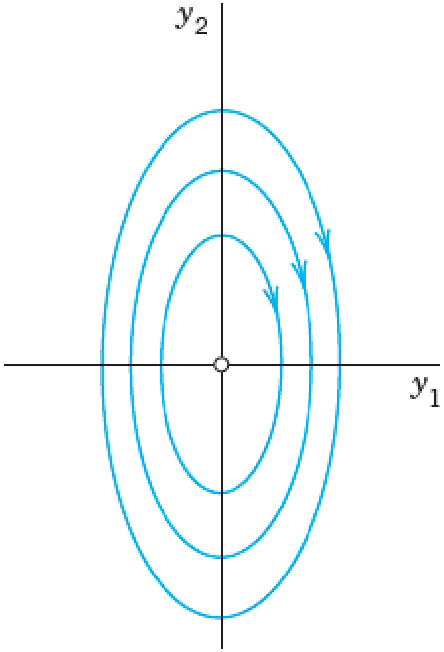
Characteristic equation:

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -4 & -\lambda \end{vmatrix} = \lambda^2 + 4 = 0$$

Eigenvalues and eigenvectors:

$$\lambda_1 = 2i \Rightarrow \begin{bmatrix} -2i & 1 \\ 4 & -2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{x}^{(1)} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2i \end{bmatrix}$$

$$\lambda_2 = -2i \Rightarrow \begin{bmatrix} 2i & 1 \\ 4 & 2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{x}^{(2)} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2i \end{bmatrix}$$



Trajectories (Center)

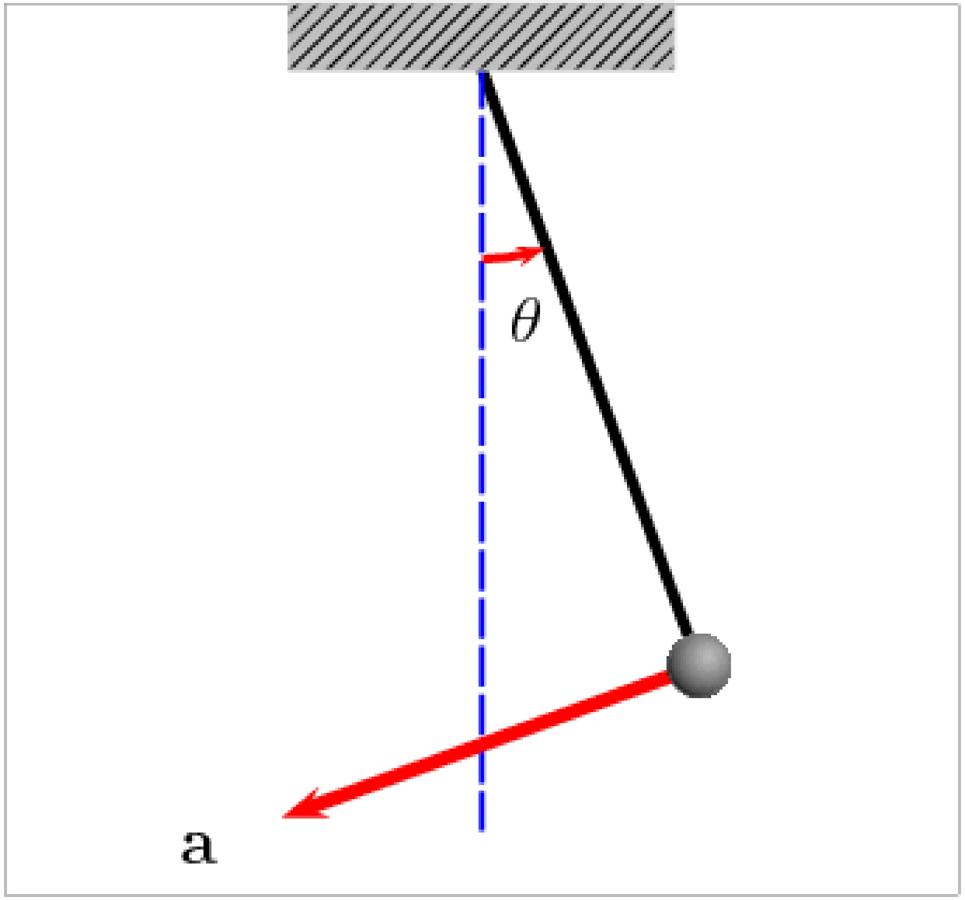
General solution:

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{2it} + c_2 \begin{bmatrix} 1 \\ -2i \end{bmatrix} e^{-2it} \Rightarrow \begin{aligned} y_1 &= c_1 e^{2it} + c_2 e^{-2it} \\ y_2 &= 2ic_1 e^{2it} - 2ic_2 e^{-2it} \end{aligned}$$

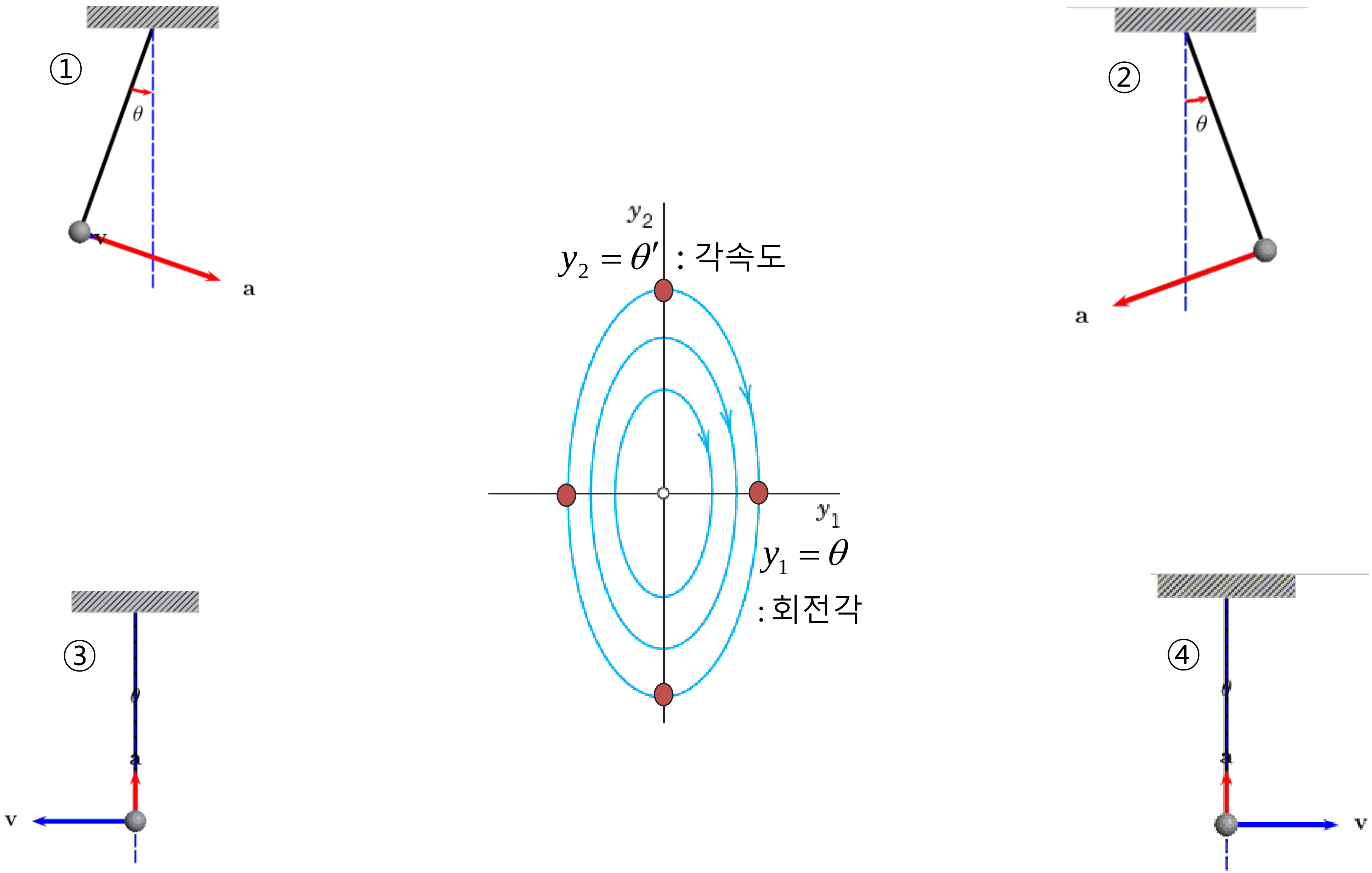
$$y_1' = y_2, \quad y_2' = -4y_1 \quad \Rightarrow \quad 4y_1 y_1' = -y_2 y_2' \quad \Rightarrow \quad 2y_1^2 + \frac{1}{2}y_2^2 = \text{const.}$$

4.3 Phase Plane Method. Pendulum Example

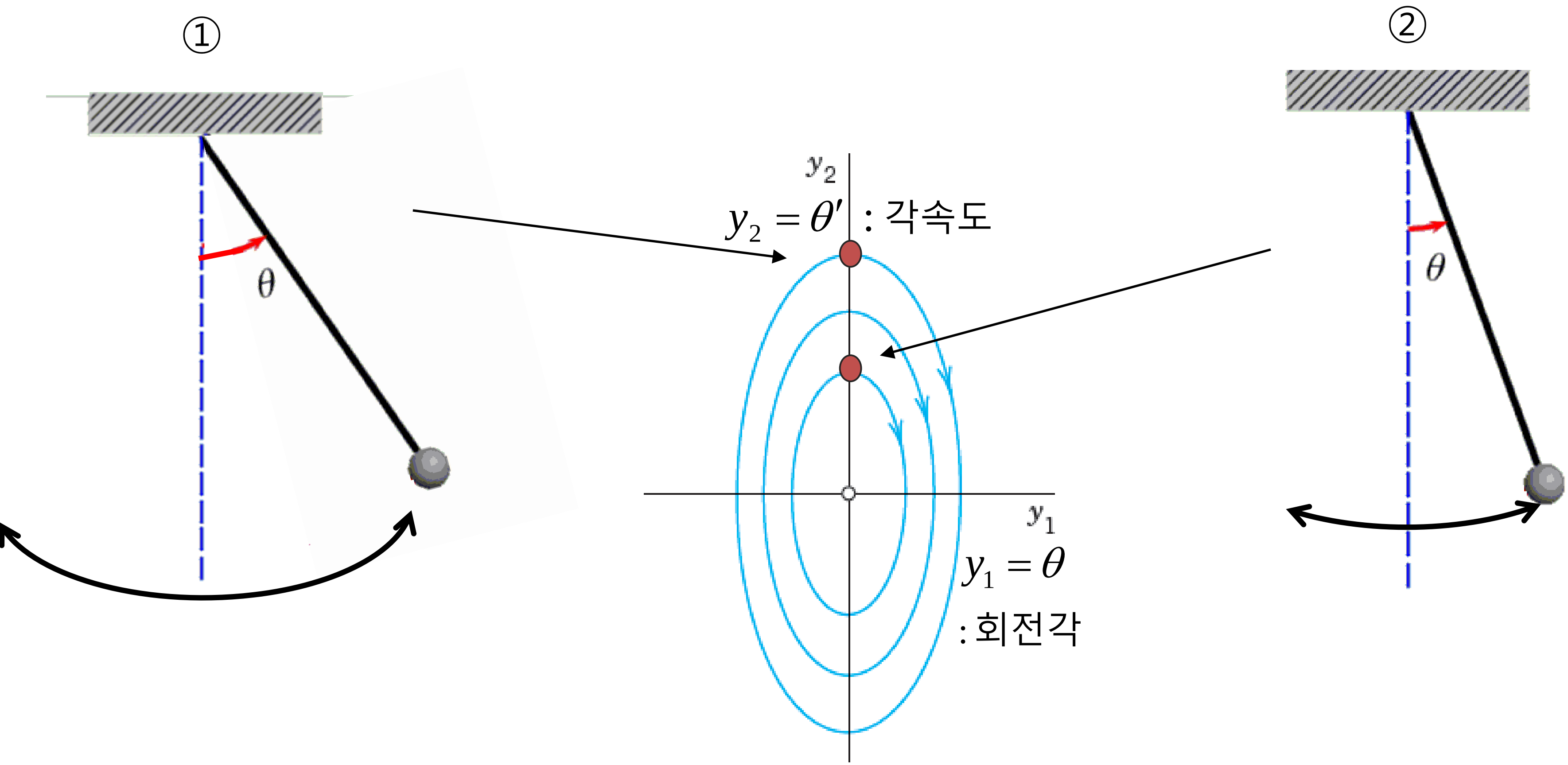
Step 4 Interpretation



4.3 Phase Plane Method. Pendulum Example



4.3 Phase Plane Method. Pendulum Example

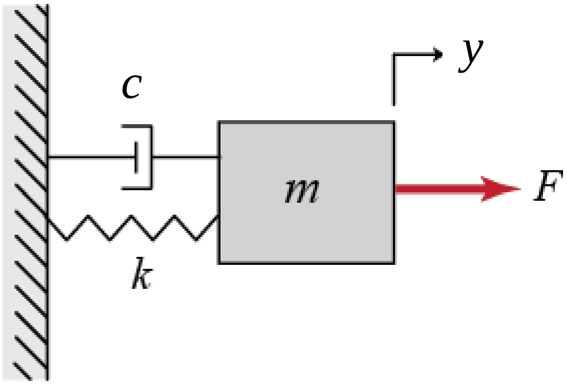


4.3 Phase Plane Method. Mass on a Spring Example

☑ Ex2 Free Motions of a Mass on a Spring

$my'' + cy' + ky = 0 \quad \rightarrow \quad y'' = -(k/m)y - (c/m)y'$

Step 1 Matrix form



$$y'' = -(k/m)y - (c/m)y'$$

$y_1 = y$: 변위 $\qquad$ $y_2 = y'$: 속도

$$\boxed{\begin{aligned} y_1' &= y_2 \\ y_2' &= y'' = -(k/m)y - (c/m)y' \\ &= -(k/m)y_1 - (c/m)y_2 \end{aligned}}$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad k, c, m > 0$$

4.3 Phase Plane Method. Mass on a Spring Example

Step 2 Solve - Undamped case: c=0

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \Rightarrow \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \Rightarrow \boxed{\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix} \mathbf{y}}$$

"The same as the free undamped pendulum problem"

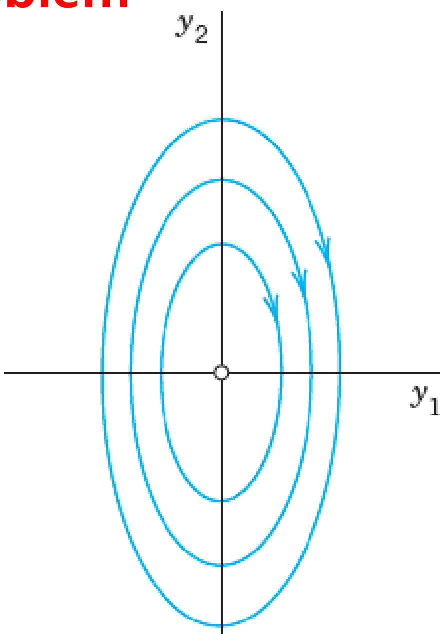
Equations of motion

$$\boxed{\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \mathbf{y}}$$

$y_1 = \theta$: 회전각

$y_2 = \theta'$: 각속도

$$y_1' = y_2, \quad y_2' = -4y_1 \quad \Rightarrow \quad 4y_1 y_1' = -y_2 y_2' \quad \Rightarrow \quad 2y_1^2 + \frac{1}{2} y_2^2 = \text{const.}$$



Trajectories (Center)

4.3 Phase Plane Method. Mass on a Spring Example

Chapter 2.2 Homogeneous Linear ODEs

☑ Ex. 5 Solve the initial value problem $y''+0.4y'+9.04y=0, \quad y(0)=0, \quad y'(0)=3$

Step 1 General solution

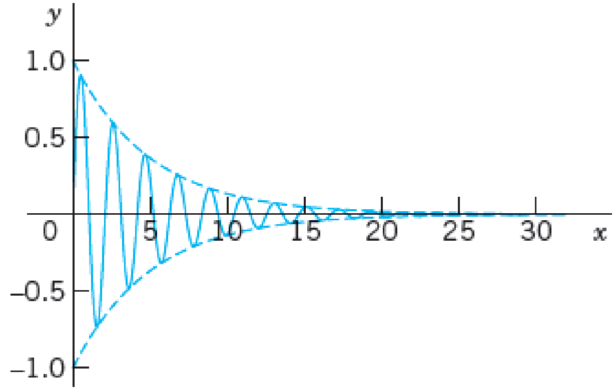
$$\lambda^2+0.4\lambda+9.04=0 \quad \text{(Characteristic equation)} \Rightarrow \quad \lambda=-0.2\pm 3i \quad \Rightarrow \quad \therefore\ y=e^{-0.2x}(A\cos 3x+B\sin 3x)$$

Step 2 Particular solution

$$y'=-0.2e^{-0.2x}(A\cos 3x+B\sin 3x)+e^{-0.2x}(-3A\sin 3x+3B\cos 3x)$$

$$\Rightarrow \quad y(0)=A=0, \quad y'(0)=-0.2A+3B=3 \quad \Rightarrow \quad A=0, \quad B=1$$

$$\Rightarrow \quad \therefore\ y=e^{-0.2x}\sin 3x$$



$my''+cy'+ky=0$

$m=1,\ c=0.4,\ k=9.04$

$c^2<4mk$

$\rightarrow$ Underdamped case

4.3 Phase Plane Method. Mass on a Spring Example

$$\mathbf{y} = \mathbf{x}e^{\lambda t}$$
$$\mathbf{y}' = \lambda \mathbf{x}e^{\lambda t}$$

Step 3 Solve - Damped case: $m=1,\ c=0.4,\ k=9.04$ **Initial condition** $y(0)=0,\ y'(0)=3$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \Rightarrow \boxed{\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ -9.04 & -0.4 \end{bmatrix}\mathbf{y}} \Rightarrow \mathbf{A}\mathbf{x}e^{\lambda t}$$

Characteristic equation:

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -9.04 & -0.4-\lambda \end{vmatrix} = \lambda^2 + 0.2\lambda + 9.04 = 0,$$

Eigenvalues and eigenvectors:

$$\Rightarrow \lambda = -0.2 \pm \sqrt{0.04 - 9.04} = -0.2 \pm 3i$$

$$\lambda_1 = -0.2+3i \Rightarrow \begin{bmatrix} 0.2-3i & 1 \\ -9.04 & -0.2-3i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{x}^{(1)} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.2+3i \end{bmatrix}$$

$$\lambda_2 = -0.2-3i \Rightarrow \begin{bmatrix} 0.2+3i & 1 \\ -9.04 & -0.2+3i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{x}^{(1)} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.2-3i \end{bmatrix}$$

General solution:

$$\mathbf{y} = c_1\begin{bmatrix} 1 \\ -0.2+3i \end{bmatrix}e^{(-0.2+3i)t} + c_2\begin{bmatrix} 1 \\ -0.2-3i \end{bmatrix}e^{(-0.2-3i)t}$$

Use initial conditions

$y(0)=0,\ y'(0)=3$

$$\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = c_1\begin{bmatrix} 1 \\ -0.2+3i \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -0.2-3i \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \Rightarrow c_1 = -0.5i,\ c_2 = 0.5i,$$

4.3 Phase Plane Method. Mass on a Spring Example

Step 3 Solve - Damped case: $m=1,\ c=0.4,\ k=9.04$ **Initial condition** $y(0)=0,\ y'(0)=3$

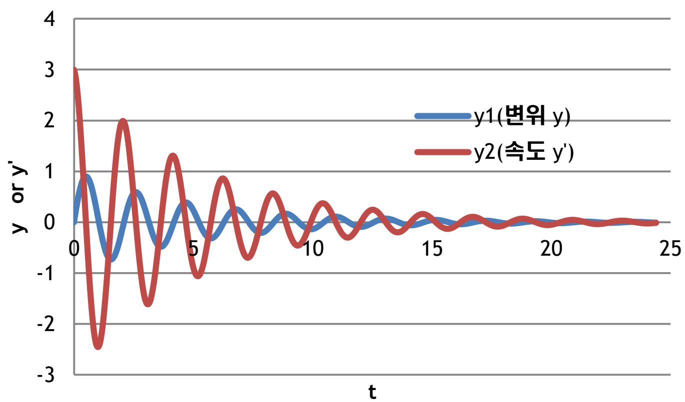
$$y_1 = c_1 e^{(-0.2+3i)t} + c_2 e^{(-0.2-3i)t} = -0.5ie^{-0.2t}(\cos 3t + i\sin 3t) + 0.5ie^{-0.2t}(\cos 3t - i\sin 3t)$$

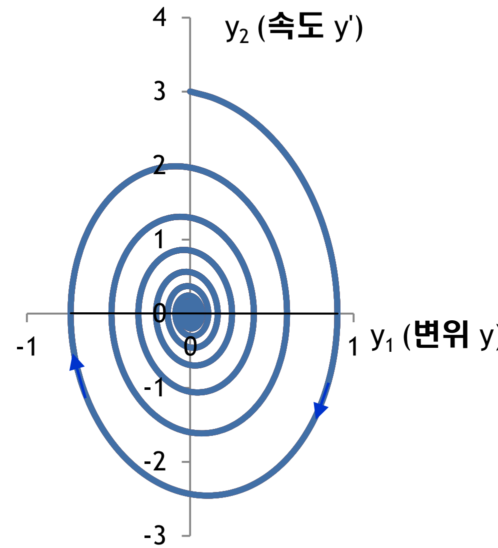
$$= \boxed{e^{-0.2t}\sin 3t}$$

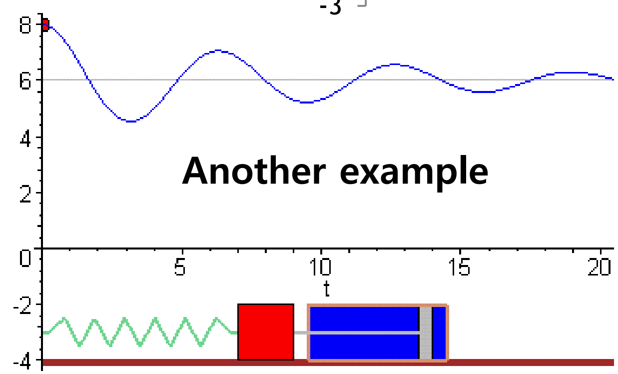
$$y_2 = y_1' = -0.2e^{-0.2t}\sin 3t + 3e^{-0.2t}\cos 3t$$

$y_1 = y$: 변위

$y_2 = y'$: 속도







at t=0, y=8

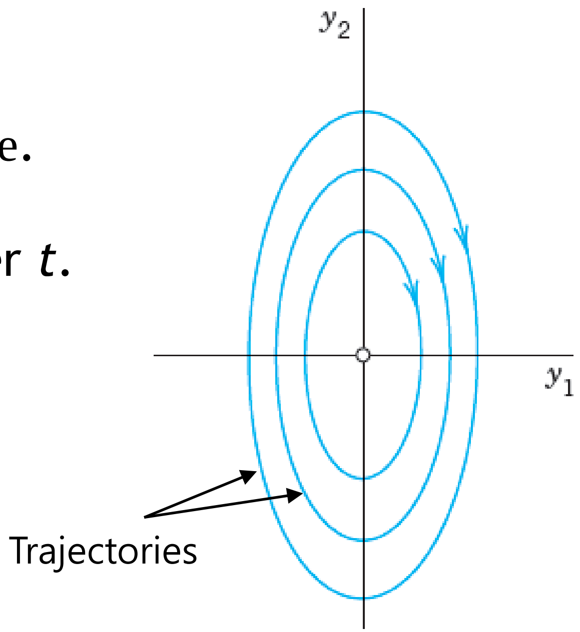
4.3 Constant-Coefficient Systems. Phase Plane Method

❖ How to Graph Solutions in the Phase Plane (상평면)

The solution of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ $\left(\text{in components,} \quad \begin{array}{l} y_1' = a_{11}y_1 + a_{12}y_2 \\ y_2' = a_{21}y_1 + a_{22}y_2 \end{array}\right)$ is $\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$.

- We can graph solutions as two curves over the *t*-axis, one for each components of $\mathbf{y}(t)$.
- We can also graph as a single curve in the $y_1 y_2$-plane.

  - This is a parametric representation with parameter *t*.
- **Trajectory** (or sometimes an **Orbit** or **Path, 궤적**):
  The single curve in the $y_1 y_2$-plane
- **Phase Plane (상평면)**: $y_1 y_2$-plane
- **Phase Portrait (상투영)**: The phase plane filled with trajectories (상평면에서의 궤적)



4.3 Constant-Coefficient Systems. Phase Plane Method

☑ **Ex. 1 Trajectories in the Phase Plane (Phase Portrait)**
(상평면에서의 궤적 (상투영))

In order to see what is going on, let us find and graph solutions of the system.

$$\begin{aligned} y_1' &= -3y_1 + y_2 \\ y_2' &= y_1 - 3y_2 \end{aligned} \quad \Leftrightarrow \quad \mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \mathbf{y}$$

Characteristic equation: $\det\left(\mathbf{A} - \lambda \mathbf{I}\right) = \begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} = \lambda^2 + 6\lambda + 8 = 0$

Eigenvalues and eigenvectors: $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x}=0$

$$\lambda_1 = -2 \Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{x}^{(1)} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -4 \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{x}^{(2)} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

General solution: $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-2t} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-4t}$

☑ Ex. 1 Trajectories in the Phase Plane (Phase Portrait)

General solution:

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \mathbf{y}^{(1)} + c_2 \mathbf{y}^{(2)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$$

- The two straight trajectories correspond to $c_1 = 0$ and $c_2 = 0$.

$$\text{When } c_1 = 0 \Rightarrow \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_2 \mathbf{y}^{(2)} = c_2 \begin{bmatrix} e^{-4t} \\ -e^{-4t} \end{bmatrix} \Rightarrow y_2 = -y_1$$

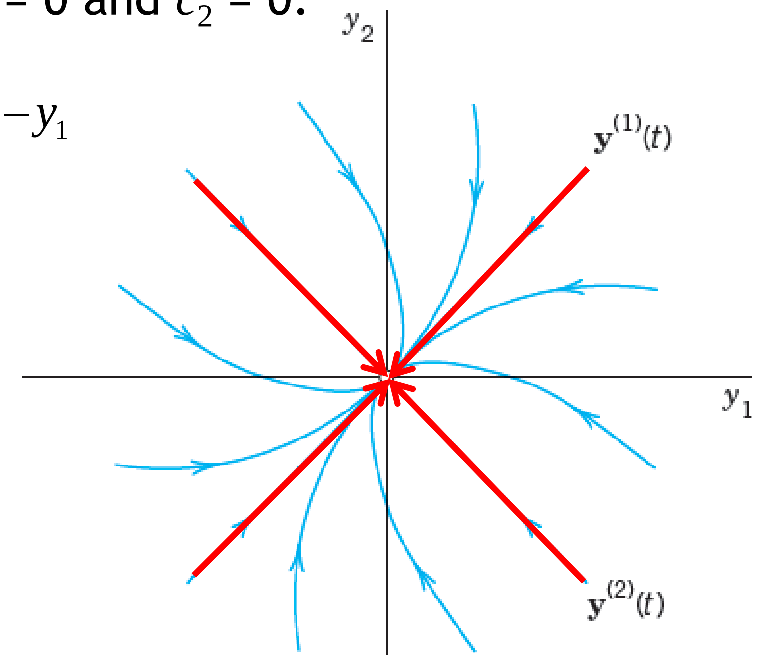
$$\text{if } c_2 > 0 \ \Rightarrow \ y_1 : +, y_2 : -$$

$$\text{if } c_2 < 0 \ \Rightarrow \ y_1 : -, y_2 : +$$

$$\text{When } c_2 = 0 \Rightarrow \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \mathbf{y}^{(1)} = c_1 \begin{bmatrix} e^{-2t} \\ e^{-2t} \end{bmatrix} \Rightarrow y_2 = y_1$$

$$\text{if } c_1 > 0 \ \Rightarrow \ y_1 : +, y_2 : +$$

$$\text{if } c_2 < 0 \ \Rightarrow \ y_1 : -, y_2 : +$$



$$as\ t\ increases \Rightarrow \ y_1 \,\&\, y_2 \to 0$$

- The others to other choices of c_1, c_2.

Trajectories (Improper node)

4.3 Constant-Coefficient Systems. Phase Plane Method

❖ Critical Points (임계점) of the system

The point $\mathbf{y} = 0$ seems to be a common point of all trajectories.

$$\mathbf{y}' = \mathbf{A}\mathbf{y} \quad \Rightarrow \quad \frac{dy_2}{dy_1} = \frac{dy_2/dt}{dy_1/dt} = \frac{y_2'}{y_1'} = \frac{a_{21}y_1 + a_{22}y_2}{a_{11}y_1 + a_{12}y_2} \quad \Leftarrow \quad \boxed{\mathbf{y}' = \mathbf{A}\mathbf{y}}$$

- dy_2/dy_1: A unique tangent direction of the trajectory passing through P:(y_1, y_2) except for the $P = P_0{:}(0, 0)$ where the right side becomes 0/0.
- **Critical point:** The point at which of dy_2/dy_1 becomes undetermined.

❖ Five Types of Critical Points

Depending on the geometric shape of the trajectories near them

Improper Node (비고유마디점), Proper Node (고유마디점),
Saddle Point (안장점), Center (중심점), Spiral Point(나선점)

4.3 Constant-Coefficient Systems. Phase Plane Method

☑ **Ex. 1 Improper Node (비고유마디점)**

: A critical point at which all the trajectories, except for two of them, have the same limiting direction of the tangent

- The two exceptional trajectories also have a limiting direction of the tangent.
- A limiting direction is different.

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\mathbf{y}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-2t} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-4t}$$

- The common limiting direction at $\mathbf{0}$ ($y_1 = y_2 = 0$) is $\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (or $\mathbf{x}^{(1)} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$)

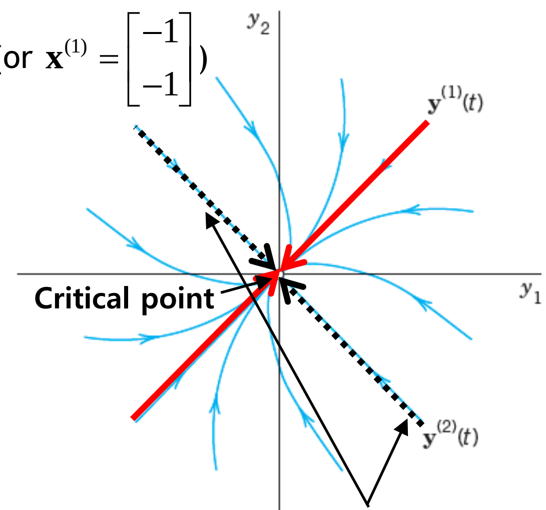
  because e^{-4t} goes to zero faster than e^{-2t} as t increases.

- The two exceptional limiting tangent directions are

$$\mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad -\mathbf{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{when } c_1 = 0.$$

$$\text{if } c_2 > 0 \;\Rightarrow\; y_1 : +,\, y_2 : -$$

$$\text{if } c_2 < 0 \;\Rightarrow\; y_1 : -,\, y_2 : +$$



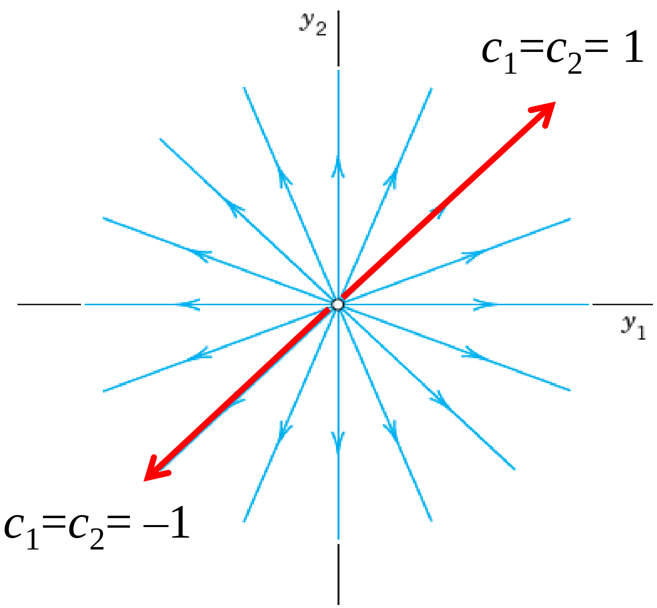
Two exceptional limiting tangent directions

4.3 Constant-Coefficient Systems. Phase Plane Method

☑ **Ex. 2 Proper Node (고유마디점)**

: A critical point at which every trajectory has a definite limiting direction and for any given direction **d** at P_0 there is a trajectory having **d** as its limiting direction.

$$\mathbf{y}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{y}, \quad \left(\text{thus} \quad \begin{matrix} y_1' = y_1 \\ y_2' = y_2 \end{matrix} \right)$$



Trajectories (Proper node)

Characteristic equation:

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = 0 \;,\; \lambda = 1$$

Eigenvector: Any $\mathbf{x} \neq 0$, we can take $[1\ 0]^T$, $[0\ 1]^T$.

General solution:

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t$$

$$\Rightarrow \quad \begin{matrix} y_1 = c_1 e^t \\ y_2 = c_2 e^t \end{matrix} \quad \Rightarrow \quad c_1 y_2 = c_2 y_1$$

4.3 Constant-Coefficient Systems. Phase Plane Method

☑ **Ex. 2 Proper Node - If we take different eignenvalues?**

Eigenvector: Any $\mathbf{x} \neq 0$, we can take $[1\ 0]^T$, $[0\ 1]^T$.

What happen if we take other eingenvector?

Ex) $[1\ 2]^T$, $[3\ 1]^T$

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^t + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^t = \begin{bmatrix} c_1 + 3c_2 \\ 2c_1 + c_2 \end{bmatrix} e^t = \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} e^t$$

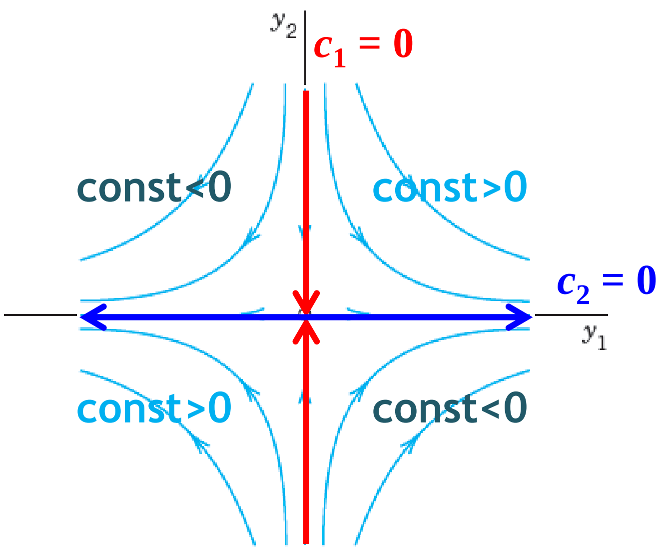
$$\Rightarrow \begin{matrix} y_1 = c_1' e^t \\ y_2 = c_2' e^t \end{matrix} \quad \Rightarrow c_1' y_2 = c_2' y_1$$ (The result is same.)

4.3 Constant-Coefficient Systems. Phase Plane Method

☑ Ex. 3 Saddle Point (안장점)

: A critical point at which there are **two incoming trajectories**, **two outgoing trajectories**, and **all the other trajectories in a neighborhood of P_0 bypass P_0**.

$$\mathbf{y}' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{y}, \quad \left(\text{thus} \quad \begin{aligned} y_1' &= y_1 \\ y_2' &= -y_2 \end{aligned} \right)$$



Trajectories (Saddle Point)

Characteristic equation:

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) = 0 \,,$$

$$\lambda_1 = 1, \quad \lambda_2 = -1$$

Eigenvector: $[1\ 0]^T$ for $\lambda_1=1$, $[0\ 1]^T$ for $\lambda_2=-1$

General solution:

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t}$$

$$\Rightarrow \quad \begin{aligned} y_1 &= c_1 e^{t} \\ y_2 &= c_2 e^{-t} \end{aligned} \quad \Rightarrow \quad y_1 y_2 = \text{const.}$$

- This is a family of hyperbolas (and the coordinate axes).

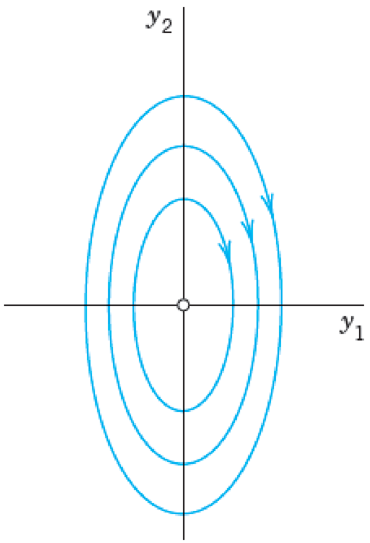
4.3 Constant-Coefficient Systems. Phase Plane Method

☑ Ex. 4 Center (중심점)

: A critical point that is enclosed by infinitely many closed trajectories.

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \mathbf{y}, \quad \left(\text{thus} \quad \begin{aligned} y_1' &= y_2 \\ y_2' &= -4y_1 \end{aligned} \right)$$

Characteristic equation: $\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -4 & -\lambda \end{vmatrix} = \lambda^2 + 4 = 0$



Trajectories (Center)

Eigenvalues and eigenvectors:

$$\lambda_1 = 2i \Rightarrow \begin{bmatrix} -2i & 1 \\ 4 & -2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{x}^{(1)} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2i \end{bmatrix}$$

$$\lambda_2 = -2i \Rightarrow \begin{bmatrix} 2i & 1 \\ 4 & 2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{x}^{(2)} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2i \end{bmatrix}$$

General solution: $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{2it} + c_2 \begin{bmatrix} 1 \\ -2i \end{bmatrix} e^{-2it} \Rightarrow \begin{aligned} y_1 &= c_1 e^{2it} + c_2 e^{-2it} \\ y_2 &= 2ic_1 e^{2it} - 2ic_2 e^{-2it} \end{aligned}$

$$y_1' = y_2, \quad y_2' = -4y_1 \quad \Rightarrow \quad 4y_1 y_1' = -y_2 y_2' \quad \Rightarrow \quad 2y_1^2 + \frac{1}{2} y_2^2 = \text{const.}$$

4.3 Constant-Coefficient Systems. Phase Plane Method

☑ Ex. 5 Spiral Point (나선점)

: A critical point about which the trajectories spiral, approaching P_0 as t→∞ (or tracing these spirals in the opposite sense, away from P_0).

$$\mathbf{y}' = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}\mathbf{y}, \left(\text{thus} \quad \begin{aligned} y_1' &= -y_1 + y_2 \\ y_2' &= -y_1 - y_2 \end{aligned} \right)$$

Characteristic equation: $\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -1-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 2 = 0$

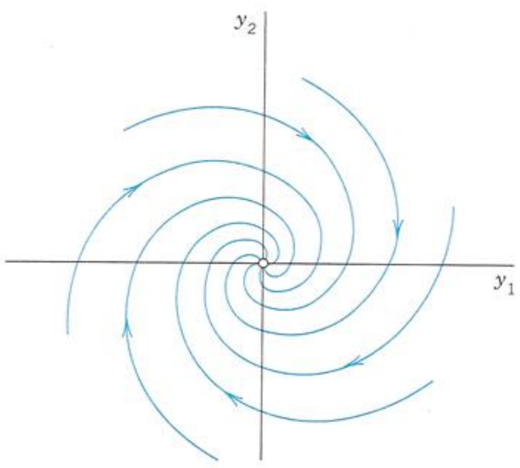
Eigenvalues and eigenvectors:

$$\lambda_1 = -1+i \Rightarrow \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{x}^{(1)} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\lambda_2 = -1-i \Rightarrow \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{x}^{(2)} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

General solution:

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(-1+i)t} + c_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(-1-i)t}$$



Trajectories (Spiral point)

4.3 Constant-Coefficient Systems. Phase Plane Method

- **Transform this complex solution to a real general solution by a trick.**

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(-1+i)t} + c_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(-1-i)t}$$

$$y_1' = -y_1 + y_2$$
$$y_2' = -y_1 - y_2$$

$$\rightarrow \quad y_1 y_1' + y_2 y_2' = -\left(y_1^2 + y_2^2\right)$$

$$\boxed{\begin{aligned} & y_1(-y_1 + y_2) + y_2(-y_1 - y_2) \\ & = -y_1^2 - y_2^2 \end{aligned}}$$

$$\xrightarrow{r^2 = y_1^2 + y_2^2} \quad \frac{1}{2}(r^2)' = -r^2$$

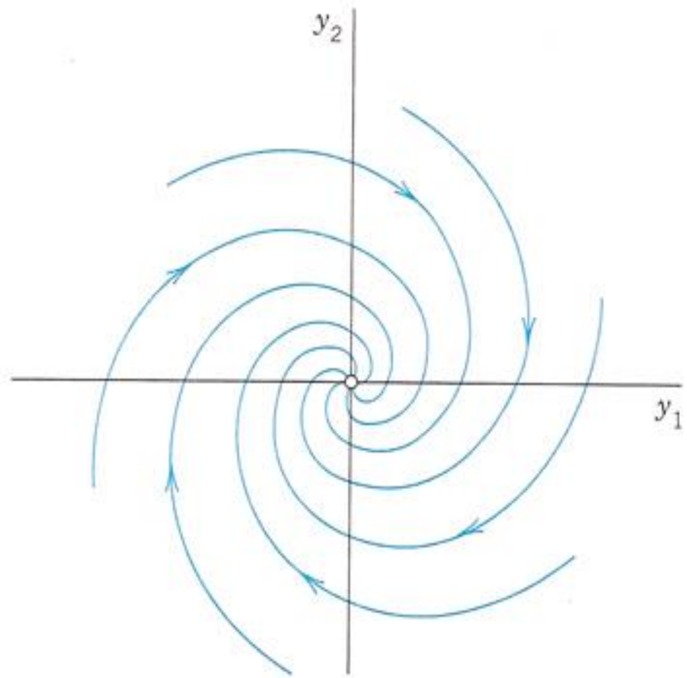
$$(r^2)' = 2rr'$$
$$= 2y_1 y_1' + 2y_2 y_2' = -2r^2$$

$$\xrightarrow{(r^2)' = 2rr'} \quad rr' = -r^2$$

$$\boxed{\frac{r'}{r} = -1, \quad \frac{dr}{r} = -dt}$$

$$\rightarrow \quad \ln r = -t + \tilde{c} \quad \rightarrow \quad r = ce^{-t}$$

$$\therefore \sqrt{y_1^2 + y_2^2} = ce^{-t}$$



Trajectories (Spiral point)

4.3 Constant-Coefficient Systems. Phase Plane Method

☑ **Ex. 6 No Basis of Eigenvectors Available. Degenerate Node (퇴화마디점).**

$$\mathbf{y}' = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} \mathbf{y}$$

Characteristic equation: $\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 4-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 9 = 0$

Eigenvalue and eigenvector: $\lambda = 3 \Rightarrow \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{x}^{(1)} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\boxed{\mathbf{y}^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t}}$

λ double root $\Rightarrow \mathbf{y}^{(2)} = \mathbf{x}^{(1)} t e^{\lambda t} + \mathbf{u} e^{\lambda t}, \quad \mathbf{u} = [u_1 \; u_2]^T$ xt - term alone would not be enough.

$$\mathbf{y}^{(2)}{}' = \mathbf{x}^{(1)} e^{\lambda t} + \cancel{\lambda \mathbf{x}^{(1)} t e^{\lambda t}} + \lambda \mathbf{u} e^{\lambda t} = \mathbf{A}\mathbf{y}^{(2)} = \cancel{\mathbf{A}\mathbf{x}^{(1)} t e^{\lambda t}} + \mathbf{A}\mathbf{u} e^{\lambda t} \Leftarrow \boxed{\mathbf{y}' = \mathbf{A}\mathbf{y}}$$

here $\mathbf{A}\mathbf{x}^{(1)} = \lambda \mathbf{x}^{(1)} \Rightarrow \lambda \mathbf{x}^{(1)} t e^{\lambda t} = \mathbf{A}\mathbf{x}^{(1)} t e^{\lambda t}$

$$\mathbf{x}^{(1)} e^{\lambda t} + \lambda \mathbf{u} e^{\lambda t} = \mathbf{A}\mathbf{u} e^{\lambda t} \Rightarrow \mathbf{x}^{(1)} + \lambda \mathbf{u} = \mathbf{A}\mathbf{u} \Rightarrow \mathbf{A}\mathbf{u} - \lambda \mathbf{u} = \mathbf{x}^{(1)} \Rightarrow (\mathbf{A} - \lambda I)\mathbf{u} = \mathbf{x}^{(1)}$$

$$\lambda = 3, \quad \mathbf{x}^{(1)} = [1 \; -1]^T \Rightarrow (\mathbf{A} - 3\mathbf{I})\mathbf{u} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \mathbf{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \boxed{\mathbf{y}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} t e^{3t} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{3t}}$$

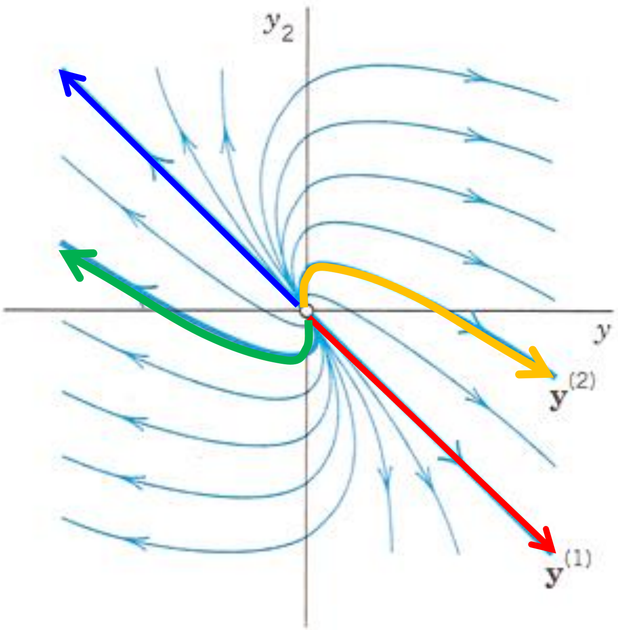
4.3 Constant-Coefficient Systems. Phase Plane Method

☑ Ex. 6 No Basis of Eigenvectors Available. Degenerate Node.

General solution:

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t} + c_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^{3t}$$

- $c_1 \boldsymbol{y}^{(1)}$ gives the heavy straight line,
  - ✓ $c_1 > 0$ the lower part
  - ✓ $c_1 < 0$ the upper part
- $\boldsymbol{y}^{(2)}$ gives the right part of the heavy curve from 0 second, first, and – finally – fourth quadrants.
- $-\boldsymbol{y}^{(2)}$ gives the other part of that curve.



Trajectories (Degenerate node)

4.3 Constant-Coefficient Systems. Phase Plane Method

☑ **Ex.** Find a real general solution of the following systems. Q Solve.

$$\mathbf{y}' = \begin{bmatrix} 7 & 3 \\ 3 & 7 \end{bmatrix} \mathbf{y}$$

4.3 Summary

❖ **Phase plane method**

The solution of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ $\left(\text{in components, } \begin{array}{l} y_1' = a_{11}y_1 + a_{12}y_2 \\ y_2' = a_{21}y_1 + a_{22}y_2 \end{array}\right)$ is $\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$.

- Families of solution curves if we represent them parametrically as

  $\mathbf{y}(t) = [y_1(x) \;\; y_2(x)]^{\mathrm{T}}$

- A **trajectory** of $\mathbf{y}(t)$: a curve of y(t) in the $y_1\, y_2$ -plane **(Phase Plane)**
- **Phase Plane:** $y_1\, y_2$ -plane
- **Phase Portrait:** The phase plane filled with trajectories

❖ **Critical point → Determine a general form of the phase portrait**

$$\mathbf{y} = \mathbf{x}e^{\lambda t} \Rightarrow \mathbf{y}' = \boxed{\lambda\mathbf{x}\cancel{e^{\lambda t}}} = \mathbf{A}\mathbf{y} = \boxed{\mathbf{A}\mathbf{x}\cancel{e^{\lambda t}}} \Rightarrow \mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

$$\mathbf{y}' = \mathbf{A}\mathbf{y} \Rightarrow \frac{dy_2}{dy_1} = \frac{dy_2 / dt}{dy_1 / dt} = \frac{y_2'}{y_1'} = \frac{a_{21}y_1 + a_{22}y_2}{a_{11}y_1 + a_{12}y_2}$$ → **becomes undetermined, 0/0**

4.4 Criteria for Critical Points. Stability

❖ **Criteria for Types of Critical Points (임계점 유형의 판별기준)**

$\lambda_1,\ \lambda_2$ are eigenvalues of $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Characteristic equation: $\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = \lambda^2 - (a_{11} + a_{22})\lambda + \det\mathbf{A} = 0$

$\Rightarrow$ $\lambda^2 - p\lambda + q = 0 \qquad p = a_{11} + a_{22}, \quad q = \det\mathbf{A} = a_{11}a_{22} - a_{12}a_{21}, \quad \Delta = p^2 - 4q$

$\lambda_1 = \frac{1}{2}(p + \sqrt{\Delta}), \quad \lambda_2 = \frac{1}{2}(p - \sqrt{\Delta}),$

Name	$p=\lambda_1+\lambda_2$	$q=\lambda_1\lambda_2$	$\Delta=(\lambda_1-\lambda_2)^2$	Comments on $\lambda_1\ \lambda_2$
(a) Node		$q>0$	$\Delta \ge 0$	Real, same sign
(b) Saddle Point		$q<0$		Real, opposite sign
(c) Center	$p=0$	$q>0$		Pure imaginary
(d) Spiral Point	$p \ne 0$		$\Delta<0$	Complex, not pure imaginary

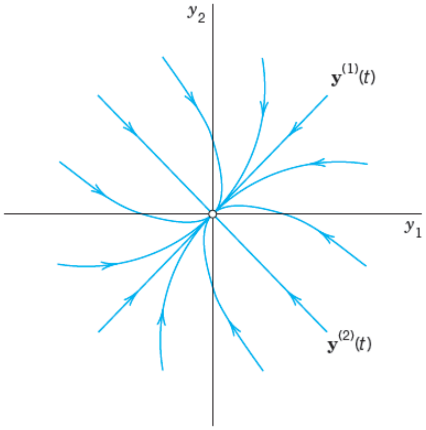
Eigenvalue Criteria for Critical Points

4.4 Criteria for Critical Points. Stability – Node

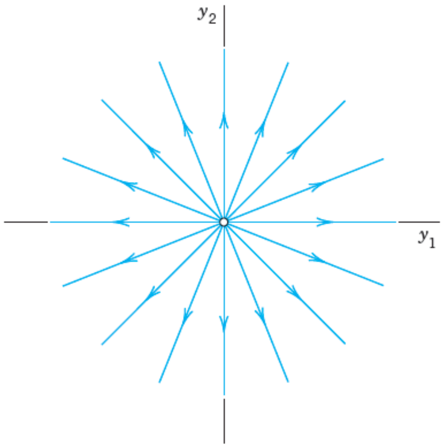
Name	$p=\lambda_1+\lambda_2$	$q=\lambda_1\lambda_2$	$\Delta=(\lambda_1-\lambda_2)^2$	Comments on $\lambda_1\ \lambda_2$
(a) Node (마디점)		$q>0$	$\Delta\geq 0$	Real, same sign

$$\lambda=-2,-4,\ q=8,\ \Delta>0$$
$$\mathbf{y}=c_1\begin{bmatrix}1\\1\end{bmatrix}e^{-2t}+c_2\begin{bmatrix}0\\1\end{bmatrix}e^{-4t}$$

$$\lambda=1,\ q=1,\Delta=0$$
$$\mathbf{y}=c_1\begin{bmatrix}1\\0\end{bmatrix}e^{t}+c_2\begin{bmatrix}0\\1\end{bmatrix}e^{t}\ \Rightarrow\ \begin{matrix}y_1=c_1e^t\\ y_2=c_2e^t\end{matrix}\ \Rightarrow\ c_1y_2=c_2y_1$$



Improper node

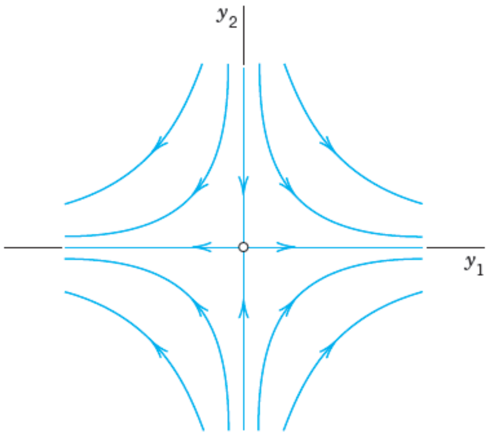


Proper node

4.4 Criteria for Critical Points. Stability – Saddle Point

Name	$p=\lambda_1+\lambda_2$	$q=\lambda_1\lambda_2$	$\Delta=(\lambda_1-\lambda_2)^2$	Comments on λ_1 λ_2
(b) Saddle Point (안장점)		$q<0$		Real, opposite sign

$$\boxed{\begin{aligned}&\lambda = 1,-1,\ \ q=-1\\ &\mathbf{y} = c_1\begin{bmatrix}1\\0\end{bmatrix}e^{t} + c_2\begin{bmatrix}0\\1\end{bmatrix}e^{-t}\end{aligned}} \Rightarrow \begin{aligned}y_1 &= c_1 e^{t}\\ y_2 &= c_2 e^{-t}\end{aligned} \Rightarrow y_1 y_2 = \text{const.}$$



Saddle point

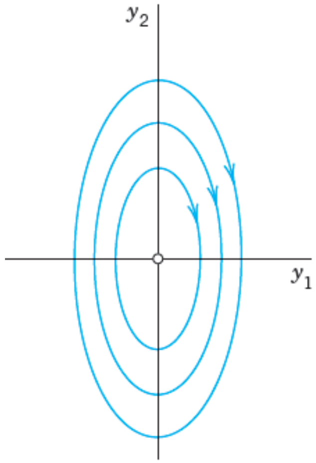
4.4 Criteria for Critical Points. Stability – Center

Name	$p=\lambda_1+\lambda_2$	$q=\lambda_1\lambda_2$	$\Delta=(\lambda_1-\lambda_2)^2$	Comments on λ_1 λ_2
(c) Center (중심점)	$p = 0$	$q > 0$		Pure imaginary

$$\lambda = 2i, -2i$$

$$\mathbf{y} = c_1\begin{bmatrix}1\\2i\end{bmatrix}e^{2it} + c_2\begin{bmatrix}0\\1\end{bmatrix}e^{-2it}$$

$$y_1' = y_2,\ y_2' = -4y_1 \quad\Rightarrow\quad 4y_1y_1' = -y_2y_2' \quad\Rightarrow\quad 2y_1^2 + \frac{1}{2}y_2^2 = \text{const.}$$



Center

$$\lambda^2 - p\lambda + q = 0,\ \ p = \lambda_1 + \lambda_2,\ q = \lambda_1\lambda_2,\ \ \Delta = p^2 - 4q = (\lambda_1 - \lambda_2)^2$$

$$\lambda_1 = \frac{1}{2}(p + \sqrt{\Delta}),\quad \lambda_2 = \frac{1}{2}(p - \sqrt{\Delta}),$$

- ✓ If p = 0, $\lambda_2 = -\lambda_1$
- ✓ If also q > 0 → $\lambda_1^2 = -q < 0$ → $\lambda_2 = -\lambda_1$ are pure imaginary
  → periodic solutions, trajectories are closed curves around P_0.

4.4 Criteria for Critical Points. Stability – Spiral Point

Name	$p=\lambda_1+\lambda_2$	$q=\lambda_1\lambda_2$	$\Delta=(\lambda_1-\lambda_2)^2$	Comments on $\lambda_1\ \lambda_2$
(d) Spiral Point (나선점)	$p \neq 0$		$\Delta < 0$	Complex, not pure imaginary

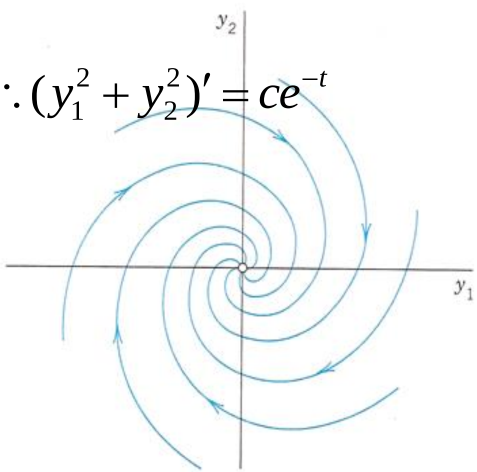
$$\lambda = -1+i, -1-i, \ \ \Delta < 0$$

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(-1+i)t} + c_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(-1-i)t}$$

$$\lambda^2 - p\lambda + q = 0, \ \ p = \lambda_1 + \lambda_2, \ q = \lambda_1\lambda_2, \ \ \Delta = p^2 - 4q = (\lambda_1 - \lambda_2)^2$$

$$\lambda_1 = \frac{1}{2}(p + \sqrt{\Delta}), \quad \lambda_2 = \frac{1}{2}(p - \sqrt{\Delta}),$$

$$\therefore (y_1^2 + y_2^2)' = ce^{-t}$$



Spiral point

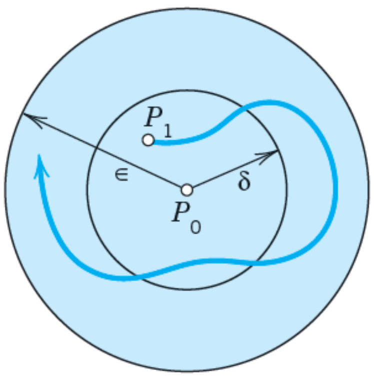
- ✓ If $\Delta < 0$, the eigenvalues are complex conjugates,
  $\rightarrow \lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$
- ✓ If also $p = \lambda_1 + \lambda_2 = 2\alpha < 0 \rightarrow$ a spiral point: **stable and attractive**
- ✓ If also $p = \lambda_1 + \lambda_2 = 2\alpha > 0 \rightarrow$ a spiral point: **unstable**

4.4 Criteria for Critical Points. Stability

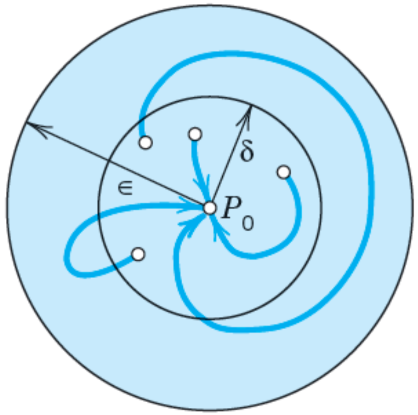
- Critical points may also be classified in terms of their stability.
- **Stability (안정성)**: A small change (a small disturbance) of a physical system at some instant changes the behavior of the system only slightly at all future times *t*.

❖ **Definitions Stable, Unstable, Stable and Attractive**

- A critical point P_0 is called **stable** if, roughly, all trajectories that at some instant are close to P_0 remain close to P_0 at all future times.
- A critical point is called **unstable** if the critical point is not stable.
- P_0 is call **stable and attractive** if P_0 is stable and every trajectory that has a point approaches P_0 as $t \to \infty$.



Stable critical point
The trajectory initiating at P_1 stays in the disk of radius ε.



Stable and attractive critical point

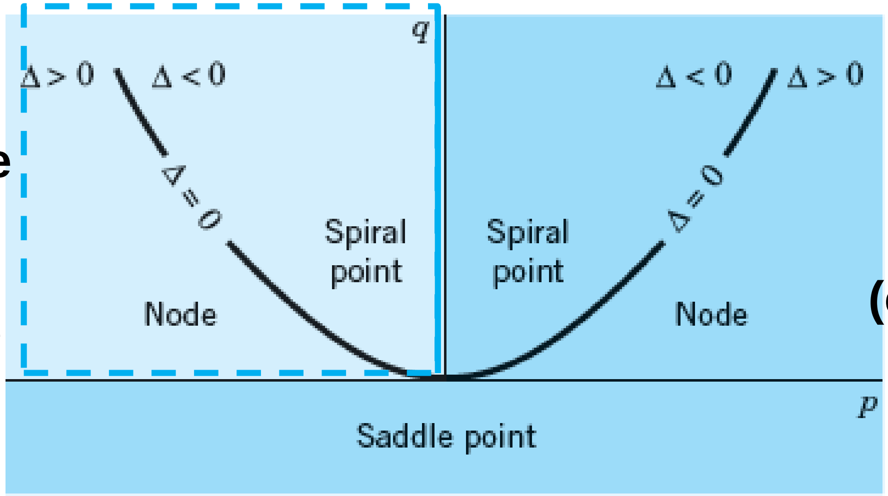
4.4 Criteria for Critical Points. Stability

❖ Stability Criteria for Critical Points

Type of Stability	$p=\lambda_1+\lambda_2$	$q=\lambda_1\lambda_2$
(a) Stable and attractive	$p < 0$	$q > 0$
(b) Stable	$p \leq 0$	$q > 0$
(c) Unstable	$p > 0$ or	$q < 0$

(a) Stable and attractive

(b) Stable



(c) Unstable

Stability chart of the system

4.4 Criteria for Critical Points. Stability – Stable

Type of Stability	$p=\lambda_1+\lambda_2$	$q=\lambda_1\lambda_2$
(a) Stable and attractive	$p<0$	$q>0$
(b) Stable	$p\le 0$	$q>0$

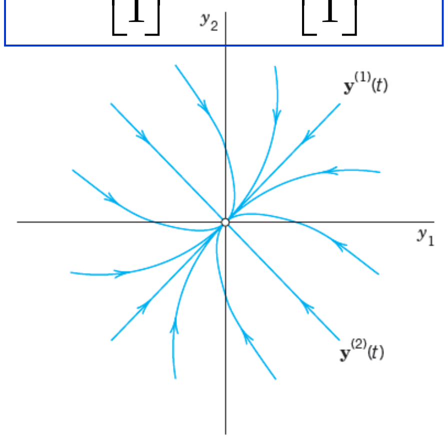
$$\lambda^2 - p\lambda + q = 0,\ \ p=\lambda_1+\lambda_2,\ q=\lambda_1\lambda_2,\ \ \Delta = p^2-4q=(\lambda_1-\lambda_2)^2$$

If $p=\lambda_1+\lambda_2<0$ and $q=\lambda_1\lambda_2>0$

$\rightarrow$ **λ_1, λ_2 are both are negative or have a negative real part.**

$$\lambda=-2,-4,\ \ p=-6, q=8$$

$$\mathbf{y}=c_1\begin{bmatrix}1\\1\end{bmatrix}e^{-2t}+c_2\begin{bmatrix}0\\1\end{bmatrix}e^{-4t}$$

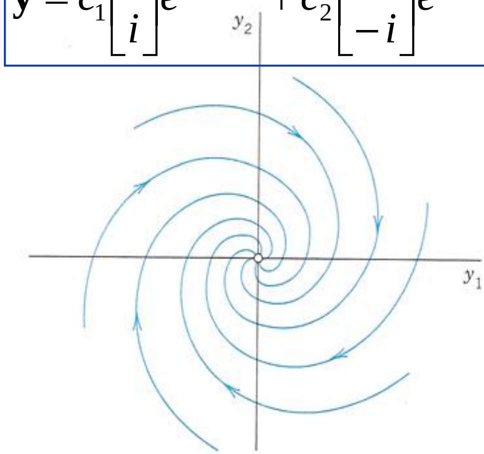


Improper node point

- Stable and attractive

$$\lambda=-1+i,-1-i,\ \ p=-2,\ q=2$$

$$\mathbf{y}=c_1\begin{bmatrix}1\\i\end{bmatrix}e^{(-1+i)t}+c_2\begin{bmatrix}1\\-i\end{bmatrix}e^{(-1-i)t}$$

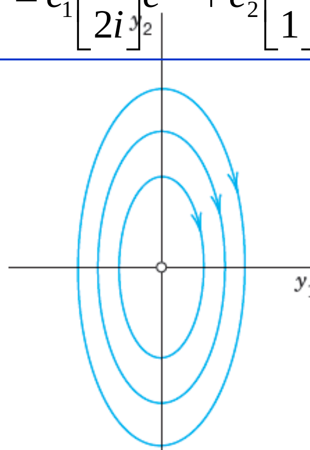


Spiral point

- Stable and attractive

$$\lambda=2i,-2i,\ \ p=0, q=4$$

$$\mathbf{y}=c_1\begin{bmatrix}1\\2i\end{bmatrix}e^{2it}+c_2\begin{bmatrix}0\\1\end{bmatrix}e^{-2it}$$



Center

- Stable

4.4 Criteria for Critical Points. Stability – Unstable

Type of Stability	$p=\lambda_1+\lambda_2$	$q=\lambda_1\lambda_2$
(c) Unstable	$p > 0$ or	$q < 0$

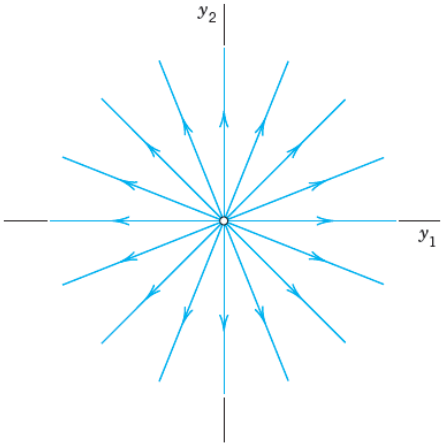
$$\lambda^2 - p\lambda + q = 0,\ \ p = \lambda_1 + \lambda_2,\ q = \lambda_1\lambda_2,\ \ \Delta = p^2 - 4q = (\lambda_1 - \lambda_2)^2$$

If $p=\lambda_1+\lambda_2>0$ or $q=\lambda_1\lambda_2<0$ $\rightarrow$ **λ_1, λ_2 are both positive or opposite.**

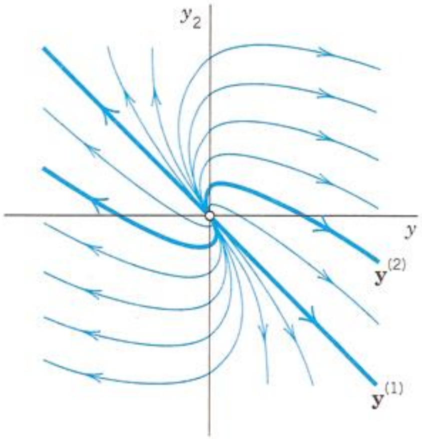
$$\lambda = 1,\ \ p = 2, q = 1$$
$$\mathbf{y} = c_1\begin{bmatrix}1\\0\end{bmatrix}e^t + c_2\begin{bmatrix}0\\1\end{bmatrix}e^t$$

$$\lambda = 3,\ \ p = 6, q = 9$$
$$\mathbf{y} = c_1\begin{bmatrix}1\\-1\end{bmatrix}e^{3t} + c_2\left(\begin{bmatrix}1\\-1\end{bmatrix}t + \begin{bmatrix}0\\1\end{bmatrix}\right)e^{3t}$$

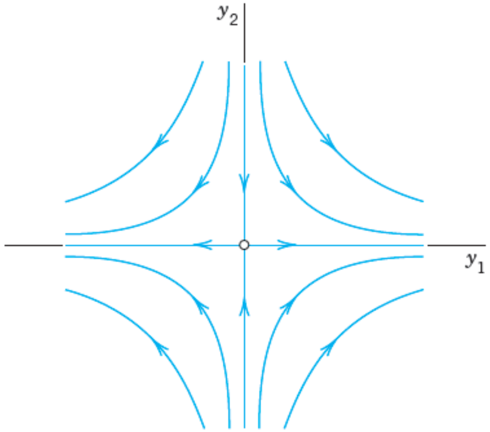
$$\lambda = 1,-1,\ \ p = 0,\ q < -1,$$
$$\mathbf{y} = c_1\begin{bmatrix}1\\0\end{bmatrix}e^t + c_2\begin{bmatrix}0\\1\end{bmatrix}e^{-t}$$



Proper node



Degenerate node



Saddle point

4.5 Qualitative Methods for Nonlinear Systems

Qualitative Method (정성법)

- Method of obtaining qualitative information on solutions without actually solving a system.
- These method is particularly valuable for systems whose solution by analytic methods is difficult or impossible.

Nonlinear systems

$$\mathbf{y}' = \mathbf{f}(\mathbf{y}), \text{ thus } \begin{array}{l} y_1' = f_1(y_1, y_2) \\ y_2' = f_2(y_1, y_2) \end{array}$$

Linearization

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{h}(\mathbf{y}), \text{ thus } \begin{array}{l} y_1' = a_{11}y_1 + a_{12}y_2 + h_1(y_1, y_2) \\ y_2' = a_{21}y_1 + a_{22}y_2 + h_2(y_1, y_2) \end{array}$$

4.5 Qualitative Methods for Nonlinear Systems

❖ Several Critical Method

- If a critical point P_0 is not at the origin (0, 0)

→ we shall move this point to the origin before analyzing the point.

- P_0 is a critical point with (a, b) not at the origin (0, 0), then we apply the translation.

→ $\tilde{y}_1 = y_1 - a, \tilde{y}_2 = y_2 - b$, which moves P_0 to $(0, 0)$.

- We can assume P_0 to be the origin $(0, 0)$ and we continue to write y_1, y_2 (instead of $\tilde{y}_1, \tilde{y}_2$).

4.5 Qualitative Methods for Nonlinear Systems

❖ **Theorem 1 Linearization (선형화)**

If f_1 and f_2 are continuous and have continuous partial derivatives in a neighborhood of the critical point (0, 0),

and if $\det \mathbf{A} \neq 0$, then the kind and stability of the critical point of nonlinear systems are the same as those of the linearized system

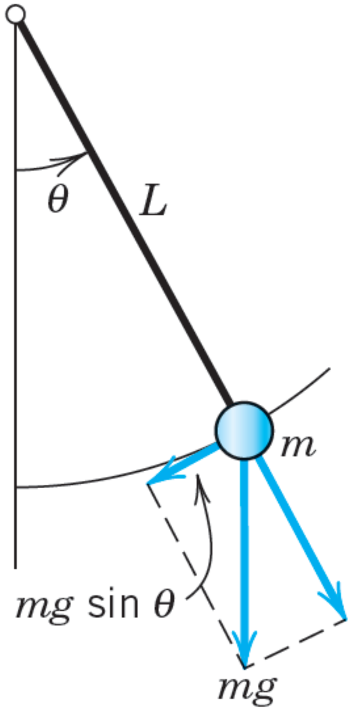
$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \quad \text{thus} \quad \begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 \\ y_2' &= a_{21}y_1 + a_{22}y_2. \end{aligned}$$

Exceptions occur if A has equal or pure imaginary eigenvalues; then the nonlinear system may have the same kind of critical points as linearized system or a spiral point.

4.5 Qualitative Methods for Nonlinear Systems

☑ Ex. 1 Free Undamped Pendulum, Linearization

Figure (a) shows a pendulum consisting of a body of mass *m* (the bob) and a rod of length *L*. Determine the locations and types of the critical points. Assume that the mass of the rod and air resistance are negligible.



Pendulum

Step 1 Setting up the mathematical model

θ: the angular displacement

mg: the weight of the bob

A restoring force tangent to the curve of motion: $mg\sin\theta$

= the force of acceleration: $mL\theta''$ ($L\theta''$: acceleration)

$$\therefore\ mL\theta'' + mg\sin\theta = 0 \quad \rightarrow \quad \theta'' + k\sin\theta = 0 \quad \left(k = \frac{g}{L}\right)$$

4.5 Qualitative Methods for Nonlinear Systems

Name	$p=\lambda_1+\lambda_2$	$q=\lambda_1\lambda_2$	$\Delta=(\lambda_1-\lambda_2)^2$	Comments on $\lambda_1\ \lambda_2$
(c) Center (중심점)	$p=0$	$q>0$		Pure imaginary

Step 2 Critical Points (0, 0), (±2π, 0), (±4π, 0), …. Linearization

$$\boxed{\theta''+k\sin\theta=0} \xrightarrow[y_2=\theta'\ :\text{각속도}]{y_1=\theta\ :\text{회전각}} \boxed{\begin{aligned} y_1' &= y_2 \\ y_2' &= -k\sin y_1 \end{aligned}}$$

$y_2=0,\quad \sin y_1=0 \quad \rightarrow \quad$ infinitely many critical points : $(n\pi\ ,\ 0),\quad n=0,\quad \pm1,\quad \pm2,\quad \cdots$

Type of Stability	$p=\lambda_1+\lambda_2$	$q=\lambda_1\lambda_2$
(b) Stable	$p\le 0$	$q>0$

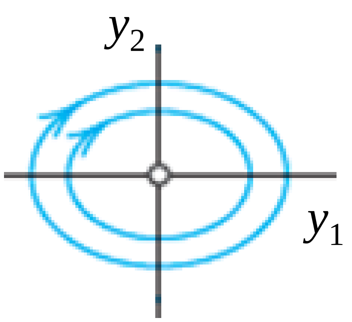
✓ **Consider (0, 0)**

Maclaurin series $\sin y_1 = y_1 - \dfrac{1}{6}y_1^{\,3} + - \cdots \approx y_1$

* A Maclaurin series is a Taylor series expansion of a function about 0.

$$\longrightarrow \quad \mathbf{y}'=\mathbf{A}\mathbf{y}=\begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix}\mathbf{y}, \quad \text{thus} \quad \begin{aligned} y_1' &= y_2 \\ y_2' &= -ky_1 \end{aligned}$$

$$p=a_{11}+a_{22}=0,\quad q=\det\mathbf{A}=k=\frac{g}{L},\qquad \Delta=p^2-4q=-4k$$



→ (0, 0) is a center, which is always stable.

* Taylor series expansion for sin(x)

$$\sin(x) = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

[Reference] Maclaurin Series

Taylor series expansion

$$T_f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{6}f'''(a)(x-a)^3 + \cdots$$

Maclaurin series is a Taylor series expansion of a function about 0, that is, a = 0

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots.$$

4.5 Qualitative Methods for Nonlinear Systems

Step 3 Critical Points $(\pm\pi, 0), (\pm 3\pi, 0), (\pm 5\pi, 0), \ldots$. Linearization

Name	$p=\lambda_1+\lambda_2$	$q=\lambda_1\lambda_2$	$\Delta=(\lambda_1-\lambda_2)^2$	Comments on λ_1 λ_2
(b) Saddle Point (안장점)		$q<0$		Real, opposite sign

✓ **Consider $(\pi, 0)$**

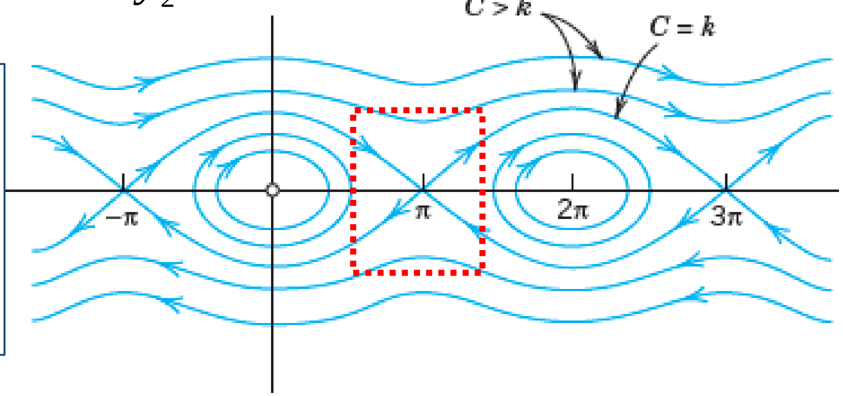
$$\theta'' + k\sin\theta = 0 \xrightarrow[\sin\theta = \sin(y_1+\pi) = -\sin y_1 = -y_1 + \frac{1}{6}y_1^3 - + \cdots \approx -y_1]{\text{set } y_1 = \theta-\pi,\ y_2 = (\theta-\pi)' = \theta'} \mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ k & 0 \end{bmatrix}\mathbf{y}$$

$$p = 0,\ \ q = -k\,(<0),\quad \Delta = p^2 - 4q = 4k \quad \Rightarrow$$ Critical points are all saddle points.

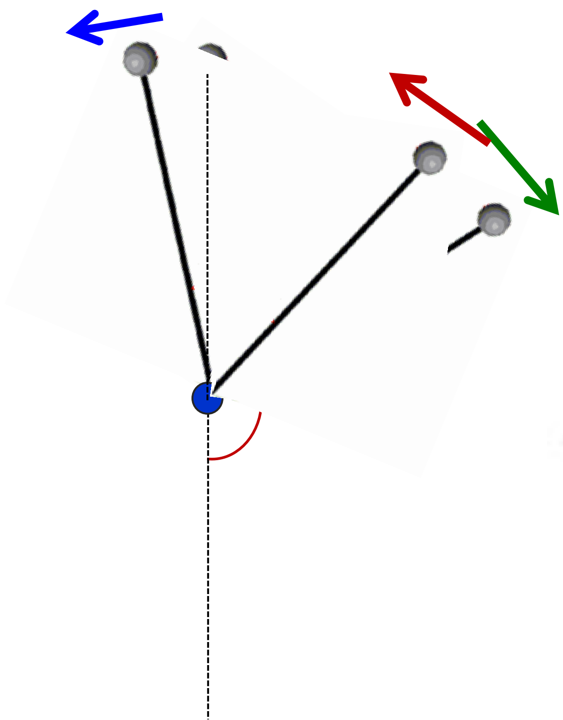
☑ **Saddle Point → Unstable**

: A critical point at which there are **two incoming trajectories**, **two outgoing trajectories**,

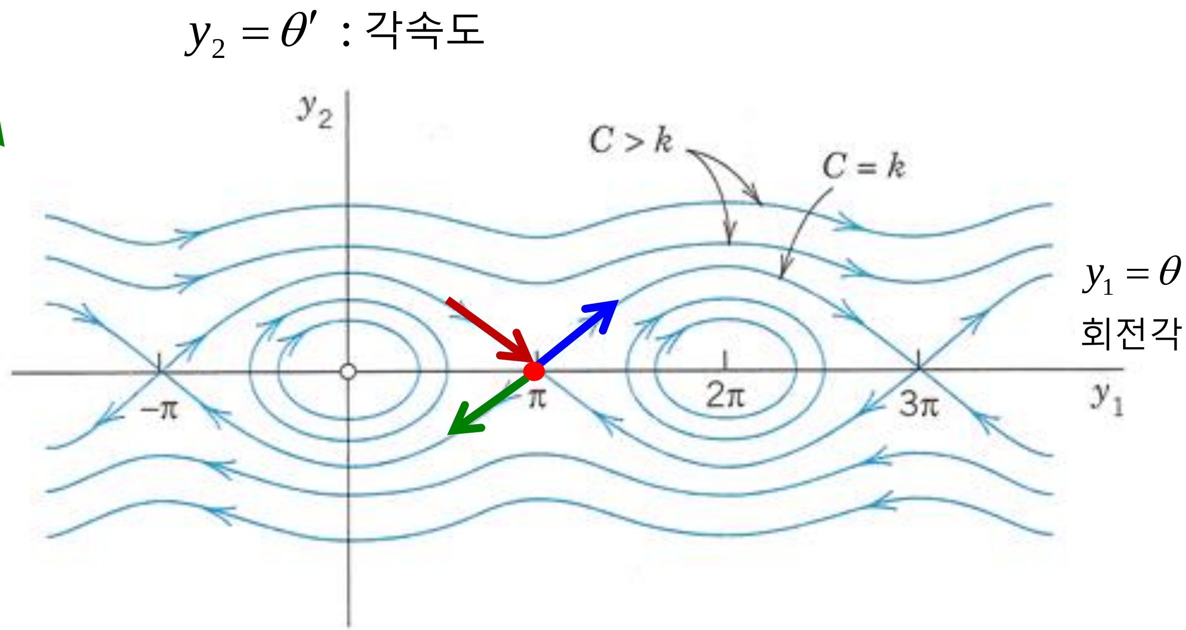
and **all the other trajectories in a neighborhood of P_0 bypass P_0**



4.5 Qualitative Methods for Nonlinear Systems



Pendulum



Solution curves in the phase plane

4.5 Qualitative Methods for Nonlinear Systems

❖ **Transformation to a First-Order Equation in the Phase Plane**

$$F\left(y, y', y''\right) = 0 \xrightarrow{\text{set } y = y_1,\ \ y' = y_2} F\left(y_1, y_2, \frac{dy_2}{dy_1} y_2 \right) = 0$$

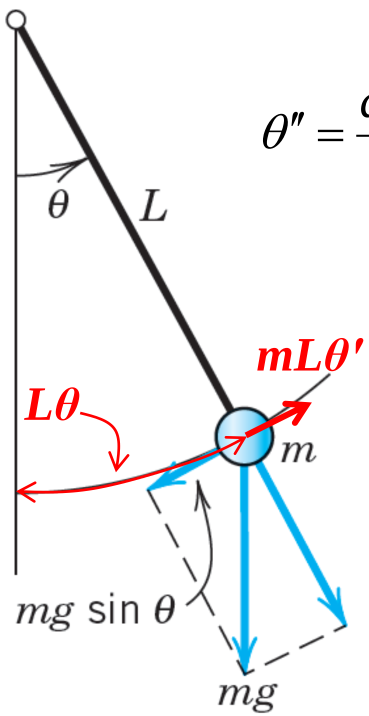
$$y'' = y_2' = \frac{dy_2}{dt} = \frac{dy_2}{dy_1}\frac{dy_1}{dt} = \frac{dy_2}{dy_1} y_2$$

4.5 Qualitative Methods for Nonlinear Systems

☑ Ex. 8 An ODE for the Free Undamped Pendulum

$$\therefore\ mL\theta'' + mg\sin\theta = 0 \quad \rightarrow \quad \theta'' + k\sin\theta = 0 \quad \left(k = \frac{g}{L}\right)$$

$$\boxed{\theta'' + k\sin\theta = 0} \xrightarrow[y_2 = \theta' \ :\text{각속도}]{y_1 = \theta \ :\text{회전각}} \boxed{\begin{aligned} y_1' &= y_2 \\ y_2' &= -k\sin y_1 \end{aligned}}$$



Pendulum

$$\theta'' = \frac{dy_2}{dt} = \frac{dy_2}{dy_1}\frac{dy_1}{dt} = \frac{dy_2}{dy_1}y_2 \Rightarrow \frac{dy_2}{dy_1}y_2 = -k\sin y_1$$

$$y_2 dy_2 = -k\sin y_1 dy_1$$

$$\frac{1}{2}y_2^2 = k\cos y_1 + C$$

✓ **Multiplying by mL^2 adding mgL to both sides**

$$\frac{1}{2}m(Ly_2)^2 - mL^2 k\cos y_1 + mgL = mL^2 C + mgL \qquad \boxed{k = \frac{g}{L}}$$

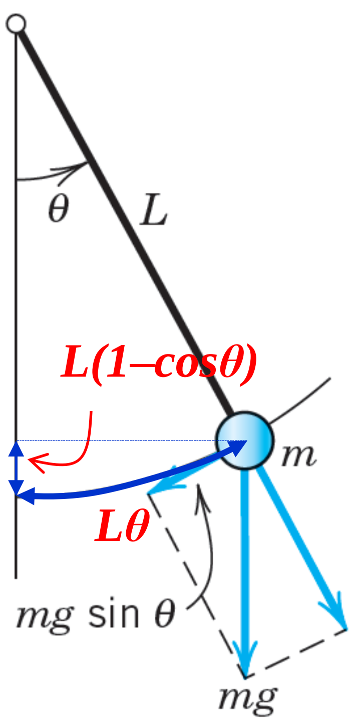
$$\frac{1}{2}m(L\theta')^2 + mgL - mgL\cos\theta = mgL + mL^2 C$$

4.5 Qualitative Methods for Nonlinear Systems

☑ Ex. 8 An ODE for the Free Undamped Pendulum

$$\frac{1}{2}m(L\theta')^2 + mgL(1-\cos\theta) = mL(g+LC)$$

Kinematic energy | Potential energy | Total energy



Pendulum

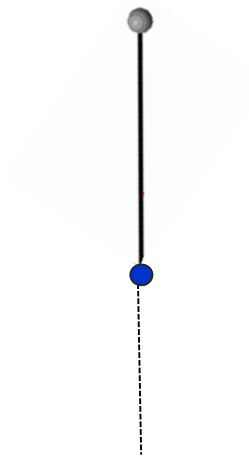
✓ **If $C = k$ (= g/L)**

$$\frac{1}{2}m(L\theta')^2 + mgL(1-\cos\theta) = mL(g+L\frac{g}{L}) = 2mgL$$

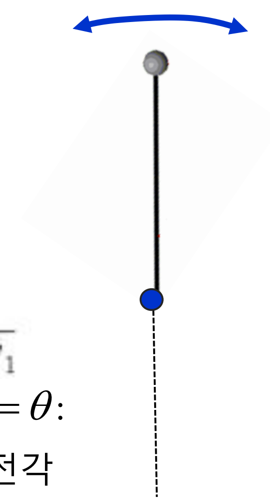
✓ **If $C > k$**

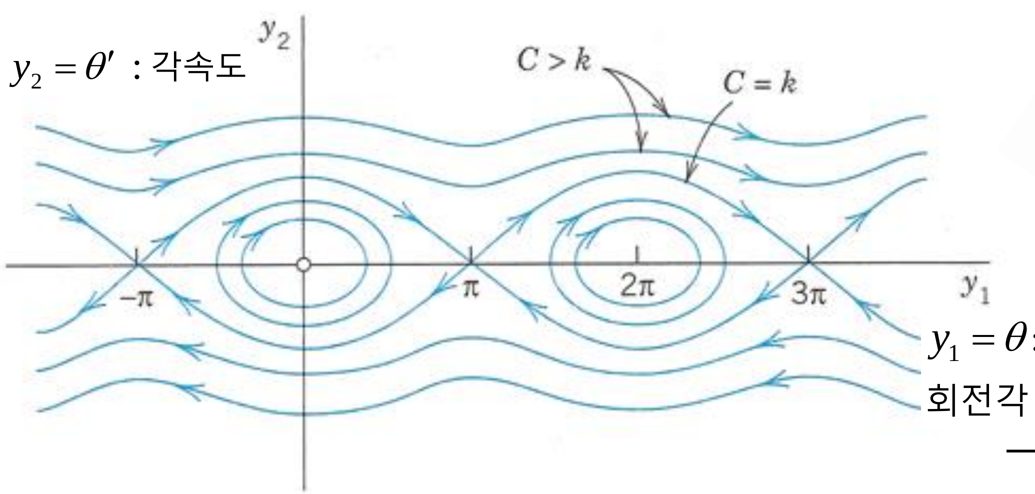
$$\frac{1}{2}m(L\theta')^2 + mgL(1-\cos\theta) = mL(g+LC) > 2mgL$$

$L\theta' = 0$



$L\theta' \neq 0$





4.5 Qualitative Methods for Nonlinear Systems

☑ Ex. 2 Linearization of the Damped Pendulum Equation

We add damping term $c\theta'$ (proportional to the angular velocity).

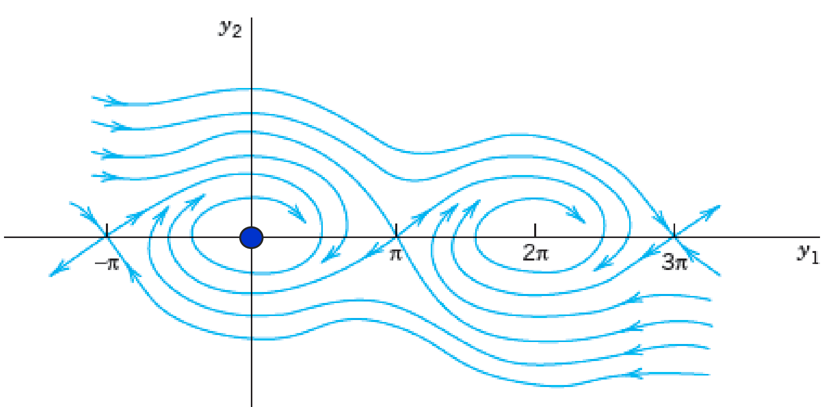
$$\boxed{\theta'' + c\theta' + k\sin\theta = 0} \xrightarrow[y_2 = \theta' \ : \text{각속도}]{y_1 = \theta \ : \text{회전각}} \boxed{\begin{aligned} y_1' &= y_2 \\ y_2' &= -k\sin y_1 - cy_2 \end{aligned}}$$

✓ **Consider (0, 0) and $\sin y_1 \approx y_1$**

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix}\mathbf{y}$$

- $p = a_{11} + a_{22} = -c < 0$
- $q = \det A = k > 0$
- $\Delta = p^2 - 4q = c^2 - 4k$
  - ✓ *if $c^2 > 4k \rightarrow$ Node*
  - ✓ *if $c^2 < 4k \rightarrow$ Spiral point*

Name	$p=\lambda_1+\lambda_2$	$q=\lambda_1\lambda_2$	$\Delta=(\lambda_1-\lambda_2)^2$
(a) Node		$q > 0$	$\Delta \geq 0$
(b) Saddle Point		$q < 0$	
(c) Center	$p = 0$	$q > 0$	
(d) Spiral Point	$p \neq 0$		$\Delta < 0$



4.5 Qualitative Methods for Nonlinear Systems

☑ Ex. 2 Linearization of the Damped Pendulum Equation

✓ **Consider $(\pi, 0)$ and $y_1 = \theta - \pi$, $y_2 = \theta'$**

$\rightarrow$ **$\sin\theta \approx \sin(y_1 + \pi) = -\sin y_1 \approx -y_1$**

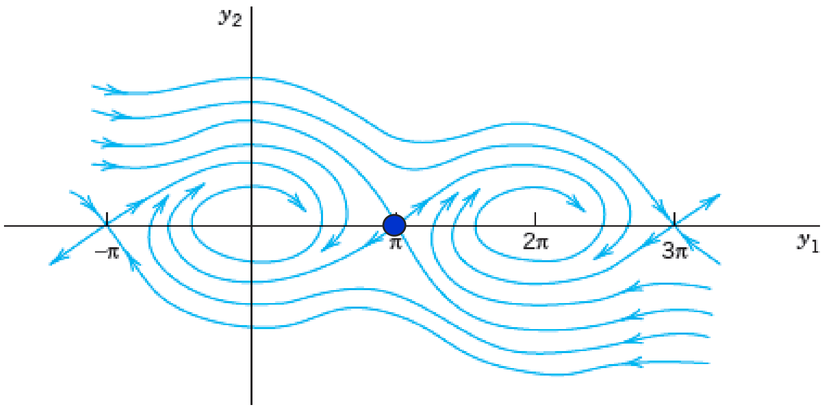
$$\boxed{\theta'' + c\theta' + k\sin\theta = 0}$$

$$y_1' = y_2$$
$$y_2' = -k\sin\theta - c\theta' = ky_1 - cy_2$$

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ k & -c \end{bmatrix}\mathbf{y} \quad \Rightarrow$$

- $p = a_{11} + a_{22} = -c < 0$
- $q = \det A = -k < 0$
- $\Delta = p^2 - 4q = c^2 + 4k > 0$

Name	$p=\lambda_1+\lambda_2$	$q=\lambda_1\lambda_2$	$\Delta=(\lambda_1-\lambda_2)^2$
(a) Node		$q > 0$	$\Delta \geq 0$
(b) Saddle Point		$q < 0$	
(c) Center	$p = 0$	$q > 0$	
(d) Spiral Point	$p \neq 0$		$\Delta < 0$



4.6 Nonhomogeneous Linear Systems of ODEs

❖ Nonhomogeneous of Linear Systems: $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g},\ \mathbf{g} \neq \mathbf{0}$

Assume $\mathbf{g}(t)$ and the entries of the $n \times n$ matrix $\mathbf{A}(t)$ to be continuous on some interval J of the t-axis.

⇨ General solution : $\mathbf{y} = \mathbf{y}^{(h)} + \mathbf{y}^{(p)}$

- $\mathbf{y}^{(h)}$: A general solution of the homogeneous system of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ on J
- $\mathbf{y}^{(p)}$: A particular solution (containing no arbitrary constants) of $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}$ on J

❖ Methods for obtaining particular solutions

- Method of Undetermined Coefficients (미정계수법)
- Method of the Variation of Parameter (매개변수변환법)

4.6 Nonhomogeneous Linear Systems of ODEs

❖ **Method of Undetermined Coefficients** (미정계수법)

If components of $\mathbf{g}$: (1) constants (2) positive integer powers of t

(3) exponential functions (4) cosines and sines.

→ $y^{(\mathrm{p})}$ is assumed in a form similar to $\mathbf{g}$.

☑ **Ex. 1 Method of Undetermined Coefficients. Modification Rule**

Find a general solutions of $\underline{\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} y + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$

A general solution of the homogeneous system: $\mathbf{y}^{(h)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$

Apply the Modification Rule by setting $\mathbf{y}^{(p)} = \mathbf{u}te^{-2t} + \mathbf{v}e^{-2t}$

$$\boxed{\mathbf{y}^{(p)'} = \mathbf{u}e^{-2t} \boxed{-2\mathbf{u}te^{-2t}} - 2\mathbf{v}e^{-2t} = \boxed{\mathbf{A}\mathbf{u}te^{-2t}} + \mathbf{A}\mathbf{v}e^{-2t} + \mathbf{g}}$$

Equating the te^{-2t} terms on both sides: $-2\mathbf{u} = \mathbf{A}\mathbf{u} \quad \Rightarrow \quad \mathbf{u} = a\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (\text{with any } a \neq 0)$

→ $\mathbf{u}$: eigenvector of A corresponding to $\lambda = -2$

[Reference] Nonhomogeneous Linear ODEs

❖ Method of Undetermined Coefficients (미정계수법)

$$y^{(n)} + p_{n-1}(x)\,y^{(n-1)} + \cdots + p_1(x)\,y' + p_0(x)\,y = r(x), \quad r(x) \neq 0$$

❖ Choice Rules for the Method of Undetermined Coefficients

b. **Modification Rule.** If a term in your choice for y_p is a solution of the homogeneous ODE corresponding to $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = r(x)$ then multiply this term by x^k, where k is the smallest positive integer such that this term times x^k is not a solution of the homogeneous ODE.

Table 2.1 Method of Undetermined Coefficients

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$	$K \cos \omega x + M \sin \omega x$
$k \sin \omega x$	
$ke^{\alpha x} \cos \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$
$ke^{\alpha x} \sin \omega x$	

4.6 Nonhomogeneous Linear Systems of ODEs

❖ **Method of Undetermined Coefficients** (미정계수법)

If components of $\mathbf{g}$: (1) constants (2) positive integer powers of t

(3) exponential functions (4) cosines and sines.

→ $y^{(\mathrm{p})}$ is assumed in a form similar to $\mathbf{g}$.

☑ **Ex. 1 Method of Undetermined Coefficients. Modification Rule**

Find a general solutions of $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} y + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$

A general solution of the homogeneous system: $\mathbf{y}^{(h)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$

Apply the Modification Rule by setting $\mathbf{y}^{(p)} = \mathbf{u}te^{-2t} + \mathbf{v}e^{-2t}$

$$\boxed{\mathbf{y}^{(p)'} = \mathbf{u}e^{-2t} - 2\mathbf{u}te^{-2t} - 2\mathbf{v}e^{-2t} = \mathbf{A}\mathbf{u}te^{-2t} + \mathbf{A}\mathbf{v}e^{-2t} + \mathbf{g}}$$

Q ? If $\mathbf{v}$ is not included?

$\mathbf{y}^{(p)'} = \mathbf{u}e^{-2t} - 2\mathbf{u}te^{-2t} = \mathbf{A}\mathbf{u}te^{-2t} + \mathbf{g}$ ⇨ $-2\mathbf{u} = \mathbf{A}\mathbf{u}\ \&\ \mathbf{u}e^{-2t} = \mathbf{g}$ ⇨ Inconsistent!

4.6 Nonhomogeneous Linear Systems of ODEs

☑ Ex. 1 Method of Undetermined Coefficients. Modification Rule

$$\boxed{\mathbf{y}^{(p)'} = \mathbf{u}e^{-2t} - 2\mathbf{u}te^{-2t} - 2\mathbf{v}e^{-2t} = \mathbf{Au}te^{-2t} + \mathbf{Av}e^{-2t} + \mathbf{g}}$$

Equating the other terms:

$$\mathbf{u} - 2\mathbf{v} = \mathbf{Av} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} a \\ a \end{bmatrix} - \begin{bmatrix} 2v_1 \\ 2v_2 \end{bmatrix} = \begin{bmatrix} -3v_1 + v_2 \\ v_1 - 3v_2 \end{bmatrix} + \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

$$v_1 - v_2 = -a - 6$$
$$-v_1 + v_2 = -a + 2$$

$$0 = -2a - 4 \Rightarrow a = -2$$

$$v_2 = v_1 + 4, \ \Rightarrow v_1 = k, \ v_2 = k + 4$$

If we simply choose k= –2, General solution:

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} - 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-2t} + \begin{bmatrix} -2 \\ 2 \end{bmatrix} e^{-2t}$$

4.6 Nonhomogeneous Linear Systems of ODEs

❖ Method of the Variation of Parameter (매개변수 변환법)

: It yields a particular solution $\mathbf{y}^{(p)}$ on some open interval J on the t-axis if a general solution of the homogeneous system on J is known.

- Explain the method

General solution of the homogeneous system:

$$\mathbf{y}^{(h)} = c_1\mathbf{y}^{(1)} + \cdots + c_n\mathbf{y}^{(n)} = \mathbf{Y}(t)\mathbf{c} \qquad (\therefore \;\; \mathbf{Y}' = \mathbf{A}\mathbf{Y})$$

Particular solution: $\mathbf{y}^{(p)} = \mathbf{Y}(t)\mathbf{u}(t)$

$$\left(\mathbf{y}^{(p)}\right)' = \mathbf{Y}'\mathbf{u} + \mathbf{Y}\mathbf{u}' \quad \Rightarrow \quad \mathbf{Y}'\mathbf{u} + \mathbf{Y}\mathbf{u}' = \mathbf{A}\mathbf{Y}\mathbf{u} + \mathbf{g}$$

$$\Rightarrow \quad \mathbf{Y}\mathbf{u}' = \mathbf{g} \quad \Rightarrow \quad \mathbf{u}' = \mathbf{Y}^{-1}\mathbf{g} \quad \Rightarrow \quad \mathbf{u} = \int_{t_0}^{t} \mathbf{Y}^{-1}(\tilde{t})\,\mathbf{g}(\tilde{t})\,d\tilde{t}$$

$$\mathbf{u}' = \mathbf{Y}^{-1}\mathbf{g} \Rightarrow \mathbf{u} = \int_{t_0}^{t} \mathbf{Y}^{-1}(\tilde{t})\mathbf{g}(\tilde{t})\,d\tilde{t}$$

☑ Ex. 2 Solution by the Method of Variation of Parameters

Solve $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} y + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$

$$\mathbf{y}^{(h)} = \begin{bmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \mathbf{Y}(t)\mathbf{c}$$

$$\mathbf{Y}^{-1} = \frac{1}{-2e^{-6t}} \begin{bmatrix} -e^{-4t} & -e^{-4t} \\ -e^{-2t} & e^{-2t} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{2t} & e^{2t} \\ e^{4t} & -e^{4t} \end{bmatrix}$$

$$\mathbf{u}' = \mathbf{Y}^{-1}\mathbf{g} = \frac{1}{2} \begin{bmatrix} e^{2t} & e^{2t} \\ e^{4t} & -e^{4t} \end{bmatrix} \begin{bmatrix} -6e^{-2t} \\ 2e^{-2t} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -4 \\ -8e^{2t} \end{bmatrix} = \begin{bmatrix} -2 \\ -4e^{2t} \end{bmatrix}$$

$$\mathbf{u} = \int_0^t \begin{bmatrix} -2 \\ -4e^{2\tilde{t}} \end{bmatrix} d\tilde{t} = \begin{bmatrix} -2t \\ -2e^{2t} + 2 \end{bmatrix}$$

$$\mathbf{y}^{(p)} = \mathbf{Y}\mathbf{u} = \begin{bmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{bmatrix} \begin{bmatrix} -2t \\ -2e^{2t} + 2 \end{bmatrix} = \begin{bmatrix} -2te^{-2t} - 2e^{-2t} + 2e^{-4t} \\ -2te^{-2t} + 2e^{-2t} - 2e^{-4t} \end{bmatrix} = \begin{bmatrix} -2t - 2 \\ -2t + 2 \end{bmatrix} e^{-2t} + \boxed{\begin{bmatrix} 2 \\ -2 \end{bmatrix} e^{-4t}}$$

$$\therefore\ y = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + \boxed{c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}} - 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-2t} + \begin{bmatrix} -2 \\ 2 \end{bmatrix} e^{-2t}$$

absorbed

4.6 Nonhomogeneous Linear Systems of ODEs

☑ **Example**

Q ? How to assume $y^{(p)}$

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} y + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$$

$$\mathbf{y}' = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} e^{t} \\ e^{-t} \end{bmatrix}$$

$$\mathbf{y}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} \sin t \\ 0 \end{bmatrix}$$