Topics in Ship Structures

07 Linear Fracture Mechanics

Reference : Fracture Mechanics by T.L. Anderson

Lecture Note of Eindhoven University of Technology

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Historical Overview

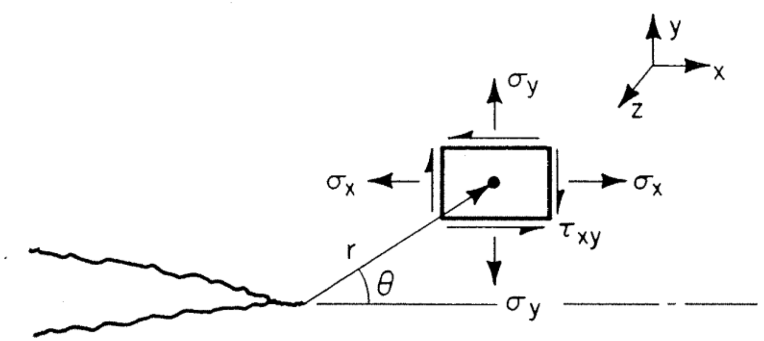
❖Griffith theory for brittle material (1920's)

- "A crack in a component will propagate if the total energy of the system is lowered with crack propagation."
- " if the change in elastic strain energy due to crack extension > the energy required to create new crack surfaces, crack propagation will occur"

❖Irwin (1940's) extended the theory for ductile materials.

- " the energy due to plastic deformation must be added to the surface energy associated with creation of new crack surfaces"
- " local stresses near the crack tip are of the general form"

$$\sigma_{ij} = \frac{K}{\sqrt{2\pi r}} f_{ij}(\theta) + \cdots$$



Minimum total potential energy principle

- Fundamental concept used in physics, chemistry, biology, and engineering.
- It dictates that a structure or body shall deform or displace to a position that (locally) minimizes the total potential energy, with the lost potential energy being converted into kinetic energy (specifically heat).

- The potential energy of an elastic body, Π, is defined as follows:

$$\Pi = U - F$$

Here, U *: strain energy stored in the body*

F *: the work done by external loads*

- Examples
  - ✓ A rolling ball will end up stationary at the bottom of a hill, the point of minimum potential energy. It rolls downward under the influence of gravity, friction produced by its motion transfers energy in the form of heat of the surroundings with an attendant increase in entropy.
  - ✓ Deformation of spring under gravity stretches and vibrates and finally stops due to the structural damping.

Minimum total potential energy principle

- This energy is at a stationary position when an infinitesimal variation from such position involves no change in energy.

$$\delta \Pi = \delta(U - F)$$

- The principle of minimum total potential energy may be derived as a special case of the virtual work principle for elastic systems subject to conservative forces.
- The equality between external and internal virtual work (due to virtual displacements) is:

$$\int_{S_t} \delta \mathrm{u}^T \, \mathrm{T} dS + \int_{S_t} \delta \mathrm{u}^T \, \mathrm{f} dV = \int_V \delta \varepsilon^T \sigma dV$$

u = vector of displacements

T = vector of distributed forces acting on the part of the surface

f = vector of body forces

- In the special case of elastic bodies

$$\delta \mathrm{U} = \int_V \delta \varepsilon^T \sigma dV, \delta \mathrm{F} = \int_{S_t} \delta u^T \, T dS + \int_V \delta u^T \, f dV$$

$$\therefore \delta U = \delta F$$

$\Rightarrow$ The basis for developing the finite element method.

An atomic view of fracture

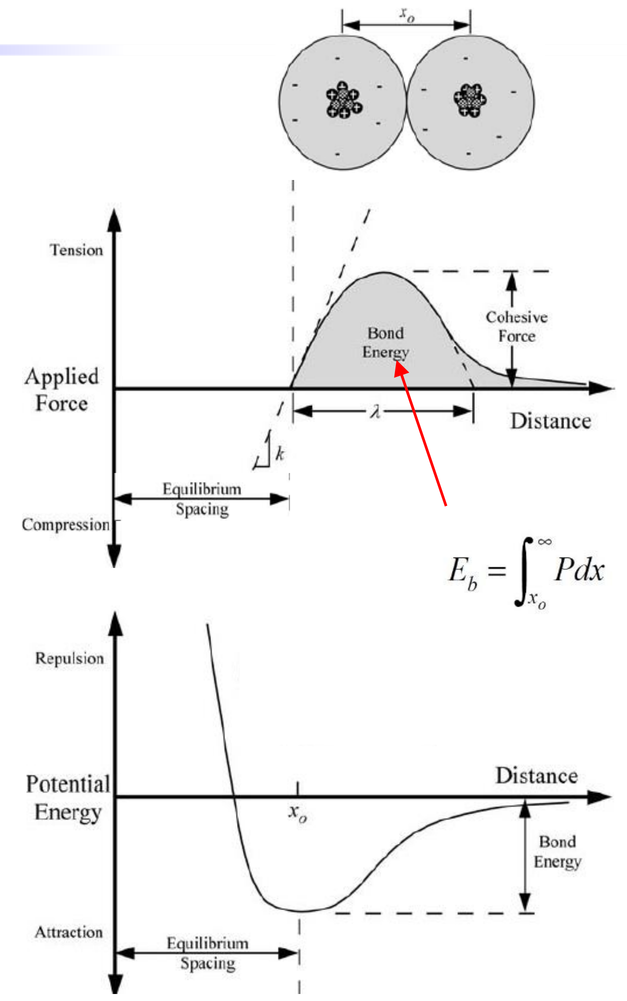
❖Physical meaning of fracture

- A material fractures when sufficient stress and work are applied at the atomic level to break the bonds that hold atoms together.
- The bond strength is supplied by the attractive forces between atoms.
- A tensile force is required to increase the separation distance from the equilibrium value, this force must exceed the cohesive force to sever the bond.
- The bond energy is given by

$$E_b = \int_{x_o}^{\infty} P dx$$

✓ x_0 : the equilibrium spacing

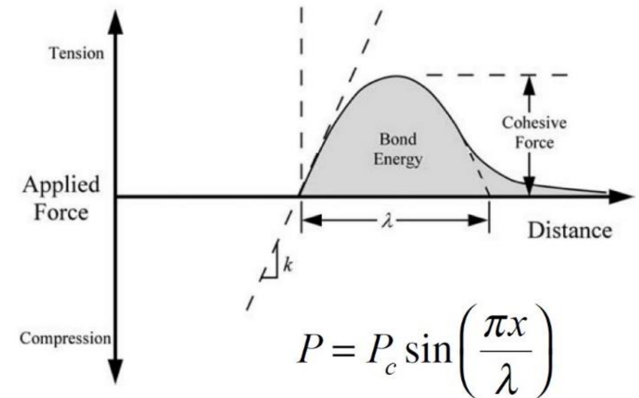
✓ P : the applied force



Calculation of cohesive stress

- Idealizing the interatomic force-displacement relationship as a sine wave.

$$P = P_c \sin\left(\frac{\pi x}{\lambda}\right)$$



- For small displacements,

$$P = P_c\left(\frac{\pi x}{\lambda}\right) \quad \Rightarrow k = P_c\left(\frac{\pi}{\lambda}\right) \quad (P = kx)$$

- Assume n atomics within unit area A(=1), nk → stiffness

- Left side $\dfrac{n k x_o}{A(=1)} = E$ Right side $\dfrac{n P_c x_o}{A(=1)}\left(\dfrac{\pi}{\lambda}\right) = \sigma_c\left(\dfrac{\pi x_o}{\lambda}\right) \Rightarrow \sigma_c = \dfrac{E\lambda}{\pi x_o}$

$$\left(\boldsymbol{\delta = \frac{PL}{EA} \to P = \frac{EA}{L}\delta,\ \ k = \frac{EA}{L} \to E = \frac{kL}{A}}\right)$$

- When λ is assumed to be equal to the atomic spacing

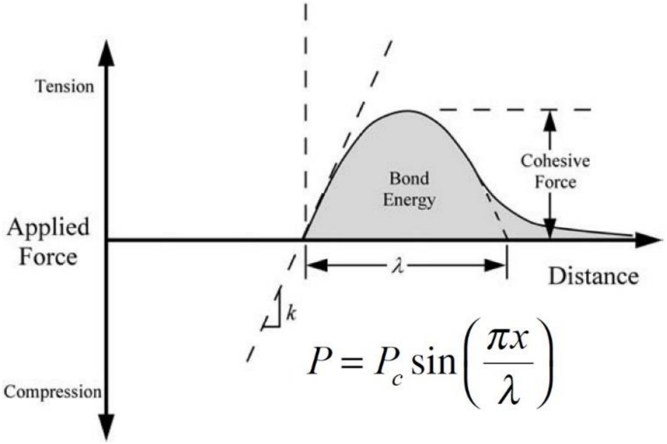
$$\sigma_c \approx \frac{E}{\pi}$$

Surface energy

- Surface energy

$$\gamma_s = \frac{1}{2}\int_0^{\lambda} \sigma_c \sin\left(\frac{\pi x}{\lambda}\right) dx = \sigma_c \frac{\lambda}{\pi}$$

force per unit area distance



- Cohesive stress in terms of surface energy

$$\sigma_c = \frac{E\lambda}{\pi x_o}, \quad \lambda = \frac{\sigma_c \pi x_o}{E}$$

$$\gamma_s = \sigma_c^2 \frac{x_o}{E} \Rightarrow \sigma_c = \sqrt{\frac{E\gamma_s}{x_o}}$$

γ_s : the surface energy per unit area,
= one-half of the fracture energy

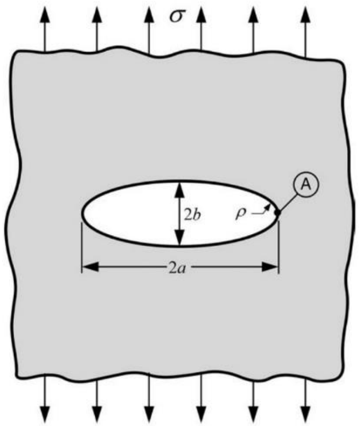
✓ because tension should be applied two atoms in opposite direction and surface energy of $2\gamma_s$ is required to split the atoms.

Stress concentration effect of flaws

❖Effect of flaws

- The theoretical cohesive strength of a material is approximately $\sigma_c = E/\pi$
- Experimental fracture strengths for brittle materials are typically three or four orders of magnitude below this value.
- The discrepancy between the actual strengths of brittle materials and theoretical estimates was due to flaws.
- Fracture occurs when the stress at the atomic level > cohesive strength of material.
- Flaws lower the global strength by magnifying the stress locally.
- The first quantitative evidence for the stress concentration effect of flaws was provided by Inglis

Stress concentration effect of flaws



❖Stress concentration

- The hole is not influenced by the plate B.C.
- i.e. plate width >> 2a, plate height >>2b

$$\sigma_A = \sigma\left(1 + \frac{2a}{b}\right)$$

- Where the ratio σ_A/σ is defined as stress concentration factor k_t. When a=b, the hole is circular and k_t=3.0.

- As the major axis a↑, relative to b, the elliptical hole → a sharp crack.

Inglis ⇒ $$\sigma_A = \sigma\left(1 + 2\sqrt{\frac{a}{\rho}}\right) \qquad \rho = \frac{b^2}{a}$$

When a>>b, $$\sigma_A = \sigma\left(1 + 2\sqrt{\frac{a}{\rho}}\right) \quad \Rightarrow \quad \sigma_A = 2\sigma\sqrt{\frac{a}{\rho}}$$

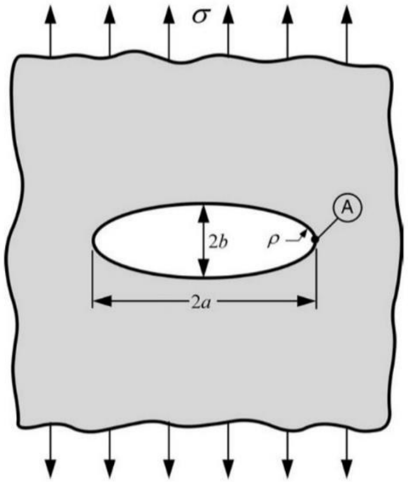
Calculation of global strength

❖ Calculation of remote stress at failure (by Inglis)

- **An infinite stress at the tip of an infinitely sharp crack, where** $\rho = 0$.
- Paradox: A material containing a sharp crack should theoretically fail upon the application of an infinitesimal(극소의) load.

  ⇒ this paradox of a sharp crack motivated Griffith to develop a fracture theory based on **energy rather than local stress**.
- **Metals deform plastically, which causes an initially sharp crack to blunt.**
- **In the absence of plastic deformation, the minimum radius a crack tip can have is on the order of atomic radius** ⇒ $\rho = x_0$
- Local stress concentration at the tip of an atomically sharp crack:

Stress at A : $\sigma_A = 2\sigma\sqrt{\frac{a}{x_o}}$ Cohesive stress : $\sigma_c = \sqrt{\frac{E\gamma_s}{x_o}}$



- Crack/fracture occurs when

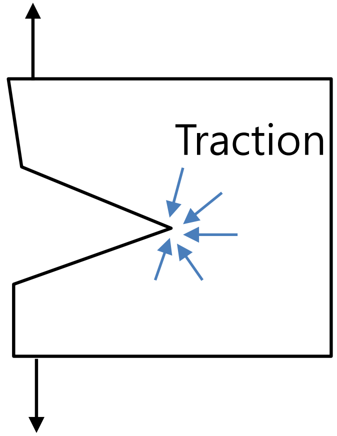
$$\sigma_A = 2\sigma\sqrt{\frac{a}{x_o}} = \sigma_c = \sqrt{\frac{E\gamma_s}{x_o}} \Rightarrow \sigma_f = \left(\frac{E\gamma_s}{4a}\right)^{1/2}$$

The Griffith energy balance

❖Griffith's idea

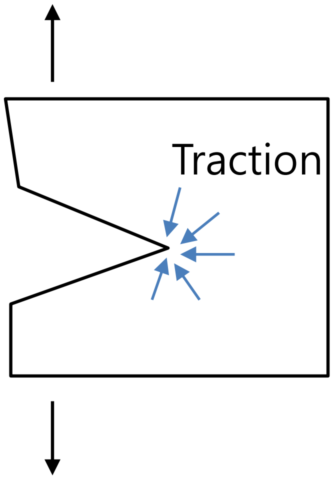
- First law of thermodynamics : when a system goes from a non equilibrium state to equilibrium, there is a net decrease in energy.
- The crack is formed by the sudden annihilation (소멸) of the tractions acting on its surface.
- The new state is not under equilibrium, then by the theorem of minimum potential energy, the potential energy is reduced by the attainment of equilibrium.
- A crack can form only if the process causes the total energy to decrease or remain constant.
- The critical conditions for fracture can be defined as the point where crack growth occurs under equilibrium conditions, with no net change in total energy.

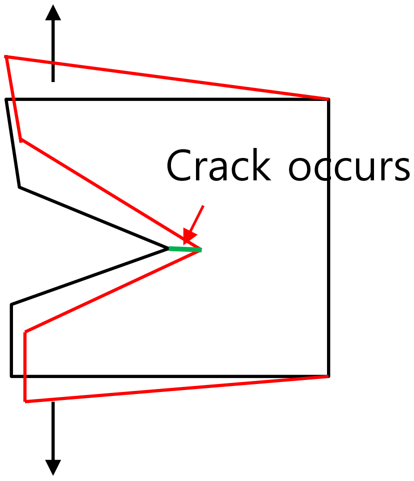
$$\frac{dE}{dA} = \frac{d\Pi}{dA} + \frac{dW_s}{dA} = 0$$



The griffith energy balance

❖Griffith's idea



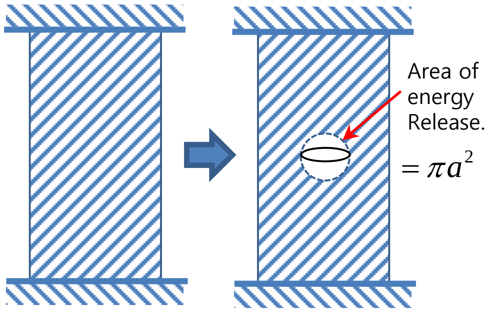


Then where the energy : $\Delta\Pi = \Pi_1 - \Pi_2$ ($\Pi_1 > \Pi_2$) gone?

⇨ The reduced potential energy is used to create new crack surfaces.

The griffith energy balance

❖The Griffith energy balance for an incremental increase in the crack area *d*A,



$$\frac{dE}{dA} = \frac{d\Pi}{dA} + \frac{dW_s}{dA} = 0 \qquad -\frac{d\Pi}{dA} = \frac{dW_s}{dA}$$

where

E = total energy

Π = potential energy supplied by the internal strain energy and external forces

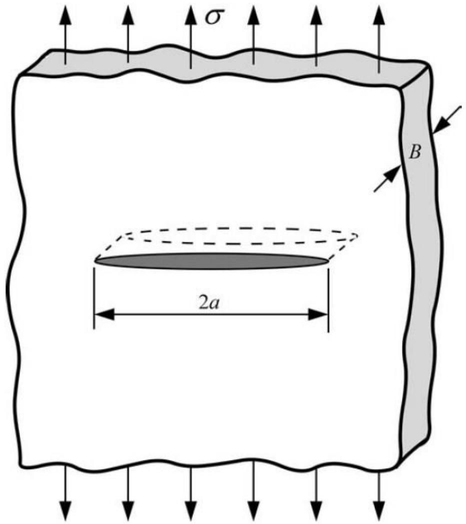
W_s = work required to create new surfaces

$$\Pi = \Pi_0 - \frac{\pi\sigma^2 a^2 B}{E} \qquad A = 2aB$$

$$\Rightarrow \quad -\frac{d\Pi}{dA} = -\frac{da}{dA}\frac{d\Pi}{da} = -\frac{d}{dA}\left(\frac{A}{2B}\right)\left(\frac{d\Pi}{da}\right) = -\frac{1}{2B}\left(\frac{d\Pi}{da}\right) = \frac{\pi\sigma^2 a}{E}$$

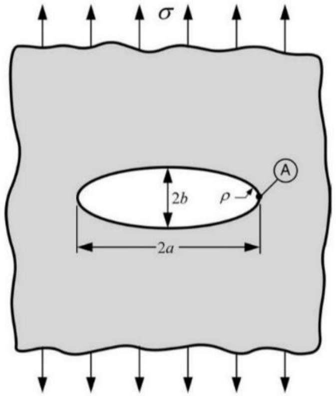
$$W_s = 4aB\gamma_s \quad \Rightarrow \quad \frac{dW_s}{dA} = \frac{1}{2B}\left(\frac{dW_s}{da}\right) = 2\gamma_s$$

$$\frac{\pi\sigma_f^2 a}{E} = 2\gamma_s \quad \Rightarrow \quad \sigma_f = \left(\frac{2E\gamma_s}{\pi a}\right)^{1/2}$$



A through-thickness crack in an infinitely wide plate subjected to a remote tensile stress

Comparison of two approaches



- Inglis's approach

$$\sigma_f = \left(\frac{E\gamma_s}{4a}\right)^{1/2}$$

- Griffith' approach

$$\sigma_f = \left(\frac{2E\gamma_s}{\pi a}\right)^{1/2}$$

- When crack-tip radius is equaling to atomic space order, these two approaches are consistent with one another, at least in the sharp crack in an ideally brittle solid.

- But, when the crack-tip radius (ρ)is significantly greater than the atomic space, there is an apparent contradiction between two approaches. However,

$$\Pi = \Pi_0 - \frac{\pi\sigma^2 a^2 B}{E}$$

  - ✓ Griffith model is insensitive to the notch radius as long as a>>b
  - ✓ According to the Inglis stress analysis, in order for σ_c to be attained at the tip of notch, σ_f must vary with $(1/\rho)^{1/2}$.

$$\sigma_c = \sqrt{\frac{E\gamma_s}{x_o}}$$

Comparison of two approach

❖Inglis's approach v.s. Griffith' approach

Example:

Consider a crack with $\rho = 5 \times 10^{-6}$m. Such a crack would appear sharp under a light microscope, but ρ would be four orders of magnitude larger than the atomic spacing in a typical crystalline solid. Thus the local stress approach would predict a global fracture strength 100 times larger than the Griffith equation. The actual material behavior is somewhere between these extremes; fracture stress does depend on notch root radius, but not to the extent implied by the Inglis stress analysis.

$$\sigma_A = 2\sigma\sqrt{\frac{a}{\rho}} \Rightarrow \rho = x_0 \qquad \sigma_A = 2\sigma\sqrt{\frac{a}{x_o}} = \sigma_c = \sqrt{\frac{E\gamma_s}{x_o}} \Rightarrow \sigma_f = \left(\frac{E\gamma_s}{4a}\right)^{1/2}$$

$$\sigma_f = \left(\frac{E\gamma_s\rho}{4ax_0}\right)^{1/2}$$

Energy balance

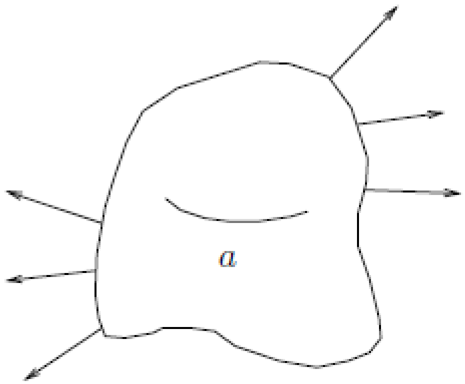


Plate with a line crack of length a.

❖ Energy types

E : total energy

Π : potential energy supplied by the internal strain energy and external forces

W_s : work required to create new free surfaces

U_k : kinetic energy resulted from material velocity

U_d : dissipated energy due to friction and plastic deformation

$\dot{E} = \dot{\Pi} + \dot{W}_s + \dot{U}_d + \dot{U}_k$	Energy balance in terms of time derivative
$\frac{d}{dt}(\ ) = \frac{da}{dt}\frac{d}{da}(\ ) = \dot{a}\frac{d}{da}(\ )$	Tran from time derivative to state variable derivative
$\frac{dE}{da} = \frac{d\Pi}{da} + \frac{dW_s}{da} + \frac{dU_d}{da} + \frac{dU_k}{da}$	Energy balance in terms of derivative of state variable, a

Griffith energy balance

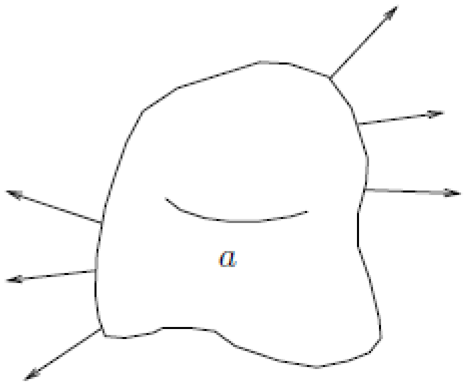


Plate with a line crack of length a.

❖ **Energy types**

E : total energy

Π : potential energy supplied by the internal strain energy and external forces

W_s : work required to create new free surfaces

U_k : kinetic energy resulted from material velocity (→ 0)

U_d : dissipated energy due to friction and plastic deformation (→ 0)

❖ Energy balance equation with all energy terms

$$\frac{dE}{da} - \frac{d\Pi}{da} = \frac{dW_s}{da} + \frac{dU_d}{da} + \frac{dU_k}{da}$$

(with $\frac{dU_d}{da} \to 0$, $\frac{dU_k}{da} \to 0$)

Assumption for Griffith energy balance ①

$$\frac{dE}{da} - \frac{d\Pi}{da} = \frac{dW_s}{da}$$

(with $\frac{dE}{da} \to 0$)

Assumption for Griffith energy balance ②

- Griffith energy balance equation

$$-\frac{d\Pi}{da} = \frac{dW_s}{da}$$

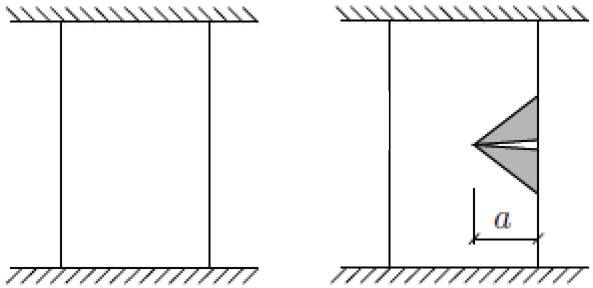
Griffith energy balance

$$W_s = 4aB\gamma_s \qquad \frac{dW_s}{da} = 4B\gamma_s \qquad -\frac{d\Pi}{dA} = \frac{\pi\sigma^2 a}{E}$$

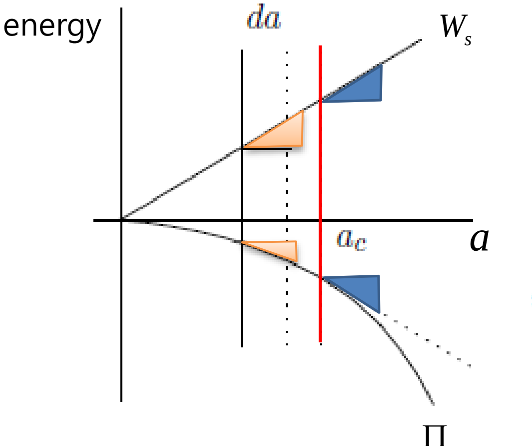
$$-\frac{d\Pi}{da} = \frac{dW_s}{da}$$ Griffith energy balance equation

$$G = -\frac{1}{B}\frac{d\Pi}{da}$$ Left side : energy release rate

$$R = \frac{1}{B}\left(\frac{dW_s}{da}\right)$$ Right side : crack resistance force



A plate loaded in tension and fixed at its edges



Surface and internal energy vs. crack length

Crack growth criterion by the relation of a with critical crack length a_c

$$a < a_c \quad \left(-\frac{d\Pi}{da} < \frac{dW_s}{da}\right)$$: no crack growth

$$a = a_c \quad \left(-\frac{d\Pi}{da} = \frac{dW_s}{da}\right)$$: unstable crack growth

$$a > a_c \quad \left(-\frac{d\Pi}{da} > \frac{dW_s}{da}\right)$$: critical crack length

⇒ For a crack increase in size, sufficient potential energy must be available in the plate to overcome the surface energy

Modified Griffith equation

❖Original Griffith equation

$$\sigma_f = \left(\frac{2E\gamma_s}{\pi a} \right)^{1/2}$$

- This equation is valid only for ideally brittle solids.
- A good agreement is obtained between the equation and the experimental fracture strength of glass.
- However, severely underestimates the fracture strength of metals.

❖Modified Griffith equation

- Irwin and Orowan independently modified the Griffith equation to account for materials that are capable of plastic flow.

$$\sigma_f = \left(\frac{2E(\gamma_s + \gamma_p)}{\pi a} \right)^{1/2}$$

- γ_p : plastic work per unit area of surface created.
- γ_s : the total energy of broken bonds in a unit area (ideally brittle solid)
- $\gamma_p >> \gamma_s$

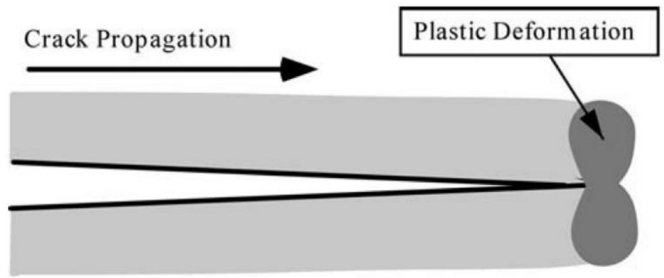
Modified Griffith equation

❖A Generalized Griffith model

- Account for any type of energy dissipation
- w_f : Fracture energy which could include plastic, viscoelastic, viscoplastic effects, crack meandering and branching (which increase the surface area), etc.

$$\sigma_f = \left(\frac{2Ew_f}{\pi a} \right)^{1/2}$$

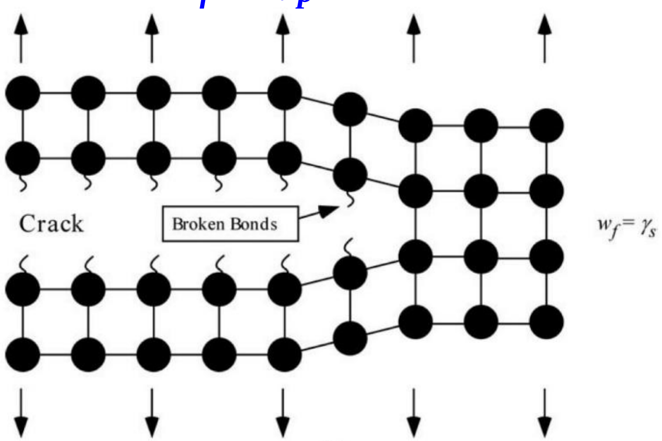
$$w_f = \gamma_p + \gamma_s$$



(b)

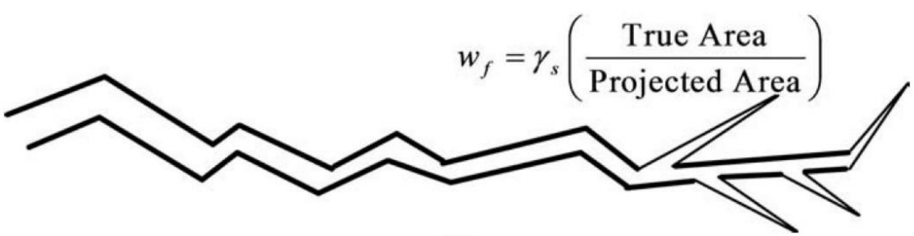
quasi-brittle elastic-plastic material

$$w_f = \gamma_p$$



(a)

ideally brittle material



(c)

brittle material with crack meandering (구불구불한 길)and branching

Modified Griffith equation

❖A Caution of applying Griffith equation

- The Griffith model only could apply to linear elastic material behavior.
- The global behavior of structure must be elastic.
- Nonlinear effects such as plasticity, must be confined to a small region near the crack tip

$$\Pi = \Pi_0 - \frac{\pi\sigma^2 a^2 B}{E} \qquad A = 2aB$$

- W_f is assumed as constant, however, in many ductile materials fracture energy increases with crack growth.

$$\sigma_f = \left(\frac{2Ew_f}{\pi a}\right)^{1/2}$$

Modified Griffith equation

❖Energy release rate (by Irwin)

- a measure of the energy available for an increment of crack extension

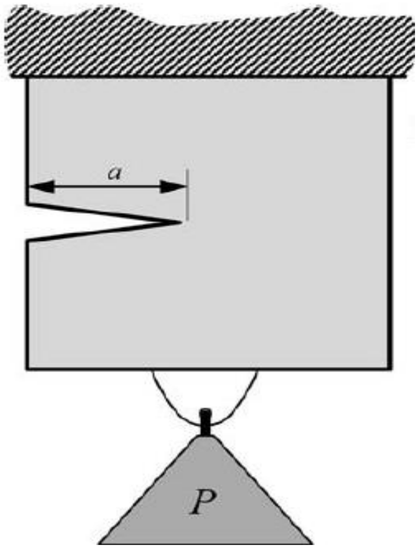
$$\mathcal{G} = -\frac{d\Pi}{dA}$$

- the rate of change in potential energy with the crack area (Not with time).
- *crack extension force* or the *crack driving force*
- The potential energy of an elastic body, Π, is defined as follows:

$$\Pi = U - F$$

- *U : strain energy stored in the body*
- *F : the work done by external loads*

Cracked Plate at a fixed load P



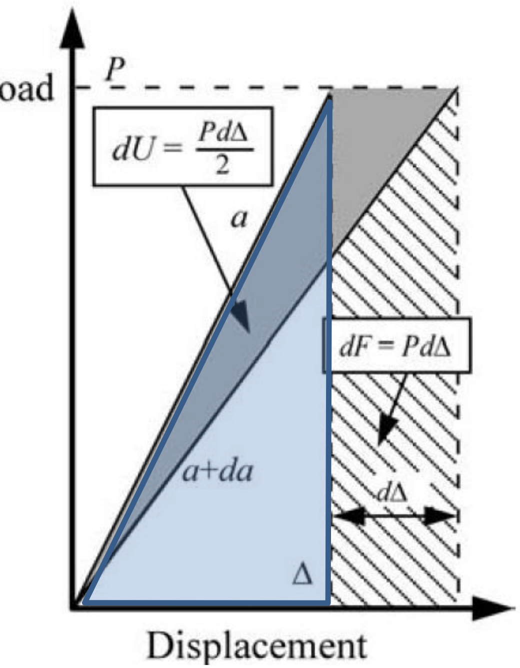
$$F = P\Delta \qquad U = \frac{P\Delta}{2}$$

$$\Pi = U - F = \frac{P\Delta}{2} - P\Delta = -\frac{P\Delta}{2}$$

$$\rightarrow \Pi = -U$$

$$\mathcal{G} = -\frac{1}{B}\left(\frac{d\Pi}{da}\right)_P = \frac{1}{B}\left(\frac{dU}{da}\right)_P = \frac{P}{2B}\left(\frac{d\Delta}{da}\right)_P$$

Crack extension *da* results in a net increase in strain energy



From $\Pi = -\frac{P\Delta}{2}$

$$(d\Pi)_p = -\frac{Pd\Delta}{2}$$

$$(dU)_p = dF + d\Pi$$

$$(dU)_p = Pd\Delta - \frac{Pd\Delta}{2} = \frac{Pd\Delta}{2}$$

From left figure

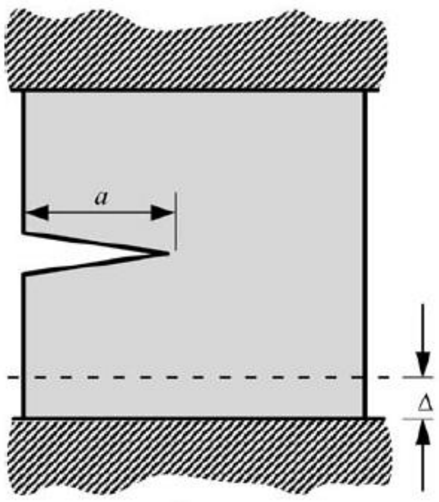
$$(dF)_p = Pd\Delta$$

$$(dU)_p = \frac{P(\Delta + d\Delta)}{2} - \frac{P\Delta}{2} = \frac{Pd\Delta}{2}$$

$$(d\Pi)_p = \frac{Pd\Delta}{2} - Pd\Delta = -\frac{Pd\Delta}{2}$$

$$i.e.\ \Pi = \frac{P(\Delta + d\Delta)}{2} - P(\Delta + d\Delta) = -\frac{P\Delta}{2} - \frac{Pd\Delta}{2}$$

Cracked Plate at a fixed displacement Δ

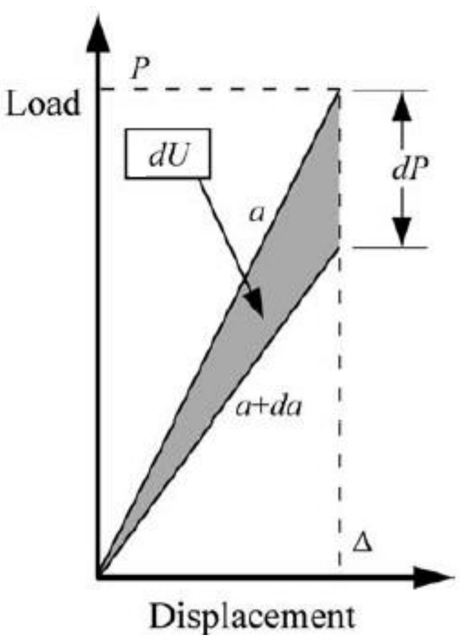


$$F = 0 \qquad U = \frac{P\Delta}{2}$$

$$\Pi = U - F = \frac{P\Delta}{2} - 0 = \frac{P\Delta}{2} \rightarrow \Pi = U$$

$$\mathcal{G} = -\frac{1}{B}\left(\frac{d\Pi}{da}\right)_{\Delta} = -\frac{1}{B}\left(\frac{dU}{da}\right)_{\Delta} = -\frac{\Delta}{2B}\left(\frac{dP}{da}\right)_{\Delta}$$

Crack extension da results in a net decrease in strain energy since $dP<0$



From $\Pi = \dfrac{P\Delta}{2}$

$$(d\Pi)_{\Delta} = \frac{dP\Delta}{2}$$

$$(dU)_{\Delta} = dF + d\Pi$$

$$(dU)_{\Delta} = 0 + \frac{\Delta dP}{2} = \frac{\Delta dP}{2}$$

From left figure

$$(dF)_{\Delta} = 0$$

$$(dU)_{\Delta} = \frac{(P+dP)\Delta}{2} - \frac{P\Delta}{2} = \frac{\Delta dP}{2} < 0$$

$$(d\Pi)_{\Delta} = \frac{\Delta dP}{2} - 0 = \frac{\Delta dP}{2}$$

$$i.e.\ \ \Pi = \frac{\Delta(P+dP)}{2} = \frac{P\Delta}{2} + \frac{\Delta dP}{2}$$

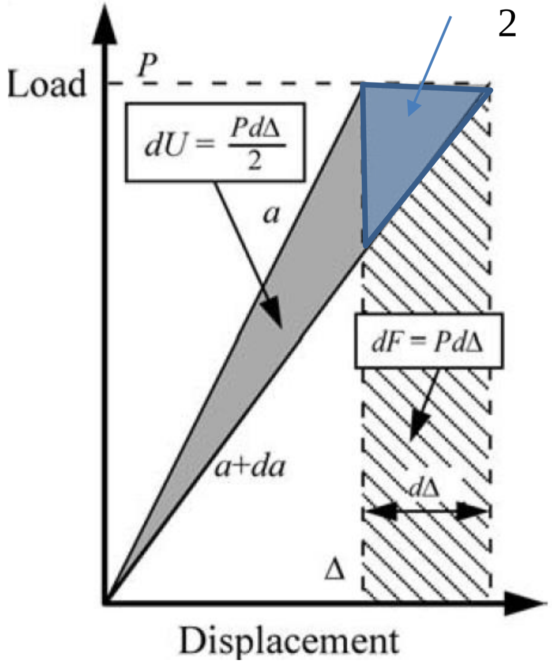
Cracked Plate at a fixed load P

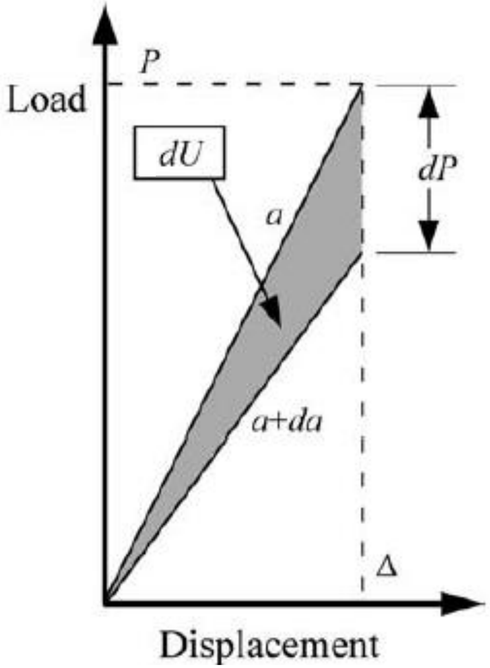
- The absolute values of these energies differ by the amount $dPd\Delta/2$

$$(dU)_p = \frac{Pd\Delta}{2} \qquad (dU)_\Delta = \frac{\Delta dP}{2}$$

$$\frac{Pd\Delta}{2} = -\left(\frac{\Delta dP}{2} + \frac{d\Delta dP}{2}\right) \Rightarrow (dU)_p = -(dU)_\Delta$$

$$\frac{d\Delta dP}{2}$$





Comparison of two approaches

❖ introduce the compliance, which is the inverse of the plate stiffness

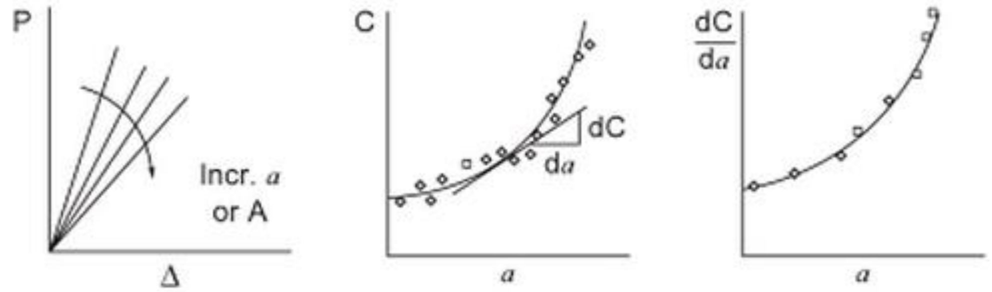
$$C = \frac{\Delta}{P}$$

	Fixed Load	Fixed Displacement
Energy Release rate	$\mathcal{G}_P = \frac{P}{2B}\left(\frac{d\Delta}{da}\right)_P$	$\mathcal{G}_\Delta = -\frac{\Delta}{2B}\left(\frac{dP}{da}\right)_\Delta$
	$\Delta = PC \rightarrow d\Delta = PdC$	$P^{-1} = \frac{1}{\Delta}C \rightarrow dP = -\frac{P^2}{\Delta}dC$
	$\mathcal{G}_\Delta = \mathcal{G}_P = \mathcal{G} = \frac{P^2}{2B}\frac{dC}{da}$	
Strain Energy	$(dU)_p = \frac{Pd\Delta}{2}(>0)$	$(dU)_\Delta = \frac{\Delta dP}{2}(<0)$
	$-(dU)_\Delta = (dU)_p$	

"에너지 해방률은 결국 하중의 제곱에 비례하고, Crack이 커지면서 유연성이 커지는 비율에 비례한다"

Calculation of energy release rate from experiment

❖ Calculation of energy release rate by Load-displacement curve obtained from a experiment



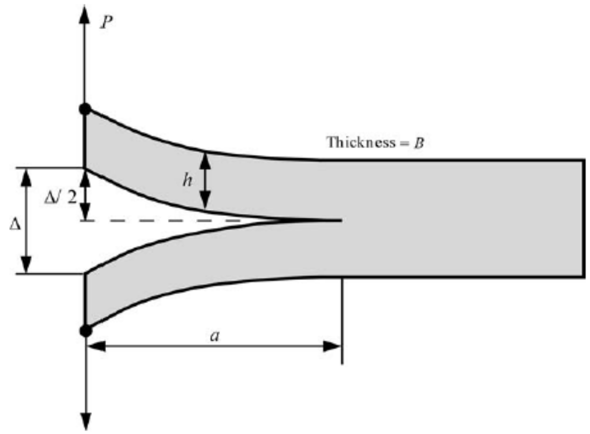
(1) (2) (3)

- Step 1 : Load – displacement curves are plotted with increasing crack length a (1)
- Step 2 : Calculate compliance $c = \frac{\Delta}{P}$ for different a.
- Step 3 : Plot compliance c versus crack length a. (2)
- Step 4 : Calculate $\frac{dC}{da}$ by differentiating compliance curve c in terms of a (3).
- Step 5 : For specific load P and crack length a, calculate energy release.

$$G = \frac{P^2}{2B}\frac{dC}{da}$$

EX 2.2 : Beam with a central crack loaded in Mode I

Determine the energy release rate for a double cantilever beam (DCB) specimen



- From Beam theory : $$\frac{\Delta}{2} = \frac{Pa^3}{3EI}, \quad I = \frac{Bh^3}{12}$$
- Compliance $$C = \frac{\Delta}{P} = \frac{8a^3}{EBh^3} \qquad \frac{dC}{da} = \frac{24a^2}{EBh^3}$$
- Energy Release Rate

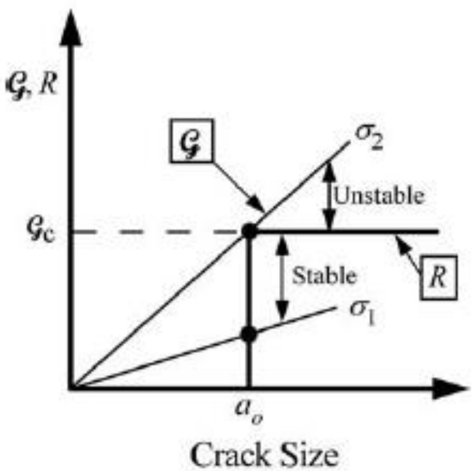
$$\mathcal{G} = \frac{P^2}{2B}\frac{dC}{da} = \frac{P^2}{2B}\frac{24a^2}{EBh^3} = \frac{12P^2a^2}{EB^2h^3}$$

Calculation of energy release rate from experiment

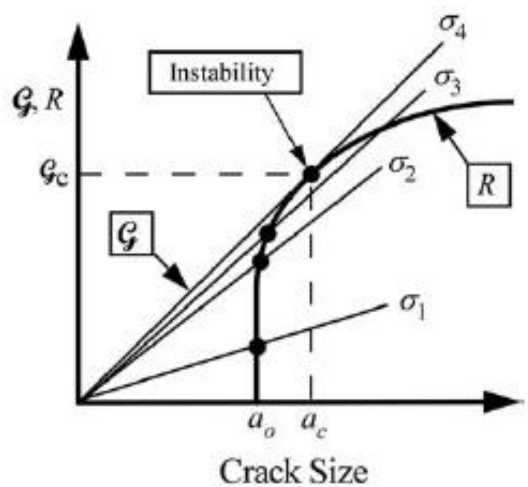
- Driving force curve : $\mathcal{G}$ vs. crack size
- Resistance curve : R vs. crack size.
- Stable condition : $\mathcal{G}$= R and $\frac{d\mathcal{G}}{da} \le \frac{dR}{da}$
- Unstable crack growth : $\frac{d\mathcal{G}}{da} > \frac{dR}{da}$
- A flat *R* curve : the material resistance is constant with crack growth.

$\Rightarrow$ Fracture occurs when the stress reaches σ_2. Unstable propagation.

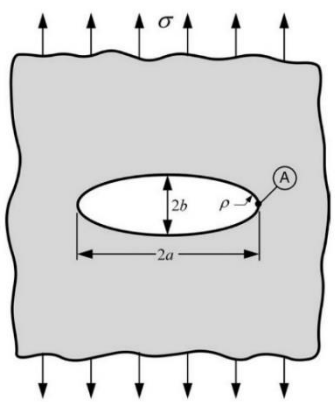
- A rising R curve : for larger σ_4, the plate is unstable with further crack growth.



Flat *R* curve

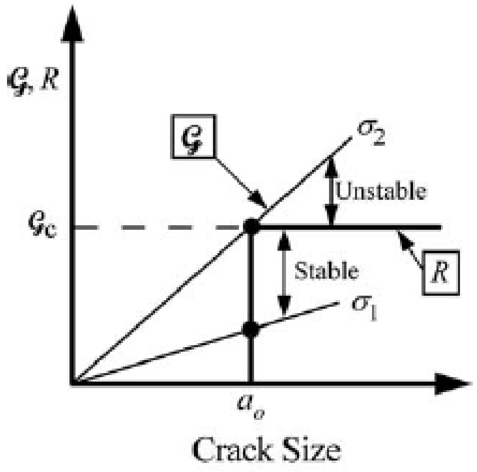


Rising *R* curve

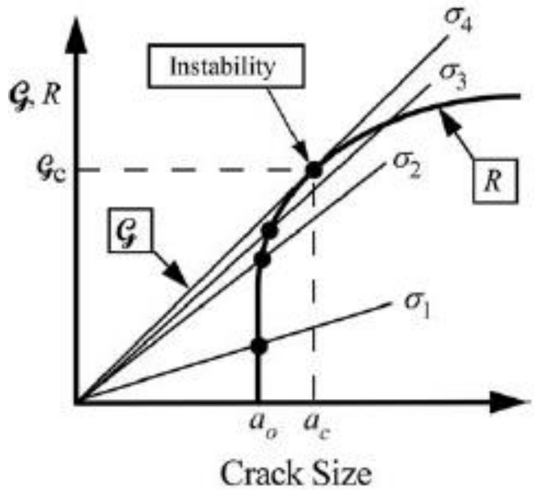


Reasons for the R Curve Shape

- The shape of the *R* curve depends on the material behavior and, to a lesser extent, on the configuration of the cracked structure.
- Ideally brittle material : flat R curve because the surface energy is an invariant material property.
- Nonlinear material behavior : *R* curve can take on a variety of shapes.
  - ✓ ex) ductile fracture in metals : rising *R* curve ⇐ a plastic zone at the tip of the crack increases in size as the crack grows. The driving force must increase to maintain the crack growth.
  - ✓ If the cracked body is infinite (i.e., if the plastic zone is small compared to the body) the plastic zone size and *R* eventually reach steady-state values ⇒ flat with further growth.



Flat *R* curve



Rising *R* curve

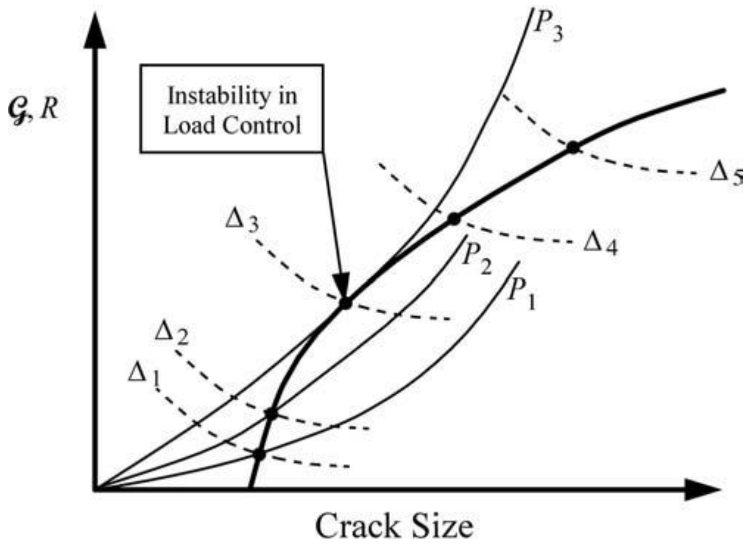
Reasons for the R Curve Shape

- Ideally, the *R* curve, as well as other measures of fracture toughness, should be a property only of the material and not depend on the size or shape of the cracked body.
- Much of fracture mechanics is predicated on the assumption that fracture toughness is a material property.
- However, the size and geometry of the cracked structure can exert some influence on the shape of the *R* curve.
- A crack in a thin sheet ⇒ a steeper *R* curve than a crack in a thick plate because of a low degree of stress triaxiality (3축) at the crack tip in the thin sheet, while the material near the tip of the crack in the thick plate may be in plane strain.
- The *R* curve can also be affected if the growing crack approaches a free boundary in the structure. A wide plate may exhibit a somewhat different crack growth resistance behavior than a narrow plate of the same material.

Load control vs. Displacement Control

- The stability of crack growth depends on the rate of change in i.e., the second derivative of potential energy. Although the driving force $\mathcal{G}$ is the same for both load control and displacement control, the *rate of change* of the driving force curve depends on how the structure is loaded.
- Displacement control tends to be more stable than load control. The driving force actually decreases with crack growth in displacement control.

- Ex) A cracked structure subjected to a load P_3 and a displacement Δ_3.
- Load controlled, it is at the point of instability where the driving force curve is tangent to the *R* curve.
- Displacement control : the structure is stable because the driving force decreases with crack growth; the displacement must be increased for further crack growth.



Driving force/*R* curve diagram for load(P) and displacement control (Δ)

Load control vs. Displacement Control

- **Ex 2.3)** Evaluate the relative stability of a DCB specimen in load control and displacement control.

$$\mathcal{G} = \frac{P^2}{2B}\frac{dC}{da} = \frac{P^2}{2B}\frac{24a^2}{EBh^3} = \frac{12P^2a^2}{EB^2h^3}$$

$$C = \frac{\Delta}{P} = \frac{8a^3}{EBh^3} \qquad \frac{dC}{da} = \frac{24a^2}{EBh^3}$$

Sol)

- **Load Control** : the slope of the driving force is given by (Ex.2.2)

$$\mathcal{G} = \frac{P}{2B}\left(\frac{d\Delta}{da}\right)_P = \frac{P}{2B}\frac{6Pa^2}{3EI} = \frac{P^2a^2}{BEI} \qquad \left(\frac{d\mathcal{G}}{da}\right)_P = \frac{2P^2a}{BEI} = \frac{2\mathcal{G}}{a}$$

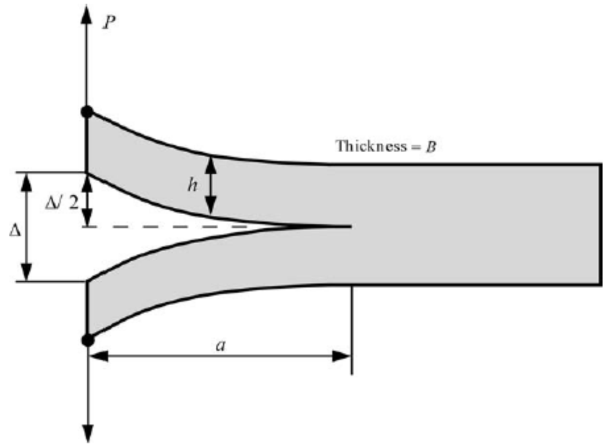
$$\frac{\Delta}{2} = \frac{Pa^3}{3EI}, \quad I = \frac{Bh^3}{12}$$

- **Displacement control** :

express $\mathcal{G}$ in terms of Δ and a.

$$P = \frac{3\Delta EI}{2a^3} \qquad \mathcal{G} = -\frac{\Delta}{2B}\left(\frac{dP}{da}\right)_\Delta = -\frac{\Delta}{2B}\left(-3\frac{3\Delta EI}{2a^4}\right) = \frac{9\Delta^2 EI}{4Ba^4}$$

$$\left(\frac{d\mathcal{G}}{da}\right)_\Delta = -\frac{9\Delta^2 EI}{Ba^5} = -\frac{4\mathcal{G}}{a}$$



Double cantilever beam (DCB) specimen.

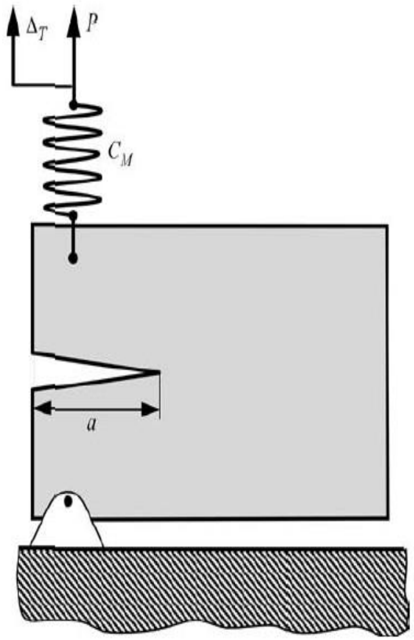
Structures with Finite Compliance

- Most real structures are subject to conditions between load control and pure displacement control.
- Pure displacement control : infinite spring, $C_m = 0$.
- Load control : an infinitely soft spring, $C_m = \infty$.
- At the moment of instability.

$$\mathcal{G} = R$$

$$\left(\frac{d\mathcal{G}}{da}\right)_{\Delta_T} = \frac{dR}{da}$$

$$\left(\frac{d\mathcal{G}}{da}\right)_{\Delta_T} = \left(\frac{\partial\mathcal{G}}{\partial a}\right)_P - \left(\frac{\partial\mathcal{G}}{\partial P}\right)_a \left(\frac{\partial\Delta}{\partial a}\right)_P \left[C_m + \left(\frac{\partial\Delta}{\partial P}\right)_a\right]^{-1}$$



A cracked structure with finite compliance, represented schematically by a spring in series.

Introduction

- For certain cracked configurations subjected to external forces, it is possible to derive closed-form expressions for the stresses in the body, assuming isotropic linear elastic material behavior
- the stress field in any linear elastic cracked body is given by

$$\sigma_{ij} = \underbrace{\left(\frac{k}{\sqrt{r}}\right) f_{ij}(\theta)}_{\text{leading term}} + \underbrace{\sum_{m=0}^{\infty} A_m r^{\frac{m}{2}} g_{ij}^{(m)}(\theta)}_{\text{high order term}}$$

where

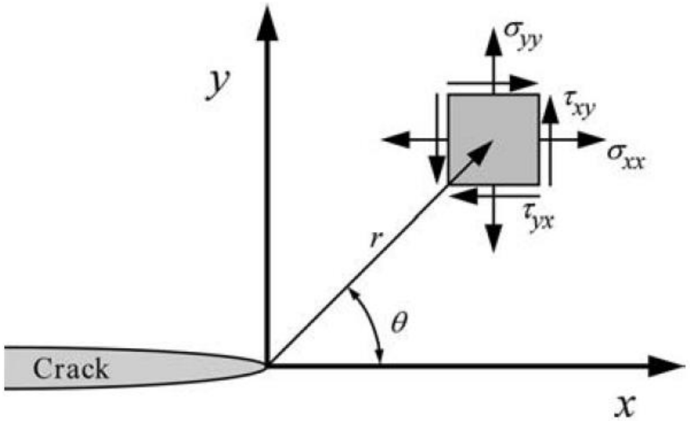
σ_{ij} : *stress tensor*

K : *constant*

f_{ij} : dimensionless function of θ

A_m : amplitude

$g_{ij}^{(m)}$: dimensionless function of θ for the m^{th} term



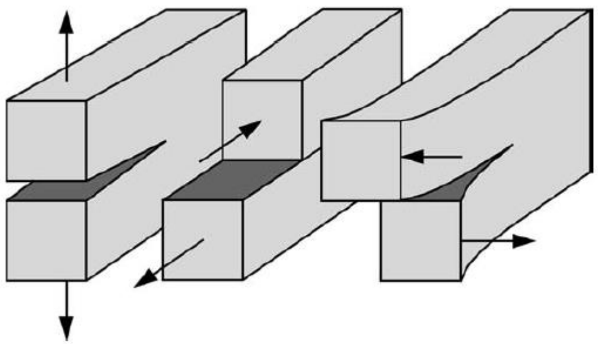
Definition of the coordinate axis ahead of a crack tip. The *z* direction is normal to the page.

- Stress near the crack tip varies with $1/\sqrt{r}$

$\Rightarrow$ Stress singularity

The Stress Intensity Factor

❖ Stress intensity factor : K_I, K_{II}, K_{III}



The stress fields ahead of a crack tip i(n an isotropic linear elastic material are

$$\lim_{r\to 0}\sigma_{ij}^{(I)} = \frac{K_I}{\sqrt{2\pi r}} f_{ij}^{(I)}(\theta)$$

$$\lim_{r\to 0}\sigma_{ij}^{(II)} = \frac{K_{II}}{\sqrt{2\pi r}} f_{ij}^{(II)}(\theta)$$

$$\lim_{r\to 0}\sigma_{ij}^{(III)} = \frac{K_{III}}{\sqrt{2\pi r}} f_{ij}^{(III)}(\theta)$$

Stress Fields Ahead of a Crack Tip(Linear Elastic, Isotropic Material)

	Mode I	Mode II
σ_{xx}	$\frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\left[1-\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]$	$-\frac{K_{II}}{\sqrt{2\pi r}}\sin\left(\frac{\theta}{2}\right)\left[2+\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{3\theta}{2}\right)\right]$
σ_{yy}	$\frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\left[1+\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]$	$\frac{K_{II}}{\sqrt{2\pi r}}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{3\theta}{2}\right)$
τ_{xy}	$\frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{3\theta}{2}\right)$	$\frac{K_{II}}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\left[1-\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]$
σ_{zz}	0 (Plane stress) $\nu(\sigma_{xx}+\sigma_{yy})$ (Plane strain)	0 (Plane stress) $\nu(\sigma_{xx}+\sigma_{yy})$ (Plane strain)
τ_{xz}, τ_{yz}	0	0

Mode III

$$\tau_{xz} = -\frac{K_{III}}{\sqrt{2\pi r}}\sin\left(\frac{\theta}{2}\right)$$

$$\tau_{yz} = \frac{K_{III}}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)$$

Note: υ is Poisson's ratio.

The Stress Intensity Factor

- In a mixed-mode problem (i.e., when more than one loading mode is present), the individual contributions to a given stress component are additive from the principle of linear superposition.

$$\sigma_{ij}^{(\text{total})} = \sigma_{ij}^{(I)} + \sigma_{ij}^{(II)} + \sigma_{ij}^{(III)}$$

	Mode I	Mode II	Mode III
u_x	$\frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}\cos\left(\frac{\theta}{2}\right)\left[\kappa-1+2\sin^2\left(\frac{\theta}{2}\right)\right]$	$\frac{K_{II}}{2\mu}\sqrt{\frac{r}{2\pi}}\sin\left(\frac{\theta}{2}\right)\left[\kappa+1+2\cos^2\left(\frac{\theta}{2}\right)\right]$	$u_z = \frac{2K_{III}}{\mu}\sqrt{\frac{r}{2\pi}}\sin\left(\frac{\theta}{2}\right)$
u_y	$\frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}\sin\left(\frac{\theta}{2}\right)\left[\kappa+1-2\cos^2\left(\frac{\theta}{2}\right)\right]$	$-\frac{K_{II}}{2\mu}\sqrt{\frac{r}{2\pi}}\cos\left(\frac{\theta}{2}\right)\left[\kappa-1-2\sin^2\left(\frac{\theta}{2}\right)\right]$	

Note: μ is the shear modulus. $\kappa = 3 - 4\nu$ (plane strain) and $\kappa = (3 - \nu)/(1 + \nu)$ (plane stress).

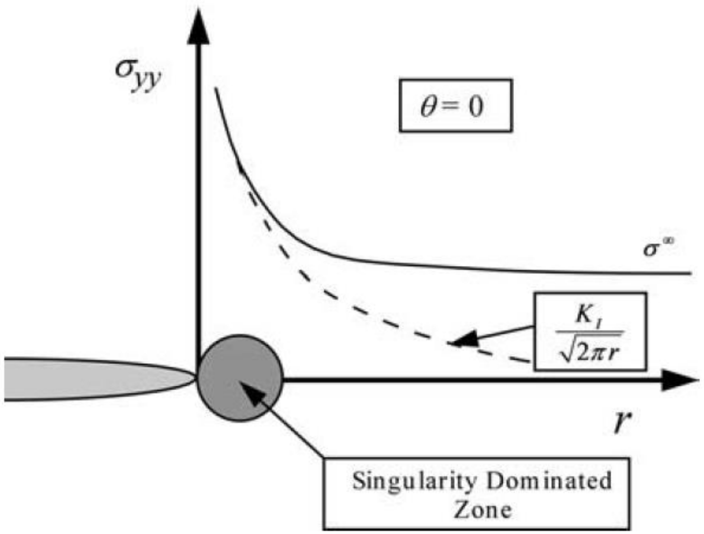
Crack-Tip Displacement Fields (Linear Elastic, Isotropic Material)

The Stress Intensity Factor

- Consider the Mode I singular field on the crack plane, where $\theta = 0$.

$$\sigma_{xx} = \sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}$$

- The shear stress is zero, which means that the crack plane is a principal plane for pure Mode I loading.
- Near the crack tip, where the singularity dominates the stress field.
- Stresses far from the crack tip are governed by the remote boundary conditions.



- K is known, it is possible to solve for all components of stress, strain, and displacement as a function of r and θ.

- This single-parameter description of crack tip conditions turns out to be one of the most important concepts in fracture mechanics

Stress normal to the crack plane in Mode I

Relationship between *K* and Global Behavior

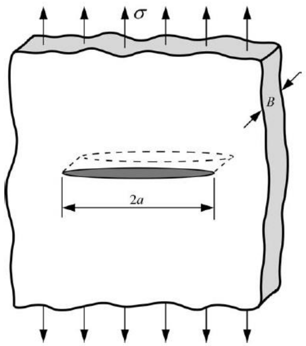
- The crack-tip stresses must be proportional to the remote stress $\Rightarrow K_I \propto \sigma$.
- Stress intensity has units of $stress\sqrt{length}$.

$$\sigma_{xx} = \sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}$$

$$\Rightarrow \quad K_I = O\left(\sigma\sqrt{a}\right)$$

- This means, the amplitude of the crack-tip singularity for this configuration is proportional to the remote stress and the square root of the crack size.
- Through Crack

$$K_I = \sigma\sqrt{\pi a}$$

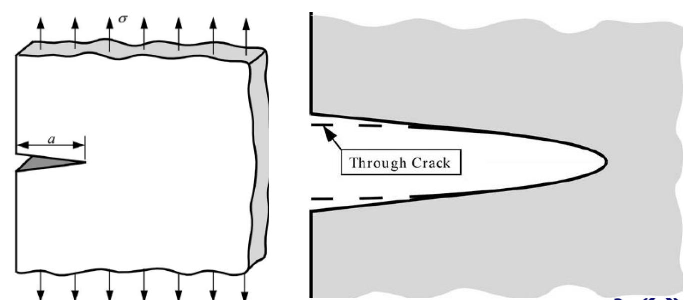


Through crack of elliptical shape.

- The edge crack

$$K_I = 1.12\sigma\sqrt{\pi a}$$

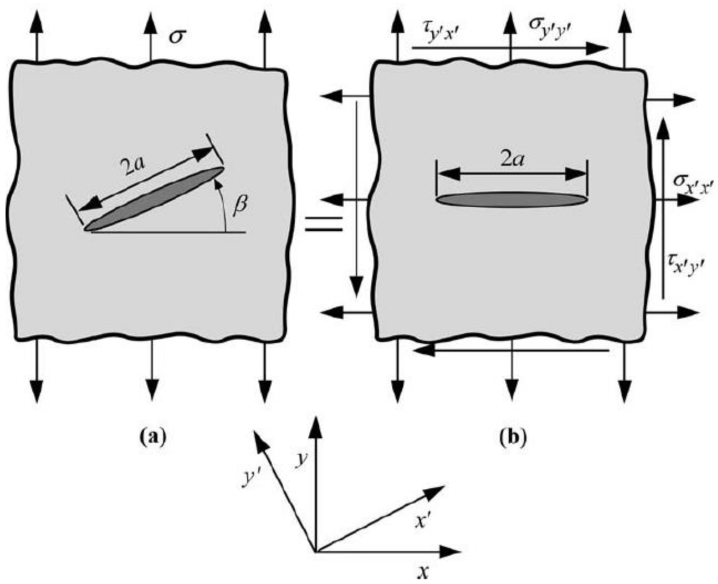
- The edge crack opens more because it is less restrained than the through crack.



Edge crack of V shape.

Relationship between *K* and Global Behavior

- A through crack in an infinite plate where the normal to the crack plane is oriented at an angle β with the stress axis.
- Redefine the coordinate axis to coincide with the crack orientation, the applied stress can be resolved into normal and shear components.



Through crack in an infinite plate for the general case where the principal stress is not perpendicular to the crack plane.

By using Mohr's circle and formulas of trigonometric function

$$\frac{1+\cos 2\theta}{2} = \cos^2\theta$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$

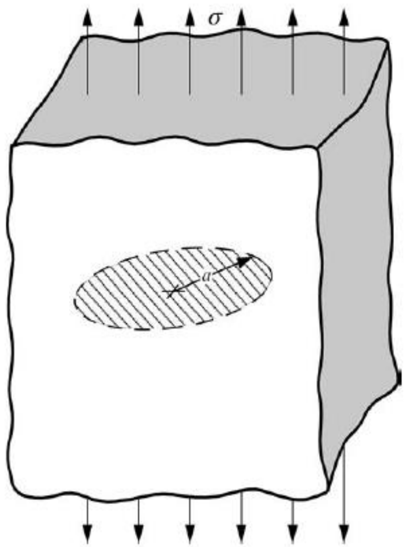
$$K_I = \sigma_{y'y'}\sqrt{\pi a}$$
$$= \sigma\cos^2(\beta)\sqrt{\pi a}$$

$$K_{II} = \tau_{x'y'}\sqrt{\pi a}$$
$$= \sigma\sin(\beta)\cos(\beta)\sqrt{\pi a}$$

Relationship between *K* and Global Behavior

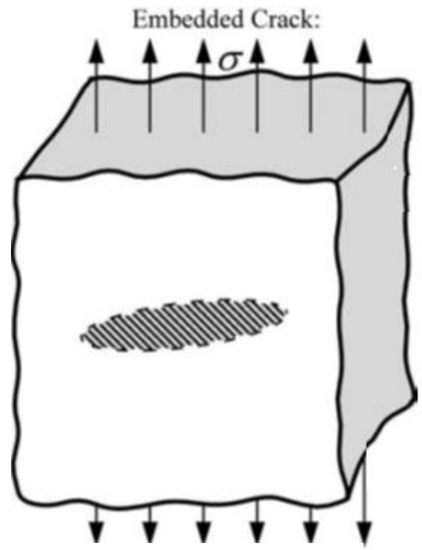
- The penny-shaped crack in an infinite medium

A penny-shaped crack



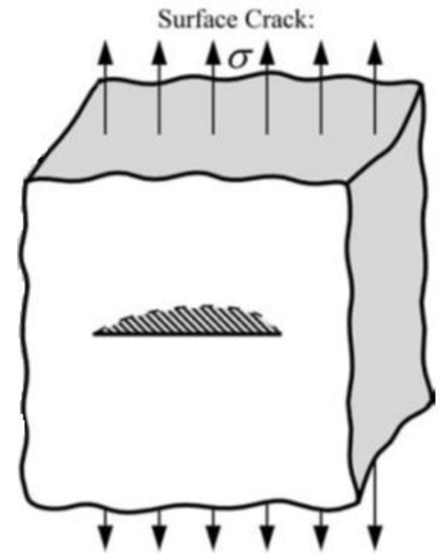
$$K_I = \frac{2}{\pi}\sigma\sqrt{\pi a}$$

Elliptical cracks

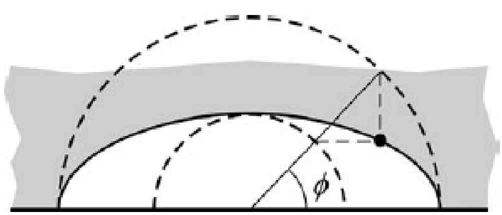


$$K_I = \sigma\sqrt{\frac{\pi a}{Q}}f(\phi)$$

Semielliptical cracks



$$K_I = \lambda_s\sigma\sqrt{\frac{\pi a}{Q}}f(\phi)$$



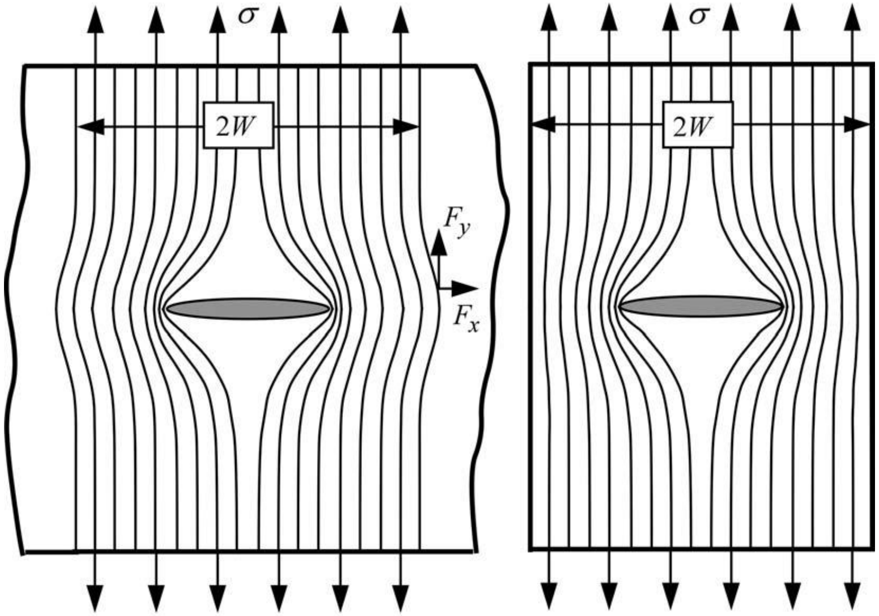
$$Q = 1 + 1.464\left(\frac{a}{c}\right)^{1.65}$$

$$\lambda_s = \left[1.13 - 0.09\left(\frac{a}{c}\right)\right]\left[1 + 0.1(1-\sin\phi)^2\right]$$

$$f(\phi) = \left[\sin^2(\phi) + \left(\frac{a}{c}\right)^2\cos^2(\phi)\right]^{1/4}$$

Effect of Finite Size

- The crack dimensions are small compared to the size of the plate; the crack-tip conditions are not influenced by external boundaries.
- As the crack size increases, or as the plate dimensions decrease, the outer boundaries begin to exert an influence on the crack tip.



Stress concentration effects infinite plate and finite plate

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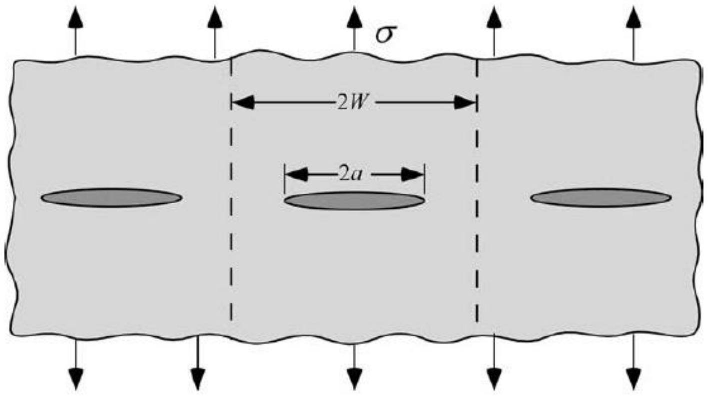


Collinear cracks in an infinite plate

$$K_I = \sigma\sqrt{\pi a}\left[\frac{2W}{\pi a}\tan\left(\frac{\pi a}{2W}\right)\right]^{1/2}$$

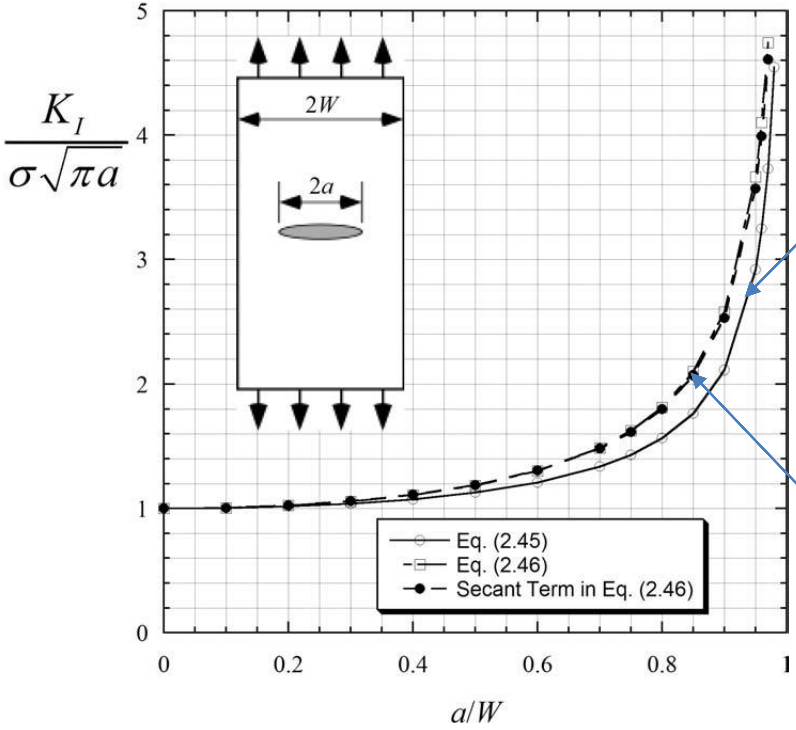
As $a/W \to 0,\ K_I \to \sigma\sqrt{\pi a}$ (Infinite plate)

As $a/W \to 1,\ K_I \to \infty$

Effect of Finite Size

- More accurate solutions for a through crack in a finite plate from finite-element analysis and fit to a polynomial expression.

$$K_I = \sigma\sqrt{\pi a}\left[\sec\left(\frac{\pi a}{2W}\right)\right]^{1/2}\left[1-0.025\left(\frac{a}{W}\right)^2+0.06\left(\frac{a}{w}\right)^4\right] \qquad (2.46)$$



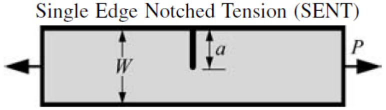
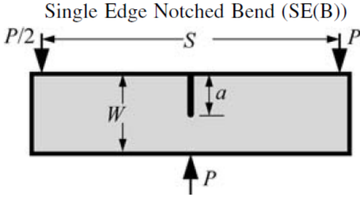
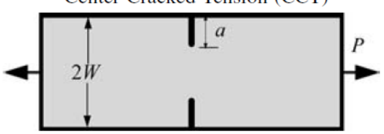
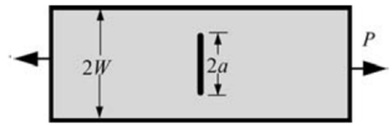
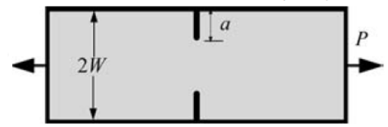
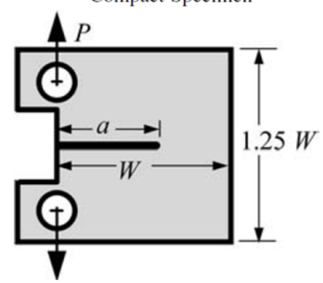
$$K_I = \sigma\sqrt{\pi a}\left[\frac{2W}{\pi a}\tan\left(\frac{\pi a}{2W}\right)\right]^{1/2} \qquad (2.45)$$

$$K_I = \sigma\sqrt{\pi a}\left[\sec\left(\frac{\pi a}{2W}\right)\right]^{1/2}\left[1-0.025\left(\frac{a}{W}\right)^2+0.06\left(\frac{a}{w}\right)^4\right]$$

Effect of Finite Size

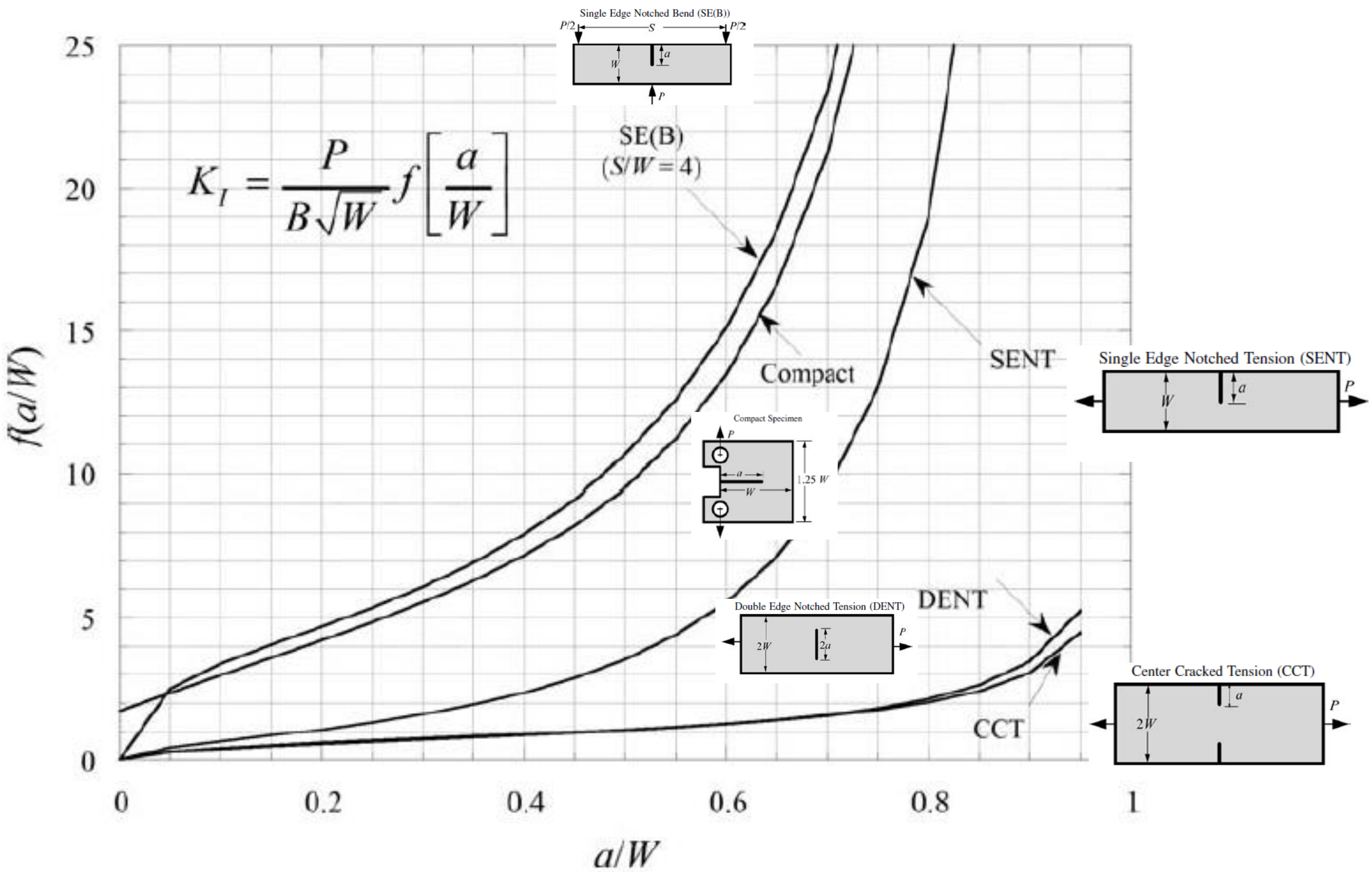
- K_I Solutions for Common Test Specimens

$$^{*}K_I = \frac{P}{B\sqrt{W}} f\left(\frac{a}{W}\right) \quad \text{where } B \text{ is the specimen thickness.}$$

GEOMETRY	$f\left(\frac{a}{W}\right)^{*}$
Single Edge Notched Tension (SENT) 	$\frac{\sqrt{2\tan\frac{\pi a}{2W}}}{\cos\frac{\pi a}{2W}}\left[0.752+2.02\left(\frac{a}{W}\right)+0.37\left(1-\sin\frac{\pi a}{2W}\right)^3\right]$
Single Edge Notched Bend (SE(B)) 	$\frac{3\frac{S}{W}\sqrt{\frac{a}{W}}}{2\left(1+2\frac{a}{W}\right)\left(1-\frac{a}{W}\right)^{3/2}}\left[1.99-\frac{a}{W}\left(1-\frac{a}{W}\right)\left\{2.15-3.93\left(\frac{a}{W}\right)+2.7\left(\frac{a}{W}\right)^2\right\}\right]$
Center Cracked Tension (CCT) 	$\sqrt{\frac{\pi a}{4W}\sec\left(\frac{\pi a}{2W}\right)}\left[1-0.025\left(\frac{a}{W}\right)^2+0.06\left(\frac{a}{W}\right)^4\right]$
Center Cracked Tension (CCT) 	$\sqrt{\frac{\pi a}{4W}\sec\left(\frac{\pi a}{2W}\right)}\left[1-0.025\left(\frac{a}{W}\right)^2+0.06\left(\frac{a}{W}\right)^4\right]$
Double Edge Notched Tension (DENT) 	$\frac{\sqrt{\frac{\pi a}{2W}}}{\sqrt{1-\frac{a}{W}}}\left[1.122-0.561\left(\frac{a}{W}\right)-0.205\left(\frac{a}{W}\right)^2+0.471\left(\frac{a}{W}\right)^3+0.190\left(\frac{a}{W}\right)^4\right]$
Compact Specimen 	$\frac{2+\frac{a}{W}}{\left(1-\frac{a}{W}\right)^{3/2}}\left[0.886+4.64\left(\frac{a}{W}\right)-13.32\left(\frac{a}{W}\right)^2+14.72\left(\frac{a}{W}\right)^3-5.60\left(\frac{a}{W}\right)^4\right]$

Effect of Finite Size

- Plot of stress intensity solutions



Effect of Finite Size

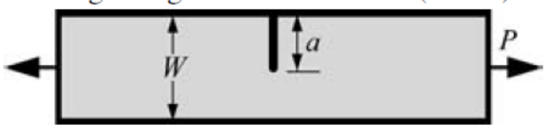
- K can always be related to the through crack with an appropriate correction factor.

$$K_{(I,II,III)} = Y\sigma\sqrt{\pi a}$$

- EX. 2.4) Show that the *KI* solution for the single edge notched tensile panel reduces to $K_I = 1.12\sigma\sqrt{\pi a}$ when *a* << *W*.

GEOMETRY

Single Edge Notched Tension (SENT)



$$f\left(\frac{a}{W}\right)^{*}$$

$$\frac{\sqrt{2\tan\frac{\pi a}{2W}}}{\cos\frac{\pi a}{2W}}\left[0.752 + 2.02\left(\frac{a}{W}\right) + 0.37\left(1-\sin\frac{\pi a}{2W}\right)^{3}\right]$$

$$\frac{P}{B\sqrt{W}} f\left(\frac{a}{w}\right) = \frac{P}{BW} f\left(\frac{a}{w}\right)\sqrt{\frac{W}{\pi a}}\sqrt{\pi a} = Y\sigma\sqrt{\pi a} \qquad Y = f\left(\frac{a}{W}\right)\sqrt{\frac{W}{\pi a}}$$

$$\lim_{a/W\to 0} f\left(\frac{a}{W}\right) = \sqrt{\frac{\pi a}{W}}(0.752 + 0.37)$$

$$\lim_{a/W\to 0} Y = 1.12$$

Principle of Superposition

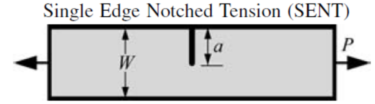
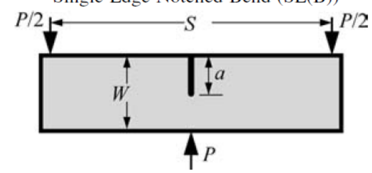
- For linear elastic materials, individual components of stress, strain, and displacement are additive.
- Stress intensity factors are additive as long as the mode of loading is consistent.

$$K_I^{(\text{total})} = K_I^{(A)} + K_I^{(B)} + K_I^{(C)} \qquad K_{(\text{total})} \neq K_I + K_{II} + K_{III}$$

- Ex) An edge-cracked panel subject to combined membrane (axial) loading P_m, and three-point bending P_b.

$$K_I^{(\text{total})} = K_I^{(\text{membrane})} + K_I^{(\text{bending})}$$

$$= \frac{1}{B\sqrt{W}}\left[P_m f_m\left(\frac{a}{W}\right) + P_b f_b\left(\frac{a}{W}\right)\right]$$

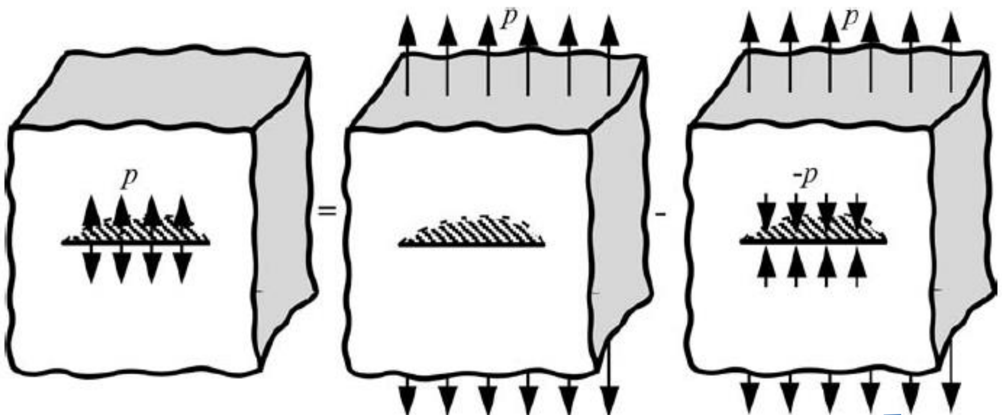
GEOMETRY	$f\left(\frac{a}{W}\right)^*$
Single Edge Notched Tension (SENT) 	$\frac{\sqrt{2\tan\frac{\pi a}{2W}}}{\cos\frac{\pi a}{2W}}\left[0.752+2.02\left(\frac{a}{W}\right)+0.37\left(1-\sin\frac{\pi a}{2W}\right)^3\right]$
Single Edge Notched Bend (SE(B)) 	$\frac{3\frac{S}{W}\sqrt{\frac{a}{W}}}{2\left(1+2\frac{a}{W}\right)\left(1-\frac{a}{W}\right)^{3/2}}\left[1.99-\frac{a}{W}\left(1-\frac{a}{W}\right)\left\{2.15-3.93\left(\frac{a}{W}\right)+2.7\left(\frac{a}{W}\right)^2\right\}\right]$

Principle of Superposition

❖ **Ex. 2.5**) Determine the stress intensity factor for a semielliptical surface crack subjected to an internal pressure *p*.

$$K_I^{(a)} = K_I^{(b)} - K_I^{(c)}$$

$$= \lambda_s p \sqrt{\frac{\pi a}{Q}} f(\phi) - 0 = \lambda_s p \sqrt{\frac{\pi a}{Q}} f(\phi)$$



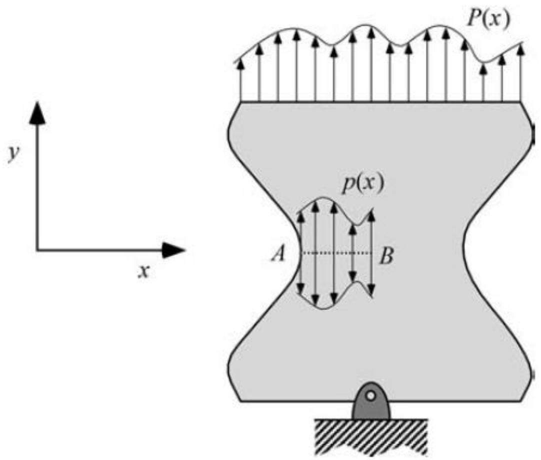
$$K_I = \lambda_s \sigma \sqrt{\frac{\pi a}{Q}} f(\phi)$$

K_I = 0 because the crack faces close, and the plate behaves as if the crack were not present

K_I for a semielliptical surface crack under internal pressure *p* by means of the principle of superposition.

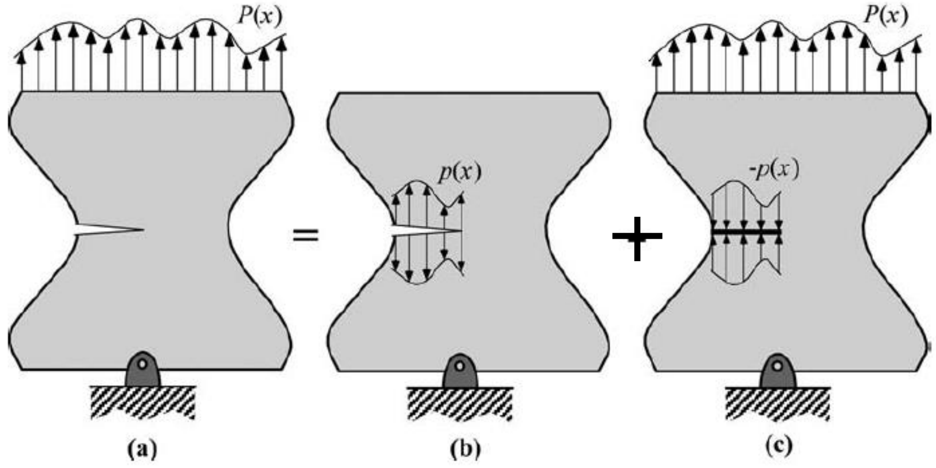
Principle of Superposition

- Stresses acting on the boundary (i.e., tractions) can be replaced with tractions that act on the crack face.
- Mode I is assumed and no shear stresses act on Plane A-B.



Uncracked body subject to an arbitrary boundary traction $P(x)$, which results in a normal stress distribution $p(x)$ acting on Plane A-B.

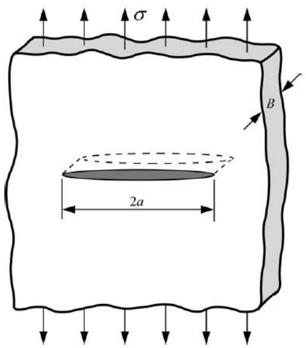
$$K_I^{(a)} = K_I^{(b)} + K_I^{(c)} = K_I^{(b)} \qquad \left(\text{since } K_I^{(c)} = 0\right)$$



$K_I = 0$ because the crack faces close, and the plate behaves as if the crack were not present

Relationship between K and $\mathcal{G}$

- $\mathcal{G}$: the energy release rate, the net change in potential energy that accompanies an increment of crack extension.
- K : quantity characterizes the stresses, strains, and displacements near the crack tip.
- For a through crack



$$\mathcal{G} = \frac{\pi\sigma^2 a}{E}$$

$$K_I = \sigma\sqrt{\pi a}$$

$$\mathcal{G} = \frac{K_I^2}{E}$$

- For plane strain conditions, *E* must be replaced by $E/(1-\nu^2)$.

$E' = E$ for plane stress

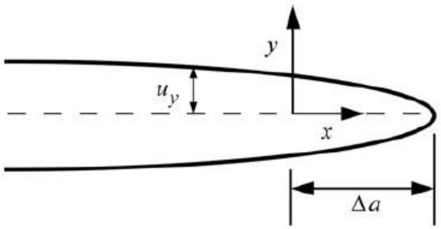
$E' = \dfrac{E}{1-\nu^2}$ for plane strain

$$\mathcal{G} = \frac{K_I^2}{E'}$$

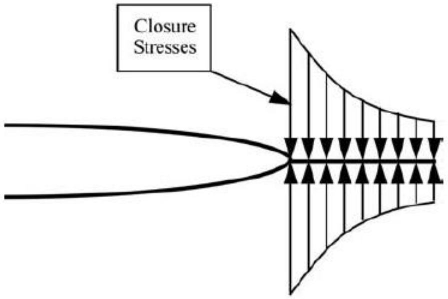
Relationship between K and $\mathcal{G}$

- Is it general relationship that applies to all configurations?
- The work required to close the crack at the tip is related to the energy release rate.

$$\mathcal{G} = \lim_{\Delta a \to 0} \left(\frac{\Delta U}{\Delta a} \right)_{\text{fixed load}}$$



(a)



- ΔU is the work of crack closure, which is equal to the sum of contributions to work from $x = 0$ to $x = \Delta a$.

$$\Delta U = \int_{x=0}^{x=\Delta a} dU(x)$$

$$dU(x) = 2 \times \frac{1}{2} F_y(x) u_y(x) = \sigma_{yy}(x) u_y(x) dx$$

Application of closure stresses which shorten a crack by Δa

- The factor of 2 on the work is required because both crack faces are displaced an absolute distance $u_y(x)$.

$$u_y = \frac{(\kappa+1) K_I (a+\Delta a)}{2\mu} \sqrt{\frac{\Delta a - x}{2\pi}}$$

Mode I

$$u_x \quad \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}\cos\left(\frac{\theta}{2}\right)\left[\kappa - 1 + 2\sin^2\left(\frac{\theta}{2}\right)\right]$$

$$u_y \quad \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}\sin\left(\frac{\theta}{2}\right)\left[\kappa + 1 - 2\cos^2\left(\frac{\theta}{2}\right)\right]$$

$\theta = \pi.$

Relationship between K and $\mathcal{G}$

- The normal stress required to close the crack is related to K_I for the shortened crack.

$\theta = \pi.$

$$\sigma_{yy} = \frac{K_I(a)}{\sqrt{2\pi x}}$$

$$u_y = \frac{(\kappa+1)K_I(a+\Delta a)}{2\mu}\sqrt{\frac{\Delta a - x}{2\pi}}$$

Mode I

σ_{xx}	$\frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\left[1-\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]$
σ_{yy}	$\frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\left[1+\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]$

$$\mathcal{G} = \lim_{\Delta a \to 0} \frac{(\kappa+1)K_I(a)K_I(a+\Delta a)}{4\pi\mu\Delta a}\int_0^{\Delta a}\sqrt{\frac{\Delta a - x}{x}}dx$$

$$= \frac{(\kappa+1)K_I^2}{8\mu} = \frac{(\kappa+1)K_I^2}{\mu} = \frac{4(1-\nu)K_I^2}{8\frac{E}{2(1+\nu)}} = \frac{K_I^2}{\frac{E}{(1-\nu^2)}} = \frac{K_I^2}{E'}$$

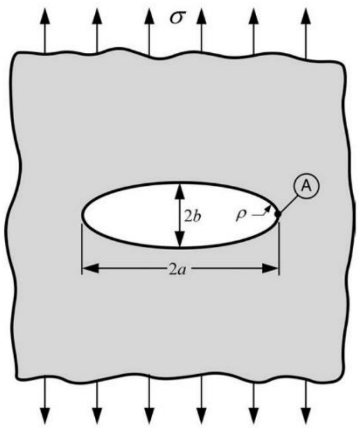
$$\kappa = (3-4\nu), \quad \mu = G = \frac{E}{2(1+\nu)}$$

- Energy release rate, like energy, is a scalar quantity.

$$\mathcal{G} = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}$$

Relationship between K and $\mathcal{G}$

$$\sigma_A = \sigma\left(1 + 2\sqrt{\frac{a}{\rho}}\right)$$



- Linear elastic stress analysis of sharp cracks predicts infinite stresses at the crack tip.
- In real materials, however, stresses at the crack tip are finite because the crack-tip radius must be finite.
- The elastic stress analysis becomes increasingly inaccurate as the inelastic region at the crack tip grows.
- Simple corrections to linear elastic fracture mechanics (LEFM) are available when moderate crack-tip yielding occurs.
- The size of the crack-tip-yielding zone can be estimated by two methods:
  - ✓ the Irwin approach, where the elastic stress analysis is used to estimate the elastic-plastic boundary,
  - ✓ the strip-yield model.

The Irwin Approach

- On the crack plane (θ = 0), the normal stress σ_{yy} in a linear elastic material is given by

Mode I

$$\sigma_{xx} \quad \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\left[1-\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]$$

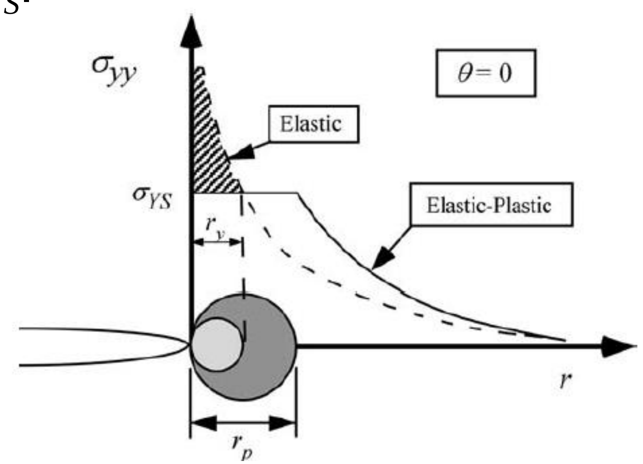
$$\sigma_{yy} \quad \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\left[1+\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]$$

$$\tau_{xy} \quad \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{3\theta}{2}\right)$$

$$\sigma_{xx} = \sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}$$

- the boundary between elastic and plastic behavior is assumed to occur when the stresses satisfy a yield criterion, $\sigma_{yy} = \sigma_{YS}$.

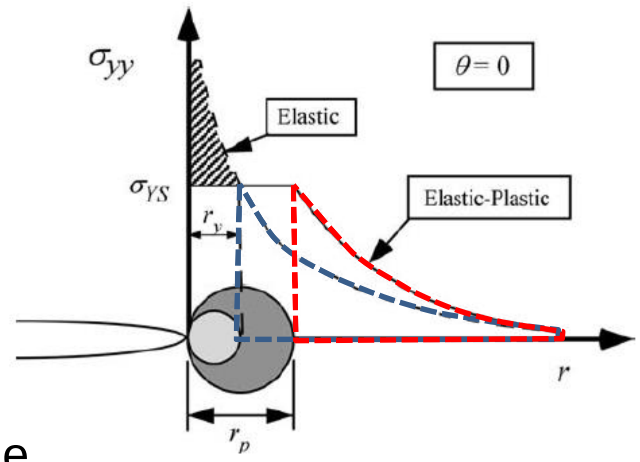
$$r_y = \frac{1}{2\pi}\left(\frac{K_I}{\sigma_{YS}}\right)^2$$



First-order and second-order estimates of plastic zone size

The Irwin Approach

- The simple analysis in the preceding paragraph is not strictly correct because it was based on an elastic crack-tip solution. When yielding occurs, stresses must redistribute in order to satisfy equilibrium.
- The cross-hatched region represents forces that would be present in an elastic material but cannot be carried in the elastic-plastic material because the stress cannot exceed the yield.
- The plastic zone must increase in size in order to accommodate these forces. A simple force balance leads to a second-order estimate of the plastic zone size r_p :



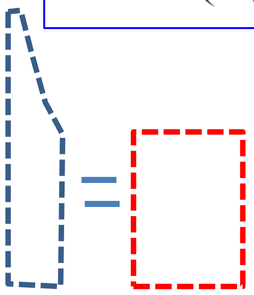
First-order and second-order estimates of plastic zone size

$$r_y = \frac{1}{2\pi}\left(\frac{K_I}{\sigma_{YS}}\right)^2$$

$$\sigma_{YS} r_p = \int_0^{r_y} \sigma_{yy}\, dr = \int_0^{r_y} \frac{K_I}{\sqrt{2\pi r}}\, dr$$

$$= \frac{K_I}{\sqrt{2\pi}} 2 r^{1/2}\Big|_0^{r_y} = \frac{K_I}{\sqrt{2\pi}} 2 r_y^{\,1/2} = \frac{K_I}{\sqrt{2\pi}} 2 \frac{1}{\sqrt{2\pi}} \frac{K_I}{\sigma_{YS}} = \frac{1}{\pi} \frac{K_I^2}{\sigma_{YS}}$$

$$r_p = \frac{1}{\pi}\left(\frac{K_I}{\sigma_{YS}}\right)^2$$



The Irwin Approach

- effective crack length that is slightly longer than the actual crack size

$$a_{eff} = a + r_y$$

- For plane stress

$$r_y = \frac{1}{2\pi}\left(\frac{K_I}{\sigma_{YS}}\right)^2$$

For plane strain

$$r_y = \frac{1}{6\pi}\left(\frac{K_I}{\sigma_{YS}}\right)^2$$

- The effective stress intensity

$$\sigma_e = \frac{1}{\sqrt{2}}\left[(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2\right]^{1/2}$$

$$K_{eff} = Y(a_{eff})\sigma\sqrt{\pi a_{eff}}$$

- Since the effective crack size is taken into account in the geometry correction factor Y, an iterative solution is usually required to solve for K_{eff}.

$$K_{eff} \Rightarrow r_y = \frac{1}{2\pi}\left(\frac{K_I}{\sigma_{YS}}\right)^2 \Rightarrow a_{eff} = a + r_y \Rightarrow K_{eff} = Y(a_{eff})\sigma\sqrt{\pi a_{eff}}$$

The Irwin Approach

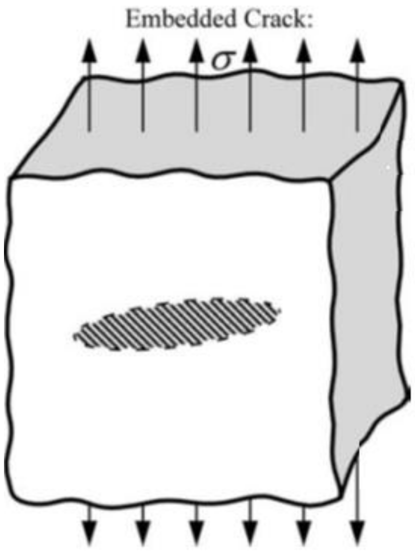
- In certain cases, this iterative procedure is unnecessary because a closed-form solution is possible. For example, the effective Mode I stress intensity factor for a through crack in an infinite plate in plane stress is given by

$$K_{eff} = \frac{\sigma\sqrt{\pi a}}{\sqrt{1 - \frac{1}{2}\left(\frac{\sigma}{\sigma_{YS}}\right)^2}}$$

- Elliptical and semielliptical flaws also have an approximate closed-form plastic zone correction, provided the flaw is small compared to the plate dimensions.
- In the case of the embedded elliptical flaw, K_{eff} is given by

Elliptical cracks

$$K_{eff} = \sigma\sqrt{\frac{\pi a}{Q_{eff}}}\left[\sin^2(\phi) + \left(\frac{a}{c}\right)^2 \cos^2(\phi)\right]^{1/4} \qquad Q_{eff} = Q - 0.212\left(\frac{\sigma}{\sigma_{YS}}\right)^2$$

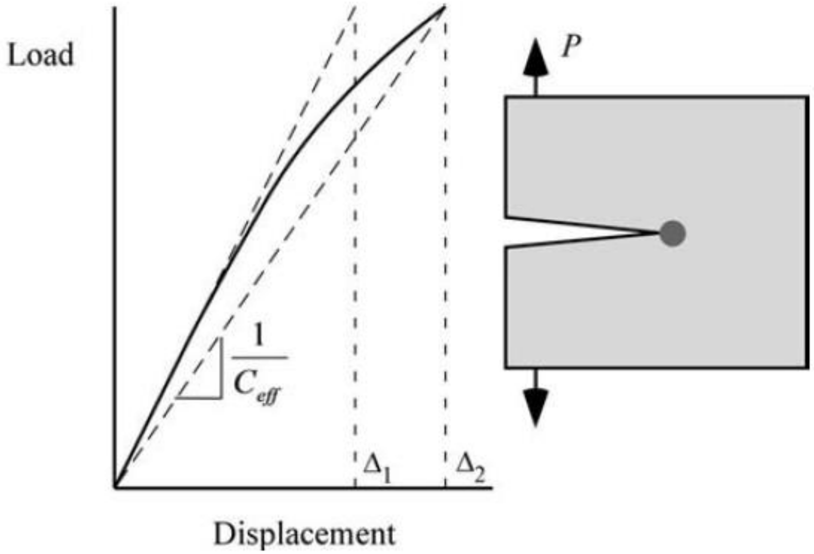


- Effective compliance

$$C_{eff} = \frac{\Delta_2}{P}$$

The Irwin Approach

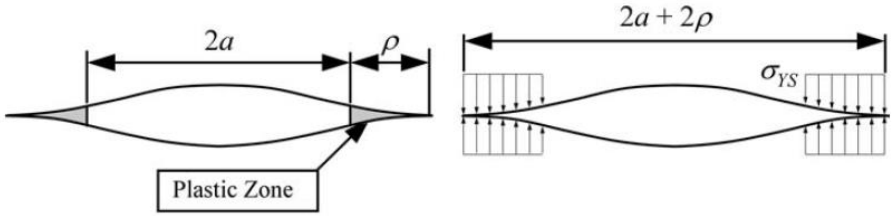
- It should be noted that the author does not recommend using the Irwin plastic zone adjustment for practical applications.
- It was presented here primarily to provide a historical context to the development of both linear and nonlinear fracture mechanics.



Definition of the effective compliance to account for crack-tip plasticity

The Strip-Yield Model, Dugdale [24] and Barenblatt [25]

- A long, slender plastic zone at the crack tip is assumed in a nonhardening material in plane stress.
- The strip-yield plastic zone is modeled by assuming a crack of length $2a + 2\rho$, where ρ is the length of the plastic zone, with a closure stress equal to σ_{YS} applied at each crack tip.



The strip-yield model.

- This model approximates elastic-plastic behavior by superimposing two elastic solutions: a through crack under remote tension and a through crack with closure stresses at the tip.
- Since the stresses are finite in the strip-yield zone, there cannot be a stress singularity at the crack tip.

$$\sigma_{ij} = \left(\frac{k}{\sqrt{r}}\right) f_{ij}(\theta) + \sum_{m=0}^{\infty} A_m r^{\frac{m}{2}} g_{ij}^{(m)}(\theta)$$

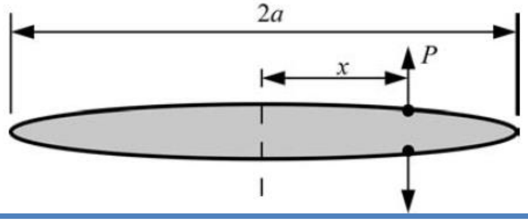
(first term crossed out → 0)

- The plastic zone length ρ must be chosen such that the stress intensity factors from the remote tension and closure stress cancel one another.

The Strip-Yield Model, Dugdale [24] and Barenblatt [25]

- The stress intensity due to the closure stress can be estimated by considering a normal force *P* applied to the crack at a distance x from the centerline of the crack.
- The stress intensities for the two crack tips are given by assuming the plate is of unit thickness.

$$K_{I(+a)} = \frac{P}{\sqrt{\pi a}}\sqrt{\frac{a+x}{a-x}} \qquad K_{I(-a)} = \frac{P}{\sqrt{\pi a}}\sqrt{\frac{a-x}{a+x}}$$



Crack-opening force applied at a distance *x* from the center-line.

- The closure force at a point within the strip-yield zone is equal to

$$P = -\sigma_{YS} dx$$

- The total stress intensity at each crack tip resulting from the closure stresses is obtained by replacing a with $a + \rho$ in Equation

$$K_{I(+a)} = \frac{P}{\sqrt{\pi a}}\sqrt{\frac{a+x}{a-x}} \qquad K_{I(-a)} = \frac{P}{\sqrt{\pi a}}\sqrt{\frac{a-x}{a+x}}$$

$$K_{\text{closure}} = -\frac{\sigma_{YS}}{\sqrt{\pi(a+\rho)}}\int_a^{a+\rho}\left\{\sqrt{\frac{a+\rho+x}{a+\rho-x}} + \sqrt{\frac{a+\rho-x}{a+\rho+x}}\right\}dx$$

$$= -2\sigma_{YS}\sqrt{\frac{a+\rho}{\pi}}\int_a^{a+\rho}\frac{dx}{\sqrt{(a+\rho)^2 - x^2}}$$

$$K_{\text{closure}} = -2\sigma_{YS}\sqrt{\frac{a+\rho}{\pi}}\cos^{-1}\left(\frac{a}{a+\rho}\right)$$

The Strip-Yield Model, Dugdale [24] and Barenblatt [25]

- The stress intensity from the remote tensile stress must balance with $K_{closure}$.

$$K_\sigma = \sigma\sqrt{\pi(a+\rho)}$$

$$K_{closure} = -2\sigma_{YS}\sqrt{\frac{a+\rho}{\pi}}\cos^{-1}\left(\frac{a}{a+\rho}\right)$$

$$K_\sigma + K_{closure} = 0 \Rightarrow \sigma\sqrt{\pi(a+\rho)} = 2\sigma_{YS}\sqrt{\frac{a+\rho}{\pi}}\cos^{-1}\left(\frac{a}{a+\rho}\right) \Rightarrow \frac{a}{a+\rho} = \cos\left(\frac{\pi\sigma}{2\sigma_{YS}}\right)$$

- ρ approaches infinity as $\sigma \rightarrow \sigma_{YS}$. Let us explore the strip-yield model further by performing a Taylor series expansion.

$$\frac{a}{a+\rho} = 1 - \frac{1}{2!}\left(\frac{\pi\sigma}{2\sigma_{YS}}\right)^2 + \frac{1}{4!}\left(\frac{\pi\sigma}{2\sigma_{YS}}\right)^4 - \frac{1}{6!}\left(\frac{\pi\sigma}{2\sigma_{YS}}\right)^6 + \cdots$$

- Neglecting all but the first two terms and solving for the plastic zone size gives. the Irwin and strip-yield approaches predict similar plastic zone sizes.

$$\rho = \frac{\pi^2\sigma^2 a}{8\sigma_{YS^2}} = \frac{\pi}{8}\left(\frac{K_I}{\sigma_{YS}}\right)^2$$

Irwin approach

$$r_p = \frac{1}{\pi}\left(\frac{K_I}{\sigma_{YS}}\right)^2$$

$$\frac{\pi}{8} = 0.392 \text{ , } \frac{1}{\pi} = 0.318$$

The Strip-Yield Model, Dugdale [24] and Barenblatt [25]

- One way to estimate the effective stress intensity with the strip-yield model is to set equal to $a + \rho$.

$$K_{eff} = \sigma \sqrt{\pi a \sec\left(\frac{\pi\sigma}{2\sigma_{YS}}\right)}$$

$$K_\sigma = \sigma\sqrt{\pi(a+\rho)}$$

$$\frac{a}{a+\rho} = \cos\left(\frac{\pi\sigma}{2\sigma_{YS}}\right) \rightarrow (a+\rho) = a\sec\left(\frac{\pi\sigma}{2\sigma_{YS}}\right)$$

- It tends to overestimate K_{eff}; the actual a_{eff} is somewhat less than $a + \rho$ because the strip-yield zone is loaded to σ_{YS}.
- Burdekin and Stone [26] obtained a more realistic estimate of K_{eff} for the strip-yield model.

$$K_{eff} = \sigma_{YS}\sqrt{\pi a}\left[\frac{8}{\pi^2}\ln\sec\left(\frac{\pi\sigma}{2\sigma_{YS}}\right)\right]^{1/2}$$

Comparison of Plastic Zone Corrections

- A pure LEFM

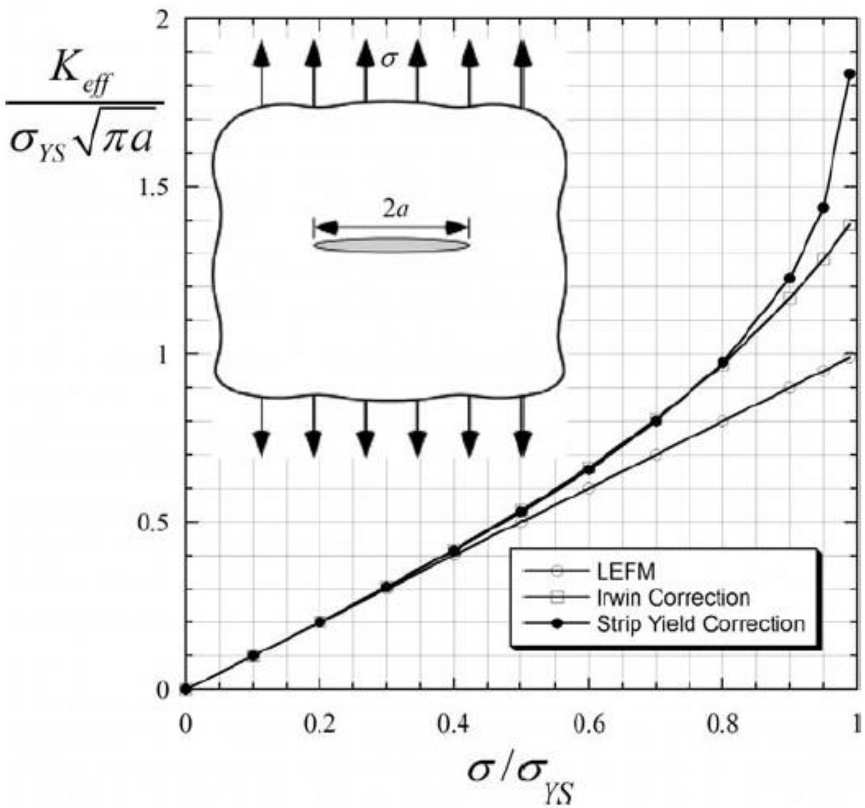
$$K_I = \sigma\sqrt{\pi a}$$

- The Irwin correction for plane stress

$$K_{eff} = \frac{\sigma\sqrt{\pi a}}{\sqrt{1-\frac{1}{2}\left(\frac{\sigma}{\sigma_{YS}}\right)^2}}$$

- The strip-yield correction on stress intens

$$K_{eff} = \sigma\sqrt{\pi a \sec\left(\frac{\pi\sigma}{2\sigma_{YS}}\right)}$$



Comparison of plastic zone corrections for a through crack in plane strain.

- Both the Irwin and strip-yield corrections deviate from the LEFM theory at stresses greater than $0.5\sigma_{YS}$.
- The two plasticity corrections agree with each other up to approximately $0.85\sigma_{YS}$.

Plastic Zone Shape

- The estimates of plastic zone size that have been presented so far consider only the crack plane $\theta = 0$.

$$\sigma_e = \frac{1}{\sqrt{2}}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2\right]^{1/2}$$

- For plane stress, Setting $\sigma_e = \sigma_{YS}$

$$\sigma_3 = 0$$

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2)^2 + (\sigma_1)^2 = 2\sigma_{YS}^2$$

Crack tip stress by principal stress

$$\sigma_1 = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\left\{1 + sin\left(\frac{\theta}{2}\right)\right\}$$

$$\sigma_2 = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\left\{1 - sin\left(\frac{\theta}{2}\right)\right\}$$

$\sigma_3 = 0$ for plane stress

$= \frac{2\upsilon K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)$ for plane strain

$$\frac{K_I^2}{2\pi r_y} cos^2\left(\frac{\theta}{2}\right)\left\{6sin^2\left(\frac{\theta}{2}\right) + 2\right\} = 2\sigma_{YS}^2$$

$$r_y = \frac{K_I^2}{2\pi\sigma_{YS}^2} cos^2\left(\frac{\theta}{2}\right)\left\{3sin^2\left(\frac{\theta}{2}\right) + 1\right\} = \frac{K_I^2}{4\pi\sigma_{YS}^2}\left[1 + \cos(\theta) + \frac{3}{2}sin^2(\theta)\right]$$

$$cos^2\left(\frac{\theta}{2}\right) = \frac{1}{2}\{1 + \cos(\theta)\},\ cos^2\left(\frac{\theta}{2}\right)sin^2\left(\frac{\theta}{2}\right) = \frac{1}{4}sin^2(\theta)$$

HW #2 : Prove this.

Plastic Zone Shape

- For plane strain, Setting $\sigma_e = \sigma_{YS}$

$$\sigma_e = \frac{1}{\sqrt{2}}\left[(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2\right]^{1/2}$$

$$\sigma_3 = \nu(\sigma_1+\sigma_2)$$

$$(\nu^2-\nu+1)({\sigma_1}^2+{\sigma_2}^2)+(2\nu^2-2\nu-1)\ \sigma_1\sigma_2=2\sigma_{YS}^2$$

Crack tip stress by principal stress

$$\sigma_1 = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\{1+sin\left(\frac{\theta}{2}\right)\}$$

$$\sigma_2 = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\{1-sin\left(\frac{\theta}{2}\right)\}$$

$\sigma_3 = 0$ for plane stress

$= \frac{2\upsilon K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)$ for plane strain

$$(\nu^2-\nu+1)\ \frac{K_I^2}{2\pi r_y}cos^2\left(\frac{\theta}{2}\right)\{2+2sin^2\left(\frac{\theta}{2}\right)\}$$

$$+\ (2\nu^2-2\nu-1)\frac{K_I^2}{2\pi r_y}cos^2\left(\frac{\theta}{2}\right)\{2-2sin^2\left(\frac{\theta}{2}\right)\}=\ 2\sigma_{YS}^2$$

$$\frac{K_I^2}{2\pi r_y}cos^2\left(\frac{\theta}{2}\right)\left[6sin^2\left(\frac{\theta}{2}\right)+2(1-2v)^2\right]=2\sigma_{YS}^2$$

$$r_y=\frac{K_I^2}{4\pi\sigma_{YS}^2}[(1-2\upsilon)^2\{1+\cos(\theta)\}+\frac{3}{2}sin^2(\theta)\}$$

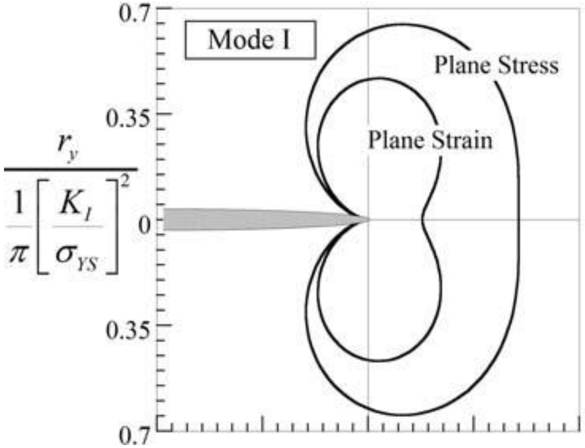
$$cos^2\left(\frac{\theta}{2}\right)=\frac{1}{2}\{1+\cos(\theta)\}, cos^2\left(\frac{\theta}{2}\right)sin^2\left(\frac{\theta}{2}\right)=\frac{1}{4}sin^2(\theta)$$

HW #2 : Prove this.

Plastic Zone Shape

❖ **Crack-tip plastic zone shapes estimated from the elastic solutions.**

- Not strictly correct because they are based on a purely elastic analysis.
- The Irwin plasticity correction, which accounts for stress redistribution by means of an effective crack length, is also simplistic and not totally correct.

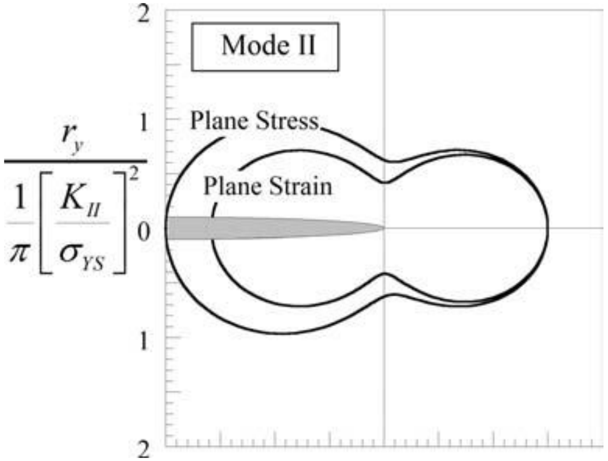


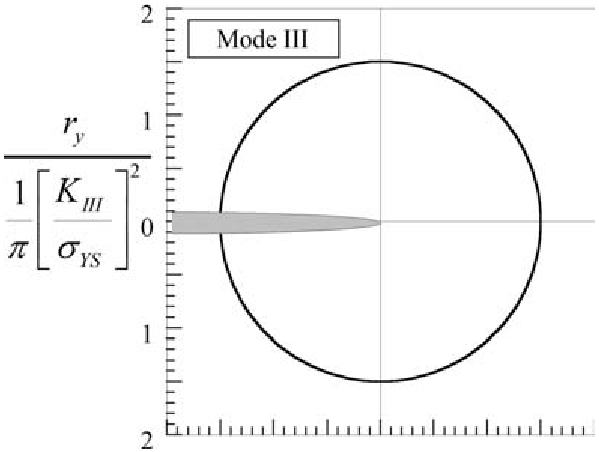
- Plane stress

$$r_y = \frac{K_I^2}{4\pi\sigma_{YS}^2}\left[1 + \cos(\theta) + \frac{3}{2}sin^2(\theta)\right]$$

- Plane strain

$$r_y = \frac{K_I^2}{4\pi\sigma_{YS}^2}\left[(1-2\upsilon)^2\{1 + \cos(\theta)\} + \frac{3}{2}sin^2(\theta)\right\}$$



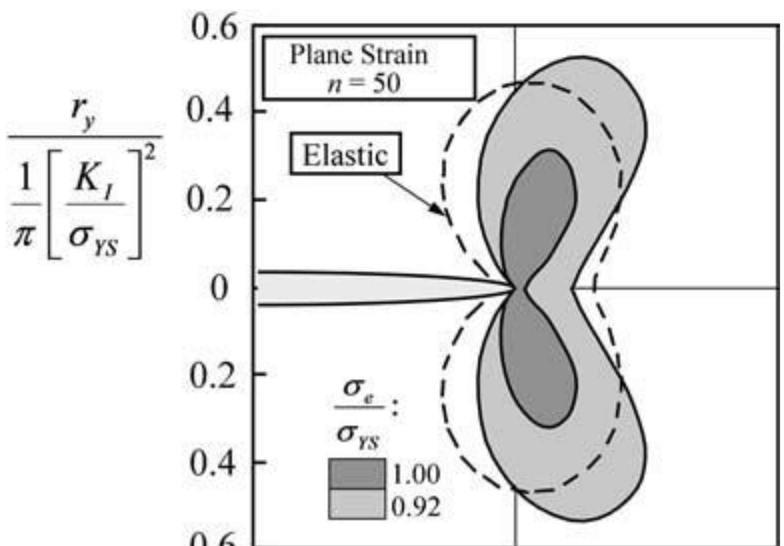


Plastic Zone Shape

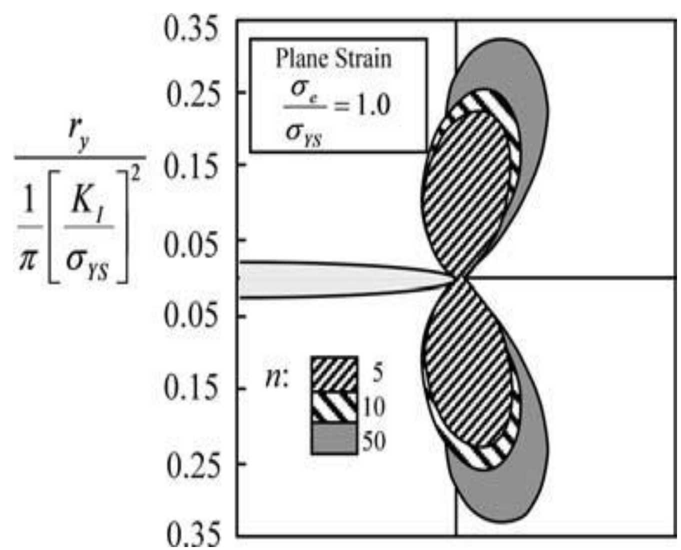
- The latter, which was published by Dodds et al. [27], assumed a material with the following uniaxial stress-strain relationship:

$$\frac{\varepsilon}{\varepsilon_o} = \frac{\sigma}{\sigma_o} + \alpha\left(\frac{\sigma}{\sigma_o}\right)^n$$

- The definition of the elastic-plastic boundary is somewhat arbitrary, since materials that can be described by the above Eq do not have a definite yield point. $\sigma_e = \sigma_{YS}$ (the 0.2% offset yield strength).



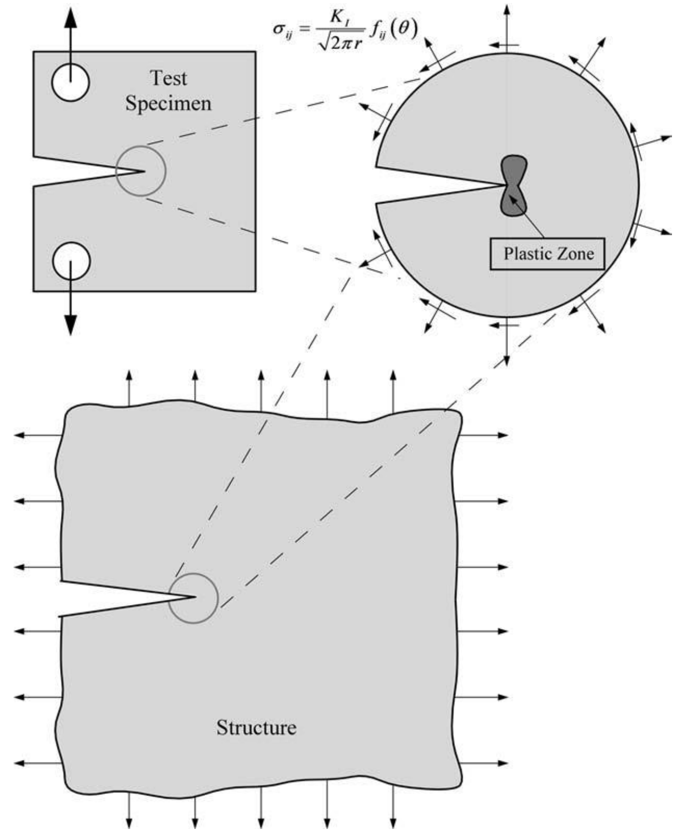
Contours of constant effective stress in Mode I, obtained from finite element analysis, n=50



Effect of strain-hardening on the Mode I plastic zone; n = 5 : a high strain hardening, n = 50 : very low hardening

Introduction

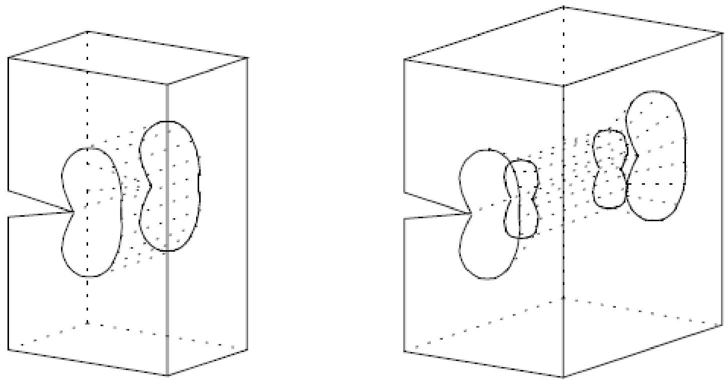
- The stresses near the crack tip in a linear elastic material vary as $1\sqrt{r}$; the stress intensity factor defines the amplitude of the singularity.
- If K> K_{crit} , cack extension must occur.
- K_{crit} : a measure of *fracture toughness*, is a material constant that is independent of the size and geometry of the cracked body
- $\mathcal{G}$ also provides a single-parameter description of crack-tip conditions, and $\mathcal{G}c$ is an alternative measure of toughness.
- Under certain conditions, *K* still uniquely characterizes crack-tip conditions when a plastic zone is present. In such cases, K_{crit} is a geometry-independent material constant.
- Assume that the plastic zone is small compared to all the length dimensions in the structure and test specimen.
- K_I characterizes crack-tip conditions. As the load is increased, both configurations (specimen and structure) will fail at the same critical stress intensity, provided the plastic zone remains small in each case.



Schematic test specimen and structure loaded to the same stress intensity. The crack-tip conditions should be identical in both configurations as long as the plastic zone is small compared to all relevant dimensions. Thus, both will fail at the same critical *K* value.

Introduction

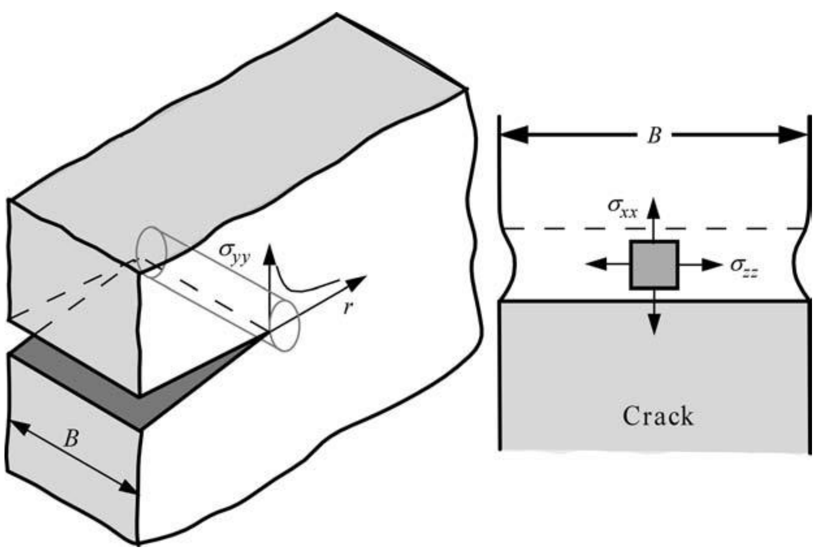
- The NASA data exhibited an apparent effect of specimen thickness on the critical stress intensity for fracture, K_{crit}.
- The explanation that was originally offered was that thin specimens are subject to plane stress loading at the crack tip, while thick specimens experience plane strain conditions.
- The biaxial stress state associated with plane stress results in a higher measured toughness than is observed in the same material when subject to a triaxial stress state.
- This section presents an updated perspective on the interrelationship between specimen dimensions, crack-tip triaxiality, and fracture toughness.



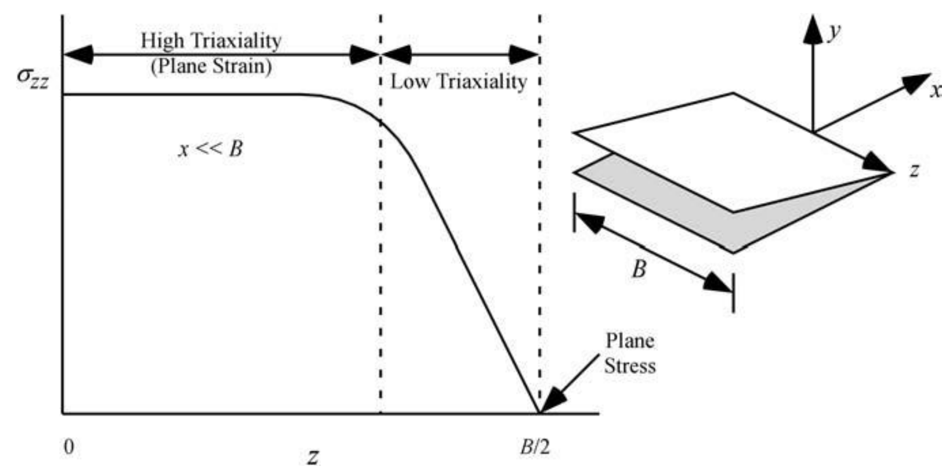
Plastic crack tip zone for a thin and a thick plate

Crack-Tip Triaxiality

- If there was no crack, the plate would be in a state of plane stress. Regions of the plate that are sufficiently far from the crack tip must also be loaded in plane stress.
- Because of the large stress normal to the crack plane, the crack-tip material tries to contract in the *x* and *z* directions, but is prevented from doing so by the surrounding material. This constraint causes a triaxial state of stress near the crack-tip.
- In the interior of the plate, the *z* stress, and therefore the level of triaxiality is high.
- The central region : plane strain.
- Near the free surface : low stress triaxiality.
- The free surface : pure plane stress.



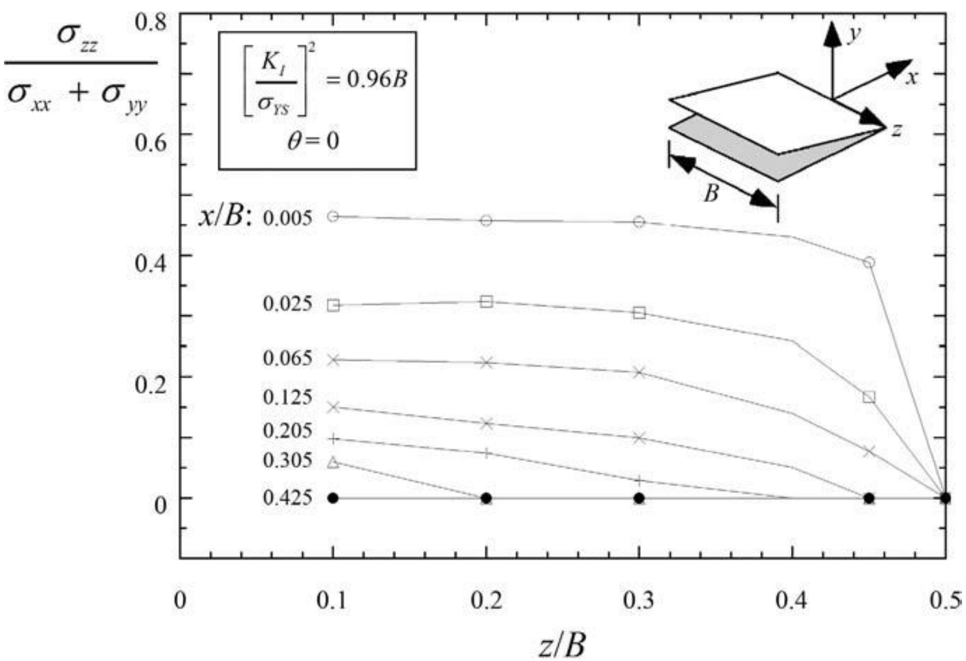
Three-dimensional deformation at the tip of a crack.



Schematic variation of transverse stress and strain through the thickness

Crack-Tip Triaxiality

- These results were obtained from a three-dimensional elastic-plastic finite element analysis performed by Narasimhan and Rosakis
- Under plane strain loading, the quantity $\sigma_{zz} /(\sigma_{xx} + \sigma_{yy})$ is equal to Poisson's ratio for elastic material behavior and is equal to 0.5 for incompressible plastic deformation.
- Material near the crack tip experiences high triaxiality, but $\sigma_{zz} = 0$ when x is a significant fraction of the plate thickness.



Transverse stress through the thickness as a function of distance from the crack tip

- Consider a point on the crack plane ($\theta = 0$) just ahead of the crack-tip.
- $\sigma_{xx} = \sigma_{yy}$ under linear elastic conditions.

$$\sigma_{xx} = \sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}$$

- Plane stress : $\sigma_{zz} = 0$
- Plane strain : $\sigma_{zz} = 2\nu\sigma_{yy}$ (ν = 0.3)

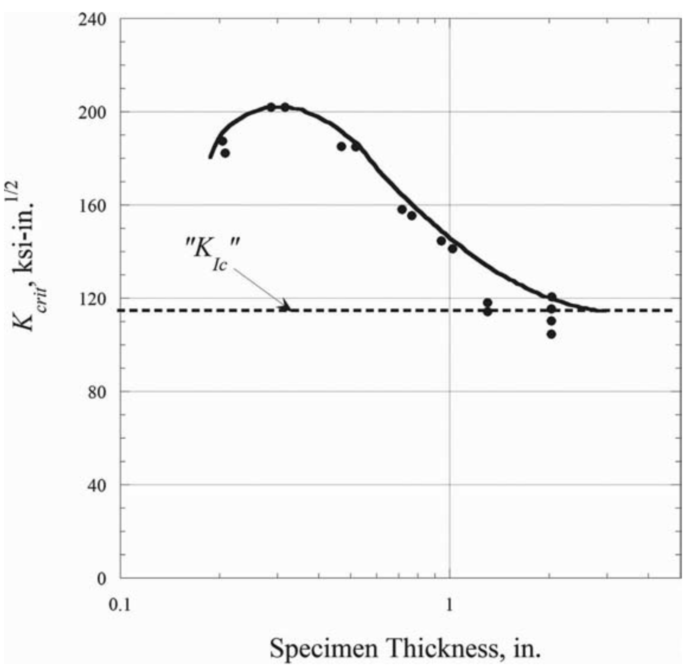
HW #2 : Prove this.

$$\sigma_{yy}(\text{at yield}) = \begin{cases} \sigma_{YS} & \text{(plane stress)} \\ 2.5\sigma_{YS} & \text{(plane strain)} \end{cases}$$

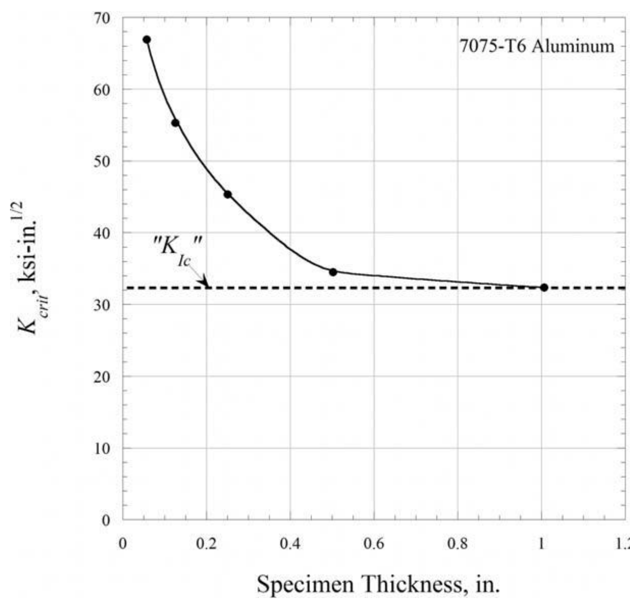
$$\sigma_e = \frac{1}{\sqrt{2}}\left[(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2\right]^{1/2}$$

Effect of thickens on Apparent Fracture toughness

- The measured K_{crit} values decrease with specimen thickness until a plateau is reached.
- This apparent asymptote in the toughness vs. thickness trend is designated by the symbol K_{Ic}, and is referred to as “plane strain fracture toughness”.
- A decrease in apparent toughness with specimen thickness, generally correspond to materials in which the crack propagation is ductile.



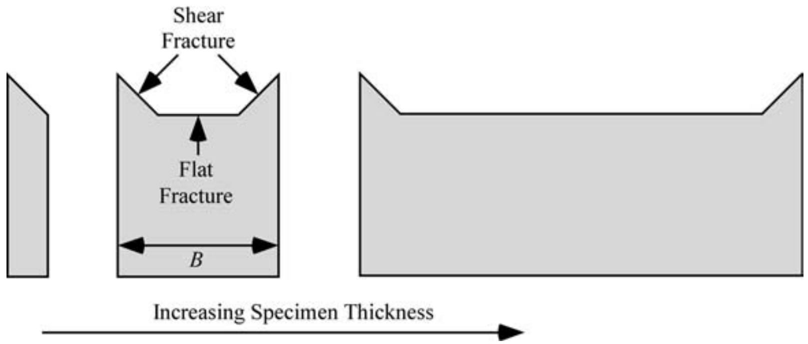
Variation of measured fracture toughness with specimen thickness for an unspecified alloy



Variation of measured fracture toughness with specimen thickness for 7075-T6 Aluminum

Effect of thickens on Apparent Fracture toughness

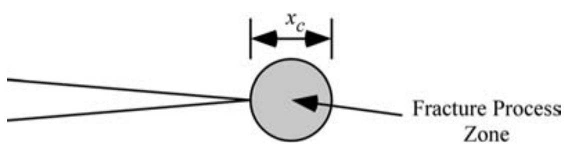
- In such tests, the crack “tunnels” through the center of the specimen.
- The crack grows preferentially in the region of high triaxiality. Crack growth on the outer regions of the specimen lags behind, and occurs at a 45° angle to the applied load. The resulting fracture surface exhibits a flat region in the central region and 45° shear lips on the edges.
- Fracture toughness tests on very thin plates or sheets typically result in a 45° shear fracture. At larger thicknesses, there is generally some mixture of shear fracture and flat fracture. The thickness effect on the apparent fracture toughness is due to the relative portions of flat and shear fracture.
- In the limit of a very thick specimen, the flat fracture mechanism dominates, and further increases in thickness have relatively little effect on the measured toughness.



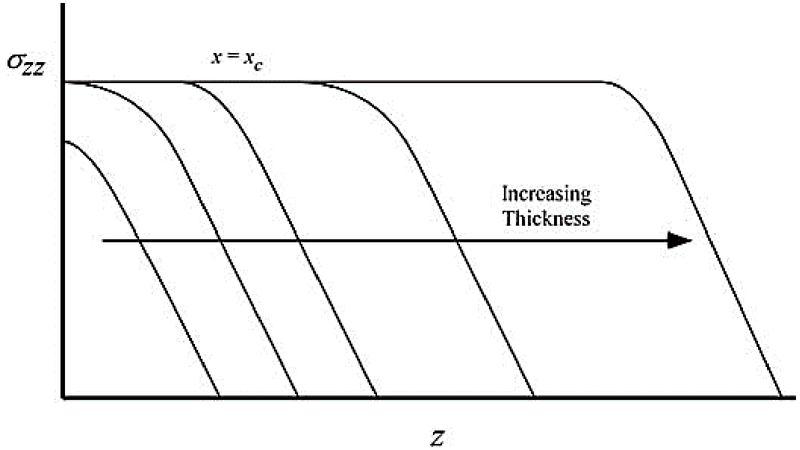
Effect of specimen thickness on fracture surface morphology for materials that exhibit ductile crack growth.

Effect of thickens on Apparent Fracture toughness

- For very thin sections, plane strain conditions do not exist at $x = x_c$. As the thickness increases, the size of the plane strain zone increases relative to the low triaxiality zone near the free surfaces.
- This trend are not indicative of a transition from "plane stress fracture" to "plane strain fracture."
- Rather, this trend reflects the differing relative contributions of two distinct fracture mechanisms.
- In fact, there is no such thing as "plane stress fracture" except perhaps in very thin foil. There is nearly always some level of triaxiality along the crack front.



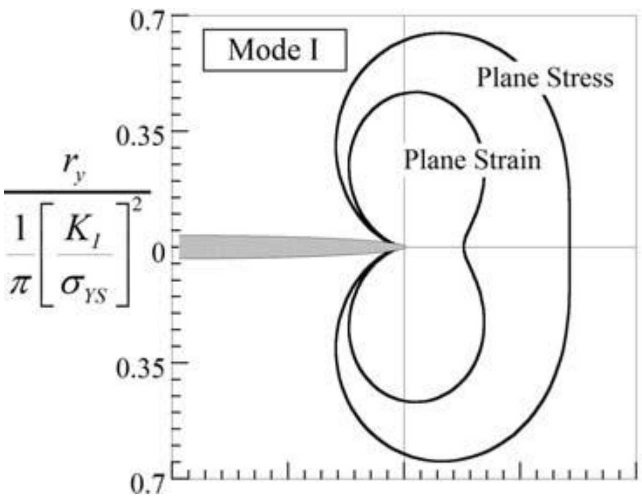
Fracture process zone at the tip of a crack tip

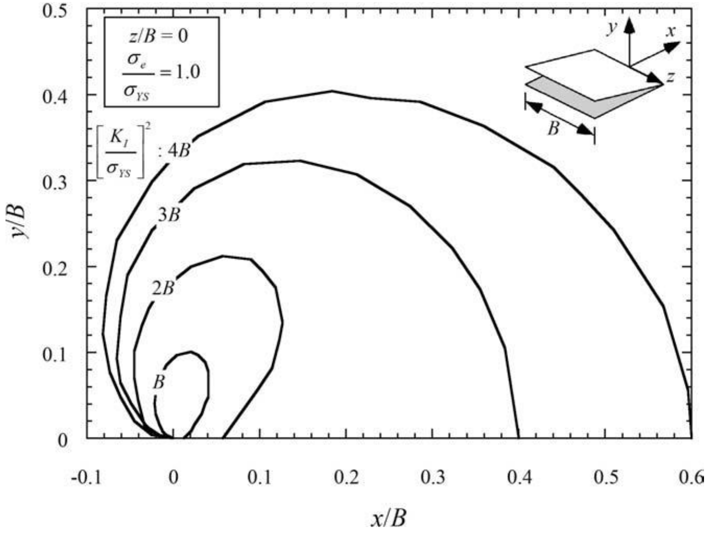


Effect of thickness on stress—the crack-tip stress state in the fracture process zone.

Plastic Zone Effects

- 3D elastic plastic finite element analyses of fracture toughness specimens ⇒ a high degree of triaxiality near the crack tip exists even when the entire cross-section has yielded.
- Although K is not valid as a characterizing parameter under fully plastic conditions, a single-parameter description of fracture toughness is still possible using the J integral, or cracktip-opening displacement.
- The evolution of the Mode I plastic zone at mid-thickness in a plate containing an edge crack. The plastic zone boundary is defined at $\sigma_e = \sigma_{YS}$.
- As the quantity increases relative to plate thickness B, the plastic zone size increases.
- At low K_I values, the plastic zone has a typical plane strain shape, but evolves into a plane stress shape at higher K_I values.





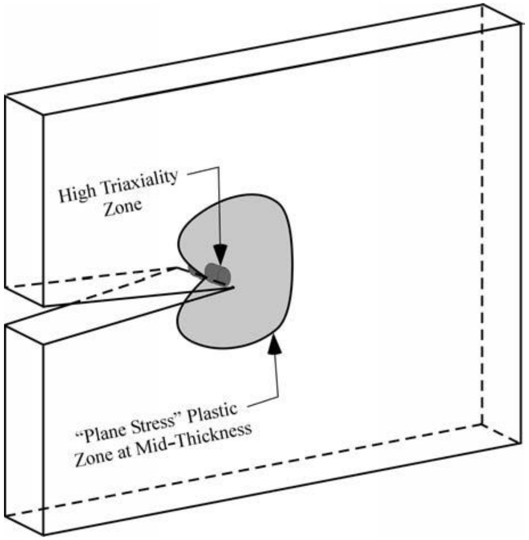
Effect of K_I, relative to thickness, of the plastic zone size and shape.

Plastic Zone Effects

- Although the stress state at the plastic zone boundary is plane stress, the material close to the crack tip is subject to a triaxial stress state. The figure depicts a plastic zone in the center of an edge-cracked plate.
- Because the plastic zone size is of the same order of magnitude as the plate thickness, the plastic zone has a plane stress shape.
- At the crack tip, however, there is a zone of high triaxiality. As stated above, the zone of high triaxiality at the crack tip can persist even in the presence of large-scale plasticity.
- When performing laboratory K_{Ic} tests on standard specimens, the following size requirements have been adopted.

$$a, B, (W-a) \geq 2.5\left(\frac{K_{Ic}}{\sigma_{YS}}\right)^2$$

- Recall that the quantity $(K_{IC}/\sigma_{YS})^2$ is proportional to the plastic zone size.
- The minimum requirements on the crack length and ligament length (W – a) are designed to ensure that the plastic zone is sufficiently small for fracture to be K-controlled. => ensure plane strain conditions along the crack front.



Cracked plate in which the plastic zone size is of the same order of magnitude as the plate thickness