Chapter 5. Series Solutions of ODEs. Special Functions

2018.5
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서유택

※ 본 강의 자료는 이규열, 장범선, 노명일 교수님께서 만드신 자료를 바탕으로 일부 편집한 것입니다.

Introduction

- In the previous chapters, linear ODEs with *constant coefficients (상수계수)* can be solved by algebraic methods (대수적인 방법), and that their solutions are elementary functions (초등함수) known from calculus (미적분).

- For ODEs with *variable coefficients (변수계수)* the situation is more complicated, and their solutions may be nonelementary functions.

- In this chapter, the three main topics are Legendre polynomials, Bessel functions, and hypergeometric functions (초기하함수).

- Legendre's ODE and Legendre polynomials are obtained by the power series method (거듭제곱급수 또는 멱급수 해법). $(1-x^2)y''-2xy'+n(n+1)y=0$

- Bessel's ODE and Bessel functions are obtained by the Frobenius method, an extension of the power series method. $x^2y''+xy'+(x^2-\nu^2)y=0$

* Elementary functions (초등함수): 다항 함수, 로그 함수, 지수 함수, 삼각 함수와 이들 함수의 합성 함수들을 총칭
* Hypergeometric functions (초기하함수): 거듭제곱 급수로 나타내지는 일련의 특수 함수들을 총칭

5.1 Power Series Method (거듭제곱급수해법, 멱급수해법)

- The **power series method** is the standard method for solving linear ODEs with ***variable*** coefficients.

- **Power Series (거듭제곱):** $\sum_{m=0}^{\infty} a_m (x-x_0)^m = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$
  - Coefficients: $a_0, a_1, a_2, \ldots$
  - Center: x_0
  - Power Series in powers of x if $x_0 = 0$: $\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \cdots$

Ex 1 Maclaurin series

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \cdots \qquad (|x| < 1, \text{ geometric series})$$

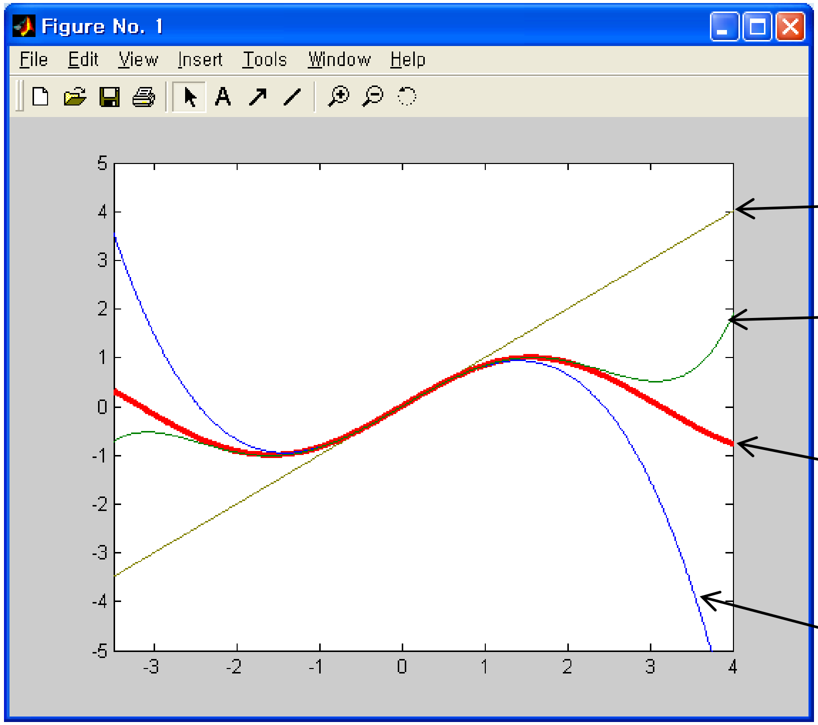
등비급수

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots$$

$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots$$

5.1 Power Series Method



$$ex^{*})\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots,$$

x

$$x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$\sin x$

$$x - \frac{x^3}{3!}$$

5.1 Power Series Method

❖ Idea of the Power Series Method

ODE $y'' + p(x)y' + q(x)y = 0$

- We represent $p(x)$ and $q(x)$ by power series in powers of x.
- We assume **a solution in the form of a power series with unknown coefficients**.

$$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

- Series obtained by termwise differentiation

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

$$y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} = 2a_2 + 3\cdot 2a_3 x + 4\cdot 3a_4 x^2 + \cdots$$

- Insert the series into the ODE.

5.1 Power Series Method

☑ Ex.2 Solve the following ODE by power series.

$$y' - y = 0$$

$$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \longrightarrow y' = \sum_{m=1}^{\infty} m a_m x^{m-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

Insert the series into the ODE $\longrightarrow$ $\left(a_1 + 2a_2 x + 3a_3 x^2 + \cdots\right) - \left(a_0 + a_1 x + a_2 x^2 + \cdots\right) = 0$

We collect like powers of x.

$$(a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \cdots = 0$$

$$(a_1 - a_0) = 0,\ (2a_2 - a_1) = 0,\ (3a_3 - a_2)x = 0, \cdots$$

$$\therefore\ a_1 = a_0,\quad a_2 = \frac{a_1}{2} = \frac{a_0}{2!},\quad a_3 = \frac{a_2}{3} = \frac{a_0}{3!},\quad a_4 = \frac{a_3}{4} = \frac{a_0}{4!},\quad \cdots$$

General solution:

Maclaurin series for e^x

$$y = a_0 + a_0 x + \frac{a_0}{2!}x^2 + \frac{a_0}{3!}x^3 + \cdots = a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) = a_0 e^x$$

5.1 Power Series Method

☑ Ex.3 A Special Legendre Equation

$$(1-x^2)y'' - 2xy' + 2y = 0$$

$$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \quad \rightarrow \quad y' = \sum_{m=1}^{\infty} m a_m x^{m-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

$$\rightarrow \quad y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} = 2a_2 + 3\cdot 2 a_3 x + 4\cdot 3 a_4 x^2 + \cdots$$

Insert the series into the ODE.

$$y'' = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + \cdots$$

$$-x^2 y'' = -2a_2 x^2 - 6a_3 x^3 - 12a_4 x^4 + \cdots$$

$$-2xy' = -2a_1 x - 4a_2 x^2 - 6a_3 x^3 - 8a_4 x^4 - \cdots$$

$$2y = 2a_0 + 2a_1 x + 2a_2 x^2 + 2a_3 x^3 + 2a_4 x^4 + \cdots$$

Sum	Power	Equations	
[0]	$[x^0]$	$a_2 = -a_0$	
[1]	$[x]$	$a_3 = 0$	
[2]	$[x^2]$	$12a_4 = 4a_2,$	$a_4 = \frac{4}{12}a_2 = -\frac{1}{3}a_0$
[3]	$[x^3]$	$a_5 = 0$	since $a_3 = 0$
[4]	$[x^4]$	$30a_6 = 18a_4,$	$a_6 = \frac{18}{30}a_4 = \frac{18}{30}(-\frac{1}{3})a_0 = -\frac{1}{5}a_0.$

⇨

General solution:

$$y = a_1 x + a_0\left(1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \cdots\right)$$

5.1 Power Series Method

Theory of the Power Series Method

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \cdots$$

- ***n*th Partial Sum:** $s_n(x) = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \cdots + a_n (x - x_0)^n$
- **Remainder:** $R_n(x) = a_{n+1}(x - x_0)^{n+1} + a_{n+2}(x - x_0)^{n+2} + \cdots$
- **For example,**

$$1 + x + x^2 + \cdots + x^n + \cdots$$

$$s_0 = 1, \qquad R_0 = x + x^2 + x^3 + \cdots,$$

$$s_1 = 1 + x, \qquad R_1 = x^2 + x^3 + x^4 + \cdots,$$

$$s_2 = 1 + x + x^2, \quad R_2 = x^3 + x^4 + x^5 + \cdots, \ etc$$

5.1 Power Series Method

❖ **Convergent (수렴)**: If this sequence converges at $x=x_1$, say,

$$\lim_{n\to\infty} s_n(x_1) = s(x_1)$$

$$s_0 = 1,$$
$$s_1 = 1 + x,$$
$$s_2 = 1 + x + x^2,$$
$$s_n = 1 + x + \cdots + x^n$$

→ the series called convergent at $x=x_1$, $s(x_1)$: convergent value or sum

$$s(x_1) = \sum_{m=0}^{\infty} a_m (x_1 - x_0)^m$$

$$s_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n$$

❖ **Divergent (발산)**: If this sequence diverges at $x=x_1$, say,

For example,

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \cdots$$

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \cdots$$

$$\sum_{m=0}^{\infty} m!\,x^m = 1 + x + 2x^2 + \cdots$$

5.1 Power Series Method

$$s(x_1) = \sum_{m=0}^{\infty} a_m (x_1 - x_0)^m$$

$$s_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n$$

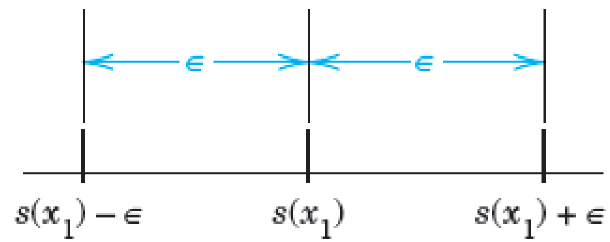
$$R_n(x) = a_{n+1}(x - x_0)^{n+1} + a_{n+2}(x - x_0)^{n+2} + \cdots$$

- **In the case of convergence,**

  → for every n, $s(x_1) = s_n(x_1) + R_n(x_1)$

- If all $s_n(x_1)$ with $n > \mathrm{N}$ lie between $s(x_1) - \varepsilon$ and $s(x_1) + \varepsilon \rightarrow s(x_1) \approx s_n(x_1)$,

$$|R_n(x_1)| = |s(x_1) - s_n(x_1)| < \varepsilon \qquad \text{for all } n > N$$

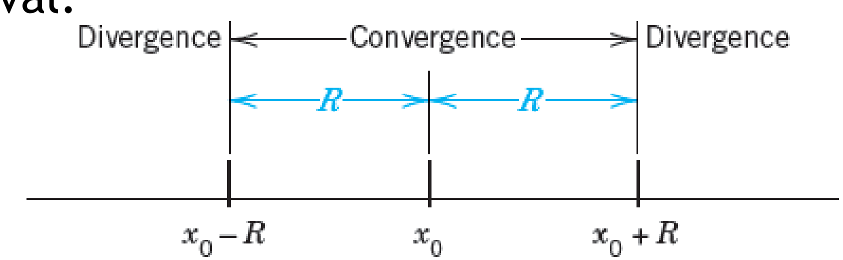


- And at $x = x_0$, $s(x_0) = a_0$ → the series converges at x_0.
- If there are other values of x for which the series converges besides x_0, there values form an interval → "convergence interval (수렴구간)"
- The series converges for all x in the interval:

$$|x - x_0| < R$$

  The series diverges for all x:

$$|x - x_0| > R$$



5.1 Power Series Method

❖ **Convergence Interval (수렴구간). Radius of Convergence (수렴반지름)**

$$s_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n$$

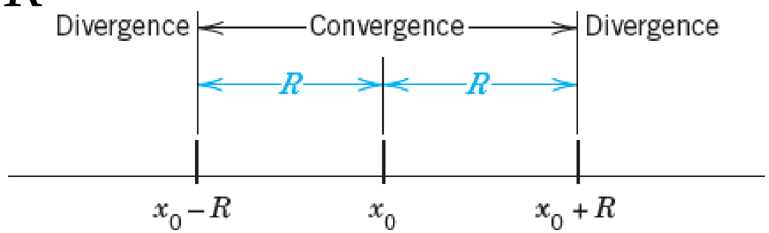
Case 1. (useless) The series always converges at the center at $x = x_0$, $s(x_0) = a_0$.

Case 2. (usual) If there are further values of x for which the series converges, these values form an interval, called the **convergence interval.**

$$\lim_{m\to\infty}\left|\frac{a_{m+1}(x-x_0)^{m+1}}{a_m(x-x_0)^m}\right| = |x-x_0|\lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right| = L \begin{cases} L<1 & \text{: converges} \\ L>1 & \text{: diverges} \\ L=1 & \text{: inconclusive} \end{cases}$$

Radius of convergence of the series (R): $|x - x_0| < R$

$$|x-x_0|\lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right| < 1 \quad \therefore |x-x_0| < \frac{1}{\lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right|} \rightarrow R$$



5.1 Power Series Method

❖ Convergence Interval. Radius of Convergence

Case 3. (best) The convergence interval may sometimes be infinite, that is, the series converges for all x.

$$|x - x_0| \lim_{m\to\infty} \left|\frac{a_{m+1}}{a_m}\right| < 1 \quad \Rightarrow \quad \therefore |x - x_0| < \frac{1}{0} = \infty$$

$$\lim_{m\to\infty} \left|\frac{a_{m+1}}{a_m}\right| \rightarrow 0$$

$$\therefore R = \infty$$

5.1 Power Series Method

For example,

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \cdots \qquad \left| \frac{a_{m+1}}{a_m} \right| = \frac{1/(m+1)!}{1/m!} = \frac{1}{m+1} \to 0,\ R = \infty$$

$$\lim_{m\to\infty} \left| \frac{a_{m+1}x^{m+1}}{a_m x^m} \right| = |x| \lim_{m\to\infty} \frac{1/(m+1)!}{1/m!} = |x| \lim_{m\to\infty} \frac{1}{m+1} = 0,\ \text{for all x, the series converges} \to R = \infty$$

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \cdots \qquad \left| \frac{a_{m+1}}{a_m} \right| = \frac{1}{1} = 1 \ ,\ R = 1$$

$$\lim_{m\to\infty} \left| \frac{a_{m+1}x^{m+1}}{a_m x^m} \right| = |x| \lim_{m\to\infty} \frac{1}{1} = |x| \to \text{only for } |x| < 1, \text{ the series converges} \to R = 1$$

$$\sum_{m=0}^{\infty} m!x^m = 1 + x + 2x^2 + \cdots \qquad \left| \frac{a_{m+1}}{a_m} \right| = \frac{(m+1)!}{m!} = m + 1 \to \infty \ ,\ R = 0$$

$$\lim_{m\to\infty} \left| \frac{a_{m+1}x^{m+1}}{a_m x^m} \right| = |x| \lim_{m\to\infty} \frac{(m+1)!}{m!} = |x| \lim_{m\to\infty} (m+1) = \infty,\ \text{for any x, the series diverge} \to R = 0$$

5.1 Power Series Method

☑ **Example**

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{k^m} x^{2m}$$

$$\sum_{m=0}^{\infty} \left(\frac{2}{3}\right)^m x^{2m}$$

5.1 Power Series Method

❖ **Theorem 1 Existence of Power Series Solutions**

If p, q, and r in $y''(x)+p(x)y'+q(x)y = r(x)$ are analytic at $x = x_0$, (power series representations)

then every solution is analytic and can thus be represented by a power series in powers of

$x- x_0$ with radius of convergence $R > 0$.

* 함수 f가 한 점 x_0에서 해석적이라는 것은 그 점 근방에서의 테일러 급수가 수렴하는 것과 같은 의미

5.2 Legendre's Equation. Legendre Polynomials $P_n(x)$

❖ **Legendre's equation:** $(1-x^2)y''-2xy'+n(n+1)y=0$

Substituting $y=\sum_{m=0}^{\infty} a_m x^m$ and its derivatives

$$\longrightarrow (1-x^2)\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - 2x\sum_{m=1}^{\infty} m a_m x^{m-1} + n(n+1)\sum_{m=0}^{\infty} a_m x^m = 0$$

The first expression as two separate series.

$$\longrightarrow \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1)a_m x^m - \sum_{m=1}^{\infty} 2m a_m x^m + \sum_{m=0}^{\infty} n(n+1)a_m x^m = 0$$

Set $m-2=s$ in the first series

and simply write s instead of m in the other three series.

$$\longrightarrow \sum_{s=0}^{\infty} (s+2)(s+1)a_{s+2} x^s - \sum_{s=2}^{\infty} s(s-1)a_s x^s - \sum_{s=1}^{\infty} 2s a_s x^s + \sum_{s=0}^{\infty} n(n+1)a_s x^s = 0$$

- The sum of the coefficients of each power of *x* on the left must be zero.

$$\sum_{s=0}^{\infty}(s+2)(s+1)a_{s+2}x^s - \sum_{s=2}^{\infty}s(s-1)a_s x^s - \sum_{s=1}^{\infty}2sa_s x^s + \sum_{s=0}^{\infty}n(n+1)a_s x^s = 0$$

$$x^0:\ \ 2\cdot 1\cdot a_2 x^0 + n(n+1)a_0 x^0 = 0 \Rightarrow a_2 = -\frac{n(n+1)}{2!}a_0$$

$$x^1:\ \ 3\cdot 2\cdot a_3 + [-2+n(n+1)]a_1 = 0 \Rightarrow a_3 = -\frac{(n-1)(n+2)}{3!}a_1$$

$$x^s:\ \ (s+2)(s+1)a_{s+2} + [-s(s-1)-2s+n(n+1)]a_s = 0 \Rightarrow a_{s+2} = -\frac{(n-s)(n+s+1)}{(s+2)(s+1)}a_s$$

- Recurrence relation (점화관계):

$$a_{s+2} = -\frac{(n-s)(n+s+1)}{(s+2)(s+1)}a_s \qquad (s = 0,1,\cdots)$$

5.2 Legendre's Equation. Legendre Polynomials $P_n(x)$

- We express successively except for a_0 and a_1.

$$a_{s+2} = -\frac{(n-s)(n+s+1)}{(s+2)(s+1)}a_s \qquad (s = 0,1,\cdots)$$

$$a_2 = -\frac{n(n+1)}{2!}a_0$$

$$a_4 = -\frac{(n-2)(n+3)}{4\cdot 3}a_2 = \frac{(n-2)n(n+1)(n+3)}{4!}a_0$$

$$a_3 = -\frac{(n-1)(n+2)}{3!}a_1$$

$$a_5 = -\frac{(n-3)(n+4)}{5\cdot 4}a_3 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!}a_1$$

- **General solution:**

$$y = \sum_{m=0}^{\infty} a_m x^m$$

$$y(x) = a_0 y_1(x) + a_1 y_2(x)$$

- converge for $|x|<1$
- y_1 and y_2 are independent
- $x=\pm 1 \rightarrow$ no longer analytic

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$$y_1(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - + \cdots$$

$$y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - + \cdots$$

5.2 Legendre's Equation. Legendre Polynomials

$$\left(1-x^2\right)y''-2xy'+n\left(n+1\right)y=0$$

❖ **Legendre Polynomials**

The reduction of power series ($y=\sum_{m=0}^{\infty}a_m x^m$, m→ ∞)

to polynomials (m is finite) is a great advantage

because then we have solutions for all *x*, without convergence restrictions.

$$a_{s+2}=-\frac{\left(n-s\right)\left(n+s+1\right)}{\left(s+2\right)\left(s+1\right)}a_s \qquad \left(s=0,1,\cdots\right)$$

$$y_1\left(x\right)=1-\frac{n\left(n+1\right)}{2!}x^2+\frac{\left(n-2\right)n\left(n+1\right)\left(n+3\right)}{4!}x^4-+\cdots$$

$$y_2\left(x\right)=x-\frac{\left(n-1\right)\left(n+2\right)}{3!}x^3+\frac{\left(n-3\right)\left(n-1\right)\left(n+2\right)\left(n+4\right)}{5!}x^5-+\cdots$$

If $s=n$

$$a_{n+2}=0,\ a_{n+4}=0,\ a_{n+6}=0,\ldots$$

If n is even

$$y_1=1+a_2x^2+a_4x^4+\cdots+a_{n-2}x^{n-2}+a_nx^n+a_{n+2}x^{n+2}+\cdots$$

→ Polynomials (finite terms)

"Legendre polynomials"

If n is odd

$$y_2=a_1x^1+a_3x^3+\cdots+a_{n-1}x^{n-1}+a_{n+1}x^{n+1}+a_{n+3}x^{n+3}+\cdots$$

→ Polynomials (finite terms)

5.2 Legendre's Equation. Legendre Polynomials

$$\left(1-x^2\right)y''-2xy'+n\left(n+1\right)y=0$$

❖ Legendre Polynomials

The reduction of power series ($y=\sum_{m=0}^{\infty}a_m x^m$, $m\to\infty$)

to polynomials (m is finite) is a great advantage

because then we have solutions for all *x*, without convergence restrictions.

$$a_{s+2}=-\frac{(n-s)(n+s+1)}{(s+2)(s+1)}a_s \qquad (s=0,1,\cdots)$$

$$y_1(x)=1-\frac{n(n+1)}{2!}x^2+\frac{(n-2)n(n+1)(n+3)}{4!}x^4-+\cdots$$

$$y_2(x)=x-\frac{(n-1)(n+2)}{3!}x^3+\frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5-+\cdots$$

If $s=n$

$$a_{n+2}=0,\ a_{n+4}=0,\ a_{n+6}=0,\ldots$$

If n is even

$$y_1=1+a_2x^2+a_4x^4+\cdots+a_{n-2}x^{n-2}+a_nx^n+a_{n+2}x^{n+2}+\cdots$$

→ Polynomials (finite terms)

"Legendre polynomials"

If n is odd

$$y_2=a_1x^1+a_3x^3+\cdots+a_{n-1}x^{n-1}+a_{n+1}x^{n+1}+a_{n+3}x^{n+3}+\cdots$$

→ Polynomials (finite terms)

5.2 Legendre's Equation. Legendre Polynomials

❖ Legendre Polynomials

We choose the coefficient a_n of the highest power x^n as

$$a_n = \frac{(2n)!}{2^n (n!)^2} = \begin{cases} 1 & \text{for } n = 0 \\ \dfrac{1\cdot 3\cdot 5\cdots(2n-1)}{n!} & \text{for } n \neq 0,\ n \text{ is a positive integer} \end{cases}$$

We solved for a_s in terms of a_{s+2}

$$a_s = -\frac{(s+2)(s+1)}{(n-s)(n+s+1)} a_{s+2} \qquad (s \le n-2)$$

$$a_{s+2} = -\frac{(n-s)(n+s+1)}{(s+2)(s+1)} a_s \qquad (s = 0, 1, \cdots)$$

The choice of a_n makes $p_n(1) = 1$ for every n

$s = n-2$

$$a_{n-2} = -\frac{n(n-1)}{2(2n-1)} a_n = -\frac{n(n-1)}{2(2n-1)} \frac{(2n)!}{2^n (n!)^2} = \frac{n(n-1)}{2(2n-1)} \frac{(2n)(2n-1)(2n-2)!}{2^n (n)(n-1)!\, n(n-1)(n-2)!}$$

$$a_{n-2} = -\frac{(2n-2)!}{2^n (n-1)!(n-2)!}$$

5.2 Legendre's Equation. Legendre Polynomials

❖ Legendre Polynomials

$$a_{n-2} = \frac{(2n-2)!}{2^n (n-1)!(n-2)!}$$

$s = n-4$

$$a_{n-4} = -\frac{(n-2)(n-3)}{4(2n-3)} a_{n-2} = -\frac{(2n-4)!}{2^n 2!(n-2)!(n-4)!}$$

$$a_s = -\frac{(s+2)(s+1)}{(n-s)(n+s+1)} a_{s+2} \qquad (s \le n-2)$$

When n-2$m \geq 0$

$$n-2m \geq 0, \quad a_{n-2m} = (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!}$$

$$P_n(x) = \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m} \qquad \left(M = \frac{n}{2} \text{ or } \frac{n-1}{2} \right) \quad \text{whichever is an integer}$$

$$= \frac{(2n)!}{2^n (n!)^2} x^n - \frac{(2n-2)!}{2^n 1!(n-1)!(n-2)!} x^{n-2} + \cdots$$

5.2 Legendre's Equation. Legendre Polynomials

❖ **Examples of the Legendre Polynomials**

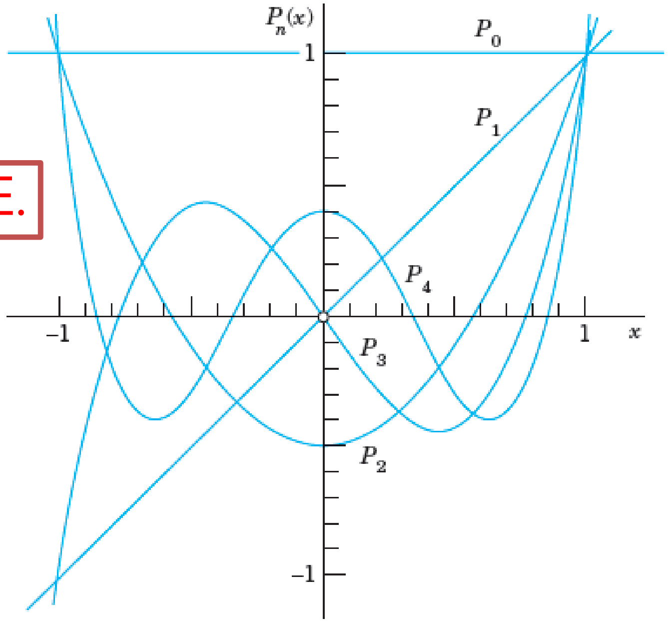
$$\left(1-x^2\right)y''-2xy'+n\left(n+1\right)y=0$$

$$P_n\left(x\right)=\sum_{m=0}^{M}\left(-1\right)^m\frac{\left(2n-2m\right)!}{2^n m!\left(n-m\right)!\left(n-2m\right)!}x^{n-2m}\qquad\left(M=\frac{n}{2}\text{ or }\frac{n-1}{2}\right)$$

$$=\frac{(2n)!}{2^n(n!)^2}x^n-\frac{(2n-2)!}{2^n 1!(n-1)!(n-2)!}x^{n-2}+\cdots$$

$$P_0\left(x\right)=\frac{0!}{2^0(0!)^2}x^0=1\qquad P_1\left(x\right)=\frac{2!}{2^1(1!)^2}x^1=x$$

Q: Derive $P_2(x)$ and prove it satisfies the ODE.



5.3 Extended Power Series Method: Frobenius Method

❖ Indicial Equation (결정방정식), Indicating the Form of Solutions

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0 \quad \xrightarrow{\text{Multiply } x^2} \quad x^2 y'' + x b(x) y' + c(x) y = 0$$

- We expand $b(x)$ and $c(x)$ in power series $\left(b(x) = b_0 + b_1 x + b_2 x^2 + \cdots, \quad c(x) = c_0 + c_1 x + c_2 x^2 + \cdots\right)$
- We differentiate $y(x) = x^r \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{m+r} = x^r \left(a_0 + a_1 x + a_2 x^2 + \cdots\right)$ term by term

$$y'(x) = \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1} = x^{r-1}[r a_0 + (r+1) a_1 x + \ldots]$$

$$y''(x) = \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-2} = x^{r-2}[r(r-1) a_0 + (r+1) r a_1 x + \ldots]$$

$m=0$ doesn't change!

$$\longrightarrow \quad x^r \left[r(r-1) a_0 + \cdots\right] + \left(b_0 + b_1 x + \cdots\right) x^r \left[r a_0 + \cdots\right] + \left(c_0 + c_1 x + \cdots\right) x^r \left(a_0 + a_1 x + \cdots\right) = 0$$

- The equation corresponding to the power x^r is $\left[r(r-1) + b_0 r + c_0\right] a_0 = 0$.
- By assumption $a_0 \neq 0$, Indicial Equation: $r(r-1) + b_0 r + c_0 = 0$

5.3 Extended Power Series Method: Frobenius Method

❖ **Theorem 2 Frobenius Method. Basis of Solutions. Three Cases**

Suppose that the ODE satisfies the assumptions in Theorem 1. Let r_1 and r_2 be the roots of the indicial equation. Then we have the following three cases.

(두 근의 차가 정수가 아닌 서로 다른 근들)

Case 1. Distinct Roots Not Differing by an Integer. A basis is

$$y_1(x) = x^{r_1}\left(a_0 + a_1 x + a_2 x^2 + \cdots\right) \text{ and } y_2(x) = x^{r_2}\left(A_0 + A_1 x + A_2 x^2 + \cdots\right)$$

(이중근)

Case 2. Double Root ($r_1 = r_2 = r$). A basis is

$$y_1(x) = x^{r}\left(a_0 + a_1 x + a_2 x^2 + \cdots\right) \text{ and } y_2(x) = y_1(x)\ln x + x^{r}\left(A_1 x + A_2 x^2 + \cdots\right)$$

(두 근의 차가 정수인 서로 다른 근들)

Case 3. Roots Differing by an Integer. A basis is

$$y_1(x) = x^{r_1}\left(a_0 + a_1 x + a_2 x^2 + \cdots\right) \text{ and } y_2(x) = k y_1(x)\ln x + x^{r_2}\left(A_0 + A_1 x + A_2 x^2 + \cdots\right)$$

where the roots are so denoted that $r_1 - r_2 > 0$.

5.3 Extended Power Series Method: Frobenius Method

☑ Ex.2 Illustration of Case 2 (Double Root)

Solve the ODE $x(x-1)y''+(3x-1)y'+y=0$.

Inserting $y(x)=x^r\sum_{m=0}^{\infty}a_m x^m = x^r\left(a_0+a_1x+a_2x^2+\cdots\right)$ and its derivatives into the ODE.

$$\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r} - \sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-1}$$

$$+3\sum_{m=0}^{\infty}(m+r)a_m x^{m+r} - \sum_{m=0}^{\infty}(m+r)a_m x^{m+r-1} + \sum_{m=0}^{\infty}a_m x^{m+r} = 0$$

Coefficients of the smallest power x^{r-1} : $\left[-r(r-1)-r\right]a_0=0 \quad \Rightarrow \quad \therefore \; r=0$ (double root)

If we take x^r, $a_0=0$. (not allowed)

- **First Solution**. We insert $r=0$.

$$\Rightarrow \sum_{m=0}^{\infty}m(m-1)a_m x^m - \sum_{m=0}^{\infty}m(m-1)a_m x^{m-1} + 3\sum_{m=0}^{\infty}m a_m x^m - \sum_{m=0}^{\infty}m a_m x^{m-1} + \sum_{m=0}^{\infty}a_m x^m = 0$$

5.3 Extended Power Series Method: Frobenius Method

☑ Ex.2 Illustration of Case 2 (Double Root)

Solve the ODE $x(x-1)y''+(3x-1)y'+y=0.$

$$\sum_{m=0}^{\infty} m(m-1)a_m x^m - \sum_{m=0}^{\infty} m(m-1)a_m x^{m-1} + 3\sum_{m=0}^{\infty} m a_m x^m - \sum_{m=0}^{\infty} m a_m x^{m-1} + \sum_{m=0}^{\infty} a_m x^m = 0$$

Coefficients of the power of x^s (Let $m = s$)

$$s(s-1)a_s - (s+1)s a_{s+1} + 3s a_s - (s+1)a_{s+1} + a_s = 0$$

$$\Rightarrow \quad (s^2-2s+1)a_s - (s^2-2s+1)a_{s+1} = 0 \Rightarrow (s-1)^2(a_s - a_{s+1}) = 0$$

$$a_{s+1} = a_s$$

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m = (1 + x + x^2 + \cdots)$$

Choose $a_0=1$.

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \cdots \quad (|x|<1, \text{ geometric series})$$

$$\therefore y_1(x) = \sum_{m=0}^{\infty} x^m = \frac{1}{1-x} \quad (|x|<1)$$

5.3 Extended Power Series Method: Frobenius Method

$$y_1(x) = \frac{1}{1-x} \qquad (|x|<1)$$

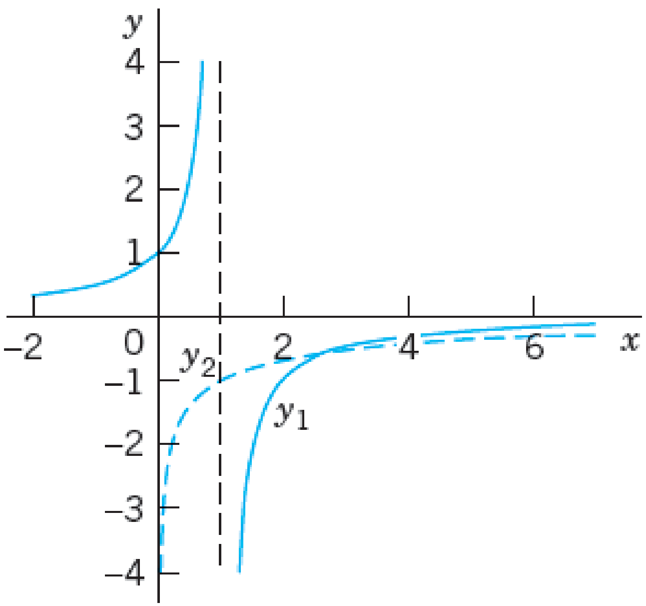
- **Second Solution.** Apply the method of reduction of order $y_2 = uy_1$

$$x(x-1)y'' + (3x-1)y' + y = 0$$

$$\rightarrow \quad -\int p\,dx = -\int \frac{3x-1}{x(x-1)}dx = -\int\left(\frac{2}{x-1}+\frac{1}{x}\right)dx = -2\ln(x-1) - \ln x = \ln\frac{1}{x(x-1)^2}$$

$$u' = y_1^{-2}e^{-\int p\,dx} = \frac{(x-1)^2}{(x-1)^2 x} = \frac{1}{x}, \qquad \boxed{u = \ln x,} \qquad \therefore y_2 = uy_1 = \frac{\ln x}{x-1}$$

These functions are linearly independent and thus form a basis on the interval $0 < x < 1$ (as well as on $1 < x < \infty$).



Reduction of order (차수축소법)

$$y'' + p(x)y' + q(x)y = 0$$

$$U = u' = \frac{1}{y_1^2}e^{-\int p\,dx}, \quad y_2 = uy_1 = y_1\int U\,dx$$

❖ Find a Basis **if One Solution Is Known**. Reduction of Order (차수축소법)

Apply **reduction of order** to the homogeneous linear ODE $y''+p(x)y'+q(x)y=0$.

$y=y_2=uy_1$ (Substitute) $\quad (y'=y_2'=u'y_1+uy_1',\ y''=y_2''=u''y_1+2u'y_1'+uy_1'')$

$$\Rightarrow \quad u''y_1+u'(2y_1'+py_1)+u(y_1''+py_1'+qy_1)=0 \quad \Rightarrow \quad u''+u'\frac{2y_1'+py_1}{y_1}=0 \quad (\because\ y_1''+py_1'+qy_1=0)$$

$$U=u',\quad U'=u'' \quad \Rightarrow \quad U'+\left(2\frac{y_1'}{y_1}+p\right)U=0$$ (Separation of variables and integration)

$$\Rightarrow \quad \frac{dU}{U}=-\left(2\frac{y_1'}{y_1}+p\right)dx \ \ \& \ \ \ln|U|=-2\ln|y_1|-\int p\,dx \quad \Rightarrow \quad \therefore\ U=\frac{1}{y_1^2}e^{-\int p\,dx},\quad y_2=uy_1=y_1\int U\,dx$$

☑ Ex. 7 Find a basis of solution of the ODE $(x^2-x)y''-xy'+y=0$

One solution: $y_1=x$

Apply reduction of order: $p=-\dfrac{x}{x^2-x}=-\dfrac{1}{x-1}$ $\quad \Rightarrow \quad U=\dfrac{1}{y_1^2}e^{-\int p\,dx}=\dfrac{1}{x^2}e^{\int \frac{1}{x-1}dx}=\dfrac{1}{x}-\dfrac{1}{x^2}$

$$\Rightarrow \quad y_2=y_1\int U\,dx=x\left(\ln|x|+\frac{1}{x}\right)=x\ln|x|+1$$

5.4 Bessel's Equation. Bessel Functions $J_\nu(x)$

❖ **Bessel's equation:** $x^2 y'' + xy' + \left(x^2 - \nu^2\right) y = 0, \text{ where } \nu \geq 0$

Apply the Frobenius method

→ Substitute the series $y = \sum_{m=0}^{\infty} a_m x^{m+r}$ with undetermined coefficients and its derivatives.

→ $$\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r} + \sum_{m=0}^{\infty}(m+r)a_m x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r+2} - \nu^2 \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

→ $$r(r-1)a_0 + ra_0 - \nu^2 a_0 = 0 \qquad (s=0)$$

$$(r+1)ra_1 + (r+1)a_1 - \nu^2 a_1 = 0 \qquad (s=1)$$

$$(s+r)(s+r-1)a_s + (s+r)a_s + a_{s-2} - \nu^2 a_s = 0 \qquad (s = 2,\ 3,\ \cdots)$$

Indicial equation: $(r+\nu)(r-\nu) = 0$

$$\therefore r_1 = \nu, r_2 = -\nu$$

5.4 Bessel's Equation. Bessel Functions $J_\nu(x)$

$$r(r-1)a_0 + ra_0 - \nu^2 a_0 = 0 \qquad (s=0)$$
$$(r+1)ra_1 + (r+1)a_1 - \nu^2 a_1 = 0 \qquad (s=1)$$
$$(s+r)(s+r-1)a_s + (s+r)a_s + a_{s-2} - \nu^2 a_s = 0 \qquad (s=2,\ 3,\ \cdots)$$

❖ Coefficient Recursion (계수 점화) for $r = r_1 = \nu$

$$(2\nu+1)a_1 = 0 \quad \Rightarrow \quad a_1 = 0\ (\because \nu \ge 0)$$
$$(s+2\nu)sa_s + a_{s-2} = 0 \quad \Rightarrow \quad a_3 = a_5 = \cdots = 0$$

For $s = 2m$, $\quad (2m+2\nu)2ma_{2m} + a_{2m-2} = 0 \quad \Rightarrow \quad a_{2m} = -\dfrac{1}{2^2 m(m+\nu)} a_{2m-2}, \quad m = 1,\ 2,\ \cdots$

$$a_2 = -\frac{1}{2^2(\nu+1)} a_0$$

$$a_4 = -\frac{1}{2^2 2(\nu+2)} a_2 = \frac{1}{2^4 2!(\nu+1)(\nu+2)} a_0$$

$$\therefore \quad a_{2m} = \frac{(-1)^m}{2^{2m} m!(\nu+1)(\nu+2)\cdots(\nu+m)} a_0, \qquad m = 1,\ 2,\ \cdots$$

5.4 Bessel's Equation. Bessel Functions $J_\nu(x)$

❖ Bessel Functions $J_\nu(x)$ for Integer $\nu = n$

$$x^2 y'' + xy' + \left(x^2 - \nu^2\right) y = 0$$

$$y'' + \frac{y'}{x} + \left(1 - \frac{n^2}{x^2}\right) y = 0$$

$$a_{2m} = \frac{(-1)^m}{2^{2m} m! (\nu+1)(\nu+2)\cdots(\nu+m)} a_0,$$

$$y = \sum_{m=0}^{\infty} a_m x^{m+r}$$

Choose $a_0 = \dfrac{1}{2^n n!}$ → $a_{2m} = \dfrac{(-1)^m}{2^{2m+n} m! (n+m)!}, \quad m = 1, 2, \cdots$

Bessel function of the first kind of order *n*:

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m! (n+m)!}$$

$$(r = n, a_1 = a_3 = a_5 = \ldots = 0)$$

(*n*차 제 1종 Bessel 함수)

5.4 Bessel's Equation. Bessel Functions $J_\nu(x)$

$$y'' + \frac{y'}{x} + \left(1 - \frac{n^2}{x^2}\right)y = 0$$

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!}$$

☑ Ex.1 Bessel Functions $J_0(x)$ and $J_1(x)$

Bessel function of order 0 (n = 0)

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + - \cdots$$

Bessel function of order 1 (n = 1)

$$J_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m+1} m!(m+1)!} = \frac{x}{2} - \frac{x^3}{2^3 1! 2!} + \frac{x^5}{2^5 2! 3!} - \frac{x^7}{2^7 3! 4!} + - \cdots$$

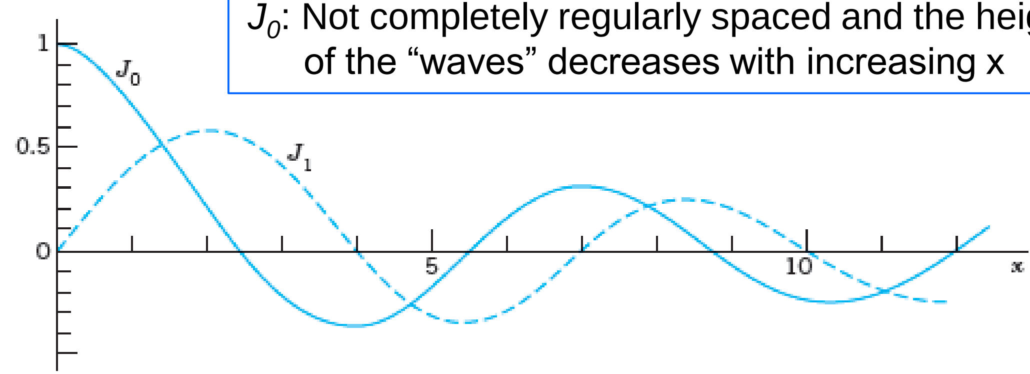
For large x

$$y'' + \frac{y'}{x} + \left(1 - \frac{n}{x^2}\right)y = 0 \quad \longrightarrow \quad y'' + y = 0$$

damping term

$$J_n(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$

J_0: Not completely regularly spaced and the height of the "waves" decreases with increasing x



5.4 Bessel's Equation. Bessel Functions $J_v(x)$

$$\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$$

❖ Bessel Functions $J_v(x)$ for any $v \ge 0$. Gamma Function.

Gamma function: $\Gamma(v) = \int_0^\infty e^{-t} t^{v-1} dt \quad (v > 0)$

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

Integration by part: $\Gamma(v+1) = \int_0^\infty t^v e^{-t} dt = -t^v e^{-t}\Big|_0^\infty + v\int_0^\infty t^{v-1} e^{-t} dt = 0 + v\Gamma(v)$

$\rightarrow$ $\Gamma(v+1) = v\Gamma(v)$ (In general, $\Gamma(n+1) = n!,\ n = 0, 1, \cdots$)

Start with $v = 1$ $\quad \Gamma(1) = \int_0^\infty e^{-t} dt = 1$

$$\Gamma(2) = 1\Gamma(1) = 1!$$
$$\Gamma(3) = 2\Gamma(2) = 2!$$
$$\Gamma(n+1) = n!$$

The gamma function generalizes the factorial function to arbitrary positive ***v***, ***v* = *n***.

$$a_0 = \frac{1}{2^n n!} \quad \rightarrow \quad a_0 = \frac{1}{2^v \Gamma(v+1)}$$

$$a_{2m} = \frac{(-1)^m}{2^{2m+n} m!(n+m)!}, \quad m = 1, 2, \cdots \quad \rightarrow \quad a_{2m} = \frac{(-1)^m}{2^{2m+v} m!\Gamma(v+m+1)}, \quad m = 1, 2, \cdots$$

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!} \quad \rightarrow \quad J_v(x) = x^v \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+v} m!\Gamma(v+m+1)}$$

(n차 제 1종 Bessel 함수) $\qquad$ (v차 제 1종 Bessel 함수)

5.4 Bessel's Equation. Bessel Functions $J_\nu(x)$

Discovery of Properties From Series

Theorem 3 Derivatives, Recursions

The derivative of $J_\nu(x)$ with respect to x can be expressed by $J_{\nu-1}(x)$ and $J_{\nu+1}(x)$ by the formulas

$$\left[x^\nu J_\nu(x)\right]' = x^\nu J_{\nu-1}(x)$$

$$\left[x^{-\nu} J_\nu(x)\right]' = -x^{-\nu} J_{\nu+1}(x)$$

Furthermore, $J_\nu(x)$ and its derivative satisfy the recurrence relations.

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x)$$

$$J_{\nu-1}(x) - J_{\nu+1}(x) = 2J_\nu'(x)$$

5.4 Bessel's Equation. Bessel Functions $J_\nu(x)$

❖ Proof

(a) We multiply by x^ν and take $x^{2\nu}$ under the summation sign.

$$J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu} m!\Gamma(\nu+m+1)} \longrightarrow x^\nu J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+2\nu}}{2^{2m+\nu} m!\Gamma(\nu+m+1)}$$

Pull $x^{2\nu-1}$ and use the functional relationship $\Gamma(\nu+m+1)=(\nu+m)\,\Gamma(\nu+m)$.

$$(x^\nu J_\nu)' = \sum_{m=0}^{\infty} \frac{(-1)^m 2(m+\nu) x^{2m+2\nu-1}}{2^{2m+\nu} m!\Gamma(\nu+m+1)} = x^\nu x^{\nu-1} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu-1} m!\Gamma(\nu+m)}$$

$$= J_{\nu-1}(x)$$

$$\therefore (x^\nu J_\nu)' = x^\nu J_{\nu-1}(x)$$

5.4 Bessel's Equation. Bessel Functions $J_\nu(x)$

(b) We multiply by $x^{-\nu}$ and use $m! = m\,(m\text{-}1)!$.

$$J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu} m!\,\Gamma(\nu+m+1)} \quad \longrightarrow \quad x^{-\nu} J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu} m!\,\Gamma(\nu+m+1)}$$

$$(x^{-\nu} J_\nu)' = \sum_{m=1}^{\infty} \frac{2m(-1)^m x^{2m-1}}{2^{2m+\nu} m(m-1)!\,\Gamma(\nu+m+1)} = \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m+\nu-1} (m-1)!\,\Gamma(\nu+m+1)}$$

$$= \sum_{s=0}^{\infty} \frac{(-1)^{s+1} x^{2s+1}}{2^{2s+\nu+1} s!\,\Gamma(\nu+s+2)} \Leftarrow s = m-1,$$

Q?

$$= -x^{-\nu} J_{\nu+1}(x)$$

$$\therefore (x^{-\nu} J_\nu)' = -x^{-\nu} J_{\nu+1}(x)$$

5.4 Bessel's Equation. Bessel Functions $J_v(x)$

(c) We perform the differentiation.

$$(x^v J_v)' = x^v J_{v-1}(x) \quad \longrightarrow \quad vx^{v-1} J_v + x^v J_v' = x^v J_{v-1}(x)$$

Divided by x^v

$$vx^{-1} J_v + J_v' = J_{v-1}(x) \qquad \textbf{(1)}$$

$$(x^{-v} J_v)' = -x^{-v} J_{v+1}(x) \quad \longrightarrow \quad -vx^{v-1} J_v + x^v J_v' = -x^v J_{v+1}(x)$$

Divided by x^v

$$-vx^{-1} J_v + J_v' = -J_{v+1}(x) \qquad \textbf{(2)}$$

(1)-(2):

$$J_{v-1}(x) + J_{v+1}(x) = \frac{2v}{x} J_v(s)$$

(1)+(2):

$$J_{v-1}(x) - J_{v+1}(x) = 2J_v'(s)$$

5.4 Bessel's Equation. Bessel Functions $J_\nu(x)$

❖ **Theorem 4 Elementary $J_\nu(x)$ for Half-Integer Order ν**

Bessel functions J_ν of orders $\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \cdots$ are elementary (초등함수); they can be expressed by finitely many cosines and sines and powers of x.

In particular,

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x, \qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

* Elementary functions (초등함수): 다항 함수, 로그 함수, 지수 함수, 삼각 함수와 이들 함수의 합성 함수들을 총칭

5.5 Bessel Functions $Y_0(x)$. General Solutions

$$x^2y''+xy'+\left(x^2-\nu^2\right)y=0$$

$$y''+\frac{y'}{x}+\left(1-\frac{n^2}{x^2}\right)y=0$$

- **Bessel Functions of the Second Kind $Y_0(x)$ with $\nu = n = 0$:**

Bessel's equation: $xy''+y'+xy=0$

Indicial equation: $(r+\nu)(r-\nu)=0 \quad \Rightarrow r^2=0 \Rightarrow \text{double root } r=0$

$$x^2y''+xy'+\left(x^2-\nu^2\right)y=0$$
$$(r+\nu)(r-\nu)=0$$

First solution: $y_1(x) = J_0(x)$

Second solution: $y_2(x)=J_0(x)\ln x+\sum_{m=1}^{\infty}A_mx^m$

$$J_n(x)=x^n\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m+n}m!(n+m)!}$$

Substitute y_2 and its derivatives into ODE.

$$y_2'=J_0'\ln x+\frac{J_0}{x}+\sum_{m=1}^{\infty}mA_mx^{m-1}$$

$$y_2''=J_0''\ln x+\frac{2J_0'}{x}-\frac{J_0}{x^2}+\sum_{m=1}^{\infty}m(m-1)A_mx^{m-2}$$

$$xy''+y'+xy=0$$

$$\Rightarrow \begin{cases} y_2'=\dfrac{J_0}{x}+\displaystyle\sum_{m=1}^{\infty}mA_mx^{m-1} \\ xy_2''=2J_0'-\dfrac{J_0}{x}+\displaystyle\sum_{m=1}^{\infty}m(m-1)A_mx^{m-1} \\ xy_2=\displaystyle\sum_{m=1}^{\infty}A_mx^{m+1} \end{cases}$$

$$xJ_0''\ln x+J_0'\ln x+xJ_0\ln x=0, \quad (xJ_0''+J_0'+xJ_0)\ln x=0, \quad \because J_0 \text{ is solution of ODE}$$

$$2J_0'+\sum_{m=1}^{\infty}m(m-1)A_mx^{m-1}+\sum_{m=1}^{\infty}mA_mx^{m-1}+\sum_{m=1}^{\infty}A_mx^{m+1}=0$$

5.5 Bessel Functions $Y_0(x)$. General Solutions

$$2J_0' + \sum_{m=1}^{\infty} m(m-1)A_m x^{m-1} + \sum_{m=1}^{\infty} mA_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0$$

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}$$

$$J_0'(x) = \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-1} m!(m-1)!} \longrightarrow \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-2} m!(m-1)!} + \sum_{m=1}^{\infty} m^2 A_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0$$

- Sum of the coefficients of even powers x^{2s} should be 0.

✓ x^0: it occurs in the second series, $A_1 \rightarrow A_1 = 0$.

✓ x^{2s}: sum of the coefficients

In the first series → None

In the second series, $m-1=2s$ → $(2s+1)^2 A_{2s+1}$

In the third series, $m+1=2s$ → A_{2s-1}

$$\therefore (2s+1)^2 A_{2s+1} + A_{2s-1} = 0 \quad s = 1,2,\cdots.$$

$$A_1 = 0, \Rightarrow A_3 = 0, A_5 = 0, \ldots, = 0$$

5.5 Bessel Functions $Y_0(x)$. General Solutions

$$J_0'(x) = \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-1} m!(m-1)!} \longrightarrow \boxed{\sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-2} m!(m-1)!}} + \boxed{\sum_{m=1}^{\infty} m^2 A_m x^{m-1}} + \boxed{\sum_{m=1}^{\infty} A_m x^{m+1}} = 0$$

- Sum of the coefficients of odd powers x^{2s+1} should be 0.

✓ x^1: it occurs in the first and second series. → $-1 + 4A_2 = 0, \quad A_2 = \frac{1}{4}$

✓ x^{2s+1}: sum of the coefficients

In the first series, $2m-1 = 2s+1$, $m = s+1$ → $\frac{(-1)^{s+1}}{2^{2s}(s+1)!s!}$

In the second series, $m-1 = 2s+1$ → $(2s+2)^2 A_{2s+2}$

In the third series, $m+1 = 2s+1$ → A_{2s}

$$\therefore \frac{(-1)^{s+1}}{2^{2s}(s+1)!s!} + (2s+2)^2 A_{2s+2} + A_{2s} = 0$$

✓ For $s = 1$

$$\frac{1}{8} + 16A_4 + A_2 = 0 \longrightarrow A_4 = -\frac{3}{128}$$

✓ In general

$$A_{2m} = \frac{(-1)^{m-1}}{2^{2m}(m!)^2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}\right), \quad m = 1, 2, \cdots,$$

5.5 Bessel Functions $Y_0(x)$. General Solutions

- ✓ In general $$A_{2m} = \frac{(-1)^{m-1}}{2^{2m}(m!)^2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}\right),\ m=1,2,\cdots,$$

$$h_1 = 1 \qquad h_m = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m},\quad m=2,3,\cdots,$$

$$y_2(x) = J_0(x)\ln x + \sum_{m=1}^{\infty} A_m x^m$$

$$y_2(x) = J_0(x)\ln x + \sum_{m=1}^{\infty}\frac{(-1)^{m-1}h_m}{2^{2m}(m!)^2}x^{2m}$$

$$= J_0(x)\ln x + \frac{1}{4}x^2 - \frac{3}{128}x^4 + \frac{11}{13{,}824}x^6 - + \cdots,$$

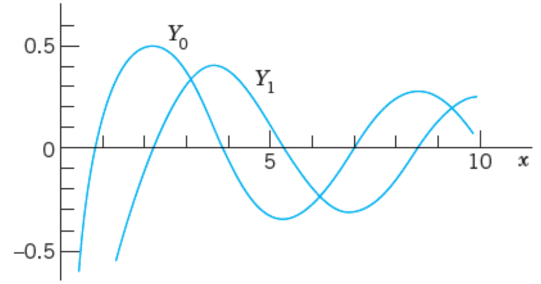


Fig. 112. Bessel functions of the second kind Y_0 and Y_1.

- ✓ J_0 and y_2 are linearly independent functions.
- ✓ Instead of y_2: $a(y_2 + bJ_0)$ (another basis called Y_0)
- ✓ It is customary to $a = 2/\pi$ and $b = \gamma - \ln 2$,

$\gamma = 0.57721566490$: Euler constant $= \lim_{s\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{s}-\ln s\right)$

❖ Bessel function of the second kind of order zero or Neumann's function of order zero

$$Y_0(x) = \frac{2}{\pi}\left[J_0\ln x + \sum_{m=1}^{\infty}A_m x^m + (\gamma - \ln 2)J_0\right] \Rightarrow Y_0(x) = \frac{2}{\pi}\left[J_0(x)\left(\ln\frac{x}{2}+\gamma\right) + \sum_{m=1}^{\infty}\frac{(-1)^{m-1}h_m}{2^{2m}(m!)^2}x^{2m}\right]$$

5.5 Bessel Functions $Y_\nu(x)$. General Solutions

❖ Bessel Functions of the Second Kind of Order n with all values of ν

$$Y_\nu(x) = \frac{1}{\sin \nu\pi}\left[J_\nu(x)\cos\nu\pi - J_{-\nu}(x)\right]$$ (ν차 제 2종 Bessel 함수)

$$Y_n(x) = \lim_{\nu\to n} Y_\nu(x)$$ (n차 제 2종 Bessel 함수)

$$J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu} m!\Gamma(\nu+m+1)}$$

$$J_{-\nu}(x) = x^{-\nu} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m-\nu} m!\Gamma(-\nu+m+1)}$$

- For noninteger order ν,
- the function $Y_\nu(x)$ is evidently a solution of Bessel's equation because $J_\nu(x)$ and $J_{-\nu}(x)$ are solutions of the equation.
- For those the solutions ν (noninteger), since $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent and Y_ν involves $J_{-\nu}(x)$,
- J_ν and Y_ν are linearly independent.

$$Y_n(x) = \lim_{\nu\to n} Y_\nu(x) = \frac{2}{\pi} J_n(x)\left(\ln\frac{x}{2} + \gamma\right) + \frac{x^n}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^{m-1}(h_m + h_{m+n})}{2^{2m+n} m!(m+n)!}x^{2m} - \frac{x^{-n}}{\pi}\sum_{m=0}^{n-1}\frac{(n-m-1)!}{2^{2m-n} m!}x^{2m}$$

$$h_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}, \quad h_{m+n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m+n}$$

5.5 Bessel Functions $Y_\nu(x)$. General Solutions

❖ **Theorem 1 General Solution of Bessel's Equation**

A general solution of Bessel's equation for all values of n (and $x > 0$) is

$$y(x) = C_1 J_\nu(x) + C_2 Y_\nu(x)$$

First kind   Second kind

$$J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu} m! \Gamma(\nu+m+1)}$$

$$J_{-\nu}(x) = x^{-\nu} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m-\nu} m! \Gamma(-\nu+m+1)}$$

$$Y_\nu(x) = \frac{1}{\sin \nu\pi}\left[J_\nu(x)\cos\nu\pi - J_{-\nu}(x)\right]$$

Summary

- In the previous chapters, linear ODEs with *constant coefficients (상수계수)* can be solved by algebraic methods (대수적인 방법), and that their solutions are elementary functions (초등함수) known from calculus (미적분).
- For ODEs with *variable coefficients (변수계수)* the situation is more complicated, and their solutions may be nonelementary functions.
- In this chapter, the three main topics are Legendre polynomials, Bessel functions, and hypergeometric functions (초기하함수).
- Legendre's ODE and Legendre polynomials are obtained by the power series method (거듭제곱급수 또는 멱급수 해법). $(1-x^2)y''-2xy'+n(n+1)y=0$
- Bessel's ODE and Bessel functions are obtained by the Frobenius method, an extension of the power series method. $x^2y''+xy'+(x^2-\nu^2)y=0$

Summary

- **Legendre's equation:** $(1-x^2)y''-2xy'+n(n+1)y=0$

  Substituting $y=\sum_{m=0}^{\infty} a_m x^m$ and its derivatives

  - **Legendre Polynomials**

$$P_n(x)=\sum_{m=0}^{M}(-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!}x^{n-2m} \qquad \left(M=\frac{n}{2} \text{ or } \frac{n-1}{2}\right)$$

- **Frobenius Method**

$$y''+\frac{b(x)}{x}y'+\frac{c(x)}{x^2}y=0$$

$$y(x)=x^r\sum_{m=0}^{\infty} a_m x^m = x^r\left(a_0+a_1x+a_2x^2+\cdots\right) \qquad (a_0\neq 0)$$

Summary

- **Bessel's equation:** $x^2 y'' + xy' + (x^2 - \nu^2) y = 0$

- **Bessel functions of the first kind $J_\nu(x)$ for Integer $\nu = n$:**

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!}$$

- **Bessel functions of the first kind $J_\nu(x)$ for any $\nu \geq 0$:**

$$J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu} m!\Gamma(\nu+m+1)}$$

- **Bessel functions of the second kind $Y_0(x)$ with $\nu = n = 0$:**

$$Y_0(x) = \frac{2}{\pi}\left[J_0(x)\left(\ln\frac{x}{2} + \gamma\right) + \sum_{m=1}^{\infty} \frac{(-1)^{m-1} h_m}{2^{2m}(m!)^2} x^{2m} \right]$$

- **Bessel functions of the second kind of order n with all values of ν:**

$$Y_\nu(x) = \frac{1}{\sin\nu\pi}\left[J_\nu(x)\cos\nu\pi - J_{-\nu}(x) \right]$$

$$Y_n(x) = \lim_{\nu\to n} Y_\nu(x) = \frac{2}{\pi} J_n(x)\left(\ln\frac{x}{2} + \gamma\right) + \frac{x^n}{\pi}\sum_{m=0}^{\infty} \frac{(-1)^{m-1}(h_m + h_{m+n})}{2^{2m+n} m!(m+n)!} x^{2m} - \frac{x^{-n}}{\pi}\sum_{m=0}^{n-1} \frac{(n-m-1)!}{2^{2m-n} m!} x^{2m}$$