# Beam Emittance

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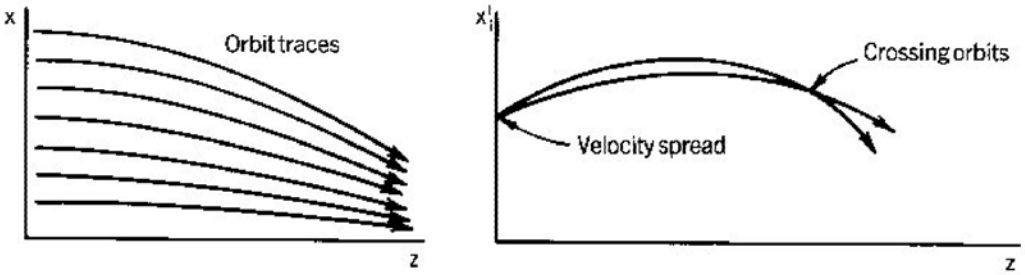
# Introduction

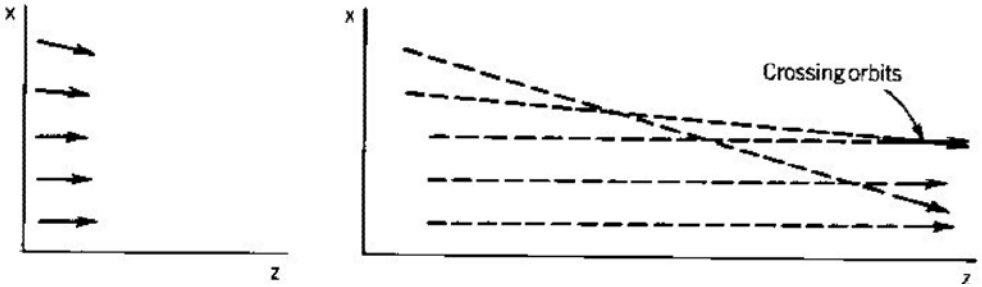
- Emittance is a measure of the parallelism of a beam — it allows us to compare the quality of beams for applications.
- Emittance is a conserved quantity in an ideal focusing systems. We can gauge imperfections in transport systems by measuring emittance growth.
- Beams with random components of transverse velocity have a spread in angle relative to the axis of propagation. The angular dispersion limits the ability to focus beams.
- We define emittance for beams where particle motions are independent in the $x$ and $y$ directions. We can find the quantity by plotting orbit vector points in a modified phase space called trace-space with axes of position and transverse angle, $[x, x']$. Emittance is proportional to the area filled by the points.
- We introduces some new quantities, including the brightness, a function of the emittance. Brightness quantifies the maximum focused power flux of a beam.

# Laminar and non-laminar beams

- Beams with good parallelism are easier to transport than beams with large random transverse velocity components. Ordered beams can focus to a small spot size.
- The ideal charged particle beam has laminar particle orbits. Orbits in a laminar beam flow in layers that never intersect. A laminar beam satisfies two conditions:
  - All particles at a position have identical transverse velocities. If this is not true, the orbits of two particles that start at the same position could separate and later cross each other.

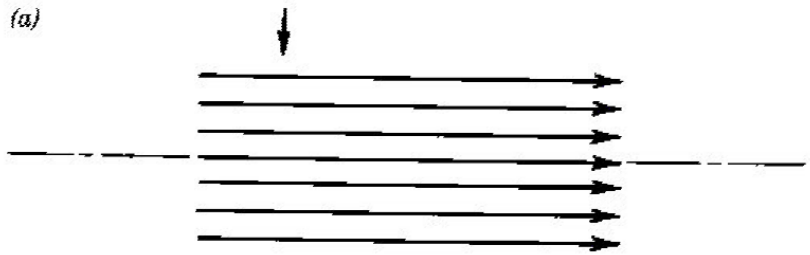



  - The magnitude of the transverse particle velocity is linearly proportional to the displacement from the axis of beam symmetry.

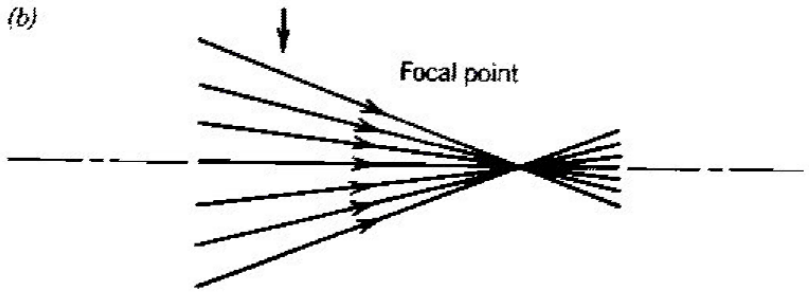


# Some examples of orbits in laminar beams

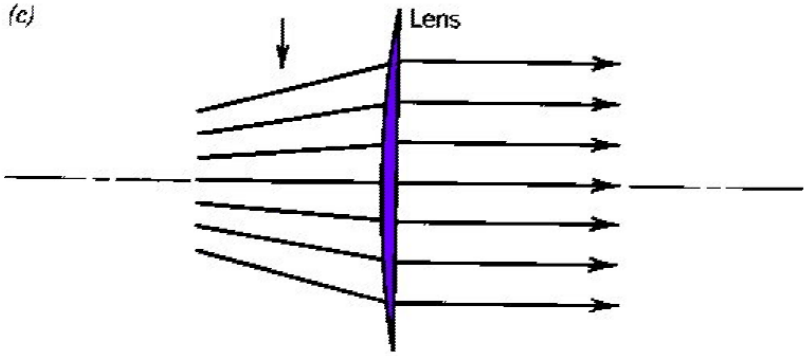
a. Ideal parallel beam (all particles have zero transverse velocity)



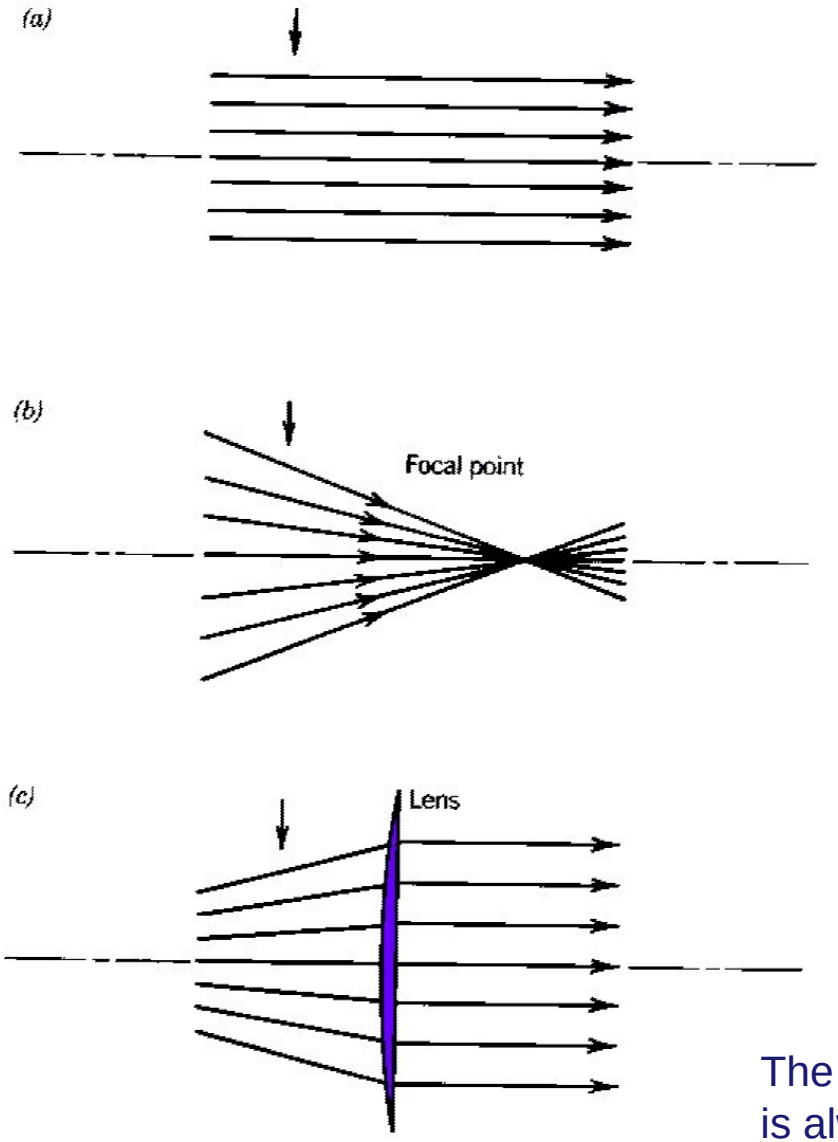

b. Converging laminar beam, where orbits pass through a common focus

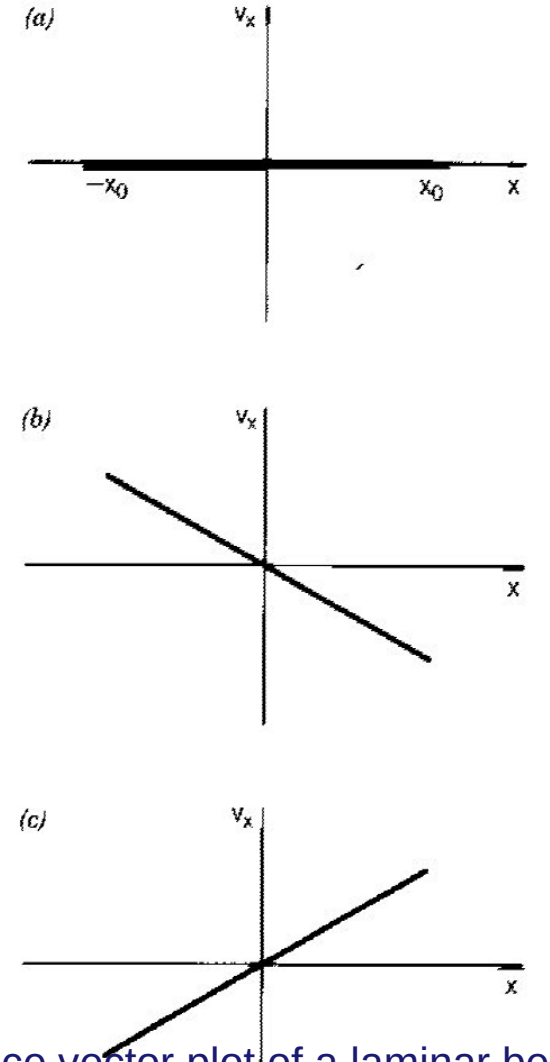



c. Diverging laminar beam converted to a parallel beam by a linear lens



# Laminar beam orbit-vector distributions viewed in transverse phase-space

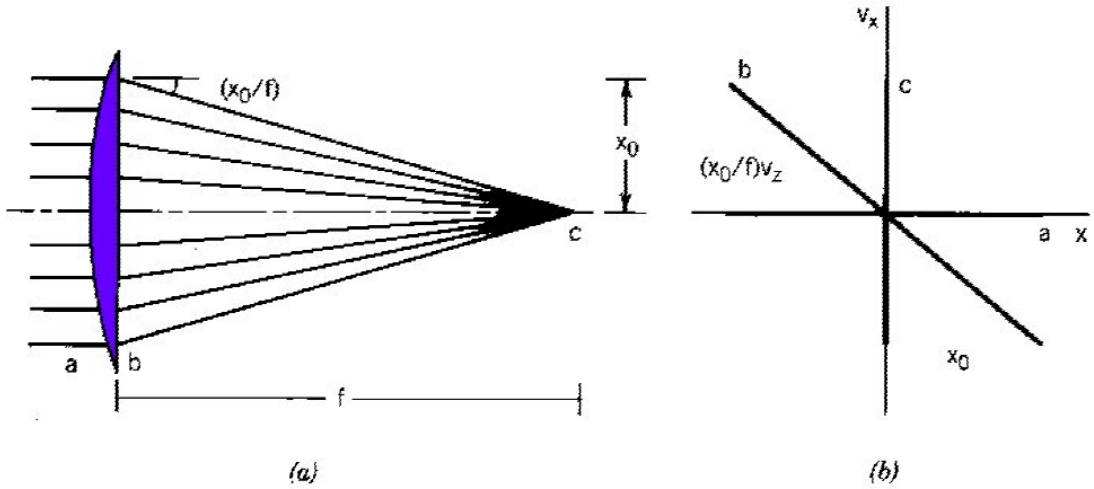



The phase space vector plot of a laminar beam is always a straight line of zero thickness

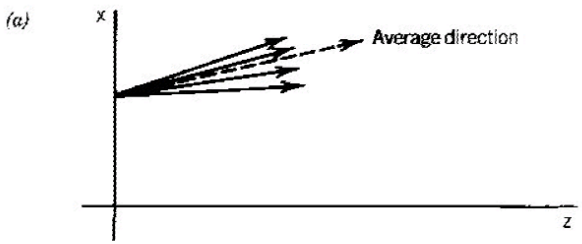
# Focusing a laminar beam

- The incident beam distribution is a straight line of length $2x_0$ aligned along the $x$ axis.
- The lens displaces the distribution in the $v_x$ direction while preserving the projected length along the $x$ axis. The velocity displacement has a maximum value of $(x_0/f)v_z$ at the beam edge.
- During subsequent transit through the drift region of length $f$, the orbit vectors converge toward $x = 0$.
- The orientation of the distribution changes until it aligns with the $v_x$ axis at the focal point. Here the distribution has dimension equal to zero in $x$ and a halfwidth along $v_x$ of $\pm(x_0/f)v_z$.



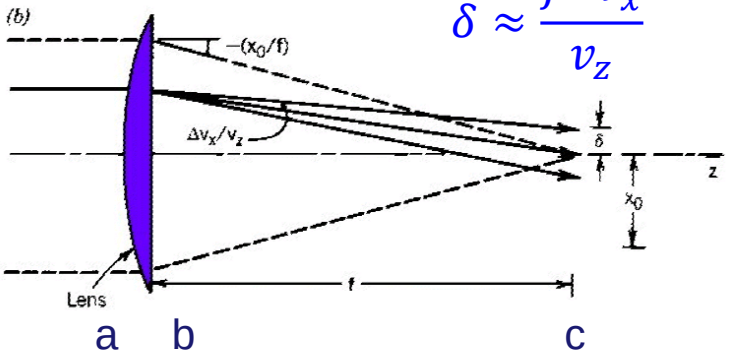
# Properties of non-laminar beams

a. Configuration-space view of particle orbits — particles at the same point move in different directions.



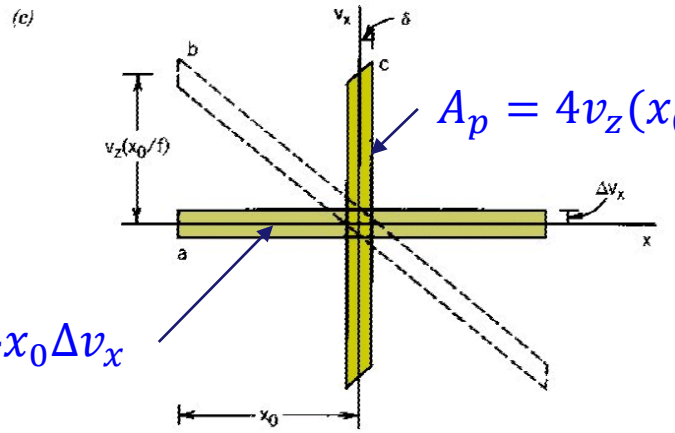

b. Configuration-space view of the transverse focusing of a non-laminar beam. The width of the focal spot is proportional to the transverse velocity spread of the incident beam.

$$\delta \approx \frac{f\Delta v_x}{v_z}$$




c. Snapshots of orbit vector distributions in phase-space at positions a, b and c.

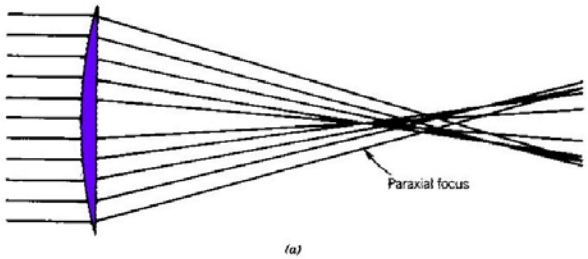
$$A_p = 4v_z(x_0/f)\delta$$



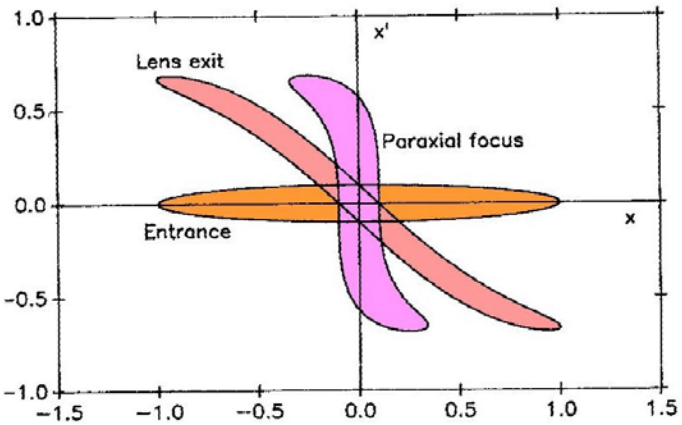

$$A_p = 4x_0\Delta v_x$$

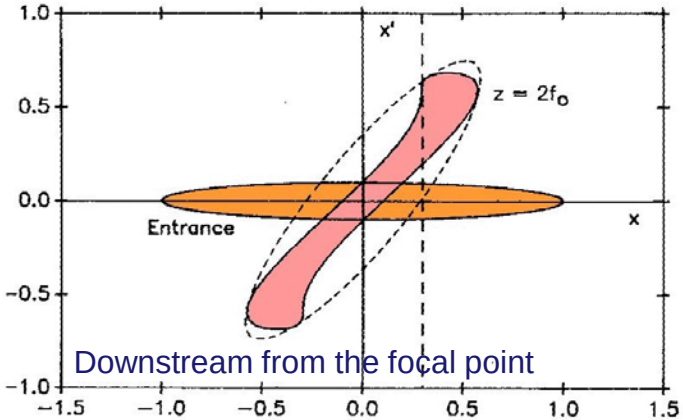
# Beam focusing by a non-linear lens with soft force

- Irreversible processes change distributions in such a way that they cannot be restored by macroscopic forces.
- The transverse force of the lens is soft; the lens force is weak at large displacement compared with a linear lens. As a result, the lens underfocuses particles on the periphery.
- There are positions in the beam, shown by dashed lines, where the particles have two different values of transverse velocity.
- Note that the phase-space area filled by the beam is unchanged. Nonetheless, the distorted distribution surrounds regions of unoccupied phase space - the effective area of the distribution is larger. If we sought to focus all particles in the distribution of Fig. (c) to a spot with an ideal linear lens, the relevant phase space area is that inside a boundary surrounding all particles. Figure (c) shows such a boundary as a dashed line.




(a)




(c)

# Emittance

- To designate the quality of a beam for an application, we must adopt a figure of merit based on the effective volume (area) occupied by the distribution. This quantity is the emittance.
- A beam will occupy a finite volume in the six-dimensional phase space, $dxdydzdp_x dp_y dp_z$, this is defined as the beam emittance.
- If the momentum in the z-direction is much greater than in the transverse direction (x-, y-direction) (paraxial beam), the radial momentum can be replaced by the orbital angle,

$$\frac{p_x}{p_z} = \frac{dx}{dz} = x', \qquad \frac{p_y}{p_z} = \frac{dy}{dz} = y'$$

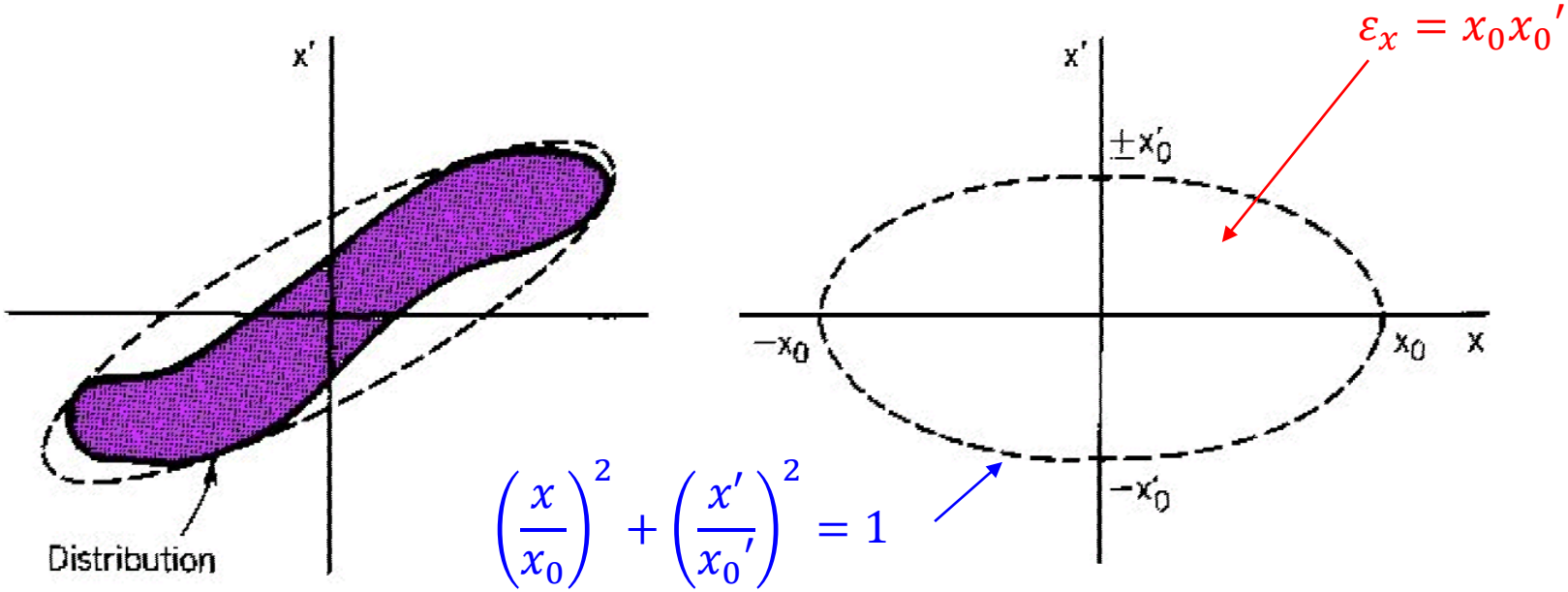
- The coordinates (x, x', y, y') are usually treated as functions of z rather than time, t. They describe the trace of a particle orbit along the axial direction, [x(z), y(z)], rather than the time-dependent position [x(t), y(t)]. Hence, the space defined by the coordinates is called trace space.
- In beam physics, it's often more convenient to work in trace space (the x-x' plane) than the x-$v_x$ or x-$p_x$ plane. This is partly because the inclination angle x' is much more useful for visualizing the shape of the beam than the transverse velocity or momentum.

# Emittance

- In many cases, the x and y motions are independent and it is convenient to consider x, x' and y, y' space projections individually. We define emittance as the area of the ellipse divided by π.

$$\varepsilon_x = \frac{1}{\pi}\iint dxdx', \qquad \varepsilon_y = \frac{1}{\pi}\iint dydy' \qquad \text{[π mm-mrad]}$$

- The smaller the phase area occupied by the beam, i.e. the smaller the beam emittance, the better the quality of the beam. Here, the term quality implies focusability or parallelism.



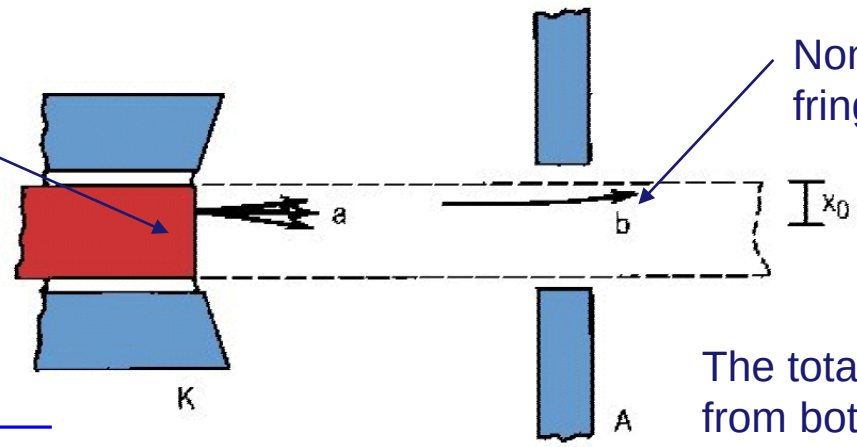
# Origins of non-zero emittance in an electron beam injector

- The final emittance of a beam represents the sum of the intrinsic emittance from the source and emittance growth during acceleration

Thermionic emitter operating at $T$

$$\overline{v_x} = \sqrt{2kT/\pi m}$$

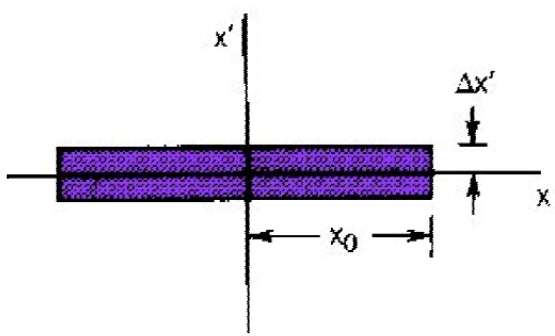
$$\Delta x' = \frac{\overline{v_x}}{v_z} = \sqrt{\frac{kT}{\pi(mv_z^2/2)}}$$



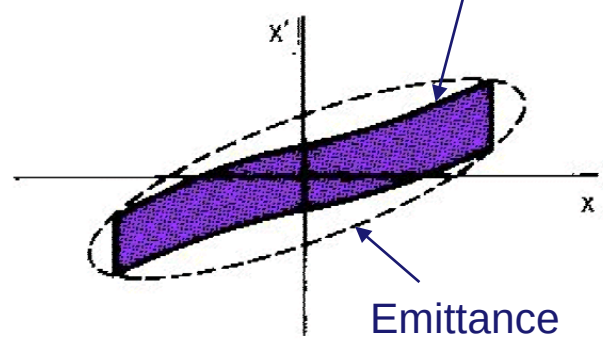

(a)

Nonlinear deflection by fringing electric field

The total area reflects contributions from both the thermal velocity spread and distortions from optical errors



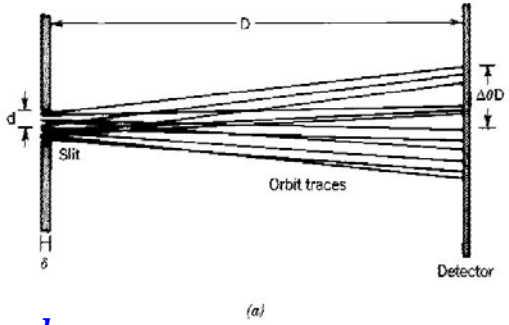

(b)




Emittance

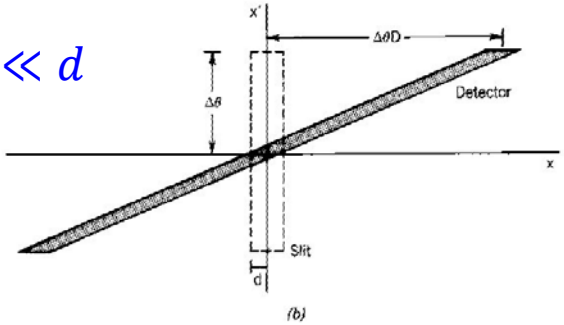
(c)

# Measurement of emittance: pepperpot diagnostic

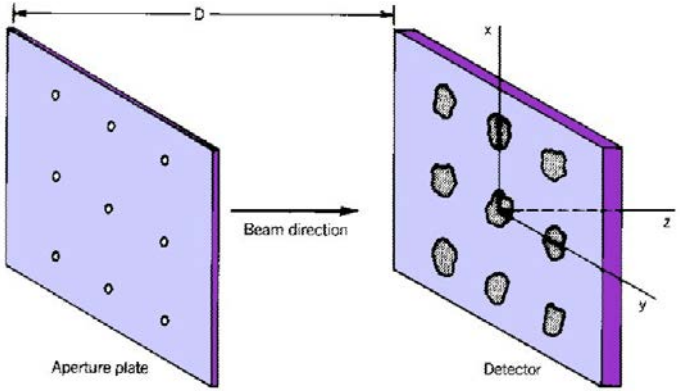



$\Delta\theta \cdot D \gg d$

$\delta \cdot \Delta\theta \ll d$

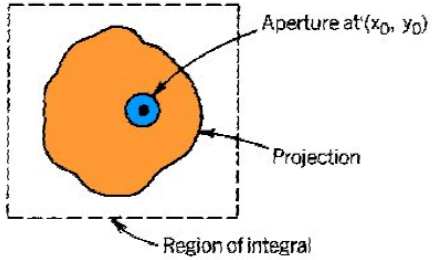


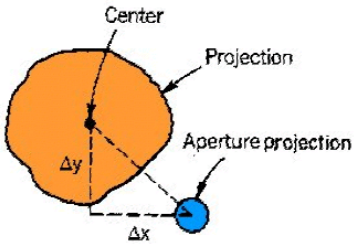

Pepperpot (imaging detector)



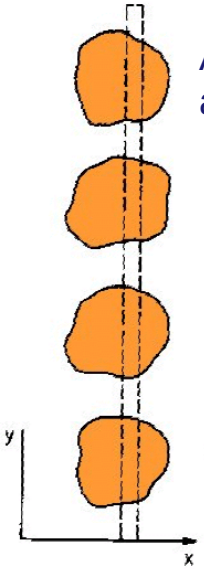

Intensity ~ Current density

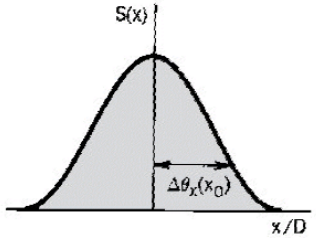
Displacement ~ Beam direction

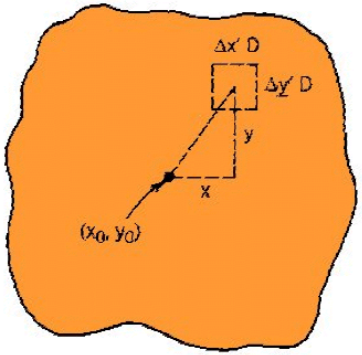



Average over y-direction ~ Divergence angle ~ Emittance in x-direction



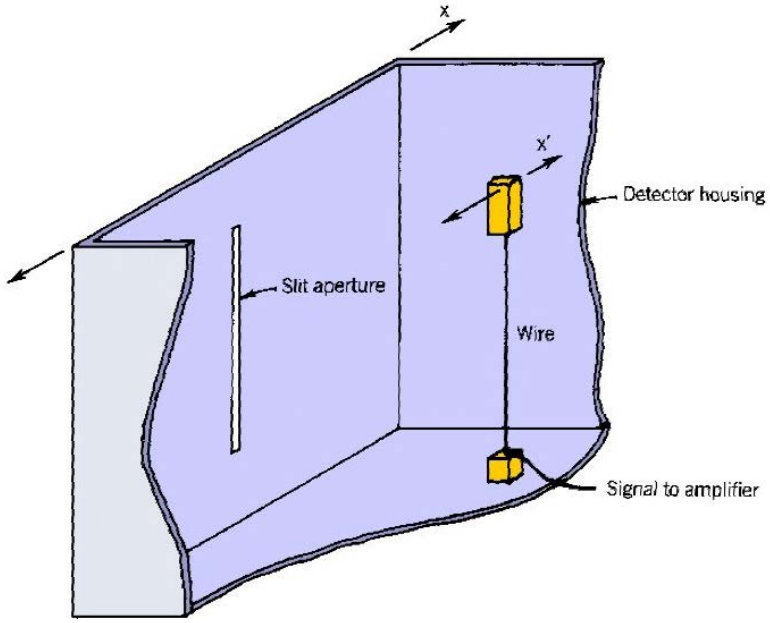



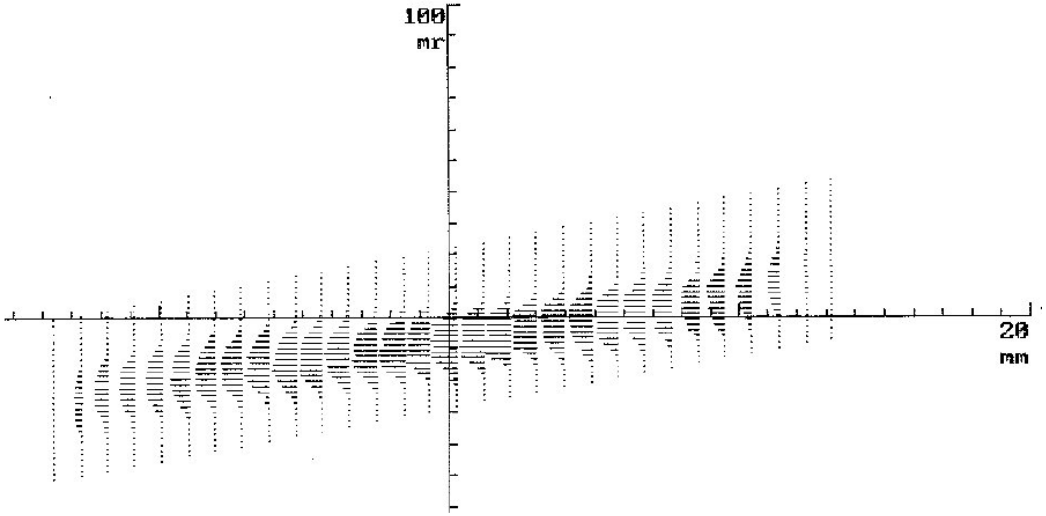



# Measurement of emittance: wire scanner diagnostic

- A detector with an analyzing slit moves over the cross section of a beam to sample different values of $x$. A wire beam collector moves within the detector to sample different values of $x'$. The extended geometry of the slit and wire gives averaging along $y$ and $y'$.
- Wire scans are useful only for steady-state or continuously pulsed beams.
- Statistical (rms) emittance:

$$\epsilon_x = 4\left[\overline{(x-\bar{x})^2}\;\overline{(x'-\bar{x'})^2} - \overline{(x-\bar{x})\,(x'-\bar{x'})}^{\,2}\right]^{1/2}$$



# Coupled transverse beam distributions

- The focusing system of a high-energy particle accelerator consists mainly of quadrupole lenses and dipole bending magnets. In these optical elements, particle motions in the $x$ and $y$ directions are independent.
- Motion is not separable in a variety of other focusing devices. Some common examples are solenoidal magnetic lenses, liquid metal lenses, or cylindrical electrostatic lenses in acceleration columns.
- Consider a paraxial beam in a focusing system where $x$ and $y$ motions couple but transverse motion is independent of axial motion. Emittances in the $x$ and $y$ directions are no longer separately-conserved quantities. Instead, the total four-dimensional trace-space volume in $(x, x', y, y')$ is constant in the absence of acceleration. The four-dimensional extension of emittance is called hyper-emittance.
- If the distribution fits into the four-dimensional ellipsoid:

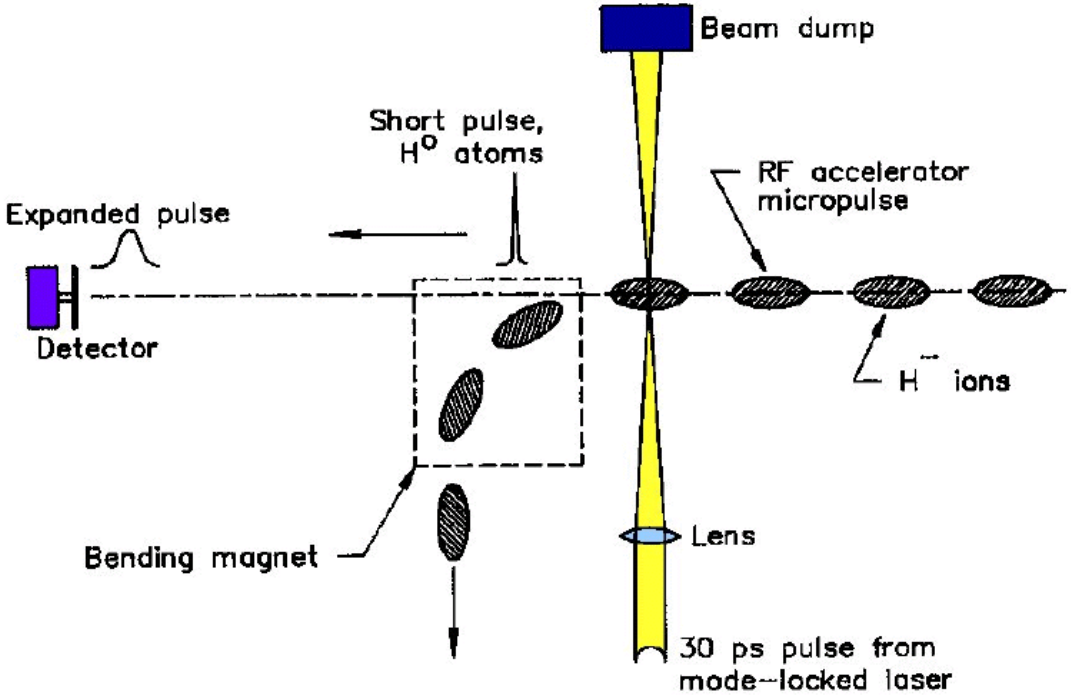
$$\left(\frac{x}{x_0}\right)^2 + \left(\frac{x'}{x_0{}'}\right)^2 + \left(\frac{y}{y_0}\right)^2 + \left(\frac{y'}{y_0{}'}\right)^2 = 1$$

then, the hyper-emittance ($\epsilon_4$) is

$$\epsilon_4 = V_4/\pi^2 = x_0 x_0' y_0 y_0' \ \ (\pi^2 - m^2 - rad^2)$$

# Longitudinal emittance

- In extending emittance to the axial direction, we recognize that angles and orbit traces are undefined. Therefore, we must plot orbit vectors to represent distributions directly in $z - p_z$ space.
- In RF accelerators, we measure and plot longitudinal distributions relative to a point of constant phase of the accelerating wave. Here, common distribution coordinates are $\phi$, the phase position of the particle relative to the wave, and $\Delta T$, the difference in kinetic energy from the average value.



# Normalized emittance

- Acceleration generally reduces emittance. The transverse momentum of particles may remain constant while the axial momentum increases, leading to a reduction in x'. We shall find it useful to designate an alternative quantity that remains constant during acceleration, the normalized emittance.
- With the effects of acceleration removed, changes in the normalized emittance indicate a degradation of beam quality resulting from non-linear forces or beam perturbations.
- Although the trace-space volume of a beam decreases during acceleration, we know that the phase-space volume stays constant in a linear focusing system. The transverse momentum is related to the inclination angle by

$$p_x = mv_x = \gamma m_0 x' v_z = \gamma m_0 x' \beta c = x'(\beta\gamma)(m_0 c)$$

- The normalized emittance of a relativistic paraxial beam is: $[x, \beta\gamma x']$

$$\epsilon_{nx} = (\beta\gamma)\epsilon_x = (Area\ in\ x - p_x\ space)/\pi m_0 c \quad (\pi - m - rad)$$

- For non-relativistic beams:

$$\epsilon_{nx} = \beta\epsilon_x = (v_x \epsilon_x)/c = (Area\ in\ x - v_x\ space)/\pi c \quad (\pi - m - rad)$$

# Brightness

- The quantity brightness was adopted from conventional optics where it characterizes the quality of light sources.
- In charged particle beam applications, beam brightness is the current density per unit solid angle in the axial direction. Bright beams have high current density and good parallelism.
- The brightness is defined as

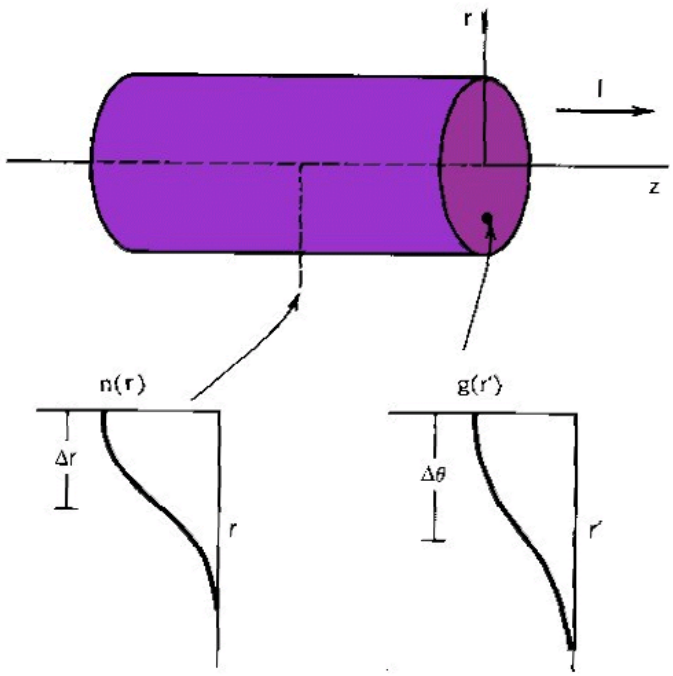
$$B \cong \frac{I}{(\pi\Delta r^2)(\pi\Delta\theta^2)} = \frac{j_b}{\pi\Delta\theta^2} \quad (A/(m-rad)^2)$$

- When beams have Cartesian symmetry in the transverse direction, we can write an expression for brightness in terms of the emittances:

$$B \cong \frac{I}{(\pi x_0 x_0')(\pi y_0 y_0')} = \frac{I}{\pi^2 \epsilon_x \epsilon_y} = \frac{I}{\pi^2 \epsilon^2}$$

- The normalized brightness (relativistic):

$$B_n = \frac{I}{\pi^2 \epsilon_n^2} = \frac{B}{(\beta\gamma)^2}$$




Note that if $\epsilon$ is constant, the beam brightness is also a conserved quantity.

# Transverse orbits in a continuous linear focusing force: betatron oscillation

- In many instances, particle motion transverse to a beam axis is separable along two Cartesian coordinates. This applies to motion in a magnetic gradient field and in an array of quadrupole lenses.
- Consider one-dimensional transverse paraxial particle motion along the z axis in the presence of a linear force, $F_x = -F_0(x/x_0)$.
- The equation of motion in the paraxial approximation:

$$\frac{d^2}{dt^2}(\gamma m_0 x) = -F_0 \frac{x}{x_0} \xrightarrow{\frac{d}{dt} = v_z \frac{d}{dz}} \frac{d^2 x}{dz^2} = -\left(\frac{F_0}{\gamma m_0 v_z^2 x_0}\right) x$$

- Particle motion is harmonic: All particle orbits have the same wavelength; they differ only in amplitude and phase.

$$x(z) = X \cos\left(\frac{2\pi z}{\lambda_z} + \varphi\right) \qquad \lambda_z = 2\pi \left(\frac{\gamma m_0 v_z^2 x_0}{F_0}\right)^{1/2}$$

- Transverse particle motions of this type in accelerators are usually referred to as betatron oscillations since they were first described during the development of the betatron [D.W. Kerst and R. Serber, Phys. Rev. 60, 53 (1941)]. The quantity $\lambda_z$ is called the betatron wavelength.

# Particle orbits in an array of uniform, equally spaced, thin lenses

- Consider lenses of focal length $f$ and axial spacing $d$ in the limit that $d \ll f$ (thin-lens approximation). We want to calculate the change in $x$ and $v_x$ passing through one drift space and one lens. If $v_x$ is the transverse velocity in the drift region, then

$$\Delta x = \left(\frac{v_x}{v_z}\right) d \qquad \frac{\Delta v_x}{v_z} = -\frac{x}{f} \quad \text{(Definition of the focal length)}$$

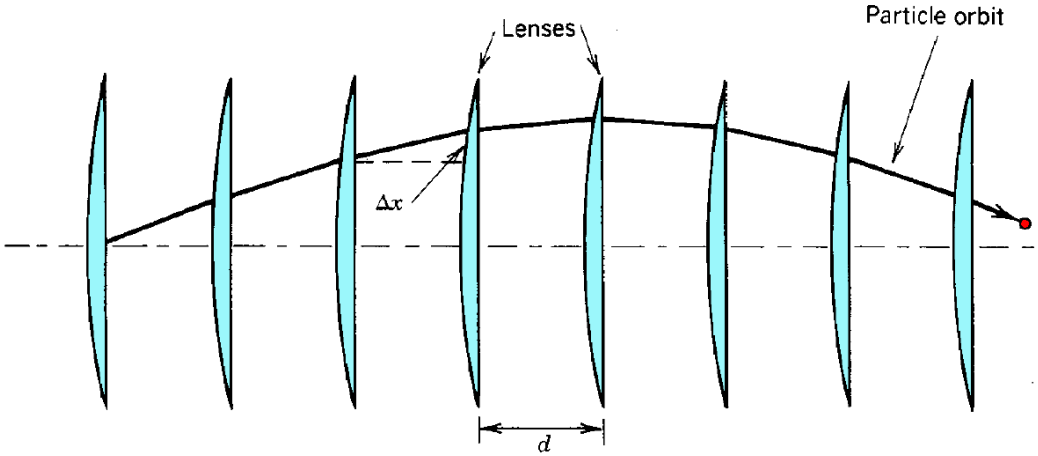
- Above equations can be converted to differential equations by associating $\Delta z$ with $d$ and letting $\Delta z \to 0$,

$$\frac{dx}{dz} = \frac{v_x}{v_z} \qquad \frac{dv_x}{dz} = -\frac{v_z}{fd} x \quad \Longrightarrow \quad \frac{d^2x}{dz^2} = -\left(\frac{1}{fd}\right) x$$

- The solution is harmonic, with $\lambda_z = (fd)^{1/2}$.

- Averaging the transverse force over many lenses gives

$$\bar{F}_x = \left(\frac{\gamma m_0 v_z^2}{fd}\right) x$$



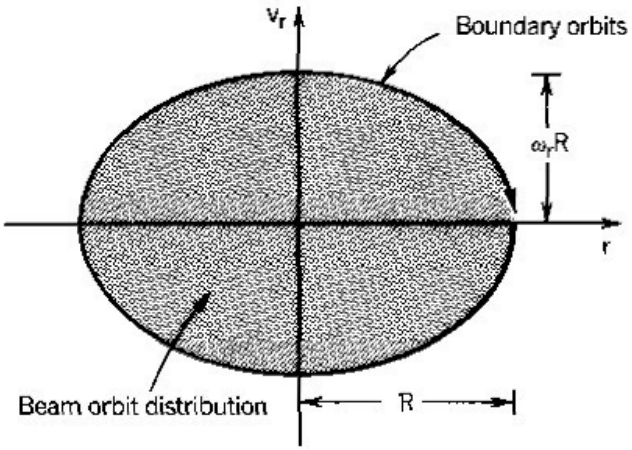
# Emittance force

- An ideal laminar beam with parallel orbits propagates indefinitely with no change in radius. In contrast, a beam with non-zero emittance expands — some of the particles are aimed outward.
- To maintain a constant radius for a beam with emittance, focusing forces must be applied to reverse the outwardly directed particles. In a sense, we can view non-zero emittance in terms of an outward force that balances the focusing force to maintain a constant radius beam. We can calculate the effective emittance force by seeking the focusing force that guarantees radial force balance.
- Suppose a linear, axicentered force confines the beam — the force varies in $z$ over scale lengths long compared with the envelope radius, $R$. We write the linear focusing force as:

$$F_r(r) = -F_0(r/R_0)$$

- If no other forces act on the beam, the orbit vector points of individual particles follow ellipses in trace-space as the particles perform radial oscillations. The oscillation frequency for all particles is

$$\omega_r = \sqrt{F_0/(\gamma m_0 R_0)}$$



# Emittance force

- By definition, the radial emittance of the beam equals the product of the maximum displacement and angle of the boundary orbit:

$$\epsilon_r = R\frac{v_{ro}}{v_z} = \frac{\omega_r R^2}{v_z} \quad \Rightarrow \quad \omega_r = \frac{\epsilon_r v_z}{R^2}$$

- By equating two $\omega_r$ values at $R$, we can find the focusing force needed to balance emittance on the beam envelope:

$$F_r(R) = -F_0\frac{R}{R_0} = \epsilon_r^2\frac{\gamma m_0 v_z^2}{R^3}$$

Effective emittance force

- In the quasi-static limit, the following approximate equation describes changes in the envelope radius of a beam with non-zero emittance subject to a linear focusing force:

$$\frac{d^2}{dt^2}(\gamma m_0 R) = -F_0\frac{R}{R_0} + \epsilon_r^2\frac{\gamma m_0 v_z^2}{R^3}$$

- Envelope equation:

$$\frac{d}{dt} = v_z\frac{d}{dz}$$

$$\boxed{R'' = \frac{d^2R}{dz^2} = -\frac{F_0(r/R_0)}{\gamma m_0\beta^2c^2} + \frac{\epsilon_r^2}{R^3}}$$

# Free-space expansion of a cylindrical beam with non-zero emittance

- Without focusing force, the envelope equation is

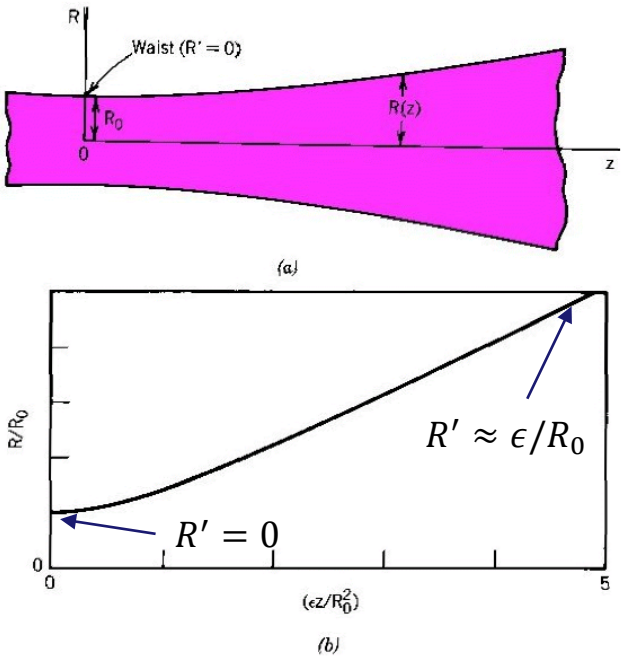
$$R'' = \frac{\epsilon^2}{R^3}$$

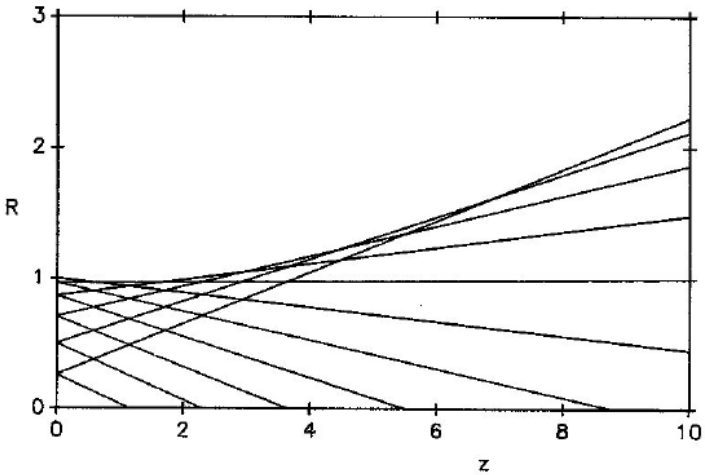
- By multiplying $2R'$ and integrating, we obtain

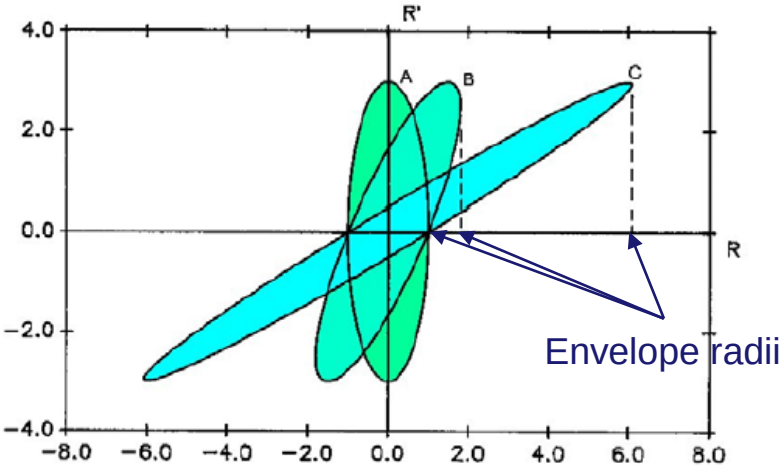
$$(R')^2 = \epsilon^2(1/R_0^2 - 1/R^2)$$

- We finally obtain

$$R(z)^2 = R_0^2 + \epsilon^2 z^2 / R_0^2$$



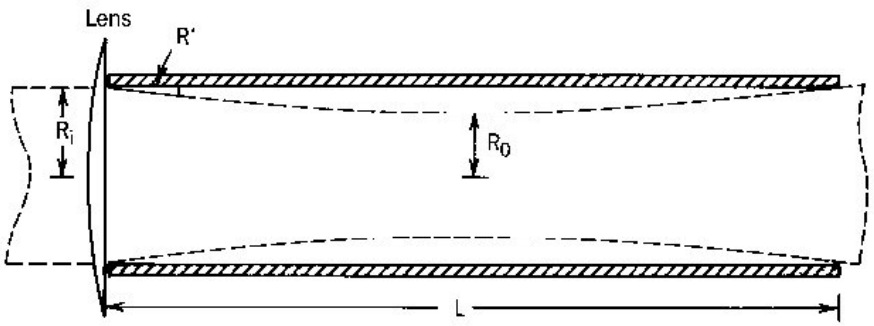



# Field-free propagation of a beam through a tube

- The envelope equation for converging or diverging beams that have a waist point at any position $z_0$:

$$R(z)^2 = R_0^2 + \epsilon^2 (z - L/2)^2 / R_0^2$$




- Find the maximum length of the pipe such that the beam can traverse without losses.
- We assume that a focusing lens at the entrance allows us to adjust the input convergence angle. Different angles give different values of the waist radius, $R_0$. We seek an maximum tube length by expressing $L$ as a function of $R_0$ and then setting $dL/dR_0$ equal to zero.
- At $z = 0$, we obtain $L^2 = (4/\epsilon^2)(R_0^2 R_i^2 - R_0^4)$
- Taking the derivative, the value of $R_0 = R_i/\sqrt{2}$ gives the maximum value of $L$:

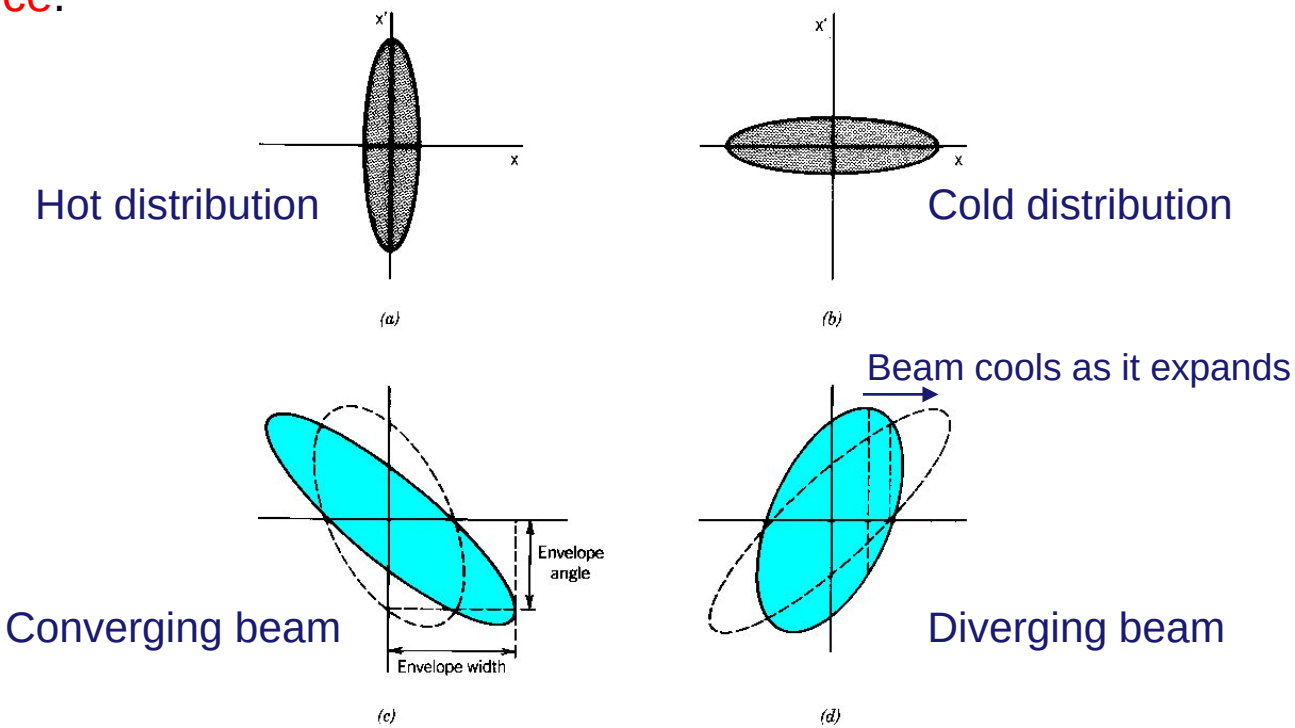
$$L = R_i^2/\epsilon$$

- The envelope angle at the pipe entrance for the optimum solution is

$$R' = -\epsilon\sqrt{1/R_0^2 - 1/R_i^2} = -\epsilon/R_i$$

# Non-laminar beams in linear focusing systems

- Accelerator transport systems combine electric or magnetic field lenses, bending elements, and drift spaces to steer beams and to confine them about an axis.
- All transport systems accept particles only within a limited range of displacement and inclination angle from the main axis. We must make certain that all particles in the beam can travel through the system without striking a boundary. We display the allowed particle orbits through a trace-space boundary called the acceptance.



# Transfer matrix

- Transfer matrix theory describes particle motion relative to a known main equilibrium orbit. The first order theory employs two assumptions:
  - Particle motions are paraxial. The inclination angles are small and all particles have approximately the same axial velocity, $v_z$, at an axial location.
  - Transverse focusing forces vary linearly with displacement from the main axis and are independent of the transverse velocity.
- In Cartesian coordinates, a particle orbit at some axial position is specified by a set of four quantities that we write as a vector:

$$\boldsymbol{x} = [x, x', y, y']$$

- Charged-particle beam focusing elements generate localized regions of transverse forces (e.g. bending magnets, quadrupole lenses and drift lengths). If the transverse forces are linear, we display the effect of an optical element on an orbit vector by:

$$\boldsymbol{x_1} = \boldsymbol{M_1 x_0}$$

The quantity $\boldsymbol{M_1}$ is a 4x4 matrix called a transfer matrix.
If there are no accelerating forces in the element, $\mathrm{Det}\, M = 1$.

- When a particle travels through two sequential optical elements, the net change of the orbit vector is

$$\boldsymbol{x_2} = \boldsymbol{M_2 x_1} = \boldsymbol{M_2}(\boldsymbol{M_1 x_0}) = (\boldsymbol{M_2} \cdot \boldsymbol{M_1})\boldsymbol{x_0}$$

# Transfer matrix

- The forces in a charged-particle transport system are often separable in the x and y directions. In this case, the transverse force along x does not depend on y or y'. Applications where forces are separable include storage rings with dipole bending magnets and quadrupole focusing magnets, electrostatic quadrupole arrays for low-energy ion beam transport, and radio-frequency quadrupole ion accelerators. The transport matrix reduces to the independent 2x2 matrices,

$$\boldsymbol{M} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \qquad \det \boldsymbol{M} = m_{11}m_{22} - m_{12}m_{21}$$

- The matrix for a thin lens with focal length $f$ is

$$\boldsymbol{M} = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$$

- The matrix of a field-free drift region of length $D$ is

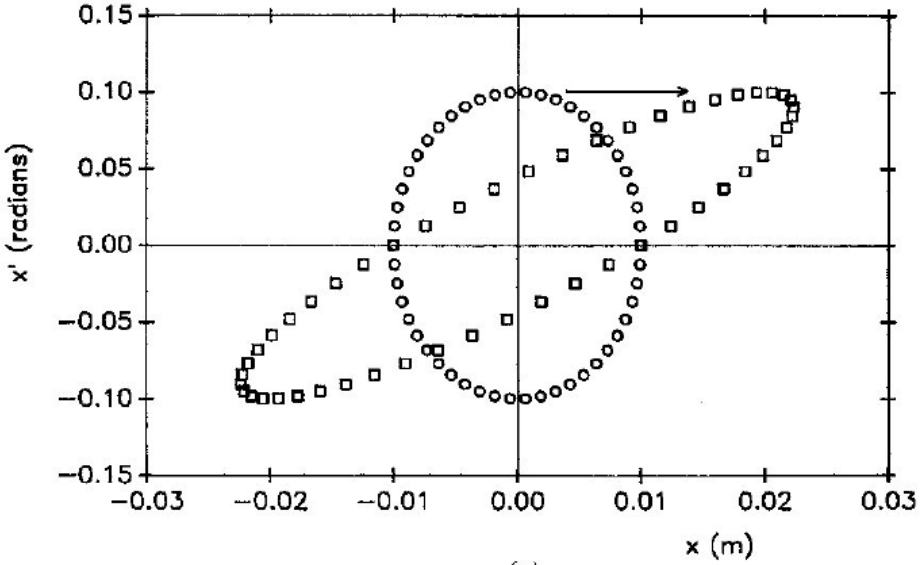
$$\boldsymbol{M} = \begin{bmatrix} 1 & D \\ 0 & 1 \end{bmatrix}$$

- We can represent the effect of each optical element in a periodic focusing cell by a transfer matrix.

$$\boldsymbol{x_n} = \boldsymbol{M_x^n}\, \boldsymbol{x_0} \qquad \boldsymbol{y_n} = \boldsymbol{M_y^n}\, \boldsymbol{y_0}$$

# Effects of linear optical elements on a beam trace-space distribution

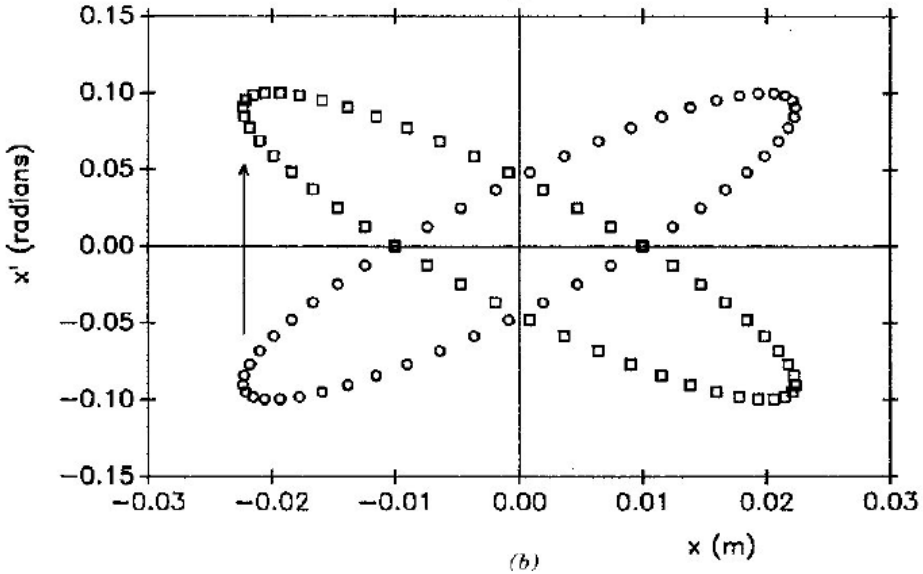
- Effect of a drift length




(a)

$$x_1 = x_0 + Dx_0'$$

- Effect of a lens

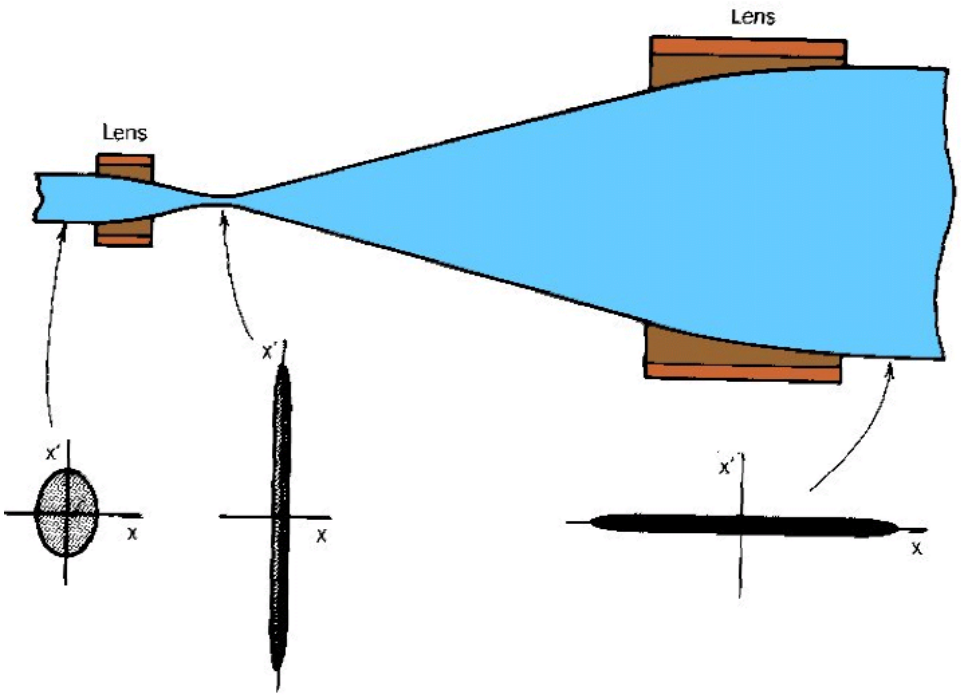



(b)

$$x_1' = x_0' - x_0/f$$

# Beam telescope

- The function of the device is to focus a beam from an accelerator to a small spot at long distance from the lens.
- If the output beam from the accelerator were focused directly by a small diameter lens, the focal spot would be large. To reduce the spot size, the telescope first expands the beam and then focuses the cooled beam with a large diameter lens.



# Beam matching

- Assume the force is uniform in $z$ and linear in $x$:

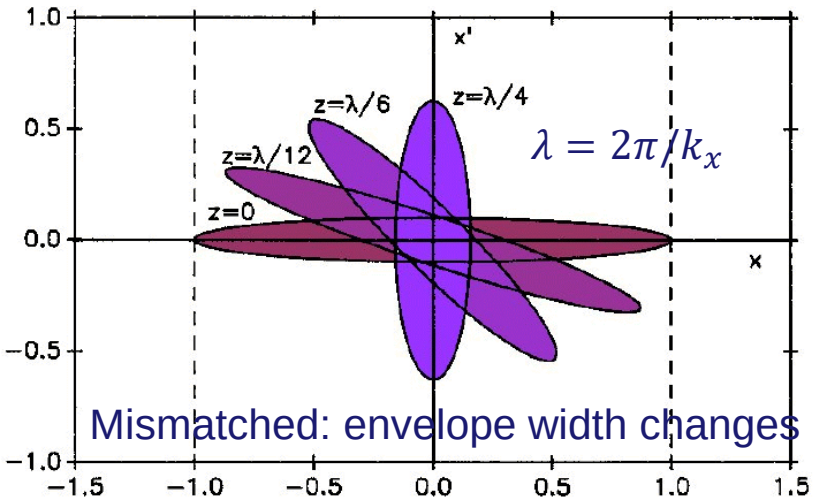
$$F_x = -Ax$$

- In the paraxial limit, individual particles follow orbits described by:

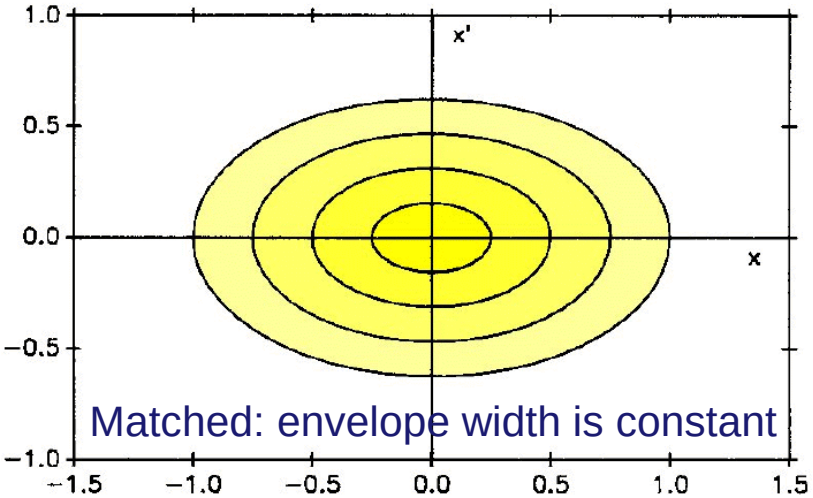
$$x_n(z) = x_{0n}\sin(k_x z + \phi_n)$$

$$x_n'(z) = x_{0n}k_x\cos(k_x z + \phi_n)$$

- Although the amplitude and phase of individual particle orbits vary, all particle orbits rotate in trace-space at the same frequency.
- A matched beam exhibits minimal envelope oscillations. The advantage of matching a beam with a given emittance to a focusing system is that the beam has the smallest possible spatial width as it propagates.




(a)




(b)

# Compression and expansion of non-laminar beams

- We can control the dimensions of a charged particle beam by changing the magnitude of focusing forces. This process is often applied to compress or to bunch beams for transfer between different types of accelerators. The velocity spread of non-laminar beams limits the dimension change for a given change in focusing force.
- To begin, we shall study compression of a one-dimensional beam normal to the direction of propagation. The term compression denotes reduction of the spatial dimension of the beam by increasing the focusing force.
- To simplify the model, we adopt the following assumptions:
  - The focusing force is almost continuous along the $z$ direction.
  - Particles sense a slowly varying focusing force compared with their betatron frequency.
  - The beam is almost in transverse equilibrium - the magnitude of the emittance force is approximately equal to the focusing force.
- Consider beam particles with uniform axial velocity $v_z$ ($v_z \gg v_x$) in a linear focusing force. The transverse motion of particles:

$$\frac{d^2x}{dt^2} = -\omega(t)^2 x$$

This quantity is proportional to the slowly-varying transverse force

# Compression and expansion of non-laminar beams

- If the time scale for the force to change ($\Delta\omega$) is long compared to the betatron period ($1/\omega$), i.e. $\Delta\omega/\omega \ll 1$, then an approximate solution should be oscillatory with a slow variation of amplitude as following:

$$x(t) \cong A(t)\sin\left[\Phi_0 + \int \omega(t)dt\right]$$

- Here, we obtain the following condition: $\omega A^2 = \text{contant}$
- The solution means that particle oscillations at time $t$ are almost harmonic with frequency equal to $\omega(t)$ and the particle trace-space trajectory is close to an ellipse with dimensions

$$x_0(t) \cong A(t)$$

$$x_0'(t) = v_{x0}(t)/v_z \cong \omega(t)A(t)/v_z$$

- The boundary trajectory encloses a distribution ellipse with constant area:

$$x_0(t)x_0'(t) = \text{constant}$$

Emittance is conserved when linear focusing forces vary slowly

# Compression and expansion of non-laminar beams

- We define the transverse beam temperature as an average over the beam velocity distribution:

$$kT_x = \gamma m_0 \overline{v_x^2}/2 \quad \Rightarrow \quad [v_{x0}(t)]^2 \sim \overline{v_x(t)^2} \sim T_x(t)$$

$$x_0(t)x_0'(t) = \text{constant} \quad \Rightarrow \quad v_{x0}(t) \sim 1/x_0(t)$$

- Combining above two equations, the transverse temperature and beam width are related by the equation:

$$T_x(t)/T_x(0) = [x_0(0)/x_0(t)]^2$$

- In an ideal one-dimensional compression, particle orbits change only in the $x$ direction. Because the beam dimensions in the $y$ and $z$ directions do not change during the compression, the beam volume, $V$, is proportional to $x_0(t)$.
- We obtain the equation of state (EOS) for a one-dimensional compression:

$$T_x(t)/T_x(0) = [V(0)/V(t)]^2$$

- In general, the EOS of isotropic beams: $T_x(t)/T_x(0) = [V(0)/V(t)]^{2/\gamma}$

# Elliptical distribution

- We have seen that beam distributions enclosed by elliptical boundaries play an important role in accelerators with linear focusing systems. Trace-space ellipses have geometric properties that lead to a compact beam transport theory.
- A linear transformation always transforms an elliptical distribution into another ellipse. Furthermore, with no acceleration, a linear transformation does not change the area of a distribution ellipse — the beam emittance is constant. If the optical element has transfer matrix $\boldsymbol{M}$, emittance conservation holds if $\det \boldsymbol{M} = 1$.
- The phase-space equation for the boundary of upright ellipse:

$$(x/X_0)^2 + (x'/X_0')^2 = 1$$

- Multiplying both sides by $X_0 X_0'$ gives a standard form for elliptical boundaries:

$$(X_0'/X_0)x^2 + (X_0/X_0')x'^2 = X_0 X_0' = \epsilon$$

- We shall now derive the general form for a skewed distribution ellipse by applying a linear transformation to an upright ellipse.

$$\begin{aligned} x_1 &= m_{11}x_0 + m_{12}x_0' \\ x_1' &= m_{21}x_0 + m_{22}x_0' \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_0 &= m_{22}x_1 - m_{12}x_1' \\ x_0' &= -m_{21}x_1 + m_{11}x_1' \end{aligned}$$

# General mathematical form for a distribution ellipse

- By arranging the equation and dropping subscript 1, we obtain the general mathematical form for a distribution ellipse:

$$\boxed{\gamma x^2 + 2\alpha x x' + \beta x'^2 = \epsilon}$$

$$\gamma = (X_o'/X_o)\, m_{22}^2 + (X_o/X_o')\, m_{21}^2 ,$$

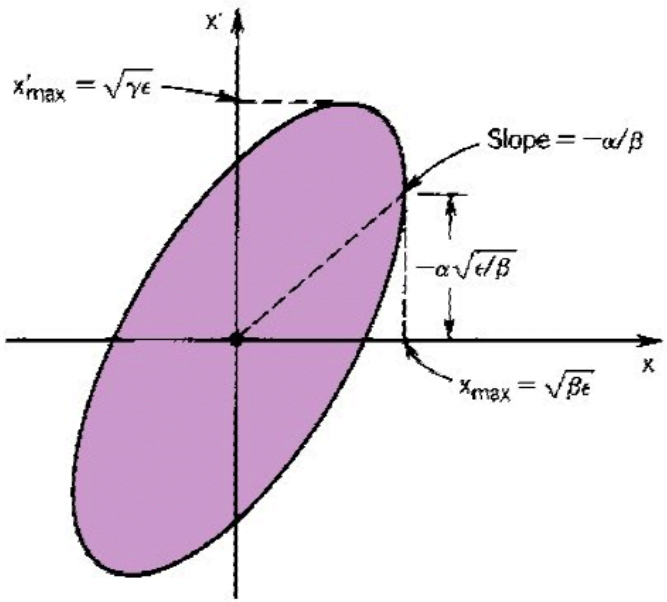
$$\alpha = -(X_o'/X_o)\, m_{12}\, m_{22} - (X_o/X_o')\, m_{11}\, m_{21} ,$$

$$\beta = (X_o'/X_o)\, m_{12}^2 + (X_o/X_o')\, m_{11}^2 .$$

- The quantities $\alpha$, $\beta$ and $\gamma$ are called the transport parameters (or sometimes Twiss parameters). Combined with the emittance, $\epsilon$, they specify the distribution at the output of the optical system.

$$\boxed{\gamma\beta - \alpha^2 = m_{11}m_{22} - m_{12}m_{21} = 1}$$

Courant-Snyder invariant




- The maximum extent in spatial dimension:

$$x_{max} = \sqrt{\beta\epsilon}$$

- The maximum extent in angular direction:

$$x'_{max} = \sqrt{\gamma\epsilon}$$

- The quantity $\alpha$ determines the envelope angle of the beam:

$$\frac{dx_{max}}{dz} = -\alpha\sqrt{\epsilon/\beta}$$