Engineering Economic Analysis

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Chap. 20

PROFIT MAXIMIZATION

Introduction

- A model of how the firm chooses the amount to produce and the method of production to employ
- Profit maximization problem of a firm that faces *competitive market* for the factors of production it uses and the output goods it produces
- Competitive market
  - A collection of well-informed consumers
  - Homogeneous product that is produced by a large number of firms
  - Price-taking behavior
    - Exogenous variable: price
    - Endogenous variable: levels of outputs and inputs

Economic Profit

- A firm uses inputs j = 1…,m to make products i = 1,….n.
- Output levels are $y_1,\ldots,y_n$.
- Input levels are $x_1,\ldots,x_m$.
- Product prices are $p_1,\ldots,p_n$.
- Input prices are $w_1,\ldots,w_{m.}$
- Profit = Revenue – Cost

$$\pi = \sum_{i=1}^{n} p_i y_i - \sum_{i=1}^{m} w_i x_i$$

  - Economic definition of profit requires that all inputs and outputs are valued at their opportunity cost

Profit Maximization

- Profit maximization

$$\operatorname*{Max}_{y_i, x_i} \quad \pi = \sum_{i=1}^{n} p_i y_i - \sum_{i=1}^{m} w_i x_i$$

  - Using production plan $\tilde{y} \in Y$, where $y_j \geq (\leq) 0$ if j is output (input)

    $$\operatorname*{Max}_{\tilde{y}} \quad \pi(\tilde{p}) = \tilde{p} \cdot \tilde{y}$$

    such that $\tilde{y} \in Y$,

    where $\tilde{p}$ is the vector of prices for all inputs and outputs

  - 1-output case, we can use the production function $y = f(\tilde{x})$

    $$\operatorname*{Max}_{x_i} \quad \pi = p \cdot f(\tilde{x}) - \sum_{i=1}^{m} w_i x_i$$

Fixed and Variable factors

- Fixed factor: a factor of production that is in a fixed amount for the firm
  - Fixed factor must be expensed even at the state of zero output
- Variable factor: a factor which can be used in different amounts
- Short run: there are some fixed factors
- Long run: all factors are variable factors
  - In the short run, the firm could make negative profits
  - But in the long run, the least profit is zero since the firm always free to decide to use zero inputs and produce zero output

Short-run Profit Maximization(1-output & 2-inputs)

- Suppose the firm is in a short-run circumstance in which $\bar{x}_2$: a fixed factor
- Its short-run production function is $f(x_1, \bar{x}_2)$
- Profit-max. problem

$$\max_{x_1} \pi = p \cdot f\left(x_1, \bar{x}_2\right) - w_1 x_1 - w_2 \bar{x}_2$$

- F.O.C.

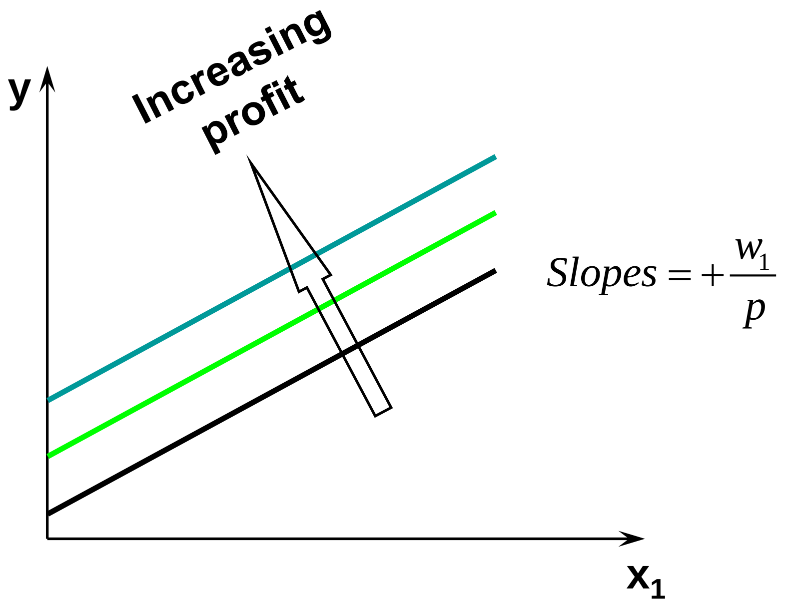
$$\frac{\partial \pi}{\partial x_1} = p \cdot \frac{\partial f(x_1^*, \bar{x}_2)}{\partial x_1} - w_1 = 0 \quad \Rightarrow \quad p \cdot MP_1\left(x_1^*, \bar{x}_2\right) = w_1$$

$x_1^*(p, w_1)$: factor demand function

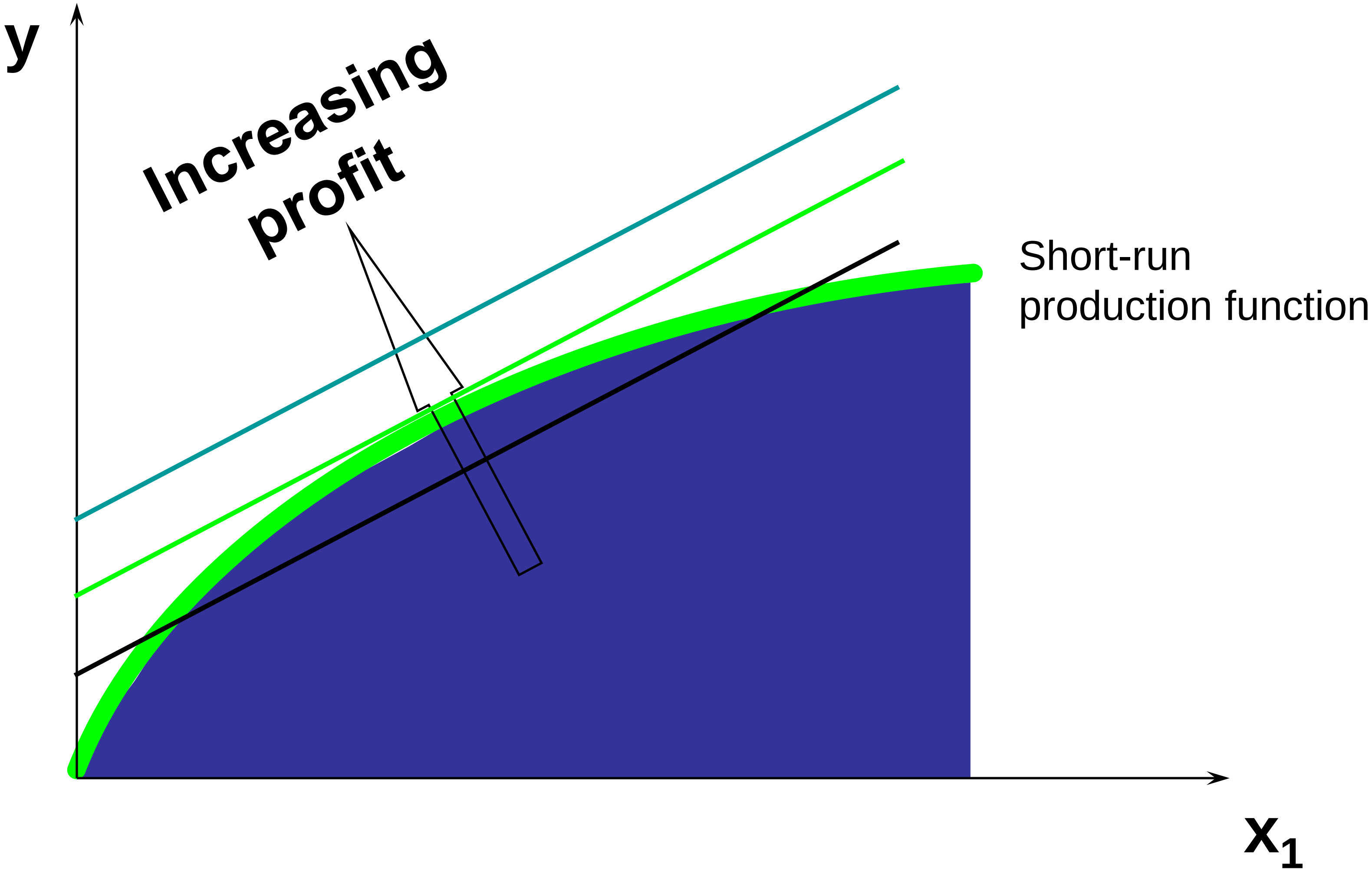
"The value of marginal product of factor 1 should equal its price"

Short-run Profit Maximization(1-output & 2-inputs)

- Iso-profit curves
  - Given profit function $\pi = py - w_1 x_1 - w_2 \bar{x}_2$
  - The level set of profit function $$y = \frac{w_1}{p} x_1 + \frac{1}{p}(\bar{\pi} + w_2 \bar{x}_2)$$
  - all combinations of inputs and outputs that give a constant level of profit



Short-Run Profit-Maximization

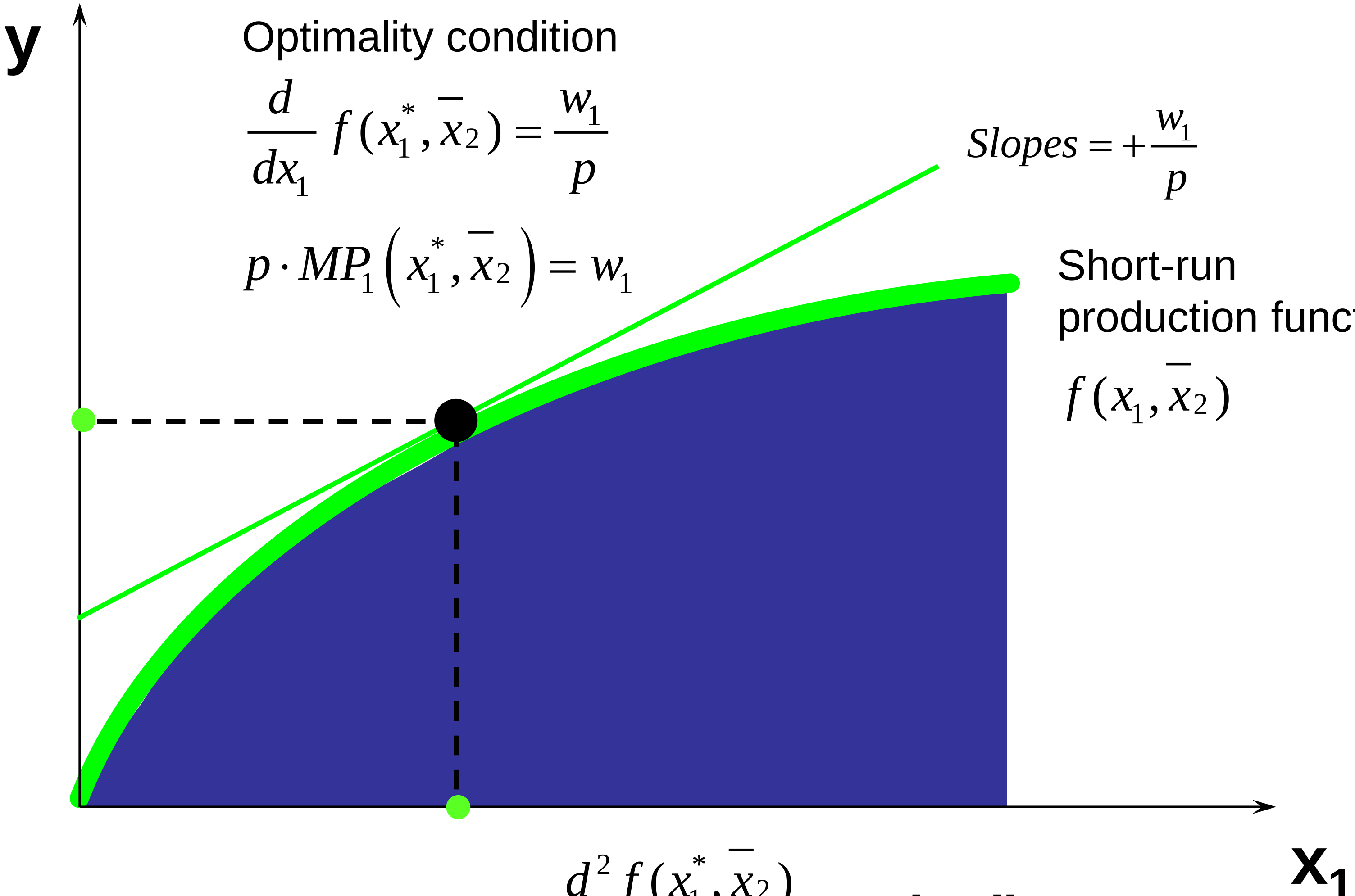


Short-Run Profit-Maximization

Optimality condition

$$\frac{d}{dx_1} f(x_1^*, \bar{x}_2) = \frac{w_1}{p}$$

$$p \cdot MP_1\left(x_1^*, \bar{x}_2\right) = w_1$$



Second order condition $\frac{d^2 f(x_1^*, \bar{x}_2)}{dx_1^2} \leq 0$: locally concave

Short-Run Profit-Maximization

- Short-run Cobb-Douglas production function

Long-Run Profit-Maximization (1-output, 2-inputs)

- Now allow the firm to vary all input levels.
- Since no input level is fixed, there are no fixed costs.
- Profit maximization

$$\max_{\{x_1, x_2\}} \pi = p \cdot f\left(x_1, x_2\right) - w_1 x_1 - w_2 x_2$$

- F.O.C.

$$\frac{\partial \pi}{\partial x_1} = p \cdot \frac{\partial f\left(x_1^*, x_2^*\right)}{\partial x_1} - w_1 = 0$$

$$\frac{\partial \pi}{\partial x_2} = p \cdot \frac{\partial f\left(x_1^*, x_2^*\right)}{\partial x_2} - w_2 = 0$$

- Optimality condition

$$p \cdot MP_1 = w_1, \; p \cdot MP_2 = w_2$$

- Solution

$$\left.\begin{array}{l} x_1^*\left(w_1, w_2, p\right) \\ x_2^*\left(w_1, w_2, p\right) \end{array}\right. : \text{Factor demand function}$$

Long-Run Profit-Maximization (1-output, 2-inputs)

- Cobb-Douglas production function $y = x_1^a x_2^b$
  - Profit-max. problem

$$\max \quad \pi(x_1, x_2) = p x_1^a x_2^b - w_1 x_1 - w_2 x_2$$

  - F.O.C.

$$\begin{cases} \dfrac{\partial \pi}{\partial x_1} = pa x_1^{a-1} x_2^b - w_1 = 0 \\ \dfrac{\partial \pi}{\partial x_2} = pb x_1^a x_2^{b-1} - w_2 = 0 \end{cases}$$

  - Multiplying x_i

$$\Rightarrow \begin{cases} pa x_1^a x_2^b - w_1 x_1 = 0 \\ pb x_1^a x_2^b - w_2 x_2 = 0 \end{cases} \Rightarrow \begin{matrix} pay = w_1 x_1 \\ pby = w_2 x_2 \end{matrix}$$

  - Factor demand function

$$x_1^*(w_1, w_2, p) = \frac{apy}{w_1}$$

$$x_2^*(w_1, w_2, p) = \frac{bpy}{w_2}$$

Long-Run Profit-Maximization (1-output, 2-inputs)

- Inserting factor demand functions into the production function gives

$$y = \left(\frac{apy}{w_1}\right)^a \left(\frac{bpy}{w_2}\right)^b = \left(\frac{ap}{w_1}\right)^a \left(\frac{bp}{w_2}\right)^b y^{a+b}$$

$$\therefore y^{1-a-b} = \left(\frac{ap}{w_1}\right)^a \left(\frac{bp}{w_2}\right)^b$$

- Supply function

$$y(p, w_1, w_2) = \left(\frac{ap}{w_1}\right)^{\frac{a}{1-a-b}} \left(\frac{bp}{w_2}\right)^{\frac{b}{1-a-b}}$$

Long-Run Profit-Maximization (1-output, n-inputs)

- Output y, Input bundle $\tilde{x}$
- Profit maximization

$$\max_{\tilde{x}} \quad \pi(\tilde{x}) = pf(\tilde{x}) - \tilde{w} \cdot \tilde{x}$$

- F.O.C.

$$p \cdot \frac{\partial f(\tilde{x})}{\partial x_i} = w_i \quad i = 1, \ldots, n$$

$$\Downarrow$$

$$\tilde{x}^*(p, \tilde{w}): \text{ factor demand function}$$

$$\Downarrow$$

$$f(\tilde{x}^*(p, \tilde{w})): \text{ supply function}$$

Long-Run Profit-Maximization (1-output, n-inputs)

- Exceptional case
  1) When production function is not differentiable
     - Leontief technology
  2) Corner (boundary) solution case $(x_i^* = 0 \quad \text{for some } i)$
     - F.O.C.

$$p \cdot \frac{\partial f(\tilde{x})}{\partial x_i} - w_i = 0 \quad \text{if } x_i^* > 0$$

$$p \cdot \frac{\partial f(\tilde{x})}{\partial x_i} - w_i \leq 0 \quad \text{if } x_i^* = 0$$

Long-Run Profit-Maximization (1-output, n-inputs)

- Optimization with constraints
  - When equality constraints

$$\max\ f(\tilde{x})$$
$$s.t.\quad h(\tilde{x}) = c$$

  - Lagrangian function

$$L(\tilde{x},\lambda) \equiv f(\tilde{x}) - \lambda\left[h(\tilde{x}) - c\right]$$

  - Kuhn-Tucker condition

Suppose that $\tilde{x}^* = \left(x_1^*, \ldots, x_n^*\right)$ is a solution.

Suppose further that $\partial h/\partial x_i\big|_{x_i = x_i^*} \neq 0$ (critical point)

Then there exits a real number λ^* such that

$$\frac{\partial L}{\partial x_i} = \frac{\partial L}{\partial \lambda} = 0 \quad \text{at}\left(\tilde{x}^*, \lambda^*\right)$$

Long-Run Profit-Maximization (1-output, n-inputs)

- When inequality constraints

$$\max f(\tilde{x})$$
$$s.t. \quad g(\tilde{x}) \le b$$

Lagrangian function

$$L(\tilde{x}, \mu) \equiv f(\tilde{x}) - \mu\left[g(\tilde{x}) - b\right]$$

- Kuhn-Tucker condition

Suppose that $\tilde{x}^* = \left(x_1^*, \ldots, x_n^*\right)$ is a solution.

If $g\left(x^*, y^*\right) = b$ (binding), then further suppose that $\partial g / \partial x_i \big|_{x_i = x_i^*} \neq 0$.

Then there is a multiplier $\mu^* \ge 0$ such that

$$\frac{\partial L}{\partial x_i} = 0 \text{ at} \left(\tilde{x}^*, \mu^*\right)$$

$$\mu^*\left[g\left(\tilde{x}^*\right) - b\right] = 0 \text{ (complementary slackness)}$$

$$g\left(x^*, y^*\right) \le b$$

Long-Run Profit-Maximization (1-output, n-inputs)

- Example

$$\max f\left(x,y\right)=xy$$
$$\text{s.t.}\quad x^2+y^2\le 1$$

Long-Run Profit-Maximization

- Generalized optimality condition for 2-input

$$\max\ p \cdot f(x_1, x_2) - (w_1 x_1 + w_2 x_2)$$
$$s.t.\quad x_i \geq 0 \ \Rightarrow\ -x_i \leq 0$$

  - Lagrangian

$$L = p \cdot f(x_1, x_2) - (w_1 x_1 + w_2 x_2) + \mu_i x_i$$

  - K-T condition

$$\frac{\partial L}{\partial x_i} = p\frac{\partial f}{\partial x_i} - w_i + \mu_i = 0,$$
$$\mu_i x_i = 0 \text{ (complementary slackness)},\quad x_i \geq 0,\quad \mu_i \geq 0$$

  - Optimality condition

$$If\ x_i^* = 0, then\ \mu_i^* \geq 0.$$
$$\text{Thus, if } x_i^* = 0, then\ p \cdot \frac{\partial f}{\partial x_i} - w_i \leq 0$$

$$If\ x_i^* > 0, then\ \mu_i^* = 0.$$
$$\text{Thus, if } x_i^* > 0, then\ p \cdot \frac{\partial f}{\partial x_i} - w_i = 0$$

Long-Run Profit-Maximization (1-output, n-inputs)

- Exceptional case

  3) No optimal solution case

  - When $f(x)=x$. If $p > w$, then $x^* = \infty$
  - When CRS technology

    Let $\tilde{x}^*$ be the optimal and assume that $p \cdot f\left(\tilde{x}^*\right) - \tilde{w} \cdot \tilde{x}^* = \pi^* > 0$

    Scale up production by $t > 1$

    Since CRS, $f\left(t\tilde{x}^*\right) = tf\left(\tilde{x}^*\right)$

    Then $p \cdot f\left(t\tilde{x}^*\right) - \tilde{w} \cdot \left(t\tilde{x}^*\right) = t\left[p \cdot f\left(\tilde{x}^*\right) - \tilde{w} \cdot \tilde{x}^*\right] = t\pi^* > \pi^*$

    Contradiction!

    $\rightarrow$*Thus the only nontrivial profit-max position for a CRS firm is zero-profits*

Long-Run Profit-Maximization (1-output, n-inputs)

- Exceptional case

  4) Multiple (Infinite) number of optimal solutions

Let $\tilde{x}^*$ be the optimal for a CRS technology which gives zero profit, i.e., $p \cdot f\left(\tilde{x}^*\right) - \tilde{w} \cdot \tilde{x}^* = \pi^* = 0$

Then scale up production by $t > 0$

$$p \cdot f\left(t\tilde{x}^*\right) - \tilde{w} \cdot \left(t\tilde{x}^*\right) = t\pi^* = 0$$

Thus $t\tilde{x}^*$ is also an optimal for any $t > 0$!!

Comparative Statics

- One-input & One-output

$$\underset{x}{Max}\ pf(x) - wx$$

$$\text{F.O.C.: } pf'\left(x^*(p,w)\right) - w = 0$$

$$\text{S.O.C.: } pf''\left(x^*(p,w)\right) \le 0$$

$\Rightarrow$ $x^*(p,w)$: factor demand function

- Differentiating F.O.C. with respect to w

$$pf''\left(x^*(p,w)\right)\frac{dx^*(p,w)}{dw} - 1 \equiv 0$$

- Assuming that $f'' \neq 0$

$$\frac{dx^*(p,w)}{dw} = \frac{1}{pf''\left(x^*(p,w)\right)}$$

$\Rightarrow$

1) Sign
   - Since $f'' < 0 \rightarrow \dfrac{\partial x^*(p,w)}{\partial w} < 0$
2) Magnitude
   - As $|f''|$ increases, $\left|\dfrac{\partial x^*}{\partial w}\right|$ decreases.

Comparative Statics

- Two-input & One-output

$$\underset{\{x_1,x_2\}}{Max}\ pf\left(x_1,x_2\right)-\left(w_1x_1+w_2x_2\right)$$

  - F.O.C.

$$p\frac{\partial f\left[x_1\left(w_1,w_2\right),x_2\left(w_1,w_2\right)\right]}{\partial x_1}\equiv w_1$$

$$p\frac{\partial f\left[x_1\left(w_1,w_2\right),x_2\left(w_1,w_2\right)\right]}{\partial x_2}\equiv w_2$$

$\Rightarrow$ $x_i^*\left(w_1,w_2\right)$: factor demand function

  - Differentiating F.O.C. with respect to w_1 and w_2 (let p=1)

$$f_{11}\frac{\partial x_1}{\partial w_1}+f_{12}\frac{\partial x_2}{\partial w_1}=1 \qquad f_{21}\frac{\partial x_1}{\partial w_1}+f_{22}\frac{\partial x_2}{\partial w_1}=0$$

and

$$f_{11}\frac{\partial x_1}{\partial w_2}+f_{12}\frac{\partial x_2}{\partial w_2}=0 \qquad f_{21}\frac{\partial x_1}{\partial w_2}+f_{22}\frac{\partial x_2}{\partial w_2}=1$$

Comparative Statics

- Rearranging by matrix form

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} \frac{\partial x_1}{\partial w_1} & \frac{\partial x_1}{\partial w_2} \\ \frac{\partial x_2}{\partial w_1} & \frac{\partial x_2}{\partial w_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- Note that $\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$ is Hessian matrix.
  - By Young's theorem, Hessian matrix is symmetric $f_{12} = f_{21}$

- Then the substitution matrix can be obtained

$$\begin{pmatrix} \frac{\partial x_1}{\partial w_1} & \frac{\partial x_1}{\partial w_2} \\ \frac{\partial x_1}{\partial w_2} & \frac{\partial x_2}{\partial w_2} \end{pmatrix} = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}^{-1} = \frac{\begin{pmatrix} f_{22} & -f_{12} \\ -f_{21} & f_{11} \end{pmatrix}}{|H|}$$

Comparative Statics

- If we assume that S.O.C. is satisfied, it is equivalent to the fact that the Hessian matrix is ND
  - Thus, $f_{11} < 0, \ |H| = f_{11} f_{22} - f_{12} f_{21} > 0$
- Comparative results
  - The changes of factor demand with respect to the change of its own price

$$\frac{\partial x_i}{\partial w_i} = \frac{f_{jj}}{|H|} < 0$$

  - The changes of factor demand with respect to the change of other price

$$\frac{\partial x_i}{\partial w_j} = -\frac{f_{ij}}{|H|} = -\frac{f_{ji}}{|H|} = \frac{\partial x_j}{\partial w_i} \text{:indeterminate and symmetric}$$