Field and Wave Electromagnetic

Chapter7

The time varying fields and Maxwell's equation

Introduction (1)

❖ Time static fields

1) Electrostatic

$\nabla \times \vec{E} = 0, \quad \nabla \bullet \vec{D} = \rho, \quad \vec{D} = \varepsilon \vec{E}$

2) Magnetostatic

$\nabla \bullet \vec{B} = 0, \quad \nabla \times \vec{H} = \vec{J}, \quad \vec{H} = \frac{1}{\mu} \vec{B}$

note) $\vec{E}$ and $\vec{D}$ are not related to $\vec{B}$ and $\vec{H}$ for time static cases

Example)

A static $\vec{E}$ field in a conducting medium $\Rightarrow$ steady current.$(\vec{J} = \sigma \vec{E})$

$\Rightarrow$ give rises to a static magnetic field:Ampere's law. But $\vec{E}$ field can be completely determined from the static electric charge or potential distributions

$\Rightarrow$ magnetic field is a consequence

Introduction (2)

❖ Time varying fields

☞ $\vec{E}$ and $\vec{D}$ are properly related to $\vec{B}$ and $\vec{H}$

1) modify $\nabla \times \vec{E}$ equation $\rightarrow$ fundamental postulate leading to Faraday's law

2) then modify the $\nabla \times \vec{H}$ equation to be consistent with the equation of continuity

$cf)$ $\nabla \bullet \vec{J} = 0$ for static. but $\nabla \bullet \vec{J} = -\dfrac{\partial \rho}{\partial t}$ for time varying

3) $\nabla \bullet \vec{D} = \rho$ and $\nabla \bullet \vec{B} = 0$ never changes.

Faraday's Law

✓ Michael Faraday $\Rightarrow$ 1831, experimental law $\Rightarrow$ postulate

✓ Definition : the quantitative relationship between the induced emf and the rate of change of flux linkage

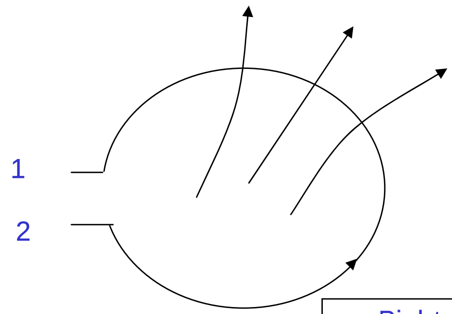
❖ Fundamental postulate for Electromagnetic Induction

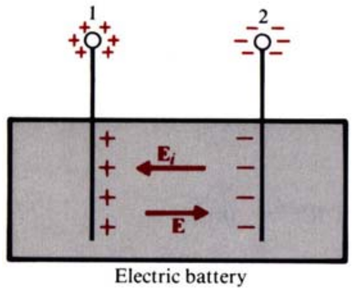
$$\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$ $\Rightarrow$ Non-conservative field cannot be expressed as the gradient of a scalar potential

$$\boxed{\oint_C \vec{E} \bullet \vec{dl} = -\int_S \frac{\partial \vec{B}}{\partial t} \bullet \vec{dS}}$$

A Stationary Circuit in a Time Varying Magnetic Field (1)

$$\oint_C \vec{E}\bullet\vec{dl} = -\frac{d}{dt}\int_S \vec{B}\bullet\vec{ds} \quad (\because \frac{\partial}{\partial t} \to \frac{d}{dt}, \text{ since stationary } \frac{\partial}{\partial t}\vec{ds} = 0)$$





Right hand rule (counter clock wise)

① $emf,\ v = \int_1^2 \vec{E}\bullet\vec{dl} = -\frac{d\Phi}{dt}$, Assume $\frac{d\Phi}{dt} > 0$

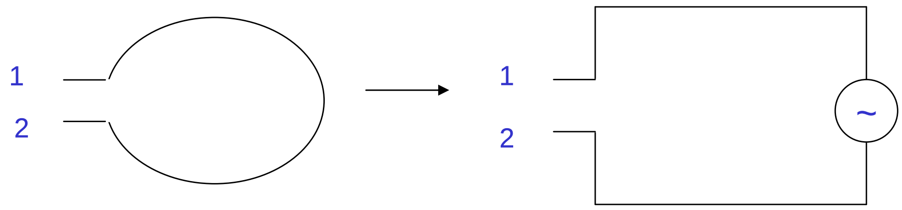
$\Rightarrow v < 0 \Rightarrow$ driving current to flow in the direction of clock wise

② potential difference of gap between terminal 1 and2

assume $V_{12} = -\int_2^1 \vec{E}\bullet\vec{dl} < 0 \qquad \because V_2 > V_1$

A Stationary Circuit in a Time Varying Magnetic Field (2)

$Define \quad v = \oint_C \vec{E}\bullet\vec{dl}$: emf induced in circuit with contour C



v : electromagnetive force driving current in the direction of right hand rule

❖ Meaning of contour integral

Field between the terminal in the gap

$$\oint_{\substack{right\\hand}} \vec{E}\bullet\vec{dl} = \int_1^2 \vec{E}\bullet\vec{dl} \quad (\because \text{ inside contour } \vec{E} = 0) = -\int_2^1 \vec{E}\bullet\vec{dl} = V_1 - V_2 = V_{12}$$

$\Rightarrow$ can be replaced with voltage source. But polarity of $v = V_{12}$ depends on the change of the flux linkage

A Stationary Circuit in a Time Varying Magnetic Field (3)

e.g) $\frac{\partial \vec{B}}{\partial t} > 0$ then v (current is in the direction of left hand rule) $i.e\ V_{12} < 0$

Define $\Phi = \int_S \vec{B} \cdot \vec{ds}$: magnetic flux crossing surface S $[Wb]$

then $v = -\frac{d\Phi}{dt} \Rightarrow$ This is valid even in the absense of a physical closed circuit

note The emf induced in a stationary loop caused by a time-varying magnetic field is a transformer emf

Ex 7-1) A Circular Loop of N Turns of Conducting Wire

A circular loop of N turns, $\vec{B} = \hat{z} B_0 \cos(\frac{\pi r}{2b}) \sin wt$

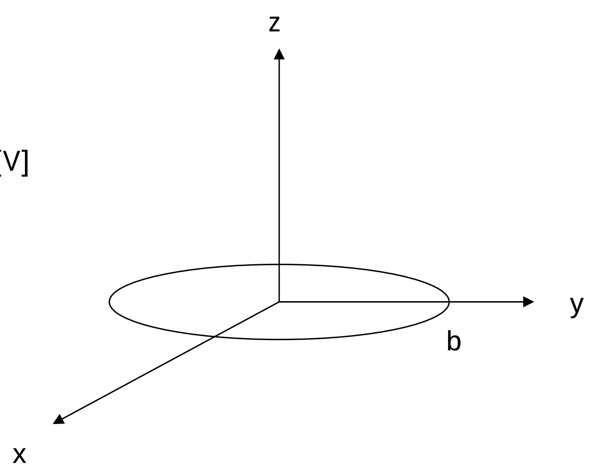
Find the emf induced in the loop

sol) each turn $\Phi = \int_S \vec{B} \cdot \vec{ds} = \int_0^b (\hat{z} B_0 \cos\frac{\pi r}{2b} \sin wt) \cdot (\hat{z} 2\pi r dr)$ $\quad cf) \int_0^{2\pi} d\phi = 2\pi$

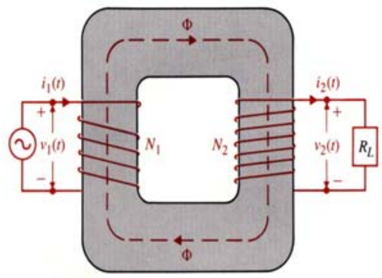
$$= \frac{8b^2}{\pi}(\frac{\pi}{2} - 1) B_0 \sin wt$$

$\therefore$ N-turns $\Rightarrow N\Phi$

$$\therefore v = -N\frac{d\Phi}{dt} = -\frac{8Nb^2}{\pi}(\frac{\pi}{2} - 1) B_0 \cos wt \ \text{[V]}$$



Transformers (1)



mmf

$$\sum_j \boxed{N_j I_j} = \sum_k \Re_k \Phi_k$$

$N_1, N_2, i_1, i_2 \Rightarrow$ number of turns and the currents

$\Re$: the reluctance of the magnetic circuit

$$\therefore \; N_1 i_1 - N_2 i_2 = \Re\Phi$$

(where $N_1 i_1$: mmf in the positive direction, $N_2 i_2$: mmf in the negative direction)

$$\Re = \frac{l}{\mu S}$$

$$\therefore \; N_1 i_1 - N_2 i_2 = \frac{l}{\mu S}\Phi$$

Transformers (2)

$a)$ Ideal transformer

$$\mu \to \infty,\; N_1 i_1 = N_2 i_2 \Rightarrow \boxed{\frac{i_1}{i_2} = \frac{N_2}{N_1}}$$

$cf)$ Faraday's law

- $v_1 = N_1 \dfrac{d\Phi}{dt}$ (No negative sign, careful of sign of flux Φ)
- $v_2 = N_2 \dfrac{d\Phi}{dt}$ (But flux is in the reverse direction) $\therefore \boxed{\dfrac{v_1}{v_2} = \dfrac{N_1}{N_2}}$
- effective load seen by the source connected to primary winding

$$(R_1)_{eff} = \frac{v_1}{i_1} = \frac{\left(\frac{N_1}{N_2}\right)v_2}{\left(\frac{N_2}{N_1}\right)i_2} = \left(\frac{N_1}{N_2}\right)^2 R_L$$

$\therefore$ Impedance transformation $(Z_1)_{eff} = \left(\dfrac{N_1}{N_2}\right)^2 Z_L$

Transformers (3)

b) Real transformer

$$N_1 i_1 - N_2 i_2 = \frac{l}{\mu s}\Phi$$

$$\Rightarrow \Lambda_1 = N_1\Phi = \frac{\mu s}{l}(N_1^2 i_1 - N_1 N_2 i_2),\ \Lambda_2 = N_2\Phi = \frac{\mu s}{l}(N_1 N_2 i_1 - N_2^2 i_2)$$

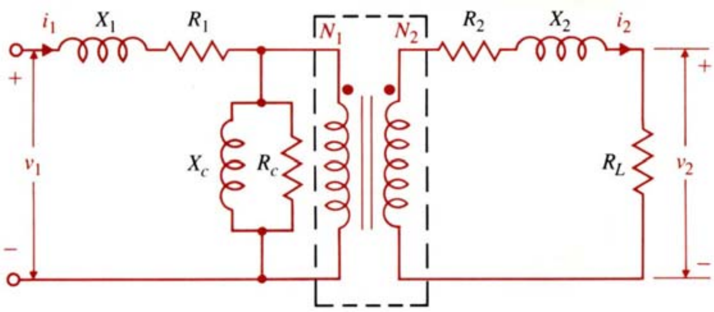
$$\because v_1 = L_1\frac{di_1}{dt} - L_{12}\frac{di_2}{dt},\ v_2 = L_{12}\frac{di_1}{dt} - L_2\frac{di_2}{dt}$$

(where $L_1 = \frac{\mu s}{l}N_1^2,\ L_2 = \frac{\mu s}{l}N_2^2,\ L_{12} = \frac{\mu s}{l}N_1 N_2$)

- For an ideal transformer $\Rightarrow$ No leakage flux $\therefore L_{12} = \sqrt{L_1 L_2}$
- For a real transformer $\therefore L_{12} = k\sqrt{L_1 L_2},\ k < 1$ (k : coefficient of coupling)

Transformers (4)

❖ Equivalent circuit



R_1, R_2 : winding resistance

X_1, X_2 : leakage inductive reactance

R_c : power loss due to hysteresis and eddy current

X_c : nonlinear inductive reactance due to the nonlinear magnetization behavior of the ferromagnetic core

A Moving Conductor in a Static Magnetic Field

$\bar{F}_m = q\bar{u} \times \bar{B}$

$\rightarrow$ Charge Seperation

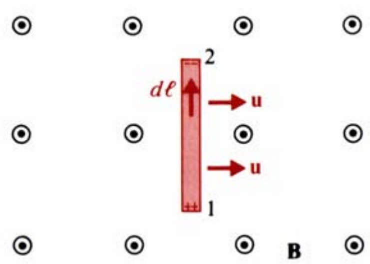
$\rightarrow$ Coulomb force of an attraction

$\rightarrow$ $\bar{F}_m$ and $\bar{F}_e$ will balance each other to be in equilibrium.

$\therefore$ Magnetic force per unit charge

$$\frac{\bar{F}_m}{q} = \bar{u} \times \bar{B},\ V = -\int \bar{E} \cdot d\bar{l},\ \bar{E} = -\frac{\bar{F}_m}{q}$$

$$\therefore\ V_{21} = \int_1^2 (\bar{u} \times \bar{B}) \cdot d\bar{l}$$



The emf generated around the closed loop is

$V' = \oint_C (\bar{u} \times \bar{B}) \cdot d\bar{l}$ $\rightarrow$ flux cutting emf

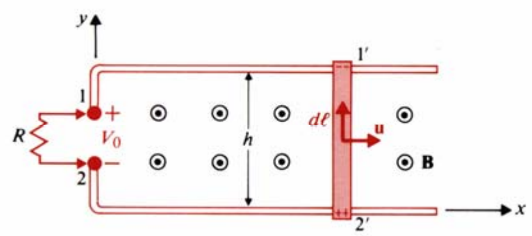
Ex 6-5) A Metal Bar Sliding Over Conducting Rails

$\bar{B} = \hat{z}B_0$, constant $\bar{u}$

a) $V_0 = V_1 - V_2 = \oint_C (\bar{u} \times \bar{B}) \cdot d\bar{l}$

$$= \int_{2'}^{1'} (\hat{x}u \times \hat{z}B_0) \cdot (\hat{y}dl)$$

$$= -uB_0 h$$



b) $I = \frac{V_0}{R},\quad P_l = I^2 R = \frac{(uB_0 h)^2}{R}$

c) mechanical power

$$F_m = I\int_{2'}^{1'} d\bar{l} \times \bar{B} = -\hat{x}IB_0 h$$

(I : negative direction to $d\bar{l}$)

$$\therefore\ P_m = \bar{F} \cdot \bar{u} = -\bar{F}_m \cdot \bar{u} = \frac{u^2 B_0^{\,2} h^2}{R}$$

A Moving Circuit in a Time Varying Magnetic Field (1)

$\bar{F}_m = q(\bar{E} + \bar{u} \times \bar{B})$

To an observer moving with C,

the force on q can be interpreted as caused by an electric field $\bar{E}'$,

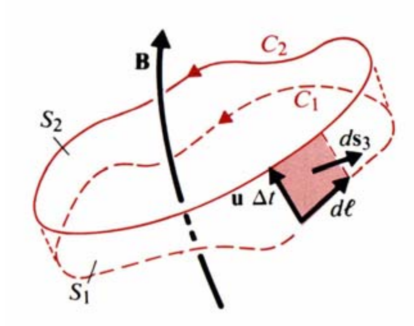
$\bar{E}' = \bar{E} + \bar{u} \times \bar{B}$

$$\oint_C \bar{E}' \cdot d\bar{l} = -\int_S \frac{\partial \bar{B}}{\partial t} d\bar{s} + \oint_C (\bar{u} \times \bar{B}) \cdot d\bar{l}$$ → General form of Faraday law

the emf induced in the moving frame of reference

transformer emf due to the time variation

motional emf due to the motion of the circuit in $\bar{B}$



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A Moving Circuit in a Time Varying Magnetic Field (2)

The time rate of chage of magnetic flux,

$$\frac{d\Phi}{dt} = \frac{d}{dt}\int_S \bar{B} \cdot d\bar{s}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\int_{S_2} \bar{B}(t+\Delta t) \cdot d\bar{s}_2 - \int_{S_1} \bar{B}(t) \cdot d\bar{s}_1\right]$$

cf) $\bar{B}(t+\Delta t) = \bar{B}(t) + \frac{\partial \bar{B}(t)}{\partial t}\Delta t + H.O.T.$ Taylor's series

$$\therefore \frac{d}{dt}\int_S \bar{B} \cdot d\bar{s} = \int_S \frac{\partial \bar{B}}{\partial t} \cdot d\bar{s} + \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\int_{S_2} \bar{B} \cdot d\bar{s}_2 - \int_{S_1} \bar{B} \cdot d\bar{s}_1 + H.O.T.\right]$$

assuming side surface S_3 as the area swept out by the conductor in time Δt

$$d\bar{S}_3 = d\bar{l} \times \bar{u}\ \Delta t$$

from divergence theorem

$$\int_V \nabla \cdot \bar{B}\ dv = \int_{S_2} \bar{B} \cdot d\bar{s}_2 - \int_{S_1} \bar{B} \cdot d\bar{s}_1 + \int_{S_3} \bar{B} \cdot d\bar{s}_3$$

$$\therefore \int_{S_2} \bar{B} \cdot d\bar{s}_2 - \int_{S_1} \bar{B} \cdot d\bar{s}_1 = -\Delta t \oint_C (\bar{u} \times \bar{B}) \cdot dl$$

$$\therefore \frac{d}{dt}\int_S \bar{B} \cdot d\bar{s} = \int_S \frac{\partial \bar{B}}{\partial t} \cdot d\bar{s} - \oint_C (\bar{u} \times \bar{B}) \cdot d\bar{l}$$

$$\therefore V' = \oint_C \tilde{E}' \bullet d\bar{l} = -\frac{d}{dt}\int_S \bar{B} \cdot d\bar{s} = -\frac{d\Phi}{dt}$$

Maxwell's Equation (1)

static	Time varying
$\nabla \times \vec{E} = 0$ $\nabla \bullet \vec{D} = \rho, \ \vec{D} = \varepsilon \vec{E}$	$\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ $\nabla \bullet \vec{D} = \rho$
$\nabla \times \vec{H} = \vec{J}$ $\nabla \bullet \vec{B} = 0, \ \vec{B} = \mu \vec{H}$	$\nabla \times \vec{H} = \vec{J} + \dfrac{\partial \vec{D}}{\partial t}$ $\nabla \bullet \vec{B} = 0$

Maxwell's Equation (2)

Note ① Continuity equation

$\nabla \bullet \vec{J} = 0$: for steady state current

$\nabla \bullet \vec{J} = -\dfrac{\partial \rho}{\partial t}$: time varying current

② Vector identity

$\nabla \bullet (\nabla \times \vec{H}) = 0 = \nabla \bullet \vec{J} \Rightarrow$ contradiction $\nabla \bullet \vec{J} = -\dfrac{\partial \rho}{\partial t}$

$\therefore \nabla \bullet (\nabla \times \vec{H}) = 0 = \nabla \bullet \vec{J} + \dfrac{\partial \rho}{\partial t} = \nabla \bullet \left(\vec{J} + \dfrac{\partial \vec{D}}{\partial t} \right)$, where $\rho = \nabla \bullet \vec{D}$

$\nabla \times \vec{H} = \vec{J} + \dfrac{\partial \vec{D}}{\partial t}$

Displacement current density. [A/m^2]
Time varying electric field and induced magnetic field→coupling

Cf) Lorentz force equation, $\vec{F} = q(\vec{E} + \vec{u} \times \vec{B})$

Integral Form of Maxwell's Equation

Cf) Differential form $\rightarrow$ Point function

$\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \Rightarrow$ Apply stokes's theorem over open surface S with contour C

$$\int_S (\nabla \times \vec{E}) \cdot \vec{ds} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot \vec{ds}$$

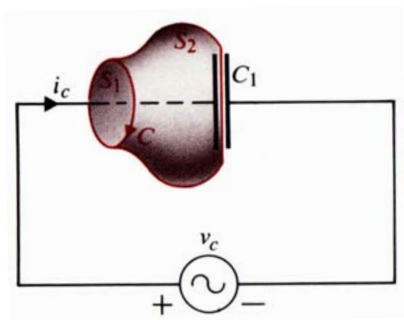
① $\oint_c \vec{E} \cdot \vec{dl} = -\oint_s \dfrac{\partial \vec{B}}{\partial t} \cdot \vec{ds} = -\dfrac{d\Phi}{dt}$: Faraday's law

② $\oint_c \vec{H} \cdot \vec{dl} = I + \oint_s \dfrac{\partial \vec{D}}{\partial t} \cdot \vec{ds}$: Ampere's circuital law

③ $\nabla \cdot \vec{D} = \rho \Rightarrow \oint_s \vec{D} \cdot \vec{ds} = Q$: Gauss law

④ $\nabla \cdot \vec{B} = 0 \Rightarrow \oint_s \vec{B} \cdot \vec{ds} = 0$: No isolated magnetic charge

Ex. 7-5



(a) Displacement current = conduction current

1 conduction current $\rightarrow$ current on the wire.

Apply circuit theorem

$$i_C = C_1 \frac{dv_c}{dt} = C_1 V_0 \omega \cos \omega t$$

2 Displacement current. Reminding

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Assuming the area A, plate separation d, permitivity μ, then $C_1 = \varepsilon \dfrac{A}{d}$

Assume $\vec{E}$ is uniform in the dielectric (ignoring fringing effects)

then

$$\vec{E} = \frac{v_c}{d}, \quad \vec{D} = \varepsilon \vec{E} = \varepsilon \frac{V_0}{d} \sin \omega t$$

$$i_D = \int_A \frac{\partial \vec{D}}{\partial t} \cdot \vec{ds} = \varepsilon \frac{A}{d} V_0 \omega \cos \omega t = C_1 V_0 \omega \cos \omega t = i_C$$

Ex. 7-5

(b) Magnetic field intensity reminding Ampere's law

$$\nabla\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t},\quad \oint_C \vec{H}\bullet\vec{dl}=I+\int_S\frac{\partial\vec{D}}{\partial t}\bullet\vec{ds}$$

① surface S1 with ring C

② surface S2 with ring C

① $\vec{D}=0,\ \oint_C \vec{H}\bullet\vec{dl}=2\pi r H_\phi$

(Symmetry around the wire along the contour C) $\Rightarrow$ constant H_ϕ

$$I=\int_S \vec{J}\bullet\vec{ds}=i_C=C_1V_0\omega\cos\omega t$$

② no conduction current, but displacement current

$$I=i_d,\quad \therefore H_\phi=\frac{C_1V_0}{2\pi r}\omega\cos\omega t$$

Potential Functions (1)

❖ Vector magnetic potential, $\vec{A}$

$\vec{B}=\nabla\times\vec{A}$ (Solenoidal nature of $\vec{B}$)

Vector identity $\nabla\bullet\vec{B}=0,\ \nabla\bullet(\nabla\times\vec{A})=0$

▷ Recall Faraday's law

Curl free

$$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}\quad \therefore\nabla\times\vec{E}=-\frac{\partial}{\partial t}(\nabla\times\vec{A})\Rightarrow\nabla\times\boxed{\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right)}=0$$

▷ Vector identity $\nabla\times(\nabla V)=0$

and reminding $\vec{E}=-\nabla V$ for electromagnetics

$\vec{E}+\frac{\partial\vec{A}}{\partial t}=-\nabla V$ for time varying i.e) $\vec{E}=-\nabla V-\frac{\partial\vec{A}}{\partial t}$ $[V/m]$

Potential Functions (2)

Cf) Static $\rightarrow \frac{\partial \vec{A}}{\partial t} = 0 \quad \therefore \vec{E} = -\nabla V$

Time varying $\vec{E}$ is induced by charge distribution ρ and time varying magnetic field $\rightarrow$ time varying current, $\vec{J}$

▷ $\vec{B}$ also depends on $\vec{A}$ $\quad \therefore \vec{E},\ \vec{B}$ are coupled

▷ $V = \frac{1}{4\pi\varepsilon_0}\int_{v'} \frac{\rho}{R} dv',\ \vec{A} = \frac{\mu_0}{4\pi}\int_{v'} \frac{\vec{J}}{R} dv'$: From the static condition

These are solution of poisson equation

$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$ and $\nabla^2 \vec{A} = -\mu_0 \vec{J}$

▷ The time-retardation effects associated with the finite velocity of propagation is neglected

Potential Functions (3)

Quasi-static fields

- ρ and $\vec{J}$ vary slowly with time
- the range of interest R is small compared to the wavelength

cf) Frequency is high, and R is large compared to wavelength

: time-retardation effect must be included.

From the equations

$$\vec{B} = \nabla\times\vec{A},\ \vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t},\ \nabla\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad (\vec{B} = \mu\vec{H},\ \vec{D} = \varepsilon\vec{E})$$

$$\boxed{\nabla\times\nabla\times\vec{A} = \mu\vec{J} + \mu\varepsilon\frac{\partial}{\partial t}\left(-\nabla V - \frac{\partial \vec{A}}{\partial t}\right)}$$

Recalling vector identity

$$\nabla\times\nabla\times\vec{A} = \nabla(\nabla\bullet\vec{A}) - \nabla^2\vec{A}$$

Potential Functions (4)

$$\therefore \nabla(\nabla \bullet \vec{A}) - \nabla^2 \vec{A} = \mu \vec{J} - \nabla\left(\mu\varepsilon \frac{\partial V}{\partial t}\right) - \mu\varepsilon \frac{\partial^2 \vec{A}}{\partial t^2}$$

$$\nabla^2 \vec{A} - \mu\varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J} + \nabla\left(\nabla \bullet \vec{A} + \mu\varepsilon \frac{\partial V}{\partial t}\right)$$

- we only designated $\nabla \times \vec{A} = \vec{B}$ but we are free to choose $\nabla \bullet \vec{A}$
- vector $\vec{A}$ will be specified by giving $\nabla \times \vec{A}$ and $\nabla \bullet \vec{A}$

- let $\boxed{\nabla \bullet \vec{A} + \mu\varepsilon \frac{\partial V}{\partial t} = 0}$, for static $\frac{\partial V}{\partial t} = 0 \;\; \therefore \nabla \bullet \vec{A} = 0$

Lorentz gauge for potentials

Potential Functions (5)

cf) For static, $\frac{\partial V}{\partial t} = 0, \;\; \frac{\partial \vec{A}}{\partial t} = 0$

$\therefore \nabla \bullet \vec{A} = 0$

then vector poisson equation

$\nabla^2 \vec{A} = -\mu_0 \vec{J}$

- Then nonhomogeneous wave equation for vector potential becomes

$$\boxed{\nabla^2 \vec{A} - \mu\varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J}}$$: Vector potential wave equation

Potential Functions(6)

∨ Scalar potential wave equation

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t},\ \nabla \bullet \vec{D} = \rho \Rightarrow -\nabla \bullet \varepsilon \left(\nabla V + \frac{\partial \vec{A}}{\partial t} \right) = \rho$$

$$\therefore \nabla^2 V + \frac{\partial}{\partial t}(\nabla \bullet \vec{A}) = -\frac{\rho}{\varepsilon},\ \nabla \bullet \vec{A} = -\mu\varepsilon \frac{\partial V}{\partial t}$$

$$\therefore \boxed{\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon}}$$

Boundary Condition (1)

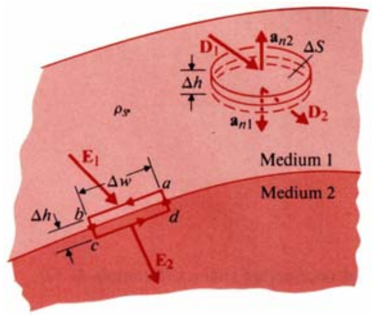
⦿ Electric field's boundary condition

$$\oint_c \vec{E} \bullet \vec{dl} = -\int_s \frac{\partial \vec{B}}{\partial t} \bullet \vec{ds} \ \ldots① \qquad \oint_s \vec{D} \bullet \vec{ds} = \int_v \rho dv \qquad \ldots②$$

From equation ①

$\int_s \frac{\partial \vec{B}}{\partial t} \bullet \vec{ds} \to 0$ when $\Delta h \to 0,$ since area $S \to 0$

$$\therefore \boxed{E_{1t} = E_{2t}} \quad (E_{1t}\Delta w - E_{2t}\Delta w = 0)$$



Boundary Condition (2)

From equation ②

$$\oint_s \vec{D}\cdot\vec{ds} = (\vec{D_1}\cdot\widehat{n_2} + \vec{D_2}\cdot\widehat{n_1})\Delta S = \widehat{n_2}\cdot(\vec{D_1}-\vec{D_2})\Delta S = \rho_s \Delta S$$

$$\therefore \boxed{\widehat{n_2}\cdot(\vec{D_1}-\vec{D_2}) = \rho_s},\ D_{2n} - D_{1n} = \rho_s$$

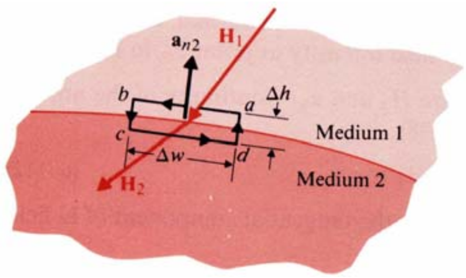
Magnetic field's boundary conditions

$$\oint_c \vec{H}\cdot\vec{dl} = \int_s \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)\cdot\vec{ds}$$

$$\therefore \vec{H_1}\cdot\vec{\Delta w} + \vec{H_2}\cdot(-\vec{\Delta w}) = J_{sn}\Delta w\ ,\ H_{1t} - H_{2t} = J_{sn}$$

$i.e)\ \widehat{n_2}\times(\vec{H_1}-\vec{H_2}) = \vec{J_s}$

cf) $\widehat{n_2}$ & $\vec{J_s}$ are perpendicular to ea



Boundary Condition (3)

note)

The tangential component of the $\vec{H}$ field is discontinuous across an interface where a free surface current exists

if both media have finite conductivity, currents are defined by volume current density

→ surface currents do not exists

→ $H_{1t} = H_{2t}$

i.e) discontinuous only for interface with an ideal perfect conductor or super conductor.

$$\nabla\cdot\vec{B} = 0 \quad \therefore \boxed{B_{1n} = B_{2n}}$$

Interface Between Two Lossless Linear Media

Linear media → permitivity : ε, permeability : μ

Lossless → σ=0

Assume, at interface, no free charge → ρ_S=0

no surface currents → J_S=0

$$E_{1t} = E_{2t} \Rightarrow \frac{D_{1t}}{D_{2t}} = \frac{\varepsilon_1}{\varepsilon_2} \quad , \quad H_{1t} = H_{2t} \Rightarrow \frac{B_{1t}}{B_{2t}} = \frac{\mu_1}{\mu_2}$$

$$D_{1n} = D_{2n} \Rightarrow \varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}, \quad B_{1n} = B_{2n} \Rightarrow \mu_1 H_{1n} = \mu_2 H_{2n}$$

Interface between a Dielectric and Perfect Conductor (1)

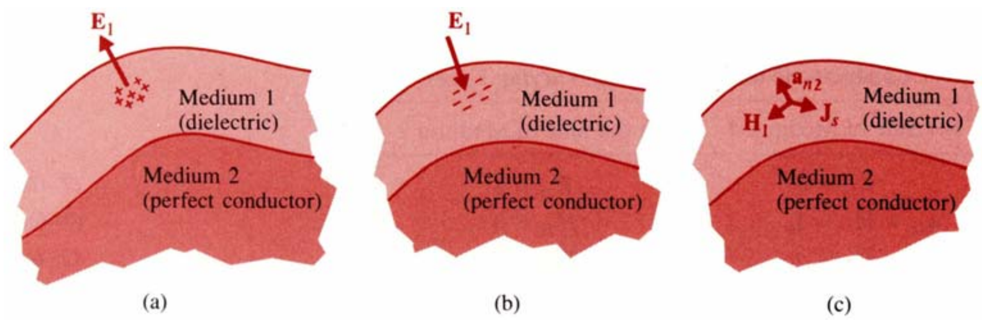
Good conductor → perfect conductor

Interior of perfect conductor (surface charge only) : $\vec{E}$

$(\vec{E}, \vec{D}) \Rightarrow (\vec{B}, \vec{H})$ are zero in the interior of a conductor

cf) In static case, $\vec{E}$, $\vec{D}$ may be zero, but $\vec{H}$, $\vec{B}$ may not be zero.

$$\vec{E}_2 = 0, \ \vec{H_2} = 0, \ \vec{D}_2 = 0, \ \vec{B_2} = 0$$



Interface between a Dielectric and Perfect Conductor (2)

① $E_{1t} = 0,\ E_{2t} = 0$

② $\widehat{n_2} \times (\overrightarrow{H_1} - \overrightarrow{H_2}) = \overrightarrow{J_s},\ H_{2t} = 0$

$H_{1t} = J_{sn}$, if $J_{sn} = 0 \rightarrow H_{1t} = 0$

③ $\widehat{n_2} \bullet (\overrightarrow{D_1} - \overrightarrow{D_2}) = \rho_s,\ D_{2n} = 0,\ D_{1n} = \rho_s$

④ $B_{1n} = 0,\ B_{2n} = 0$

note) $\widehat{n_2}$: outward normal from medium2

At an interface between a dielectric and a perfect conductor

: $\overrightarrow{E_1}$ is normal to and points away from(into) the conductor surface

$$\left|\overrightarrow{E_1}\right| = E_{1n} = \frac{\rho_s}{\varepsilon_1}$$

: $\overrightarrow{H_1}$ is tangential to the interface with a magnitude of

$$\left|\overrightarrow{H_1}\right| = \left|\overrightarrow{H_{1t}}\right| = \left|\overrightarrow{J_s}\right|$$

cf) direction $\widehat{n_2} \times (\overrightarrow{H_1} - \overrightarrow{H_2}) = \overrightarrow{J_s}$

Wave Equation and Their Solutions (1)

Wave equation : $\nabla^2 \vec{A} - \mu\varepsilon \dfrac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J}$

$$\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon}$$

Solution :

- Assume an elemental point charge at time t, $\rho(t)\Delta v'$ located at the origin of the coordinates.
- Spherical coordinates.
- V depends only on R. and t because of spherical symmetry. (No dependence on ϕ)
- Except at the origin,

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial V}{\partial R}\right) - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = 0$$

Wave Equation and Their Solutions (2)

New variable

$$V(R,t)=\frac{1}{R}U(R,t)$$

$$R^2\frac{\partial}{\partial R}\left(\frac{1}{R}U\left(R,t\right)\right)=R^2\left[-\frac{1}{R^2}U+\frac{1}{R}\frac{\partial U}{\partial R}\right]=-U+R\frac{\partial U}{\partial R}$$

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left[-U+R\frac{\partial U}{\partial R}\right]=\frac{1}{R^2}\left[-\frac{\partial U}{\partial R}+\frac{\partial U}{\partial R}+R\frac{\partial^2 U}{\partial R^2}\right]$$

$$\therefore \frac{1}{R}\frac{\partial^2 U}{\partial R^2}-\mu\varepsilon\frac{1}{R}\frac{\partial^2 U}{\partial t^2}=0,\quad \text{i.e)}\frac{\partial^2 U(R,t)}{\partial R^2}-\mu\varepsilon\frac{\partial^2 U(R,t)}{\partial t^2}=0$$

Any function of $(t-R\sqrt{\mu\varepsilon})$ or of $(t+R\sqrt{\mu\varepsilon})$ will satisfy the differential equation

$f(t-R\sqrt{\mu\varepsilon})$ is a wave equation which travels away from the origin

Wave Equation and Their Solutions (3)

$f(t+R\sqrt{\mu\varepsilon})$ is a wave equation which travels to the origin $\rightarrow$ physical nonsense

$$\therefore U(R,t)=f(t-R\sqrt{\mu\varepsilon})$$

the function at $R+\Delta R$ at a later $t+\Delta t$.

$$U(R+\Delta R,t+\Delta t)=f\left[t+\Delta t-(R+\Delta R)\sqrt{\mu\varepsilon}\right]=f(t-R\sqrt{\mu\varepsilon})$$

$$if\ \ \Delta t=\Delta R\sqrt{\mu\varepsilon}.$$

$$\frac{\Delta R}{\Delta t}=\frac{1}{\sqrt{\mu\varepsilon}}\Rightarrow \lim_{\Delta t\to 0}\frac{\Delta R}{\Delta t}=u=\frac{1}{\sqrt{\mu\varepsilon}}$$: velocity of propagation

$$V(R,t)=\frac{1}{R}f\left(t-\frac{R}{u}\right)$$

Determine $f\left(t-\frac{R}{u}\right)$

Wave Equation and Their Solutions (4)

Recall potential function induced by a static point charge $\rho(t)\Delta v$ at the origin

$$\boxed{\Delta V(R) = \frac{\rho(t)\Delta v'}{4\pi\varepsilon R}}, \quad \Delta f\left(t - \frac{R}{u}\right) = \frac{\rho\left(t - \frac{R}{u}\right)\Delta v'}{4\pi\varepsilon}$$

$$\therefore \boxed{V(R,t) = \frac{1}{4\pi\varepsilon}\int_{v'} \frac{\rho\left(t - \frac{R}{u}\right)}{R} dv'}$$: Retarded scalar potential

Scalar potential at a distance R from the source at time t

$\rightarrow$ Depends on the value of charge distribution at an earlier time $\left(t - \frac{R}{u}\right)$

Retarded vector potential

$$\vec{A}(R,t) = \frac{\mu}{4\pi}\int_{v'} \frac{\vec{J}\left(t - \frac{R}{u}\right)}{R} dv'$$

Source Free Wave Equation

If the wave is in a simple (linear, isotropic and homogeneous) non conducting medium. i.e) ε, μ (σ=0)

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}, \quad \nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \bullet \vec{E} = 0 \quad , \quad \nabla \bullet \vec{H} = 0$$

From vector identity

$$\nabla \times \nabla \times \vec{E} = -\mu \frac{\partial}{\partial t}(\nabla \times \vec{H}) = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$cf)\ \vec{A} \times \vec{B} \times \vec{C} = \vec{B}(\vec{A} \bullet \vec{C}) - \vec{C}(\vec{A} \bullet \vec{B}) = \vec{B}(\vec{A} \bullet \vec{C}) - (\vec{A} \bullet \vec{B})\vec{C}$$

$$\nabla(\nabla \bullet \vec{E}) - (\nabla \bullet \nabla)\vec{E} = -\nabla^2 \vec{E}$$

$$\therefore \nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \ \ if \ \ \mu\varepsilon = \frac{1}{u^2}$$

$$\boxed{\nabla^2 \vec{E} - \frac{1}{u^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0, \ \nabla^2 \vec{H} - \frac{1}{u^2}\frac{\partial^2 \vec{H}}{\partial t^2} = 0}$$: Homogeneous vector wave equation

Time Harmonic Fields

Maxwell's equations

- linear differential equations
- sinusoidal time variation of source functions at given frequency
- $\vec{E},\ \vec{H}$ are sinusoidal with the same frequency

Time harmonic $\rightarrow$ steady state sinusoidal

Phasors : Amplitude and phase information

$\rightarrow$ independent of time

cf) $e^{j\omega t}$: time dependent factor

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Time Harmonic Electromagnetics (1)

Vector phasors of field vectors : depend on space coordinates

$$\vec{E}(x,y,z;t) = \mathrm{Re}\left[\vec{E}(x,y,z)e^{j\omega t}\right],$$

where $\vec{E}(x,y,z)$: vector phasor : complex quantity

$$\frac{\partial}{\partial t}\vec{E}(x,y,z;t) = \mathrm{Re}\left[j\omega\vec{E}(x,y,z)e^{j\omega t}\right]$$

where $j\omega\vec{E}(x,y,z)$: vector phasor

$$\int\vec{E}(x,y,z;t)dt = \mathrm{Re}\left[\frac{\vec{E}(x,y,z)}{j\omega}e^{j\omega t}\right]$$

where $\dfrac{\vec{E}(x,y,z)}{j\omega}$: vector phasor

i.e) $\dfrac{\partial}{\partial t} \Rightarrow j\omega,\ \dfrac{\partial^2}{\partial t^2} \Rightarrow (j\omega)^2,\ \int \Rightarrow \dfrac{1}{j\omega}$

Time Harmonic Electromagnetics (2)

Maxwell's equations

Vector field phasors ($\vec{E},\ \vec{H}$)

Source phasors ($\rho,\ \vec{J}$), Simple (linear, isotropic and homogeneous) media

$$\left.\begin{aligned} &\nabla\times\vec{E} = -j\omega\mu\vec{H},\ \nabla\times\vec{H} = \vec{J} + j\omega\varepsilon\vec{E} \\ &\nabla\bullet\vec{E} = \frac{\rho}{\varepsilon},\ \nabla\bullet\vec{H} = 0 \end{aligned}\right\}\ \text{Assuming } e^{j\omega t}$$

Time harmonic wave equation for V and $\vec{A}$

$$\left.\begin{aligned} &\nabla^2 V + k^2 V = -\frac{\rho}{\varepsilon} \\ &\nabla^2\vec{A} + k^2\vec{A} = -\mu\vec{J} \end{aligned}\right\}$$

Non-homogeneous helmholtz's equations

where $k^2 = \omega^2\mu\varepsilon,\ k = \omega\sqrt{\mu\varepsilon} = \dfrac{\omega}{u}$: wave-number

Time Harmonic Electromagnetics (3)

$$cf)\ \nabla^2\vec{A} - \mu\varepsilon\frac{\partial^2\vec{A}}{\partial t^2} = -\mu\vec{J} \qquad (\nabla\bullet\vec{A} + \mu\varepsilon\frac{\partial V}{\partial t} = 0 \Rightarrow \nabla\bullet\vec{A} + j\omega\mu\varepsilon V = 0)$$

$$\nabla^2 V - \mu\varepsilon\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon}$$

$$\nabla^2\vec{E} - \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

$$\nabla^2\vec{H} - \mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2} = 0$$

Phasor solution

$$V(R,t) = \frac{1}{4\pi\varepsilon}\int_{v'}\frac{\rho e^{j\omega(t-\frac{R}{u})}}{R}dv' \Rightarrow V(R)e^{j\omega t} = \frac{1}{4\pi\varepsilon}\int_{v'}\frac{\rho e^{-j\frac{\omega}{u}R}}{R}dv'\cdot e^{j\omega t}$$

$$\left\{\begin{aligned} &V(R) = \frac{1}{4\pi\varepsilon}\int_{v'}\frac{\rho e^{-jkR}}{R}dv'\ \text{[V]} \\ &\vec{A}(R) = \frac{\mu}{4\pi}\int_{v'}\frac{\vec{J}e^{-jkR}}{R}dv'\ \text{[Wb/m]} \end{aligned}\right.$$

Expressions for the retarded scalar and vector potentials due to time harmonic sources

Time Harmonic Electromagnetics (4)

cf) $e^{-jkR} = 1 - jkR - \frac{k^2R^2}{2} + \ldots$: Taylor series expansion.

$$k = \frac{\omega}{u} = \frac{2\pi f}{u} = \frac{2\pi}{\lambda}, \ u = f\lambda$$

$if \ kR = 2\pi\frac{R}{\lambda} \ll 1 \Rightarrow e^{-jkR} = 1$, then $V(R) = \frac{1}{4\pi\varepsilon}\int_{v'}\frac{\rho}{R}dv' \Rightarrow$ static potential

Procedure for determining the electric and magnetic fields due to time harmonic charge and current distributions

1. Find phasors $V(R)$ and $A(R)$
2. Find phasors $\vec{E}(R) = -\nabla V - j\omega\vec{A}$ $\quad$ $cf)\vec{E} = -\nabla V - \frac{\partial\vec{A}}{\partial t}$

   $\vec{B}(R) = \nabla\times\vec{A}$
3. Find instantaneous $\vec{E}(R,t)$

Source-free Fields in Simple Media (1)

Source free fields in simple media

$$\begin{cases} \nabla\times\vec{E} = -j\omega\mu\vec{H} \\ \nabla\times\vec{H} = j\omega\varepsilon\vec{E} \\ \nabla\bullet\vec{E} = 0 \\ \nabla\bullet\vec{H} = 0 \end{cases}$$

$$\begin{cases} \nabla^2\vec{E} + k^2\vec{E} = 0 \\ \nabla^2\vec{H} + k^2\vec{H} = 0 \text{ and } k^2 = \omega^2\mu\varepsilon \end{cases}$$

Homogeneous vector Helmholtz's equation

Principle of duality : Source free Maxwell's equations in a simple media are invariant under the linear transformation

$$\vec{E}' = \eta\vec{H}, \ \vec{H}' = -\frac{\vec{E}}{\eta}, \ \eta = \sqrt{\frac{\mu}{\varepsilon}}$$

Source-free Fields in Simple Media (2)

If simple medium is conducting i.e) $\sigma \neq 0 \Rightarrow \vec{J} = \sigma \vec{E}$

$$\therefore \nabla \times \vec{H} = (\sigma + j\omega\varepsilon)\vec{E} = j\omega\left(\varepsilon + \frac{\sigma}{j\omega}\right)\vec{E} = j\omega\varepsilon_c \vec{E}$$

and $\boxed{\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega} \text{ [F/m]}}$: complex permitivity

cf) · out of phase polarization : power loss to overcome a fractional damping mechanism caused by the inertia the charged particle

· finite conductivity $\rightarrow$ ohmic losses

Complex permitivity

$\varepsilon_c = \varepsilon' - j\varepsilon''$ [F/m] , where ε'' : out of phase polarization and finite conductivity

$\Rightarrow$ equivalent conductivity $\boxed{\sigma = \omega\varepsilon''}$ $\leftarrow$ representing all losses

Source-free Fields in Simple Media (3)

Complex permeability : out of phase component of magnetization

$\mu = \mu' - j\mu''$, where $\mu' \gg \mu''$ for ferromagnetic materials

$\therefore \mu = \mu'$

Complex wavenumber

$k_c = \omega\sqrt{\mu\varepsilon_c} = \omega\sqrt{\mu(\varepsilon' - j\varepsilon'')}$: in a lossy dielectric

Loss tangent

$$\begin{cases} \tan\delta_c = \dfrac{\varepsilon''}{\varepsilon'} \cong \dfrac{\sigma}{\omega\varepsilon}, \text{ where } \delta_c \text{ : loss angle} \\ \text{loss tangent } \dfrac{\varepsilon''}{\varepsilon'} \end{cases}$$

Good conductor & Good insulator

$\sigma \gg \omega\varepsilon$: Good conductor

$\sigma \ll \omega\varepsilon$: Good insulator

Source-free Fields in Simple Media (4)

Cf) Electric hertz vector, $\vec{\pi}$

① $\vec{A} = \mu\varepsilon \dfrac{\partial \vec{\pi}}{\partial t}, \; V = -\nabla \bullet \vec{\pi}$

: combine the vector and scalar potential and satisfy the Lorentz condition

$$\begin{cases} \vec{E} = -\nabla V - \dfrac{\partial \vec{A}}{\partial t} \\ \nabla \bullet \vec{A} + \mu\varepsilon \dfrac{\partial V}{\partial t} = 0 \end{cases}$$

② combine continuity equation with $\vec{J}$ and ρ

$$\nabla \bullet \vec{J} + \frac{\partial \rho}{\partial t} = 0, \quad \boxed{\vec{J} = \frac{\partial \vec{P}}{\partial t}, \; \rho = -\nabla \bullet \vec{P}}$$

③ Single vector equation

$$\nabla^2 \vec{\pi} - \mu\varepsilon \frac{\partial^2 \vec{\pi}}{\partial t^2} = -\frac{\vec{P}}{\varepsilon}$$

④ $\vec{E} = \nabla(\nabla \bullet \vec{\pi}) - \mu\varepsilon \dfrac{\partial^2 \vec{\pi}}{\partial t^2} = \nabla \times \nabla \times \vec{\pi} - \dfrac{\vec{P}}{\varepsilon}$

$$\vec{H} = \varepsilon \nabla \times \frac{\partial \vec{\pi}}{\partial t}$$

The electromagnetic spectrum

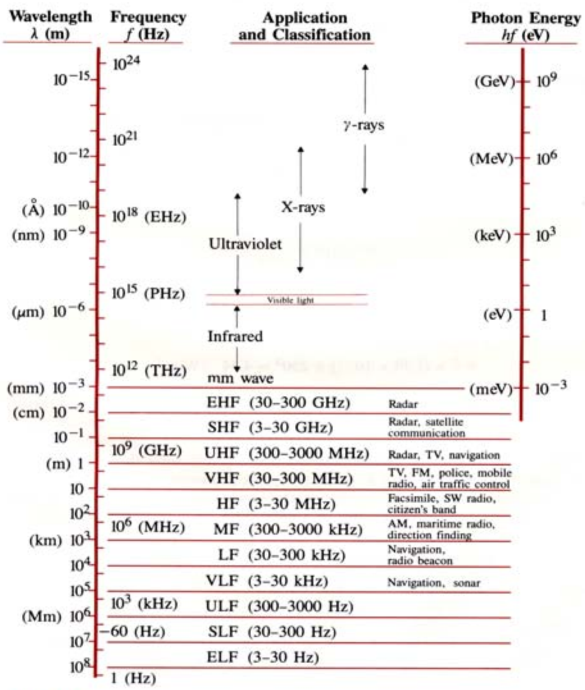

FIGURE 7-9
Spectrum of electromagnetic waves.

TABLE 7-5
Band Designations for Microwave Frequency Ranges

Old†	New	Frequency Ranges (GHz)
Ka	K	26.5–40
K	K	20–26.5
K	J	18–20
Ku	J	12.4–18
X	J	10–12.4
X	I	8–10
C	H	6–8
C	G	4–6
S	F	3–4
S	E	2–3
L	D	1–2
UHF	C	0.5–1

† Because the old band designations have been in wide use since the early days of radar, they are still in common use because of habit.