Field and Wave Electromagnetic

Chapter8

Plane Electromagnetic Waves

Introduction (1)

- Homogeneous vector wave equation

$$\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

In free space $\mu = \mu_0 = 4\pi \times 10^{-7}, \ \varepsilon = \varepsilon_0 = \frac{1}{36\pi} \times 10^{-9}$

Let $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \cong 3 \times 10^8$ m/s $\Rightarrow$ velocity of wave propagation in free space

- Time harmonic, unbounded homogeneous medium
  - · Parameter : intrinsic impedance, attenuation constant, phase constant
  - · Skin depth : the depth of wave penetration into a good conductor
  - · Poynting vector : power flux density
  - · plane wave incident normally on a plane boundary
  - · reflection and refraction on a plane boundary
  - · no reflection and total reflection

Introduction (2)

- Uniform plane wave
  : Assuming the same direction, magnitude, phase of fields in infinite $\perp$ planes the direction of propagation

Plane Wave in Lossless Media (1)

- Homogeneous vector Helmholtz's equation
  (assuming time harmonic $e^{j\omega t}$)

$$\nabla^2 \vec{E} + k_0^2 \vec{E} = 0, \text{ where } k_0 = \omega_0 \sqrt{\mu_0 \varepsilon_0} = \frac{\omega}{c} \text{ [rad/m]}$$

- In cartesian coordinates

$$\nabla^2 \vec{E} + k_0^2 \vec{E} = 0 \Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k_0^2 \right) E_x = 0$$

- Uniform plane wave→uniform E_x over plane surface $\perp$ z

$$i.e)\ \frac{\partial^2 E_x}{\partial x^2} = 0, \quad \frac{\partial^2 E_x}{\partial y^2} = 0$$

$$\therefore \frac{\partial^2 E_x}{\partial z^2} + k_o^2 E_x = 0$$: ordinary differential equation

$$\therefore E_x(z) = E_x^+(z) + E_x^-(z) = E_0^+ e^{-jk_0 z} + E_0^- e^{+jk_0 z}$$

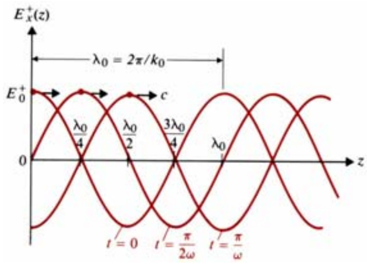
where E_0^+ and E_0^- : arbitrary constant satisfying boundary condition

Plane Wave in Lossless Media (2)

- Assume cos(ωt), E_0^+ : real (zero reference phase at z =0)

$$E_x^+(z,t) = \mathrm{Re}[E_x^+(z)e^{j\omega t}] = \mathrm{Re}[E_0^+ e^{j(\omega t - k_0 z)}] = E_0^+ \cos(\omega t - k_0 z)$$

- Phase velocity : the velocity of propagation of an equiphase front



$\omega t - k_0 z = A$: constant phase

$$\boxed{u_p = \frac{dz}{dt} = \frac{\omega}{k_0} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c}$$

: the velocity of propagation of an equiphase front=the velocity of light

- wavenumber

$$k_0 = \frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda_0}$$: number of wavelength in a complete circle

Plane Wave in Lossless Media (3)

- wave length : $\lambda_0 = \frac{2\pi}{k_0}$
- $E_0^- e^{jk_0 z}$: traveling wave in the -z direction

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \frac{\partial}{\partial z} \\ E_x^+(z) & 0 & 0 \end{vmatrix} = -j\omega\mu_0(\hat{x}H_x^+ + \hat{y}H_y^+ + \hat{z}H_z^+)$$

$$\therefore H_x^+ = 0,\ H_y^+ = -\frac{1}{j\omega\mu_0}\frac{\partial E_x^+(z)}{\partial z},\ H_z^+ = 0$$

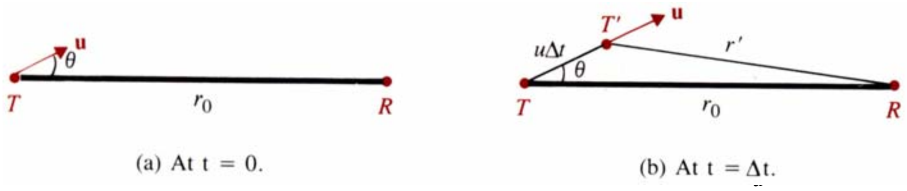
$$\frac{\partial E_x^+(z)}{\partial z} = \frac{\partial}{\partial z}(E_0^+ e^{-jk_0 z}) = -jk_0 E_x^+(z)$$

$$\therefore H_y^+(z) = \frac{k_0}{\omega\mu_0}E_x^+(z) = \frac{1}{\eta_0}E_x^+(z) \Rightarrow \boxed{\eta_0 = \frac{\omega\mu_0}{k_0}}$$

$$\therefore \boxed{\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \cong 120\pi = 377\Omega}$$: intrinsic impedance of the free space

$$\vec{H}(z,t) = \hat{y}H_y^+(z,t) = \hat{y}\,\mathrm{Re}\left[H_y^+(z)e^{j\omega t}\right] = \hat{y}\frac{E_0^+}{\eta_0}\cos(\omega t - k_0 z)$$

Doppler Effect (1)



- Assume transmitter moves at the velocity of $\vec{u}$
- Wave radiated at $t=0$ arrives at the Rx at $t=t_1$. $t_1 = \frac{r_0}{c}$
- Wave radiated at $t=\triangle t$ arrives at the Rx at $t=t_2$.

$$t_2 = \Delta t + \frac{r'}{c} = \Delta t + \frac{1}{c}\left[r_0^2 - 2r_0 u\Delta t\cos\theta + (u\Delta t)^2\right]^{1/2}$$

$$\text{if } (u\Delta t)^2 \ll r_0^2,\ t_2 \simeq \Delta t + \frac{r_0}{c}\left(1 - \frac{u\Delta t}{r_0}\cos\theta\right)$$

$$cf)\ (1-x)^{1/2} \simeq 1 - \frac{1}{2}x$$

- the time elapsed at Rx, corresponding to △t at Tx is

$$\Delta t' = t_2 - t_1 = \Delta t\left(1 - \frac{u}{c}\cos\theta\right)$$

Doppler Effect (2)

- Assume △t is period of the time harmonic source

$$i.e)\ \Delta t = \frac{1}{f}$$

then $f' = \frac{1}{\Delta t'} = \frac{f}{1 - \frac{u}{c}\cos\theta} \simeq f\left(1 + \frac{u}{c}\cos\theta\right) \quad \because \left(\frac{u}{c}\right)^2 \ll 1$

TEM Wave (1)

❖ Uniform plane wave characterized by $\vec{E} = \hat{x}E_x,\ \vec{H} = \hat{y}H_y$ propagating along *z* axis
→ particular case of a transverse electromagnetic wave

❖ General form of TEM wave

$$\vec{E}(x,y,z) = \vec{E}_0 e^{-j\vec{k}\cdot\vec{r}} = \vec{E}_0 e^{-jk_x x} e^{-jk_y y} e^{-jk_z z}$$

satisfies the homogeneous Helmholtz's equation, provided that

$$k_x^2 + k_y^2 + k_z^2 = \omega^2 \mu\varepsilon$$

$$\begin{cases} \text{Wave number vector : } \vec{k} = \hat{x}k_x + \hat{y}k_y + \hat{z}k_z = k\hat{n} \\ \text{Position vector from the origin : } \vec{r} = \hat{x}x + \hat{y}y + \hat{z}z \end{cases}$$

when $\hat{n}\cdot\vec{r} =$ constant is a plane of constant phase and uniform amplitude for the wave.

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TEM Wave (2)

❖

$\nabla\cdot\vec{E} = 0$ (Source free)

$$\nabla\cdot\vec{E} = \nabla\cdot(\vec{E}_0 e^{-j\vec{k}\cdot\vec{r}}) = (\nabla\cdot\vec{E}_0)\cdot e^{-j\vec{k}\cdot\vec{r}} + \vec{E}_0\cdot\nabla(e^{-j\vec{k}\cdot\vec{r}}) = 0$$

$\because$ plane wave $\rightarrow \vec{E}_0$ is a constant vector

$$\therefore \vec{E}_0\cdot\nabla(e^{-j\vec{k}\cdot\vec{r}}) = 0$$

$$\nabla(e^{-j\vec{k}\cdot\vec{r}}) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)(e^{-j(k_x x + k_y y + k_z z)}) = -j(\hat{x}k_x + \hat{y}k_y + \hat{z}k_z)e^{-j(k_x x + k_y y + k_z z)}$$

$$= -j\vec{k}e^{-j\vec{k}\cdot\vec{r}}$$

$$\therefore -j(\vec{E}_0\cdot\vec{k})e^{-j\vec{k}\cdot\vec{r}} = 0$$

$\therefore \vec{k}\cdot\vec{E}_0 = 0 \Rightarrow \vec{E}_0$ is transverse to the direction of propagation

TEM Wave (3)

❖ $$\vec{H}(\vec{r}) = -\frac{1}{j\omega\mu}\nabla\times\vec{E}(\vec{r}) = -\frac{1}{j\omega\mu}(-j\vec{k})\times\vec{E}(\vec{r})$$

$$= \frac{k}{\omega\mu}\hat{n}\times\vec{E}(\vec{r}) = \frac{1}{\eta}\hat{n}\times\vec{E}(\vec{r}), \quad \text{where } \eta = \frac{\omega\mu}{k} = \sqrt{\frac{\mu}{\varepsilon}}$$

$$\therefore \vec{H}(\vec{r}) = \frac{1}{\eta}(\hat{n}\times\vec{E}_0)e^{-j\vec{k}\cdot\vec{r}}$$

$cf)$

$$\nabla\times(\psi\vec{A}) = \psi\nabla\times\vec{A} + \nabla\psi\times\vec{A}$$

$$\therefore \nabla\times(\vec{E}_0 e^{-j\vec{k}\cdot\vec{r}}) = e^{-j\vec{k}\cdot\vec{r}}\nabla\times\vec{E}_0 + \nabla e^{-j\vec{k}\cdot\vec{r}}\times\vec{E}_0$$

$$= -j\vec{k}\times\vec{E}_0 e^{-j\vec{k}\cdot\vec{r}} = -jk\hat{n}\times\vec{E}(\vec{r})$$

Remark : A uniform plane wave propagating in an arbitrary direction $\hat{n}$ is a TEM wave with $\vec{E}\perp\vec{H}$ and $\vec{E}$ and $\vec{H}$ are normal to $\hat{n}$

Polarization of Plane Waves (1)

❖ Linearly polarized

if) $\vec{E}$ vector of the plane wave is fixed in the x direction

$$i.e)\ \vec{E} = \hat{x}E_x$$

❖ Superposition of two linearly polarized waves

→One in the *x*-direction. The other in the *y*-direction and lagging 90° in time phase

$$\vec{E}(z) = \hat{x}E_1(z) + \hat{y}E_2(z) = \hat{x}E_{10}e^{-jkz} - \hat{y}jE_{20}e^{-jkz} \quad \text{(j implys } 90^\circ \text{ lagging in time)}$$

$$\vec{E}(z,t) = \text{Re}\{\left[\hat{x}E_1(z) + \hat{y}E_2(z)\right]e^{j\omega t}\}$$

$$= \hat{x}E_{10}\cos(\omega t - kz) + \hat{y}E_{20}\cos\left(\omega t - kz - \frac{\pi}{2}\right)$$

set *z=0*,

$$\vec{E}(0,t) = \hat{x}E_1(0,t) + \hat{y}E_2(0,t) = \hat{x}E_{10}\cos\omega t + \hat{y}E_{20}\sin\omega t$$

Polarization of Plane Waves (2)

the tip of the vector $\vec{E}(0,t)$ $\rightarrow$ traversing an elliptical locus in the counter clockwise direction

$$\cos\omega t = \frac{E_{10}(0,t)}{E_{10}}, \quad \sin\omega t = \frac{E_2(0,t)}{E_{20}} = \sqrt{1-\cos^2\omega t} = \sqrt{1-\left[\frac{E_1(0,t)}{E_{10}}\right]^2}$$

$$\therefore \left[\frac{E_2(0,t)}{E_{20}}\right]^2 + \left[\frac{E_1(0,t)}{E_{10}}\right]^2 = 1$$

if $E_{10} \neq E_{20}$: elliptically polarized
if $E_{10} = E_{20}$: circularly polarized

When $E_{10} = E_{20}$, the instantaneous angle α

$$\alpha = \tan^{-1}\frac{E_2(0,t)}{E_1(0,t)} = \omega t$$

Polarization of Plane Waves (3)

① right hand polarized.
   if $\vec{E}_2$ is lagging 90° in time phase
② left hand polarized
   if $\vec{E}_2$ is leading 90° in time phase

cf) time lagging

$$\vec{E}(z) = \hat{x}E_{10}e^{-jkz} + \hat{y}jE_{20}e^{-jkz}$$

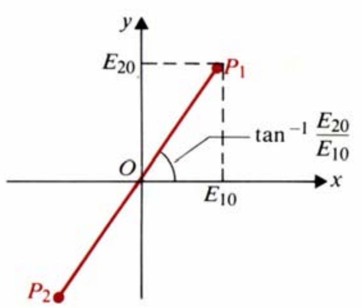
$$\vec{E}(0,t) = \hat{x}E_{10}\cos\omega t - \hat{y}E_{20}\sin\omega t$$

$$\alpha = \tan^{-1}\frac{E_2(0,t)}{E_1(0,t)} = -\omega t$$

❖ Linearly polarized

if $E_2(z)$ and $E_1(z)$ are in space quadrature , but in time phase

$$\vec{E}(0,t) = (\hat{x}E_{10} + \hat{y}E_{20})\cos\omega t$$



Plane Wave in Lossy Media (1)

- In a source-free lossy medium

$$\nabla^2 \vec{E} + k_c^2 \vec{E} = 0, \text{ where } k_c = \omega\sqrt{\mu\varepsilon_c}$$

- Conventional notation used in transmission-line theory

propagation constant

$$\gamma = jk_c = j\omega\sqrt{\mu\varepsilon_c}$$

$$\gamma = \alpha + j\beta = j\omega\sqrt{\mu\varepsilon}\left(1 + \frac{\sigma}{j\omega\varepsilon}\right)^{1/2} = j\omega\sqrt{\mu\varepsilon'}\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)^{1/2}$$

$$cf)\varepsilon_c = \varepsilon' - j\varepsilon''$$

- For a lossless medium, $\sigma = 0 \ (\varepsilon'' = 0, \varepsilon = \varepsilon')$

$$\alpha = 0, \ \beta = k = \omega\sqrt{\mu\varepsilon}$$

Then $\nabla^2 \vec{E} - \gamma^2 \vec{E} = 0$

Plane Wave in Lossy Media (2)

Solution → Uniform plane wave propagating in the $+z$ direction

$\vec{E} = \hat{x}E_x = \hat{x}E_0 e^{-\gamma z}$ assuming that $\vec{E}$ is linearly polarized in the x direction

$cf)1Np = 20\log e = 8.69dB$

Then $E_x = E_0 e^{-\alpha z} e^{-j\beta z} \Rightarrow \begin{cases} \alpha \text{ : attenuation constant (Np/m)} \\ \beta \text{ : phase constant (rad/m)} \end{cases}$

- Low loss dielectrics

: means a good but imperfect insulator, $\varepsilon'' \ll \varepsilon'$ or $\dfrac{\sigma}{\omega\varepsilon} \ll 1$

then $\gamma = \alpha + j\beta \simeq j\omega\sqrt{\mu\varepsilon'}\left[1 - j\dfrac{\varepsilon''}{2\varepsilon'} + \dfrac{1}{8}\left(\dfrac{\varepsilon''}{\varepsilon'}\right)^2\right]$

$\therefore \alpha \cong \dfrac{\omega\varepsilon''}{2}\sqrt{\dfrac{\mu}{\varepsilon'}}$ (Np/m), $\beta \cong \omega\sqrt{\mu\varepsilon'}\left[1 + \dfrac{1}{8}\left(\dfrac{\varepsilon''}{\varepsilon'}\right)^2\right]$

$cf)\varepsilon' = \varepsilon, \ \varepsilon'' = \dfrac{\sigma}{\omega}$

Plane Wave in Lossy Media (3)

- Intrinsic impedance

$$\eta_c = \sqrt{\frac{\mu}{\varepsilon'}}\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)^{-1/2} \simeq \boxed{\sqrt{\frac{\mu}{\varepsilon'}}\left(1 + j\frac{\varepsilon''}{2\varepsilon'}\right)} \Rightarrow \frac{E_x}{H_y}$$: complex number (E and H are not in time phase)

$$E = \frac{H}{\eta},\ H = \eta E = |\eta| E e^{j\phi}$$

$E \sim e^{j\omega t}, H \sim e^{j(\omega t + \phi)}$: lagging in time phase,

- phase velocity

$$u_p = \frac{\omega}{\beta} \cong \frac{1}{\sqrt{\mu\varepsilon'}}\left[1 - \frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right] \text{ (m/s)}$$

Plane Wave in Lossy Media (4)

- Good conductors

$$\frac{\sigma}{\omega\varepsilon} \gg 1$$

$$\gamma = j\omega\sqrt{\mu\varepsilon}\left(1 + \frac{\sigma}{j\omega\varepsilon}\right)^{1/2} \simeq j\omega\sqrt{\mu\varepsilon}\sqrt{\frac{\sigma}{j\omega\varepsilon}} = \sqrt{j}\sqrt{\omega\mu\sigma} = \frac{1+j}{\sqrt{2}}\sqrt{\omega\mu\sigma}$$

$$cf)\sqrt{j} = (e^{j\frac{\pi}{2}})^{1/2} = e^{j\frac{\pi}{4}} = \frac{1+j}{\sqrt{2}}$$

$$\therefore \gamma = \alpha + j\beta \simeq (1+j)\sqrt{\pi f \mu\sigma},\ \alpha = \beta \propto \sqrt{f}, \sqrt{\sigma}$$

- $\alpha = \beta = \sqrt{\pi f \mu\sigma}$

- $\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} \cong \sqrt{\frac{\mu}{-j\frac{\sigma}{\omega}}} = \sqrt{\frac{j\omega\mu}{\sigma}} = (1+j)\sqrt{\frac{\pi f \mu}{\sigma}} = (1+j)\frac{\alpha}{\sigma}\ [\Omega]$

$\eta_c = |\eta_c| e^{j\frac{\pi}{4}}$, magnetic field lags behind the electric field intensity by 45°

Plane Wave in Lossy Media (5)

- phase velocity

$$u_p = \frac{\omega}{\beta} \simeq \sqrt{\frac{2\omega}{\mu\sigma}}$$

$e.g)$ copper $\sigma = 5.8\times10^{7}$ [s/m], $\mu = 4\pi\times10^{-7}$ [H/m]

$u_p = 720$ [m/s] at 3 MHz

- wavelength

$$\lambda = \frac{2\pi}{\beta} = \frac{u_p}{f} = 2\sqrt{\frac{\pi}{f\mu\sigma}},$$

3MHz $\rightarrow \lambda = 0.24(mm)$, in the air $\lambda = 100(m)$

$\alpha = \sqrt{\pi(3\times10^{6})(4\pi\times10^{-7})(5.8\times10^{7})} = 2.64\times10^{4}$ (Np/m)

$\delta = \frac{1}{\alpha} = \frac{1}{2.62}\times10^{-4}m = 0.038mm = 38\mu m$ at 3MHz $= 0.66\mu m$ at 10GHz

$\delta = \frac{1}{\alpha} = \frac{1}{\sqrt{\pi f\mu\sigma}}$: skin depth or $\delta = \frac{1}{\beta} = \frac{\lambda}{2\pi}$ $\because \alpha = \beta$ for good conductor

Plane Wave in Lossy Media (6)

Note $H_y(z) = \frac{E_x(z)}{\eta_c}$, η_c : complex value

cf) $H_y(z,t) = \frac{E_x(z,t)}{\eta_c} \Rightarrow$ mistake

Group Velocity (1)

Phase velocity $u_p = \dfrac{\omega}{\beta}$ (m/s)

- lossless medium $\beta = \omega\sqrt{\mu\varepsilon}$ is a linear function of ω

$$u_p = \frac{1}{\sqrt{\mu\varepsilon}}$$: independent of frequency

- Information-bearing signals : a band of frequencies
  → waves of the component frequencies travel with different phase velocities → distortion in the signal wave shape → signal disperse → dispersion → lossy dielectric is a dispersive medium.

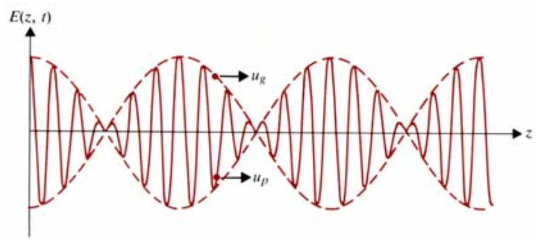
  i.e) Information-bearing signal → small spread of frequencies around a high carrier frequency → group of frequencies → wave packet

Group Velocity (2)

∴ Group velocity : the velocity of propagation of the wave-packet envelope (a group of frequencies)

- Simplest case : wave packet consisting of two traveling waves having equal amplitude and slightly different frequencies $\omega_0 + \Delta\omega,\ \omega_0 - \Delta\omega$
  phase constants $\beta + \Delta\beta,\ \beta - \Delta\beta$

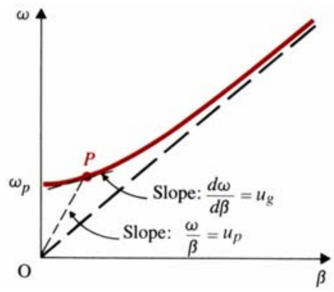
$$E(z,t) = E_0 \cos\left[(\omega_0 + \Delta\omega)t - (\beta_0 + \Delta\beta)z\right] + E_0 \cos\left[(\omega_0 - \Delta\omega)t - (\beta_0 - \Delta\beta)z\right]$$
$$= 2E_0 \cos(t\Delta\omega - z\Delta\beta)\cos(\omega_0 t - \beta_0 z),\ \Delta\omega \ll \omega_0$$



Group Velocity (3)

- Phase velocity

$$\omega_0 t - \beta_0 z = \text{constant},\ u_p = \frac{dz}{dt} = \frac{\omega_0}{\beta_0}$$



- the velocity of envelope (group velocity, u_g)

$$t\Delta\omega - z\Delta\beta = \text{constant},\ u_g = \frac{dz}{dt} = \frac{\Delta\omega}{\Delta\beta} = \frac{1}{\Delta\beta/\Delta\omega}$$

$$\Delta\omega \to 0,\ u_g = \frac{1}{d\beta/d\omega}$$

$$u_p = \frac{\omega}{\beta},\ u_g = \frac{d\omega}{d\beta},\ \frac{d\beta}{d\omega} = \frac{d}{d\omega}\left(\frac{\omega}{u_p}\right) = \frac{1}{u_p} - \frac{\omega}{u_p^2}\frac{du_p}{d\omega}$$

$$\boxed{u_g = \frac{u_p}{1 - \frac{\omega}{u_p}\frac{du_p}{d\omega}}}$$

Group Velocity (4)

① No dispersion $\frac{du_p}{d\omega} = 0$ (u_p is independent of ω, β is a linear function of ω)

$$\therefore u_g = u_p$$

② Normal dispersion

$$\frac{du_p}{d\omega} < 0,\ (u_p \text{ decreasing with } \omega)$$

$$u_g < u_p$$

③ Anomalous dispersion

$$\frac{du_p}{d\omega} > 0\ (u_p \text{ increasing with } \omega)$$

$$u_g > u_p$$

Poynting Vector (1)

$$\begin{cases} \nabla\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} \\ \nabla\times\vec{H} = \vec{J} + \dfrac{\partial\vec{D}}{\partial t} \end{cases}$$

Vector identity

$$\nabla\bullet(\vec{E}\times\vec{H}) = \vec{H}\bullet(\nabla\times\vec{E}) - \vec{E}\bullet(\nabla\times\vec{H}) = -\vec{H}\bullet\frac{\partial\vec{B}}{\partial t} - \vec{E}\bullet\frac{\partial\vec{D}}{\partial t} - \vec{E}\bullet\vec{J}$$

where $\vec{H}\bullet\dfrac{\partial\vec{B}}{\partial t} = \vec{H}\bullet\dfrac{\partial\mu\vec{H}}{\partial t} = \dfrac{1}{2}\dfrac{\partial}{\partial t}\left(\mu\vec{H}\bullet\vec{H}\right) = \dfrac{\partial}{\partial t}\left(\dfrac{1}{2}\mu H^2\right)$

$$\vec{E}\bullet\frac{\partial\vec{D}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon E^2\right)$$

$$\therefore \nabla\bullet(\vec{E}\times\vec{H}) = -\frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2 + \frac{1}{2}\varepsilon E^2\right) - \sigma E^2$$

Poynting Vector (2)

$$\oint_s (\vec{E}\times\vec{H})\bullet\vec{ds} = -\frac{\partial}{\partial t}\int_V\left(\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2\right)dv - \int_V \sigma E^2 dv$$

- $\vec{E}\times\vec{H}$: power flow per unit area
- $-\dfrac{\partial}{\partial t}\int_V\left(\dfrac{1}{2}\varepsilon E^2 + \dfrac{1}{2}\mu H^2\right)dv$: the rate of decrease of the electric and magnetic energies stored

  where, $\dfrac{1}{2}\varepsilon E^2 + \dfrac{1}{2}\mu H^2$: energy stored in electric and magnetic field

- $\int_V \sigma E^2 dv$: Ohmic power dissipation or heat

$\vec{P} = \vec{E}\times\vec{H}$ (W/m^2): Poynting vector

$\oint_S \vec{P}\bullet\vec{ds}$: the power leaving the enclosed volume (Poynting theorem)

Poynting Vector (3)

or $$-\oint_S \vec{P}\bullet\vec{ds} = \frac{\partial}{\partial t}\int_V (w_e + w_m)dv + \int_V p_\sigma dv$$

$$w_e = \frac{1}{2}\varepsilon E^2 = \frac{1}{2}\varepsilon\vec{E}\bullet\vec{E}^*, \qquad w_m = \frac{1}{2}\mu H^2 = \frac{1}{2}\mu\vec{H}\bullet\vec{H}^*$$

$$p_\sigma = \sigma E^2 = \frac{J^2}{\sigma} = \sigma\vec{E}\bullet\vec{E}^* = \frac{\vec{J}\bullet\vec{J}^*}{\sigma}$$

❖ Instantaneous and average power densities.

Assuming phasor $\vec{E}(z) = \hat{x}E_x(z) = \hat{x}E_0 e^{-j(\alpha+j\beta)z}$

Then $\boxed{\vec{E}(z,t) = \mathrm{Re}\left[\vec{E}(z)e^{j\omega t}\right] = \hat{x}E_0 e^{-\alpha z}\,\mathrm{Re}\left[e^{j(\omega t-\beta z)}\right]}$

propagating in a lossy medium in the +z direction

$$\vec{H}(z) = \hat{y}H_y(z) = \hat{y}\frac{E_0}{|\eta|}e^{-\alpha z}e^{-j(\beta z+\theta_\eta)},\ \text{where } \eta = |\eta|e^{j\theta_\eta}$$

$$\therefore \vec{H}(z,t) = \mathrm{Re}\left[\vec{H}(z)e^{j\omega t}\right] = \hat{y}\frac{E_0}{|\eta|}e^{-\alpha z}\cos(\omega t - \beta z - \theta_\eta)$$

Poynting Vector (4)

$$\vec{P}(z,t) = \vec{E}(z,t)\times\vec{H}(z,t)$$

But, $\mathrm{Re}\left[\vec{E}(z)e^{j\omega t}\right]\times\mathrm{Re}\left[\vec{H}(z)e^{j\omega t}\right] \neq \mathrm{Re}\left[\vec{E}(z)\times\vec{H}(z)e^{j\omega t}\right]$

$$cf)\ \mathrm{Re}(\vec{A})\times\mathrm{Re}(\vec{B}) = \frac{1}{2}(\vec{A}+\vec{A}^*)\times\frac{1}{2}(\vec{B}+\vec{B}^*) = \frac{1}{4}\left[(\vec{A}\times\vec{B}^* + \vec{A}^*\times\vec{B}) + (\vec{A}\times\vec{B} + \vec{A}^*\times\vec{B}^*)\right]$$
$$= \frac{1}{2}\mathrm{Re}\left[\vec{A}\times\vec{B}^* + \vec{A}\times\vec{B}\right]$$

$$\therefore \vec{P}(z,t) = \vec{E}(z,t)\times\vec{H}(z,t) = \hat{z}\frac{E_0^2}{|\eta|}e^{-2\alpha z}\cos(\omega t - \beta z)\cos(\omega t - \beta z - \theta_\eta)$$
$$= \hat{z}\frac{E_0^2}{|\eta|}e^{-2\alpha z}\left[\cos\theta_\eta + \cos(2\omega t - 2\beta z - \theta_\eta)\right]$$

$$\boxed{\vec{P}_{av}(z) = \frac{1}{T}\int_0^T \vec{P}(z,t)dt = \hat{z}\frac{E_0^2}{2|\eta|}e^{-2\alpha z}\cos\theta_\eta\ \ (\mathrm{W/m^2})}$$

Poynting Vector (5)

or $\vec{P}(z,t) = \mathrm{Re}\left[\vec{E}(z)e^{j\omega t}\right] \times \mathrm{Re}\left[\vec{H}(z)e^{j\omega t}\right] = \frac{1}{2}\mathrm{Re}\left[\vec{E}(z)\times\vec{H}^*(z) + \vec{E}(z)\times\vec{H}(z)e^{j2\omega t}\right]$

$\therefore \vec{P}_{av}(z) = \frac{1}{2}\mathrm{Re}\left[\vec{E}(z)\times\vec{H}^*(z)\right] \Rightarrow$ General form $\vec{P}_{av}(z) = \frac{1}{2}\mathrm{Re}\left[\vec{E}\times\vec{H}^*\right]$ (W/m^2)

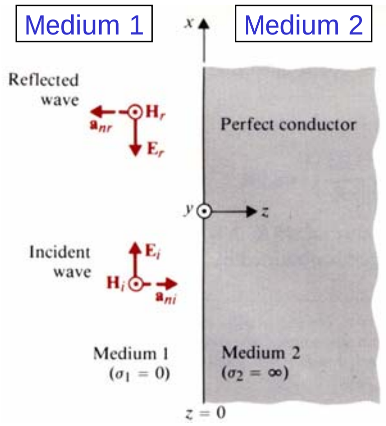
Normal Incidence at a Plane Conducting Boundary (1)

In medium 1,

$$\vec{E}_i(z) = \hat{x}E_{io}e^{-j\beta_1 z},\ \vec{H}_i(z) = \hat{y}\frac{E_{io}}{\eta_1}e^{-j\beta_1 z}$$

$$\vec{P}_i(z) = \vec{E}_i(z)\times\vec{H}_i(z)$$

In medium 2, $\vec{E}_2 = 0, \vec{H}_2 = 0$



$\therefore$ Incident wave is reflected on the boundary

$$\vec{E}_r(z) = \hat{x}E_{r0}e^{j\beta_1 z}$$

$\therefore$ Total $\vec{E}$ field in the medium1

$$\vec{E}_1(z) = \vec{E}_i(z) + \vec{E}_r(z) = \hat{x}(E_{i0}e^{-j\beta_1 z} + E_{r0}e^{j\beta_1 z})$$

$$\vec{E}_1(0) = \vec{E}_2(0) = 0\ \ (E_{t1} = E_{t2}),\quad \therefore E_{i0} = -E_{r0}$$

$$\therefore \boxed{\begin{aligned}\vec{E}_1(z) &= \hat{x}E_{i0}(e^{-j\beta_1 z} - e^{j\beta_1 z}) \\ &= -\hat{x}j2E_{i0}\sin\beta_1 z\end{aligned}}$$

Normal Incident at a Plane Conducting Boundary (2)

$$\vec{H}_r(z) = \frac{1}{\eta_1}\hat{n} \times \vec{E}_r(z) = \frac{1}{\eta_1}(-\hat{z}) \times \vec{E}_r(z)$$

$$\vec{H}_r(z) = -\hat{y}\frac{1}{\eta_1}E_{r0}e^{j\beta_1 z} = \hat{y}\frac{E_{i0}}{\eta_1}e^{j\beta_1 z}$$

$$\therefore \vec{H}_1(z) = \vec{H}_i(z) + \vec{H}_r(z) = \hat{y}2\frac{E_{i0}}{\eta_1}\cos\beta_1 z$$

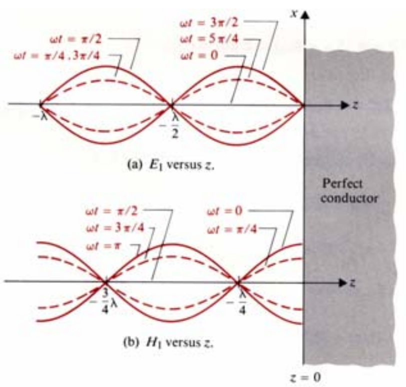
No average power since $\vec{E}_1(z)$ and $\vec{H}_1(z)$ are in phase quadrature

$$\because \vec{P}_{av} = \frac{1}{2}\mathrm{Re}(\vec{E} \times \vec{H}^*) = 0, \text{ cf) } \vec{E} \times \vec{H}^* \text{ : imaginary}$$

Normal Incident at a Plane Conducting Boundary (3)

❖ Space time behavior of the total field

$$\begin{cases} \vec{E}_1(z,t) = \hat{x}2E_{i0}\sin\beta_1 z \sin\omega t \\ \vec{H}_1(z,t) = \hat{y}2\dfrac{E_{i0}}{\eta_1}\cos\beta_1 z \cos\omega t \end{cases} \Rightarrow \text{standing wave}$$



zeros of $\vec{E}_1(z,t)$, Maxima of $\vec{H}_1(z,t)$ occur at $\beta_1 z = -n\pi$ or $z = -n\frac{\lambda}{2}$

Maxima of $\vec{E}_1(z,t)$, zeros of $\vec{H}_1(z,t)$ occur at $\beta_1 z = -(2n+1)\frac{\pi}{2}$ or $z = -(2n+1)\frac{\lambda}{4}$

Normal Incident at a Plane Conducting Boundary (4)

Note

① $\vec{E}_1$ vanishes on the conducting boundary $\Rightarrow E_{r0} = -E_{i0}$

② $\vec{E}_1$ vanishes at points of multiples of $\dfrac{\lambda}{2}$ from the boundary

③ $\vec{H}_1$ is maximum on the conducting boundary

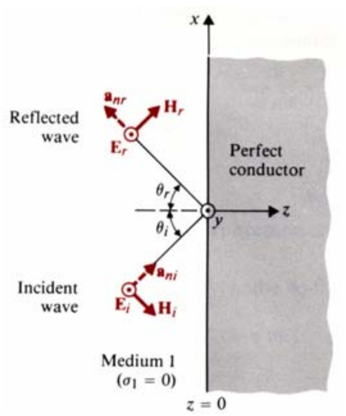
$$H_{r0} = H_{i0} = \frac{E_{i0}}{\eta_1} \Rightarrow \text{Boundary condition}$$

$$cf)P.331\ \hat{n}_2 \times \vec{H}_1 = \vec{J}_s,\ (\hat{n}_2 \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_s)$$

④ The standing waves of $\vec{E}_1$ and $\vec{H}_1$ are in time quadrature (90° phase difference) and are shifted in space by a quarter wave length

Oblique Incidence at a Plane Conducting Boundary (1)

- Depending on the polarization of incident wave
- Plane of incidence : defined by the normal vector to the boundary and the wave vector
- $\vec{E}_i$: orientation of the incident $\vec{E}$ field



$\vec{E}_i$ can be decomposed into two components : one perpendicular to, and the other parallel to the plane of incidence

$$\hat{n}_i = \hat{x}\sin\theta_i + \hat{z}\cos\theta_i$$

θ_i : angle of incidence

Perpendicular polarization

Oblique Incidence at a Plane Conducting Boundary (2)

$$\vec{E}_i(x,z) = \hat{y}E_{i0}e^{-j\beta_1 \hat{n}_i \cdot \vec{R}} = \hat{y}E_{i0}e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$= \hat{y}E_{i0}e^{-j(\beta_x x + \beta_z z)} \text{ where } \beta_x = \beta_1 \sin\theta_i,\ \beta_z = \beta_1\cos\theta_i$$

$$\vec{H}_i(x,z) = \frac{1}{\eta_1}\left[\hat{n}_i \times \vec{E}_i(x,z)\right] = \frac{E_{i0}}{\eta_1}(-\hat{x}\cos\theta_i + \hat{z}\sin\theta_i)e^{-j(\beta_x x + \beta_z z)}$$

For the reflected wave, $\begin{cases} \hat{n}_r = \hat{x}\sin\theta_r - \hat{z}\cos\theta_r \\ \theta_r \text{ : angle of reflection} \end{cases}$

$$\vec{E}_r(x,z) = \hat{y}E_{r0}e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

at z=0, $\vec{E}(x,0) = \vec{E}_i(x,0) + \vec{E}_r(x,0) = \hat{y}\left[E_{i0}e^{-j\beta_1 x\sin\theta_i} + E_{r0}e^{-j\beta_1 x\sin\theta_r}\right] = 0$

$\Rightarrow$ hold for all value of x

$E_{i0} = -E_{r0}$ and $\beta_1 x\sin\theta_i = \beta_1 x\sin\theta_r$

$\therefore \theta_r = \theta_i$: Snell's law of reflection

Oblique Incidence at a Plane Conducting Boundary (3)

$$\vec{E}_r(x,z) = -\hat{y}E_{i0}e^{-j\beta_1(x\sin\theta_i - z\cos\theta_i)}$$

$$\vec{H}_r(x,z) = \frac{1}{\eta_1}\left[\hat{n}_r \times \vec{E}_r(x,z)\right] = \frac{E_{i0}}{\eta_1}(-\hat{x}\cos\theta_i - \hat{z}\sin\theta_i)e^{-j\beta_1(x\sin\theta_i - z\cos\theta_i)}$$

$\hat{x}$ component of $\vec{H}_i$ and $\vec{H}_r$ are in the same direction
$\hat{z}$ component of $\vec{H}_i$ and $\vec{H}_r$ are in the opposite direction

The total field

$$\vec{E}_1(x,z) = \vec{E}_i(x,z) + \vec{E}_r(x,z)$$

$$= \hat{y}E_{i0}(e^{-j\beta_1 z\cos\theta_i} - e^{j\beta_1\cos\theta_i})e^{-j\beta_1 x\sin\theta_i} = -\hat{y}j2E_{i0}\sin(\beta_1 z\cos\theta_i)e^{-j\beta_1 x\sin\theta_i}$$

$$\vec{H}_1(x,z) = -2\frac{E_{i0}}{\eta_1}\left[\hat{x}\cos\theta_i\cos(\beta_1 z\cos\theta_i)e^{-j\beta_1 x\sin\theta_i} + \hat{z}j\sin\theta_i\sin(\beta_1 z\cos\theta_i)e^{-j\beta_1 x\sin\theta_i}\right]$$

Oblique Incidence at a Plane Conducting Boundary (4)

Note

① In the direction normal to the boundary $\rightarrow E_{1y}$ and H_{1x}

$\Rightarrow$ standing wave patterns according to $\sin\beta_{1z}z$ and $\cos\beta_{1z}z$ where

$\beta_{1z} = \beta_1 \cos\theta_i \rightarrow$ No average power is propagated

② In the direction parallel to the boundary $\rightarrow E_{1y}$ and H_{1z} are in both time and space phase $\rightarrow$ propagate with a phase velocity

$$u_{1x} = \frac{\omega}{\beta_{1x}} = \frac{\omega}{\beta_1 \sin\theta_i} = \frac{u_1}{\sin\theta_i}, \quad \lambda_{1x} = \frac{2\pi}{\beta_{1x}} = \frac{\lambda_1}{\sin\theta_i}$$

③ The propagating wave in the x direction is a nonuniform plane wave

④ $\vec{E}_1 = 0$ for all x when $\sin(\beta_1 z\cos\theta_i) = 0$ or $\beta_1 z\cos\theta_i = \frac{2\pi}{\lambda_1} z\cos\theta_i = -m\pi,\ m = 1, 2, 3...$

a conducting plate could be inserted at $z = -\frac{m\lambda_1}{2\cos\theta_i}$ without changing the field pattern that exists between the conducting plate and the conducting boundary at z=0

$\Rightarrow$ TE wave($E_{1x} = 0$) would bounce back and force between the conducting planes and propagate in the x direction

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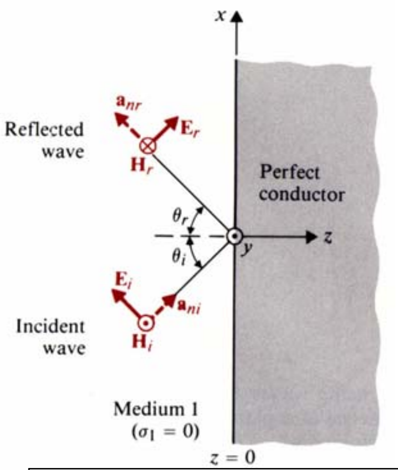
Oblique Incidence at a Plane Conducting Boundary (5)



Parallel polarization

$\vec{E}_i$ and $\vec{E}_r$ have x- and y- component

$\vec{H}_i$ and $\vec{H}_r$ have only y- component

$$\vec{E}_i(x,z) = E_{i0}(\hat{x}\cos\theta_i - \hat{z}\sin\theta_i)e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{H}_i(x,z) = \hat{y}\frac{E_{i0}}{\eta_1}e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

Oblique Incidence at a Plane Conducting Boundary (6)

The reflected wave

$$\vec{E}_r(x,z) = E_{r0}(\hat{x}\cos\theta_r + \hat{z}\sin\theta_r)e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{H}_r(x,z) = -\hat{y}\frac{E_{r0}}{\eta_1}e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

At z=0, $E_t = 0$ for all x

$$\therefore (E_{i0}\cos\theta_i)e^{-j\beta_1 x\sin\theta_i} + (E_{r0}\cos\theta_r)e^{-j\beta_1 x\sin\theta_r} = 0$$

$\therefore E_{r0} = -E_{i0}$ and $\theta_i = \theta_r$

$$\therefore \vec{E}_1(x,z) = \vec{E}_i(x,z) + \vec{E}_r(x,z)$$

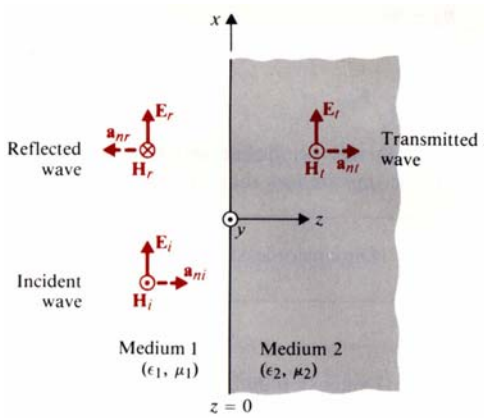
$$= \hat{x}E_{i0}\cos\theta_i(e^{-j\beta_1 z\cos\theta_i} - e^{j\beta_1 z\cos\theta_i})e^{-j\beta_1 x\sin\theta_i} - \hat{z}E_{i0}\sin\theta_i(e^{-j\beta_1 z\cos\theta_i} + e^{j\beta_1 z\cos\theta_i})e^{-j\beta_1 x\sin\theta_i}$$

$$= -2E_{i0}\left[\hat{x}j\cos\theta_i\sin(\beta_1 z\cos\theta_i) + \hat{z}\sin\theta_i\cos(\beta_1 z\cos\theta_i)\right]e^{-j\beta_1 x\sin\theta_i}$$

$$\vec{H}_1(x,z) = \vec{H}_i(x,z) + \vec{H}_r(x,z) = \hat{y}2\frac{E_{i0}}{\eta_1}\cos(\beta_1 z\cos\theta_i)e^{-j\beta_1 x\sin\theta_i}$$

Normal Incidence at a Plane Dielectric Boundary (1)

❖ Assume lossless (σ_1=0, σ_2=0) media



① Incident wave travels in +z directon

$$\vec{E}_i(z) = \hat{x}E_{i0}e^{-j\beta_1 z},\ \vec{H}_i(z) = \hat{y}\frac{E_{i0}}{\eta_1}e^{-j\beta_1 z}$$

Normal Incidence at a Plane Dielectric Boundary (2)

② The boundary surface is the plane z=0.

→ Discontinuity at z=0 cause that the incident wave is partly reflected back and partly transmitted into medium 2

③ Reflected wave ($\vec{E}_r, \vec{H}_r$)

$$\vec{E}_r(z) = \hat{x}E_{r0}e^{j\beta_1 z},\ \ \vec{H}_r(z) = -\hat{z}\times\frac{1}{\eta_1}\vec{E}_r(z) = -\hat{y}\frac{E_{r0}}{\eta_1}e^{j\beta_1 z}$$

④ Transmitted wave ($\vec{E}_t, \vec{H}_t$)

$$\vec{E}_t(z) = \hat{x}E_{t0}e^{-j\beta_2 z},\ \ \vec{H}_t(z) = \hat{z}\times\frac{1}{\eta_2}\vec{E}_t(z) = \hat{y}\frac{E_{t0}}{\eta_2}e^{-j\beta_2 z}$$

We have two unknown E_{r0}, E_{t0} thus two equations are required to determine E_{r0}, E_{t0} . Boundary conditions for the electric and magnetic fields

Normal Incidence at a Plane Dielectric Boundary (3)

$$\vec{E}_i(0) + \vec{E}_r(0) = \vec{E}_t(0),\quad E_{i0} + E_{r0} = E_{t0}$$

$$\vec{H}_i(0) + \vec{H}_r(0) = \vec{H}_t(0),\ \ \frac{1}{\eta_1}(E_{i0} - E_{r0}) = \frac{E_{t0}}{\eta_2}\quad \because \text{ no current}$$

$$\therefore E_{r0} = \boxed{\frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}}E_{i0},\ \ \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}:\ \text{Reflection coefficient}$$

$$E_{t0} = \boxed{\frac{2\eta_2}{\eta_2 + \eta_1}}E_{i0},\ \ \frac{2\eta_2}{\eta_2 + \eta_1}:\ \text{Transmission coefficient(Dimensionless)}$$

$$\Gamma = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1},\ \ \tau = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2}{\eta_2 + \eta_1}$$

- Γ can be positive or negative depending on η_2
- τ is always positive→ $\vec{E}_i$ and $\vec{E}_t$ are in same direction

Normal Incidence at a Plane Dielectric Boundary (4)

The definition for Γ and τ apply even when the media are dissipative

i.e) η_1 and/or η_2 are complex

$$cf)\ \tau > 1 \text{ if } \eta_2 > \eta_1$$

$$\left.\begin{array}{l}\text{i.e } |\vec{E}_t| > |\vec{E}_i| \\ \text{But } |\vec{H}_t| < |\vec{H}_i|\end{array}\right\} \Rightarrow |\vec{E}_t \times \vec{H}_t| < |\vec{E}_i \times \vec{H}_i|$$

$$|\vec{E}_t \times \vec{H}_t^*| = \left|\frac{2\eta_2}{\eta_2+\eta_1}E_{i0} \times \frac{2E_{i0}}{\eta_2+\eta_1}\right| = \frac{|4\eta_2|E_{i0}^2}{|\eta_2+\eta_1|^2} < \frac{E_{i0}^2}{\eta_1} \quad (\because \frac{4\eta_2}{|\eta_2+\eta_1|^2} - \frac{1}{\eta_1} < 0)$$

$$|\vec{E}_i \times \vec{H}_i^*| = \left|E_{i0} \times \frac{E_{i0}}{\eta_1}\right| = \frac{E_{i0}^2}{\eta_1}$$

Relation between reflection and transmission coefficient

$$\boxed{1+\Gamma = \tau}$$

Normal Incidence at a Plane Dielectric Boundary (5)

if $\eta_2 = 0$ (i.e medium 2 is perfect conductor)

$$\Gamma = -1, \ \tau = 0$$

if $\eta_2 \neq 0$, partial reflection

$$\vec{E}_1(z) = \vec{E}_i(z) + \vec{E}_r(z) = \hat{x}E_{i0}(e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z})$$

$$= \hat{x}E_{i0}\left[(1+\Gamma)e^{-j\beta_1 z} + \Gamma(e^{j\beta_1 z} - e^{-j\beta_1 z})\right]$$

$$= \hat{x}E_{i0}\left[(1+\Gamma)e^{-j\beta_1 z} + \Gamma(j2\sin\beta_1 z)\right]$$

$$\text{or } \vec{E}_1(z) = \hat{x}E_{i0}\left[\tau e^{-j\beta_1 z} + \Gamma(j2\sin\beta_1 z)\right]$$

Two parts :
- A traveling wave with an amplitude τE_{i0}
- A standing wave with an amplitude $2\Gamma E_{i0}$

$\therefore$ $\vec{E}_1(z)$ never goes to zero at fixed distance from the interface but has merely locations of minima and maxima

Normal Incidence at a Plane Dielectric Boundary (6)

- Location of minima and maxima.

  Assume lossless

$$\vec{E}_1(z) = \hat{x}E_{i0}e^{-j\beta_1 z}(1+\Gamma e^{j2\beta_1 z})$$

case 1, $\Gamma > 0 \ (\eta_2 > \eta_1)$

the maximum value of $|E_1(z)|$ is $E_{i0}(1+\Gamma)$

$\Rightarrow$ which occurs when $2\beta_1 z_{\max} = -2n\pi \ (\because z < 0)$

$$\therefore z_{\max} = -\frac{n\pi}{\beta_1} = -\frac{n\lambda_1}{2}, \ n = 0,1,2,\ldots$$

the minimum value of $|E_1(z)|$ is $E_{i0}(1-\Gamma)$

$\Rightarrow$ which occurs when $2\beta_1 z_{\min} = -(2n+1)\pi$

$$\therefore z_{\min} = -\frac{(2n+1)\pi}{2\beta_1} = -\frac{(2n+1)\lambda_1}{4}, \ n = 0,1,2,\ldots$$

Normal Incidence at a Plane Dielectric Boundary (7)

case 2, $\Gamma < 0 \ (\eta_2 < \eta_1)$

the maximum value of $|E_1(z)|$ is $E_{i0}(1-\Gamma)$

$\Rightarrow$ which occurs at $z_{\max} = -(2n+1)\pi$

the minimum value of $|E_1(z)|$ is $E_{i0}(1+\Gamma)$

$\Rightarrow$ which occurs at $z_{\min} = -2n\pi$

$\Rightarrow$ The location for $|E_1(z)|_{\max}$ when $\Gamma > 0$ and when $\Gamma < 0$ are interchanged

- standing wave ratio(SWR), s→expressed in dB scale ($20\log_{10}$ s)

$$s = \frac{|E|_{\max}}{|E|_{\min}} = \frac{1+|\Gamma|}{1-|\Gamma|}, \ 1 < s < \infty$$

An inverse relation $|\Gamma| = \dfrac{s-1}{s+1}, \ -1 < \Gamma < 1$

Normal Incidence at a Plane Dielectric Boundary (8)

- Magnetic field intensity

$$\vec{H}_1(z) = \hat{y}\frac{E_{i0}}{\eta_1}(e^{-j\beta_1 z} - \Gamma e^{j\beta_1 z}) = \hat{y}\frac{E_{i0}}{\eta_1}e^{-j\beta_1 z}(1-\Gamma e^{j2\beta_1 z})$$

In lossless medium, Γ is real; $|\vec{H}_1(z)|$ will be a minimum at locations where $|E_1(z)|$ is a maximum, and vice versa.

- In medium 2

$$\begin{cases} \vec{E}_t(z) = \hat{x}\tau E_{i0}e^{-j\beta_2 z} \\ \vec{H}_t(z) = \hat{y}\dfrac{\tau}{\eta_2}E_{i0}e^{-j\beta_2 z} \end{cases}$$

Normal Incidence at a Plane Dielectric Boundary (9)

- $\vec{P}_{av} = \frac{1}{2}\text{Re}\left[\vec{E}\times\vec{H}^*\right]$

$$(\vec{P}_{av})_1 = \hat{z}\frac{E_{i0}^2}{2\eta_1}\text{Re}\left[(1+\Gamma e^{j2\beta_1 z})(1-\Gamma e^{-j2\beta_1 z})\right]$$

$$= \hat{z}\frac{E_{i0}^2}{2\eta_1}\text{Re}\left[(1-\Gamma^2)+\Gamma(e^{j2\beta_1 z}-e^{-j2\beta_1 z})\right]$$

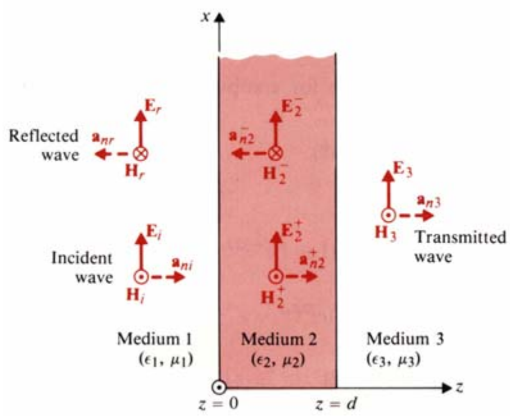
$$= \hat{z}\frac{E_{i0}^2}{2\eta_1}\text{Re}\left[(1-\Gamma^2)+j2\Gamma\sin 2\beta_1 z\right] = \hat{z}\frac{E_{i0}^2}{2\eta_1}(1-\Gamma^2)$$

$$(\vec{P}_{av})_2 = \hat{z}\frac{E_{i0}^2}{2\eta_2}(\tau^2)$$

$$(\vec{P}_{av})_1 = (\vec{P}_{av})_2 \text{ for lossless medium}$$

$$\therefore \boxed{1-\Gamma^2 = \frac{\eta_1}{\eta_2}\tau^2}$$

Normal Incidence at Multiple Dielectric Interface (1)



- Reflection occur at both *z=0* and *z=d*
- Total electric field in medium 1

$$\vec{E}_1 = \hat{x}(E_{i0}e^{-j\beta_1 z} + E_{r0}e^{j\beta_1 z})$$

Normal Incidence at Multiple Dielectric Interface (2)

- E_{r0} is no longer related to E_{i0} by $E_{r0}=\Gamma E_{i0}$, owing to the existence of a second discontinuity at *z=d.*
- The total reflected wave is the result of the initial reflected component and an infinite sequence of multiply reflected contributions within medium 2 that are transmitted back into medium 1.
- How to find the relation between E_{r0} and E_{i0}?
- One way is to write down the electric and magnetic field intensity vectors in all three regions and apply the boundary conditions.
- magnetic field in region 1

$$\vec{H}_1(z) = \hat{y}\frac{1}{\eta_1}(E_{i0}e^{-j\beta_1 z} - E_{r0}e^{j\beta_1 z})$$

- the electric and magnetic fields in region 2

$$\vec{E}_2 = \hat{x}(E_2^+ e^{-j\beta_2 z} + E_2^- e^{+j\beta_2 z})$$

$$\vec{H}_2 = \hat{y}\frac{1}{\eta_2}(E_2^+ e^{-j\beta_2 z} - E_2^- e^{+j\beta_2 z})$$

Normal Incidence at Multiple Dielectric Interface (3)

- In region 3

$$\vec{E}_3 = \hat{x}E_3^+ e^{-j\beta_3 z}$$

$$\vec{H}_3 = \hat{y}\frac{1}{\eta_3}E_3^+ e^{-j\beta_3 z}$$

- Four unknown, $E_{r0},\ E_2^+,\ E_2^-,\ E_3^+$
- Boundary conditions at two interface give birth to four equations

$$\text{At } z = 0, \begin{cases} \vec{E}_1(0) = \vec{E}_2(0) \\ \vec{H}_1(0) = \vec{H}_2(0) \end{cases}$$

$$\text{At } z = d, \begin{cases} \vec{E}_2(d) = \vec{E}_3(d) \\ \vec{H}_2(d) = \vec{H}_3(d) \end{cases}$$

Normal Incidence at Multiple Dielectric Interface (4)

- wave impedance of the total field at any plane parallel to the plane boundary

wave impedance of the field, Z(z) : the ratio of the total electric field intensity to the total magnetic field intensity for a z-dependent uniform plane wave.

$$Z(z) = \frac{Total\ E_x(z)}{Total\ H_y(z)}\ (\Omega)$$

For the normal incidence of z-dependent uniform plane wave,

$$E_{1x}(z) = E_{i0}(e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z})$$

$$H_{1y}(z) = \frac{E_{i0}}{\eta_1}(e^{-j\beta_1 z} - \Gamma e^{j\beta_1 z})$$

$$\therefore Z_1(z) = \frac{E_{1x}(z)}{H_{1y}(z)} = \eta_1 \frac{e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z}}{e^{-j\beta_1 z} - \Gamma e^{j\beta_1 z}} : \text{ function of z}$$

Normal Incidence at Multiple Dielectric Interface (5)

at $z = -l$,

$$Z_1(-l) = \frac{E_{1x}(-l)}{H_{1y}(-l)} = \eta_1 \frac{e^{j\beta_1 l} + \Gamma e^{-j\beta_1 l}}{e^{j\beta_1 l} - \Gamma e^{-j\beta_1 l}} = \eta_1 \frac{\eta_2 \cos\beta_1 l + j\eta_1 \sin\beta_1 l}{\eta_1 \cos\beta_1 l + j\eta_2 \sin\beta_1 l}$$

or $$= \eta_1 \frac{\eta_2 + j\eta_1 \tan\beta_1 l}{\eta_1 + j\eta_2 \tan\beta_1 l} \quad \text{with } \Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

cf) If $\eta_2 = 0$ and $\Gamma = -1$, then $Z(-l) = j\eta_1 \tan\beta_1 l$

wave impedance of standing wave to the left of perfect conducting boundary

- Impedance transformation

The total field in medium 2 is the result of multiple reflections of the two boundary planes at z=0 and z=d; but it can be grouped into a wave traveling in the +z direction and another traveling in the –z direction.

Normal Incidence at Multiple Dielectric Interface (6)

- The wave impedance of the total field in medium 2 at z=0

$$E_{2x}(z) = E_2^+(e^{-j\beta_2 z} + \Gamma_2 e^{j\beta_2 z}), \quad \Gamma_2 = \frac{\eta_3 - \eta_2}{\eta_3 + \eta_2}$$

$$H_{2y}(z) = \frac{E_2^+}{\eta_2}(e^{-j\beta_2 z} - \Gamma_2 e^{j\beta_2 z})$$

$$\therefore Z_2(z) = \frac{E_{2x}(z)}{H_{2y}(z)} = \eta_2 \frac{e^{-j\beta_2 z} + \Gamma_2 e^{-j\beta_2 z}}{e^{-j\beta_2 z} - \Gamma_2 e^{-j\beta_2 z}}$$

$$Z_2(0) = \eta_2 \frac{\eta_3 \cos\beta_2 d + j\eta_2 \sin\beta_2 d}{\eta_2 \cos\beta_2 d + j\eta_3 \sin\beta_2 d}$$

Note)

As far as the wave in medium 1 is concerned, it encounters a discontinuity at z=0 and the discontinuity can be characterized by an infinite medium with an intrinsic impedance $Z_2(0)$

Normal Incidence at Multiple Dielectric Interface (7)

The effective reflection coefficient at z=0 for the incident wave in medium 1 is

$$\Gamma_0 = \frac{E_{r0}}{E_{i0}} = -\frac{H_{r0}}{H_{i0}} = \frac{Z_2(0)-\eta_1}{Z_2(0)+\eta_1}$$

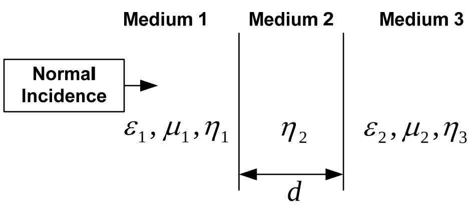
- Effect of transform η_3 to $Z_2(0)$
  : inserting a dielectric layer of thickness d and η_2 in front of medium 3
- Given η_1 and η_3, Γ_0 can be adjusted by suitable choices of η_2 and d

In many applications, Γ_0 and E_{r0} are the only quantities of interest; hence this impedance transform approach is conceptually simple and yields the desired answers in a direct manner.

If the fields E_2^+, E_2^- and E_t in medium 2 and 3 are also desired, they can be determined by boundary conditions at z=0 and z=d

Normal Incidence at Multiple Dielectric Interface (8)

Eg) No reflection condition



The condition of no reflection at interface $z = 0$

$$\Rightarrow \Gamma_0 = 0 \text{ or } Z_2(0) = \eta_1$$

$$\Rightarrow \eta_2 \frac{\eta_3 \cos\beta_2 d + j\eta_2 \sin\beta_2 d}{\eta_2 \cos\beta_2 d + j\eta_3 \sin\beta_2 d} = \eta_1$$

$$\begin{cases} \eta_3 \cos\beta_2 d = \eta_1 \cos\beta_2 d & \text{....①} \\ \eta_2^2 \sin\beta_2 d = \eta_1\eta_3 \sin\beta_2 d & \text{....②} \end{cases}$$

Normal Incidence at Multiple Dielectric Interface (9)

$① \Rightarrow \eta_3 = \eta_1 \quad \text{or} \quad \cos\beta_2 d = 0$

if $\eta_3 = \eta_1$, ② can be satisfied when $\eta_2 = \eta_3 = \eta_1$ (trivial case of no discontinuities)

$$\text{or} \quad \sin\beta_2 d = 0 \quad \text{or} \quad d = \frac{n\lambda_2}{2}$$

if $\cos\beta_2 d = 0$, i.e) $d = (2n+1)\frac{\lambda_2}{4}$, $n = 0,1,2,3,..$

$\sin\beta_2 d \neq 0$ and ② can be satisfied when $\eta_2 = \sqrt{\eta_1\eta_3}$

In summary, two possible conditions

1. when $\eta_1 = \eta_3$, $d = n\frac{\lambda_2}{2}$, $n = 0,1,2,...$

   The thickness of the dielectric layer can be a multiple of a half-wavelength in the dielectric layer $\Rightarrow$ Half-wave dielectric window (Narrow band device)

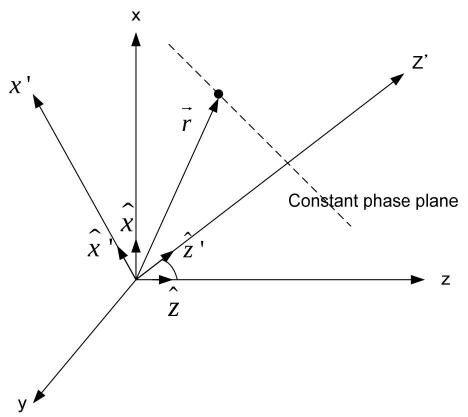
2. when $\eta_1 \neq \eta_3$, we require $\eta_2 = \sqrt{\eta_1\eta_3}$, $d = (2n+1)\frac{\lambda_2}{4}$

   Quarter-wave impedance transformer

Oblique Incidence on the Dielectric Boundary (1)

1. TE polarization

① Description of the E-field component E_y



Asssume that wave propagates along z' axis with $e^{j\omega t}$ dependence

$$E(z',t) = \text{Re}\left[E_y(z')e^{j\omega t}\right], \text{ where } E_y(z') = Ae^{-j\beta z'}$$

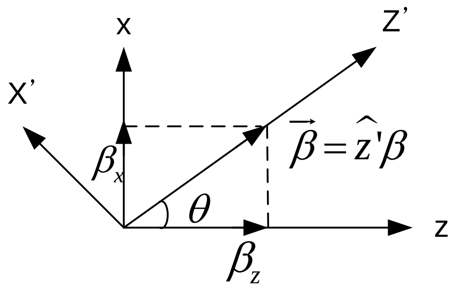
Oblique Incidence on the Dielectric Boundary (2)

- (x,y,z) $\Rightarrow$ coordinate for interface
- (x',y,z') $\Rightarrow$ coordinate for propagation of wave
- $\beta z'$ is constant over any plane perpendicular to z' $\Rightarrow$ plane wave
- the perpendicular distance z' from origin to the constant phase plane

$$z' = \hat{z}' \bullet \vec{r}$$

Assuming $\vec{r}$ displacement vector from origin to any point on a constant phase plane

- Vector wave-number : $\vec{\beta} = \hat{z}'\beta$
- Then phase of plane wave : $\beta z' = \vec{\beta} \bullet \vec{r}$



Oblique Incidence on the Dielectric Boundary (3)

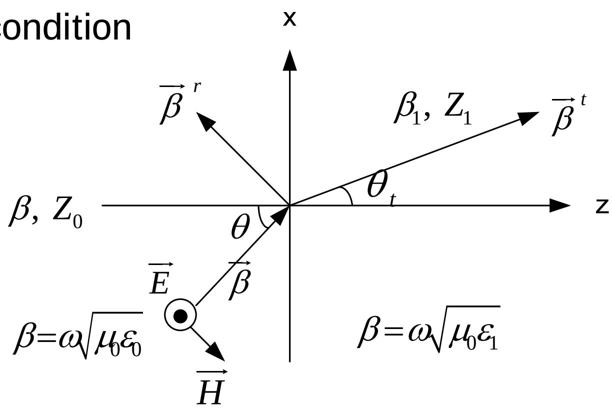
$$\vec{\beta} = \beta_x \hat{x} + \beta_z \hat{z}, \text{ where } \beta_x = \beta \sin\theta,\ \beta_z = \beta\cos\theta$$

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\therefore \beta z' = \vec{\beta} \bullet \vec{r} = \beta_x x + \beta_z z$$

$$\therefore E_y(x,y) = Ae^{-j(\beta_x x + \beta_z z)}$$

② Boundary condition



Oblique Incidence on the Dielectric Boundary (4)

$$E_y^i(x,z) = Ae^{-j(\beta_x x+\beta_z z)},\ \text{where } \beta_x = \beta\sin\theta,\ \beta_z = \beta\cos\theta$$

$$E_y^r(x,z) = Be^{-j(\beta_x^r x-\beta_z^r z)},\ \text{where } \beta_x^r = \beta\sin\theta_r,\ \beta_z^r = -\beta\cos\theta_r$$

$$E_y^t(x,z) = Ce^{-j(\beta_x^t x+\beta_z^t z)},\ \text{where } \beta_x^t = \beta_1\sin\theta_t,\ \beta_z^t = \beta_1\cos\theta_t$$

Boundary condition→Tangential component of E-field at boundary should be continuous

$$E_y(x,0^-) = E_y(x,0^+):\ \text{independent of x}$$

$$E_y(x,0^-) = E_y^i(x,0^-) + E_y^r(x,0^-)$$

$$E_y(x,0^+) = E_y^t(x,0^+)$$

$$Ae^{-j\beta_x x} + Be^{-j\beta_x^r x} = Ce^{-j\beta_x^t x}\ \text{regardless of x}$$

$$\therefore \boxed{A+B=C,\ \beta_x x = \beta_x^r x = \beta_x^t x}:\ \text{Snell's law}$$

Oblique Incidence on the Dielectric Boundary (5)

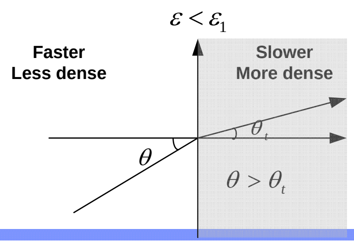
$$\boxed{A+B=C,\ \beta_x = \beta_x^r = \beta_x^t}$$

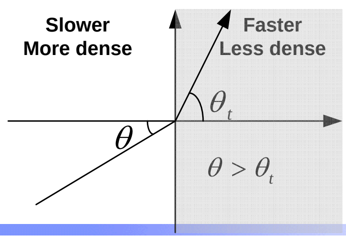
- If not, change of x leads to the situation that each term be out of phase
- β^r, β^t can not have y component since β^r and β^t must lie in the plane of incidence (Remember B.C.)
- $\beta_x = \beta_x^r = \beta_x^t$

$$\beta\sin\theta = \beta\sin\theta_r \Rightarrow \theta = \theta_r$$

$$\beta\sin\theta = \beta_1\sin\theta_t \Rightarrow \omega\sqrt{\mu_0\varepsilon_0}\sin\theta = \omega\sqrt{\mu_0\varepsilon_1}\sin\theta_t$$

$$\therefore \frac{\sin\theta}{\sin\theta_t} = \sqrt{\frac{\varepsilon_1}{\varepsilon}} = \frac{\beta_1}{\beta} = \frac{v}{v_1} = \frac{n_1}{n},\quad n = \frac{c}{u_p}$$





Oblique Incidence on the Dielectric Boundary (6)

- Critical angle

$$\theta_t = 90°,\ \sin\theta_t = 1 \qquad \therefore\ \sin\theta_c = \frac{v}{v_1} = \frac{\beta_1}{\beta}$$

$$\text{for } \theta > \theta_c, \quad \sin\theta_t = \frac{\beta}{\beta_1}\sin\theta = \frac{\sin\theta}{\sin\theta_c} > 1$$

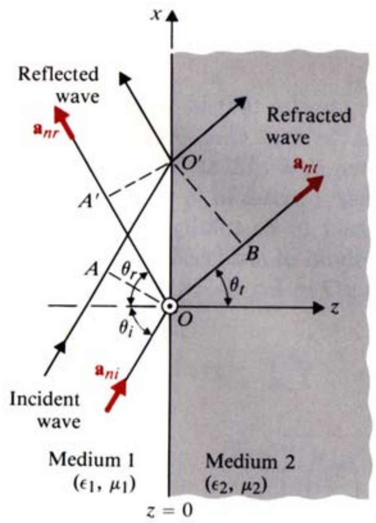
$\therefore$ θ_t is complex angle

$\Rightarrow$ This means no refracted wave. i.e) The incident wave is totally reflected

$\therefore$ The angle of incidence θ_c which corresponds to the threshold of total reflection $\theta_t = \frac{\pi}{2}$, is called the critical angle.

$$\sin\theta_c = \sqrt{\frac{\varepsilon_1}{\varepsilon}} = \frac{n_1}{n} \ \text{ for } \varepsilon > \varepsilon_1 \ \text{ or } \ \theta_c = \sin^{-1}\left(\frac{n_1}{n}\right)$$

Oblique Incidence at a Plane Dielectric Boundary (7)



- Line AO, O'A', O'B : the intersections of the wavefronts of the incident, reflected and transmitted waves respectively, on the plane of incidence

Oblique Incidence at a Plane Dielectric Boundary (8)

- $\overline{OA'} = \overline{AO'}$ since the phase velocity of the incident and the reflected wave are the same.

$$\overline{OO'}\sin\theta_r = \overline{OO'}\sin\theta_i \Rightarrow \boxed{\theta_i = \theta_r} : \text{ Snell's law of reflection}$$

cf) Boundary condition should be satisfied independent of x

$$\Rightarrow e^{-j\beta_1 x\sin\theta_i} = e^{-j\beta_1 x\sin\theta_r} = e^{-j\beta_2 x\sin\theta_t}$$

$$\therefore \beta_1 x\sin\theta_i = \beta_1 x\sin\theta_r = \beta_2 x\sin\theta_t$$

- $\dfrac{\overline{OB}}{u_{p2}} = \dfrac{\overline{AO'}}{u_{p1}}, \quad \therefore \dfrac{\overline{OB}}{\overline{AO'}} = \dfrac{\overline{OO'}\sin\theta_t}{\overline{OO'}\sin\theta_i} = \dfrac{u_{p2}}{u_{p1}}$

$$\boxed{\frac{\sin\theta_t}{\sin\theta_i} = \frac{u_{p2}}{u_{p1}} = \frac{\beta_1}{\beta_2} = \frac{n_1}{n_2}}, \text{ cf)}\beta = \frac{\omega}{u_p}, n = \frac{c}{u_p} : \text{Refraction index}$$

Oblique Incidence at a Plane Dielectric Boundary (9)

Note)

for non-magnetic media, $\mu_1 = \mu_2 = \mu_0$

$$\frac{\sin\theta_t}{\sin\theta_i} = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}} = \frac{n_1}{n_2} = \frac{\eta_2}{\eta_1}$$

→A plane wave incident oblique at an interface with a denser medium will be bent toward the normal.

→Snell's law are independent of polarization

❖ Total reflection

- For $\varepsilon_1 > \varepsilon_2$: the wave in medium1 is incident on a less dense medium 2

$$\theta_t > \theta_i$$

Oblique Incidence at a Plane Dielectric Boundary (10)

- For $\theta_i = \theta_c,\ \theta_t = \dfrac{\pi}{2}$: the refracted wave will be grazing along the interface

  $\theta_i > \theta_c$: No refracted wave

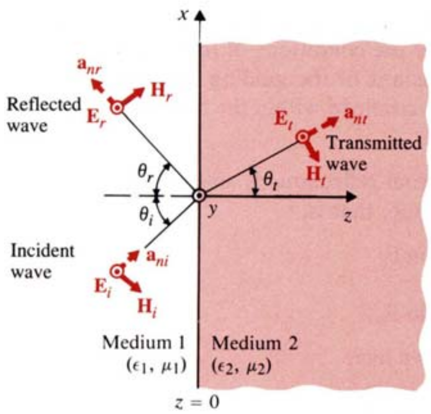
  $\Rightarrow$ the incident wave is totally reflected

$\therefore$ The angle of incidence θ_c, which corresponds to the threshold of total reflection $\theta_t = \dfrac{\pi}{2}$, is called the critical angle.

$$\text{let } \theta_t = \frac{\pi}{2},\quad \frac{\sin\theta_t}{\sin\theta_c} = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}} \Rightarrow \sin\theta_c = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \text{ or } \theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$$

Oblique Incidence at a Plane Dielectric Boundary (1)

❖ Perpendicular polarization (TE polarization)



- Incident field

$$\vec{E}_i(x,z) = \hat{y}E_{i0}e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{H}_i(x,z) = \frac{E_{i0}}{\eta_1}(-\hat{x}\cos\theta_i + \hat{z}\sin\theta_i)e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

Oblique Incidence at a Plane Dielectric Boundary (2)

- Reflected field

$$\vec{E}_r(x,z) = \hat{y}E_{r0}e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{H}_r(x,z) = \frac{E_{r0}}{\eta_1}(\hat{x}\cos\theta_r + \hat{z}\sin\theta_r)e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

- Transmitted field into medium 2

$$\vec{E}_t(x,z) = \hat{y}E_{t0}e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

$$\vec{H}_t(x,z) = \frac{E_{t0}}{\eta_2}(-\hat{x}\cos\theta_t + \hat{z}\sin\theta_t)e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

  - Unknowns : E_{r0}, E_{t0}, θ_r, θ_t
  - Boundary conditions at the boundary $z = 0$

$$E_{iy}(x,0) + E_{ry}(x,0) = E_{ty}(x,0)$$

$$\Rightarrow E_{i0}e^{-j\beta_1\sin\theta_i} + E_{r0}e^{-j\beta_1\sin\theta_r} = E_{t0}e^{-j\beta_2\sin\theta_t}$$

Oblique Incidence at a Plane Dielectric Boundary (3)

$$H_{ix}(x,0) + H_{rx}(x,0) = H_{tx}(x,0)$$

$$\Rightarrow \frac{1}{\eta_1}(-E_{i0}\cos\theta_i e^{-j\beta_1\sin\theta_i} + E_{r0}\cos\theta_r e^{-j\beta_1\sin\theta_r}) = -\frac{E_{t0}}{\eta_2}\cos\theta_r e^{-j\beta_2\sin\theta_t}$$

$\therefore$ Phase matching

$$\beta_1 x\sin\theta_i = \beta_1 x\sin\theta_r = \beta_2 x\sin\theta_t$$

$$i.e)\theta_r = \theta_c, \quad \frac{\sin\theta_t}{\sin\theta_i} = \frac{\beta_1}{\beta_2} = \frac{n_1}{n_2}$$

Amplitude matching

$$E_{i0} + E_{r0} = E_{t0}, \quad \frac{1}{\eta_1}(E_{i0} - E_{r0})\cos\theta_i = \frac{E_{r0}}{\eta_2}\cos\theta_t$$

Oblique Incidence at a Plane Dielectric Boundary (4)

- Reflection coefficient

$$\Gamma_\perp = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} = \frac{\dfrac{\eta_2}{\cos\theta_t} - \dfrac{\eta_1}{\cos\theta_i}}{\dfrac{\eta_2}{\cos\theta_t} + \dfrac{\eta_1}{\cos\theta_i}}$$

$$\tau_\perp = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} = \frac{\dfrac{2\eta_2}{\cos\theta_t}}{\dfrac{\eta_2}{\cos\theta_t} + \dfrac{\eta_1}{\cos\theta_i}}$$

Normal Incidence	Oblique Incidence (Perpendicular)
$\Gamma = \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$	$\Gamma_\perp = \dfrac{\frac{\eta_2}{\cos\theta_t} - \frac{\eta_1}{\cos\theta_i}}{\frac{\eta_2}{\cos\theta_t} + \frac{\eta_1}{\cos\theta_i}}$
η_1	$\dfrac{\eta_1}{\cos\theta_i}$
η_2	$\dfrac{\eta_2}{\cos\theta_t}$

Oblique Incidence at a Plane Dielectric Boundary (5)

- $\boxed{1+\Gamma_\perp = \tau_\perp}$
- Brewster angle : Incident angle for which the reflection coefficient be zero.

$$\eta_2\cos\theta_{B\perp} = \eta_1\cos\theta_t,\ \text{where}\ \cos\theta_t = \sqrt{1-\frac{n_1^2}{n_2^2}\sin^2\theta_i},\ n_1 = \frac{c}{u_{p1}}, n_2 = \frac{\sqrt{\mu_2\varepsilon_2}}{\sqrt{\mu_0\varepsilon_0}}, \eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}, \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}$$

$$\therefore \boxed{\sin^2\theta_{B\perp} = \frac{1-\dfrac{\mu_1\varepsilon_2}{\mu_2\varepsilon_1}}{1-\left(\dfrac{\mu_1}{\mu_2}\right)^2}}$$

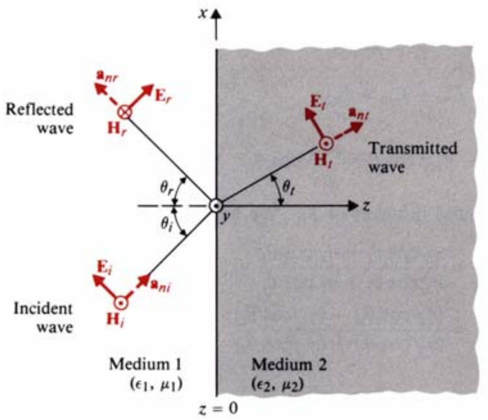
$$cf)\sin\theta_{B\perp} = \frac{1}{\sqrt{1+\dfrac{\mu_1}{\mu_2}}}\quad \text{for}\ \varepsilon_1 = \varepsilon_2,\ \mu_1 \neq \mu_2\ \text{(very very rare case)}$$

usually $\mu_1 = \mu_2 = \mu_0,\ \varepsilon_1 \neq \varepsilon_2$

$\therefore$ For the most of the case there is no Brewester angle for the TE($\perp$) polarization

Oblique Incidence at a Plane Dielectric Boundary (6)

❖ Parallel Polarization (TM polarization)



➢ Incident field

$$\vec{E}_i(x,z) = E_{i0}(\hat{x}\cos\theta_i - \hat{z}\sin\theta_i)e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{H}_i(x,z) = \hat{y}\frac{E_{i0}}{\eta_1}e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

Oblique Incidence at a Plane Dielectric Boundary (7)

➢ Reflected field

$$\vec{E}_r(x,z) = E_{r0}(\hat{x}\cos\theta_r + \hat{z}\sin\theta_r)e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{H}_r(x,z) = -\hat{y}\frac{E_{r0}}{\eta_1}e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

➢ Transmitted field

$$\vec{E}_t(x,z) = E_{t0}(\hat{x}\cos\theta_t - \hat{z}\sin\theta_t)e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

$$\vec{H}_t(x,z) = \hat{y}\frac{E_{t0}}{\eta_2}e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

- Boundary condition at $z = 0$

$$(E_{i0} + E_{r0})\cos\theta_i = E_{t0}\cos\theta_t, \quad \frac{1}{\eta_1}(E_{i0} - E_{r0}) = \frac{1}{\eta_2}E_{r0}$$

cf) The Snell's law as TE applies for TM

Oblique Incidence at a Plane Dielectric Boundary (8)

- Reflection coefficient

$$\Gamma_{\parallel} = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$

$$\tau_{\parallel} = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$

$$\therefore \boxed{1+\Gamma_{\parallel} = \tau_{\parallel}\left(\frac{\cos\theta_t}{\cos\theta_i}\right)} \quad (1+\Gamma_{\parallel} \neq \tau_{\parallel} \text{ if } \theta_i \neq 0)$$

Note)

$|\Gamma_{\perp}|^2 \geq |\Gamma_{\parallel}|^2 \Rightarrow$ example : polaroid sunglasses to reduce the sun glare

cf) E-field in parallel to the Earth's surface is predominantly reaching the eye.
Design sunglasses to filter out this component.

Oblique Incidence at a Plane Dielectric Boundary (9)

- Brewster angle

$$\eta_2 \cos\theta_t = \eta_1 \cos\theta_{B\parallel}, \quad \cos\theta_t = \sqrt{1-\frac{n_1^2}{n_2^2}\sin^2\theta_{B\parallel}}$$

$$\boxed{\sin^2\theta_{B\parallel} = \frac{1-\dfrac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}}{1-\left(\dfrac{\varepsilon_1}{\varepsilon_2}\right)^2}}$$

For many cases, $\mu_1 = \mu_2$ $\quad \therefore \sin^2\theta_{B\parallel} = \dfrac{1}{\sqrt{1+\left(\dfrac{\varepsilon_1}{\varepsilon_2}\right)}} = \dfrac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_2+\varepsilon_1}}$ for $\mu_1 = \mu_2$

or $\theta_{B\parallel} = \tan^{-1}\sqrt{\dfrac{\varepsilon_2}{\varepsilon_1}} = \tan^{-1}\left(\dfrac{n_2}{n_1}\right)$ for $\mu_1 = \mu_2$

Home work

H.W

8-28, 8-30, 8-32, 8-33, 8-36, 8-42