6.6 Torsion of Elastic Hollow Circular Shafts

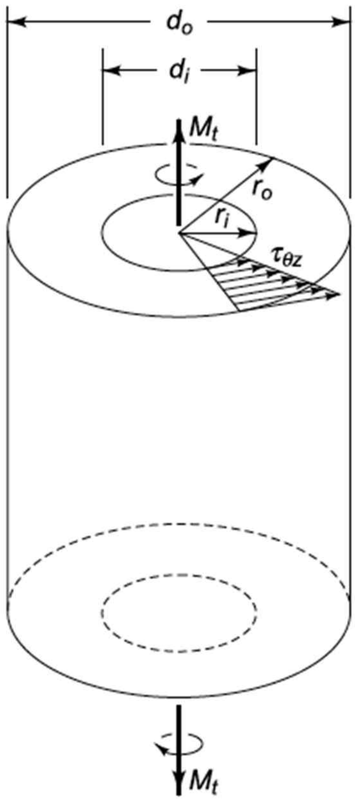


Fig. 6.12 *Stress distribution in elastic hollow circular shaft*

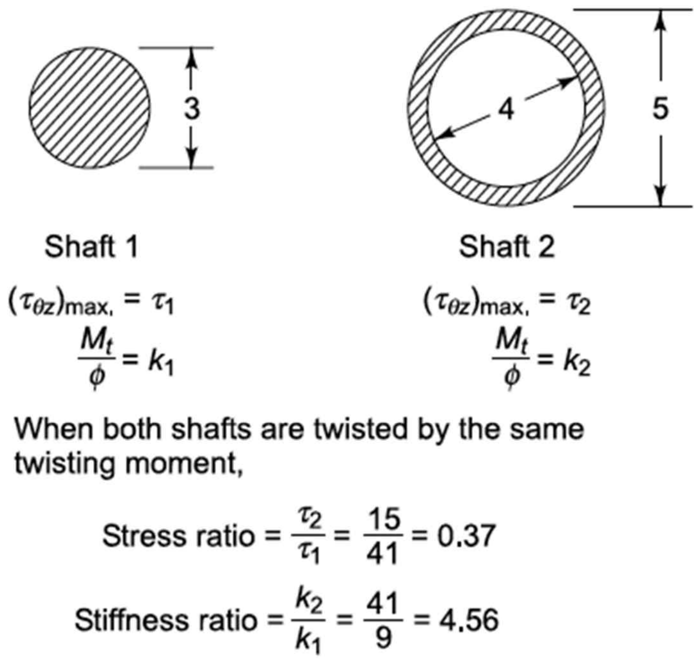


Fig. 6.13 *Illustration of advantages of hollow shaft over solid shaft of same cross-sectional area*

→ The only difference is that the integral in (6.4) now extends over an annulus instead of a complete circle.

$$I_z = \frac{\pi r_0^4}{2}\left(1 - \frac{r_L^4}{r_0^4}\right) = \frac{\pi d_0^4}{32}\left(1 - \frac{d_i^4}{d_0^4}\right) \qquad (6.11)$$

cf. $\int_A r(\tau_{\theta z} dA) = M_t \qquad (6.4)$

▶ Analysis

i) Making a concentric hole in a shaft does not reduce the torsional stiffness in proportion to the amount of material removed.

→ An element of material near the center of the shaft has a low stress and a small moment arm and thus contributes less to the twisting moment than an element near the outside of the shaft.

ii) The torsional stiffness for a given length of given material depends only on the polar moment of inertia I_Z.

iii) It is apparent that a given amount of material is used most efficiently in torsion when it is formed into a hollow shaft.

cf. There is a limit on the increase in effectiveness that can be obtained by increasing the diameter and decreasing the wall thickness. (If the wall is made too thin, the cylinder wall will buckle due to compressive stresses which act in the wall on surfaces inclined at $45°$ to the axis of the cylinder.)

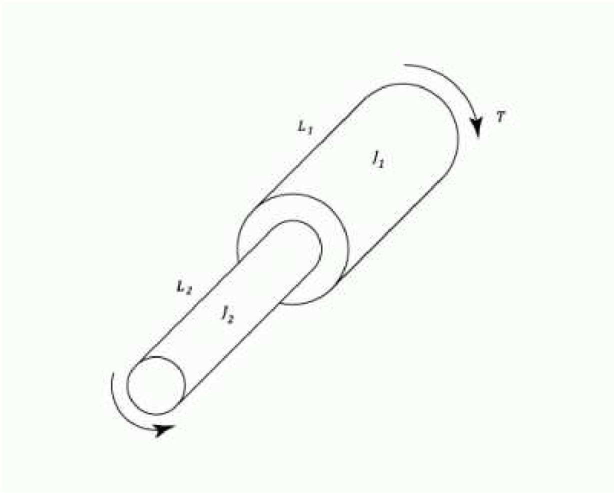
iv) Compare the hollow shaft and solid shaft in Fig 6.13 which have the same cross-sectional area but markedly different maximum stresses and deformation.

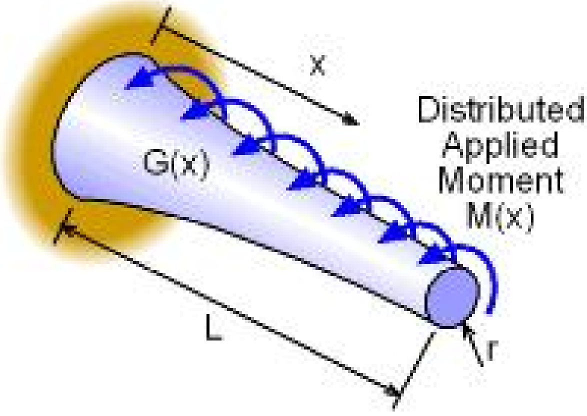
$$\frac{\tau_2}{\tau_1} = \frac{M_t r_2/I_2}{M_t r_1/I_1} = \frac{I r_2}{I_2 r_2} = \frac{(3^4)(5)}{(5^4-4^4)3^4} = \frac{135}{369} = \frac{15}{41} = 0.37$$

$$\frac{k_2}{k_1} = \frac{GI_2/L_2}{GI_1/L_1} = \frac{I_2}{I_1} = \frac{(5^4-4^4)}{3^4} = \frac{41}{9} = 4.56$$

cf. The shear-stress ratio is same with yield M_t ratio and stiffness ratio means ratio of torsion angle.

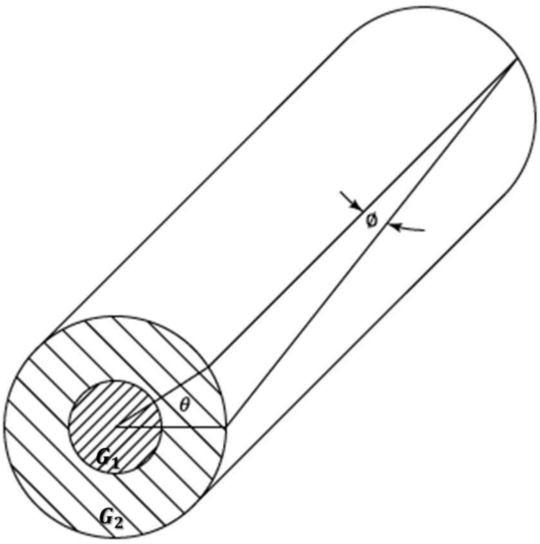
▶Non-uniform torsion examples





$$\phi = \sum_{i=1}^{n} \frac{T_i L_i}{G_i I_i} \qquad \phi = \int_0^L d\phi = \int_0^L \frac{T_x dx}{G I_x}$$

▶Composite shaft



$$\begin{cases} M_t = M_{t1} + M_{t2} \\ \phi_1 = \phi_2 = \frac{M_{t1} L}{G_1 I_1} = \frac{M_{t2} L}{G_2 I_2} \end{cases}$$

$$\begin{cases} M_{t1} = M_t \left(\frac{G_1 I_1}{G_1 I_1 + G_2 I_2} \right) \\ M_{t2} = M_t \left(\frac{G_2 I_2}{G_1 I_1 + G_2 I_2} \right) \end{cases}$$

$$\therefore \quad \phi_1 = \phi_2 = \frac{M_t L}{G_1 I_1 + G_2 I_2}$$

$$\frac{\tau_2}{\tau_1} = \frac{G_2 \gamma_2}{G_1 \gamma_1} = \frac{G_2}{G_1}$$

cf. Above ratio can be smaller than 1.

cf. Shear strains in two parts which are attached have same value, but each material has different coefficient and therefore stress is different.

6.7 Stress Analysis in Torsion; Combined Stress

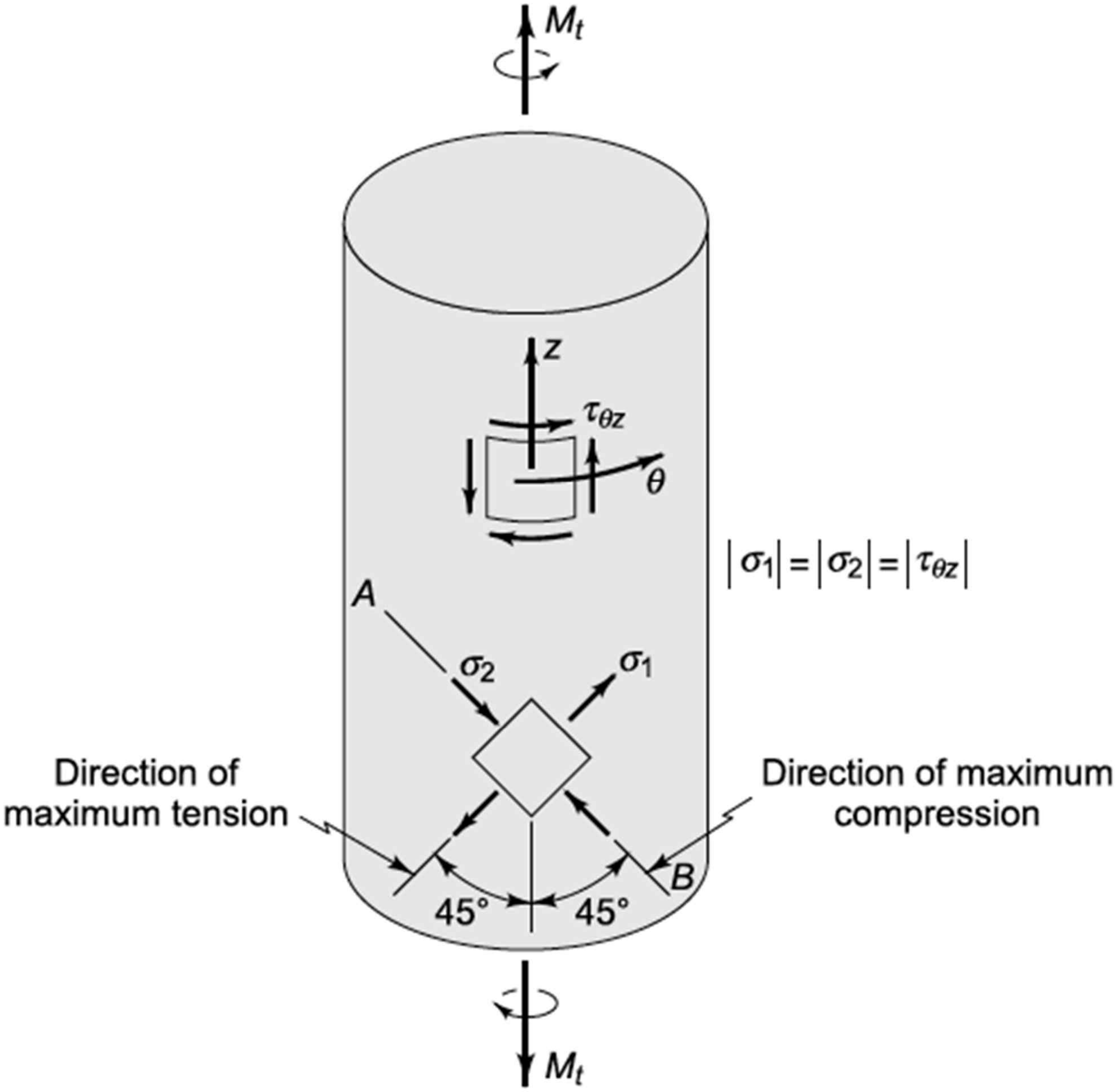


Fig. 6.14 *The principal stresses in torsion are equal tension and compression acting on faces inclined at 45° to the axis of the shaft*

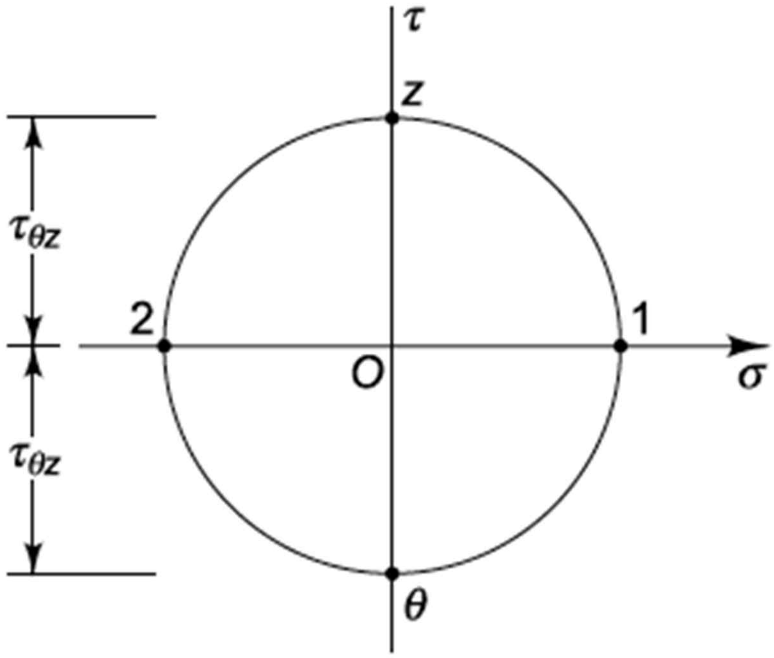


Fig. 6.15 *Mohr's circle for stress for element of shaft in torsion*

→ When shaft is twisted, it is on the pure shear stress state. And a convenient way to determine these stress components is to use Mohr's circle for stress.

cf. We may use the two-dimensional Mohr's circle because there is no stress in the r-direction.

▶ Magnitudes of principal stresses (from Mohr circle)

$|\sigma_1| = |\sigma_2| = |\tau_{\theta z}|$

$\theta_P = 45°$

cf. If a piece of chalk (which is a brittle material with a low tensile strength and much larger strength in compression and shear) is twisted, the chalk will fracture along a spiral line normal to the direction of maximum tension (e.g., along the line AB in Fig. 6.14)

▶ Combined-stress

→ The stresses and strains contributed by one form of loading are not altered by the presence of another kind of loading.

→ The justification for superposition lies in the linearity of Eqs. (5.6), (5.7), and (5.8) underlying the theory of elasticity.

▶ Example 6.3

In Fig. 6.16 (a) an uniform, homogeneous, circular shaft is shown subjected simultaneously to an axial tensile force P and a twisting moment M_t. In Fig. 6.16 (b) the individual stress distributions are sketched for the separate loads.

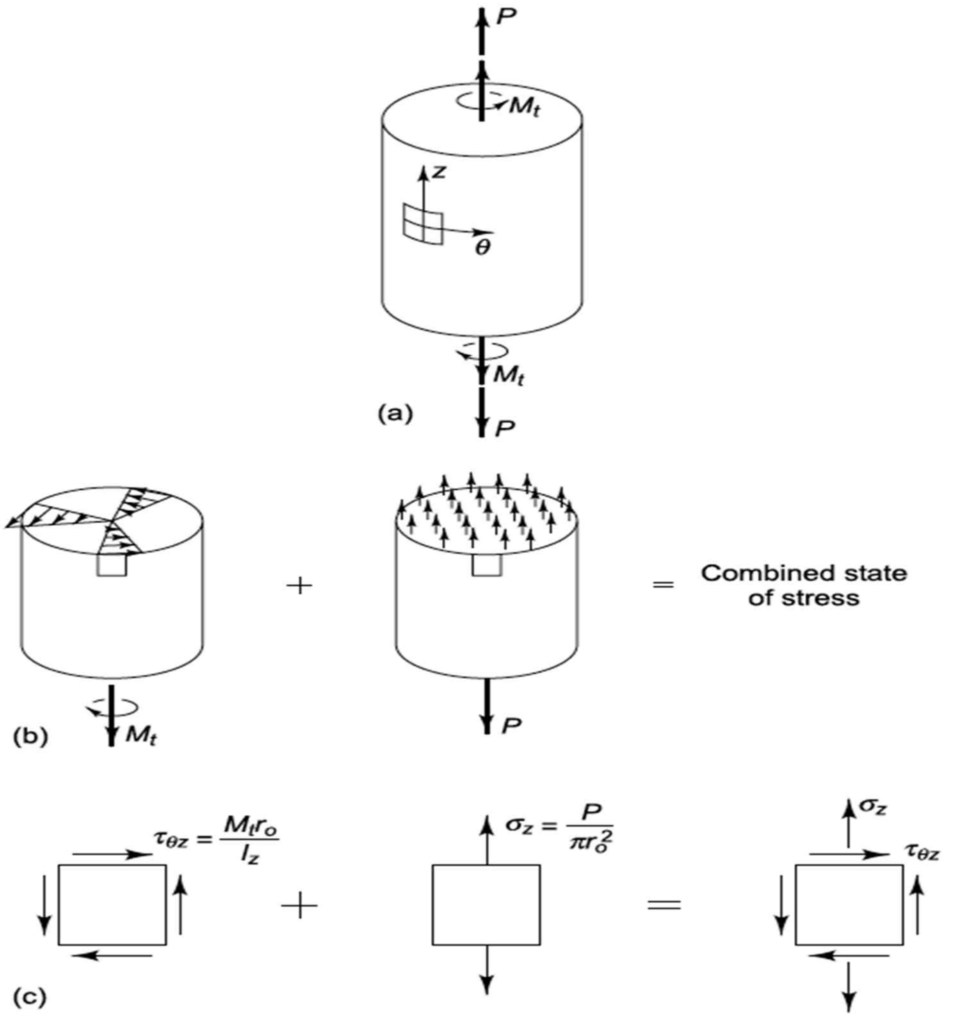


Fig. 6.16 *Example 6.3. Combined stresses due to torsion and tension*

From the Fig. 6.16 (a),

$$\tau_{\theta z} = \frac{M_t r_0}{J_z} \quad \text{(a)}$$

$$\sigma_z = \frac{P}{\pi r_0^2} \quad \text{(b)}$$

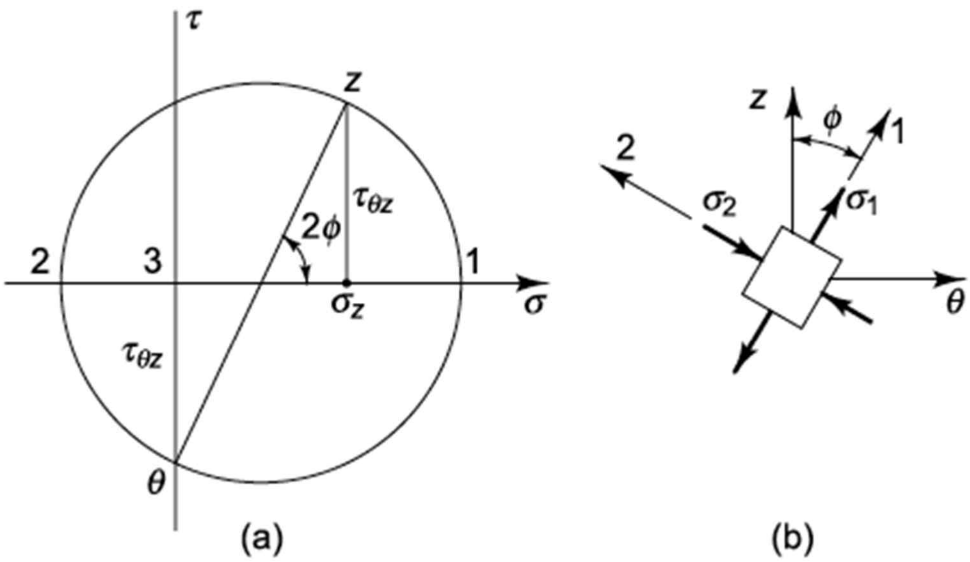


Fig. 6.17 *Example 6.3. Principal directions and principal stresses*

cf. The most convenient method of describing the combined-stress state is to use the principal stress components.

cf. Note that this element is in a state of plane stress, i.e., the third principal stress σ_3 is zero.

→ Positive shear stress τ_{xy} (see Fig. 4.11) is plotted downward at x and upward at y. Negative shear stress is plotted upward at x and downward at y.

▶ Note

▷ In pure shear state,

$$\epsilon_1 = \gamma_{\theta z}/2$$

6.8 Strain Energy Due to Torsion

→ In this section we apply that result specifically to the case of torsion of circular members and consider an example of Castigliano's theorem applied to torsional deformation.

cf. Obtaining the strain energy is important in many ways such as dynamic analysis and structure theory.

▶ **For circular shaft [Isotropic-linear-elastic]**

→ The only non-vanishing stress and strain components are $\tau_{\theta z}$ and $\gamma_{\theta z}$. The total strain energy (5.17) thus reduces to

$$U = \frac{1}{2}\int_V \tau_{\theta z}\gamma_{\theta z}\, dV \tag{6.12}$$

$$= \frac{1}{2}\int_V \frac{1}{G}\left[\frac{M_t r}{I_z}\right]^2 dV = \frac{1}{2}\int_L \frac{M_t^2}{GI_z^2} dz \int_A r^2\, dA$$

$$= \int_L \frac{M_t^2}{2GI_z}\, dz = \int_L \frac{GI_z}{2}\left(\frac{d\emptyset}{dz}\right)^2 dz \tag{6.13}$$

$$\rightarrow dU = \frac{1}{2} M_t\, d\phi = \frac{1}{2} M_t \frac{d\phi}{dz}\, dz = \frac{GI_z}{2}\left(\frac{d\phi}{dz}\right)^2 \tag{6.14}$$

▷ For uniform torsion

$$\begin{cases} u = \dfrac{U}{V} = \dfrac{\tau_{\theta z}\gamma_{\theta z}}{2} = \dfrac{\tau_{\theta z}^2}{2G} = \dfrac{G\gamma_{\theta z}^2}{2} \\ U = \dfrac{M_t^2 L}{2GI_z} = \dfrac{GI_z\emptyset^2}{2L} = \dfrac{I_z\emptyset}{2} \end{cases}$$

cf. We illustrate the application of Castigliano's theorem ($\delta_i =$

$\partial U / \partial P_i$) to a torsional system in the following example 6.4.

▶Example 6.4

Consider a closely wound coil spring of radius R loaded by a force P (Fig. 6.18 (a)). The spring consists of n turns of wire with wire radius r. We wish to find the deflection of the spring and hence the spring constant.

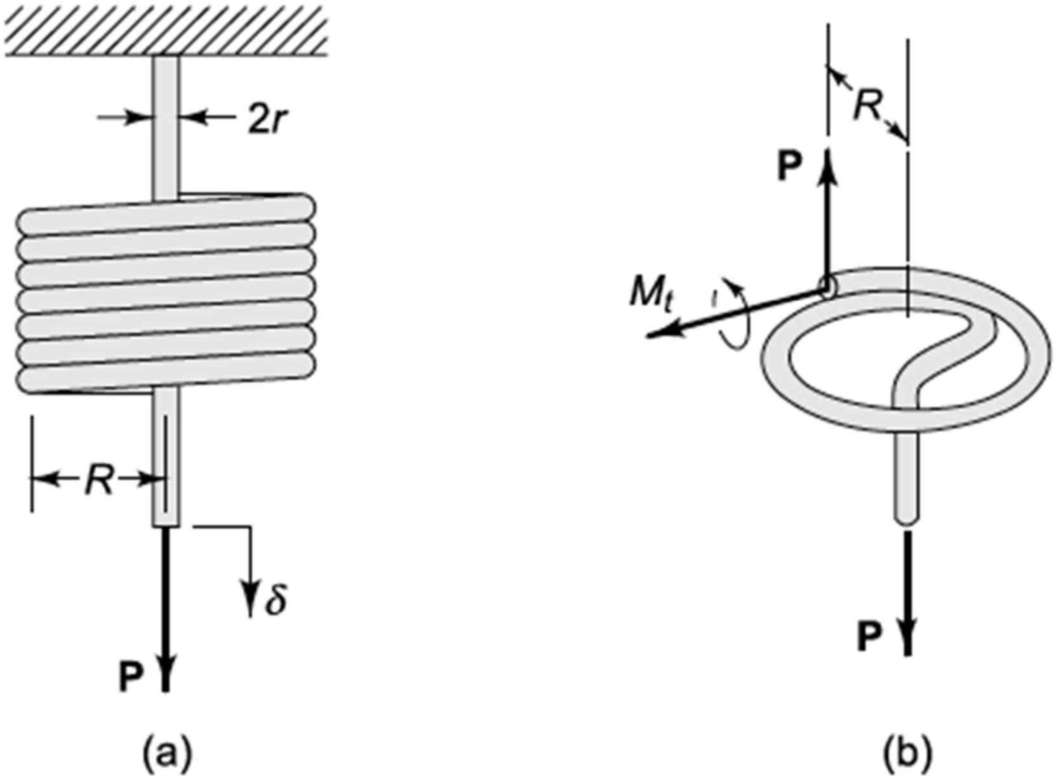


Fig. 6.18 Example 6.4

1▷ The strain energy associated with the twisting moment

$$U = \int_L \frac{P^2R^2}{2GI_z}\,dz = \int_0^{2\pi n} \frac{P^2R^2}{2GI_z} R\,d\theta = \frac{P^2R^2}{2GI_z} 2\pi n \qquad \text{(a)}$$

2▷ Strain energy due to the transverse shear force

→ There is additional strain energy in the spring due to the transverse shear force P. It can be shown, however, that the ratio of strain energy due to transverse shear to strain energy due to torsion is proportional to $\left(\frac{r}{R}\right)^2$ and hence is small for springs of usual design.

3 ▷ Application of Castigliano's theorem

$$\delta = \frac{\partial U}{\partial P} = \frac{PR^3}{GI_z} 2\pi n \qquad \text{(b)}$$

$$\rightarrow \quad \therefore \quad k = \frac{P}{\delta} = \frac{GI_z}{2\pi n R^3} \qquad \text{(c)}$$

Upon substituting for the moment of inertia I_z in (c), we find that

$$k = \frac{Gr^4}{4nR^3}$$

→ We see that the spring constant is inversely proportional to the number of coils n and directly proportional to the fourth power of the wire radius. For example, if we increase the wire radius by 19 percent, the spring constant is doubled.

6.9 The Onset of Yielding in Torsion

→ In order to apply either criterion to a particular material it is necessary to obtain (experimentally) the yield stress Y in uniaxial

→ Then, to decide whether yielding will occur in a general state of stress, we compute the equivalent or effective stress $\bar{\sigma}$ (or $\bar{\tau}$) according to the criterion employed and compare with Y.

▶ The principal stresses acting on an element of a shaft in torsion

$$\sigma_1 = \tau_{\theta z}, \qquad \sigma_2 = -\tau_{\theta z}, \qquad \sigma_3 = 0 \qquad (6.15)$$

1 ▷ Using the Mises criterion

$$\bar{\sigma} = Y = \sqrt{\frac{1}{2}[(2\tau_{\theta z})^2 + (-\tau_{\theta z})^2 + (-\tau_{\theta z})^2]} = \sqrt{3}\tau_{\theta z} \qquad (6.16)$$

thus an element of a shaft in torsion would be expected to begin yielding when

$$\therefore \quad \tau_{\theta z} = \frac{1}{\sqrt{3}} Y = 0.577\, Y \tag{6.17}$$

2 ▷ Using the maximum shear-stress criterion

$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{Y}{2} \tag{6.18}$$

The equivalent shear stress is $\bar{\tau} = \tau_{\theta z}$

$$\therefore \quad \tau_{\theta z} = \frac{1}{2} Y = 0.500Y \tag{6.19}$$

→ As can be seen from (6.17) and (6.19), this discrepancy is about 15 percent. From the point of view of the designer trying to avoid yielding, it is more conservative to design on the basis of (6 19).

cf. Since the shear stress $\tau_{\theta z}$ is proportional to the radius r in an elastic shaft, it is clear that according to either criterion the elements on the outer surface of the shaft will reach the yield condition first.

6.10 Plastic Deformations

→ It is important to remember that in passing from elastic to plastic behavior there is no alteration in the conditions of equilibrium or in the conditions of geometric compatibility. The only change is in the stress-strain relation.

cf. The only non-vanishing strain component was $\gamma_{\theta z}$ remain valid whether the material is elastic or plastic. What will be different is the relation between $\gamma_{\theta z}$ and $\tau_{\theta z}$.

▶ Two ways to obtain the relation between $\gamma_{\theta z}$ and $\tau_{\theta z}$ in plastic region

i) Direct experiment in which the material is subjected to uniform pure shear

ii) To make use of tension test data and to predict the relation between $\gamma_{\theta z}$ and $\tau_{\theta z}$ in torsion by using one of the plastic flow rules → it is less exact, but simpler.

cf. In this chapter we shall confine our analytical treatment to the elastic-perfectly plastic material. (∴ Strain hardening is not exist.)

→ In plastic region, $\tau_{\theta z} = \tau_{Y} = const$.

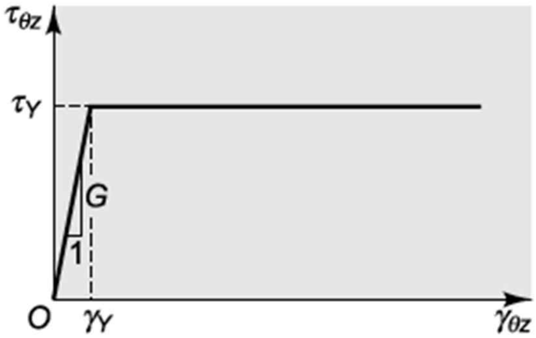


Fig. 6.19 *Shear-stress–shear-strain curve for elastic-perfectly plastic material*

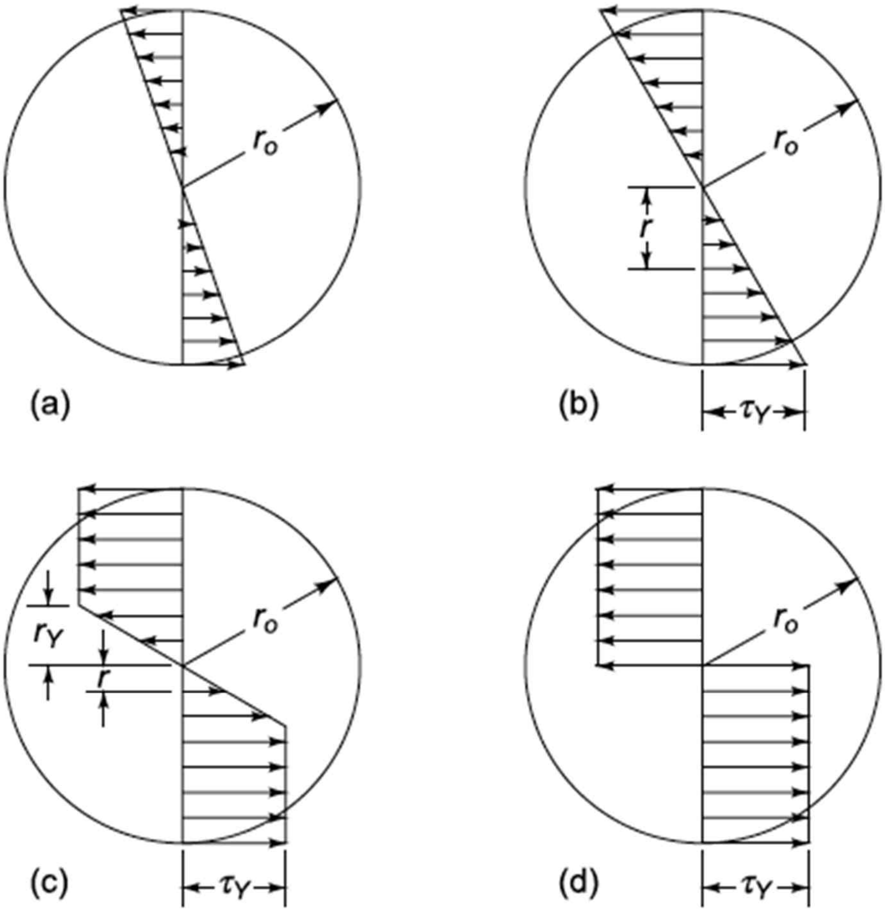


Fig. 6.20 *Shear-stress distribution in a twisted shaft of material having the stress-strain curve of Fig. 6.19. (a) Entirely elastic; (b) onset of yield; (c) partially plastic; (d) fully plastic*

▶ Analysis

1 ▷ To obtain quantitative representations of the sketches in Fig. 6 20, we proceed as follows. The elastic relations (6.8) and (6.9) apply until the yield-point situation in Fig. 6.20 (b) is reached.

$$\phi = \frac{d\phi}{dz}L = \frac{M_t L}{G I_z} \tag{6.8}$$

$$\tau_{\theta z} = Gr\left(\frac{M_t}{G I_z}\right) = \frac{M_t r}{I_z} \tag{6.9}$$

2 ▷ Let us call the twisting moment and twisting angle associated with this (b) stress distribution T_Y and ϕ_Y, respectively. Then from (6.8) and (6.9) we have

$$T_Y = \frac{\tau_Y I_z}{r_0} = \frac{\pi}{2}\tau_Y r_0^3 \quad (6.20\ a)$$

$$\therefore \phi_Y = T_Y \frac{L}{GI_z} = \left(\frac{\pi}{2}\tau_Y r_0^3\right)\frac{L}{G\pi r^4/2} = \frac{\tau_Y L}{G r_0} \quad (6.20\ b)$$

$$cf.\ \phi = \int_0^L \frac{M_t}{G\, I_z}\, dz = \frac{M_t\, L}{G\, I_z} \quad (6.8)$$

$$\tau_{\theta z} = \frac{M_t r}{I_z} \quad (6.9)$$

3 ▷ Now as the shaft is twisted further the shear strain at the outer radius becomes larger than γ_Y. We still have the geometric relation (6.1) between shear strain and twist angle

$$\gamma_{\theta z} = r\frac{d\phi}{dz} = r\frac{\phi}{L} \quad (6.21)$$

4 ▷ At some intermediate radius r_Y the strain will be just equal to γ_Y. We can solve for r_Y when $\phi > \phi_Y$

$$r_Y = \frac{L\gamma_Y}{\phi} \quad (6.22)$$

5 ▷ Using the fact that $\tau_Y = G\gamma_Y$ and introducing the second of (6.20), we find

$$r_Y = \frac{L\gamma_Y}{\phi} = r_0\frac{\phi_Y}{\phi} \quad (6.23)$$

6 ▷ Next, we obtain a quantitative representation for the stress distribution $\tau_{\theta z}$ corresponding to the strain distribution $\gamma_{\theta z}$ of (6.21) by using the stress-strain relation of Fig. 6.19. In the inner elastic core $0 < r < r_Y$,

$$\tau_{\theta z} = G\gamma_{\theta z}$$

$$= G\frac{\phi}{L}r = G\frac{[L\tau_Y/G r_Y]r}{L} = \tau_Y\frac{r}{r_Y} \quad (6.24)$$

7 ▷ In the outer plastic region $r_Y < r < r_o$,

$$\tau_{\theta z} = \tau_Y \quad (6.25)$$

8 ▷ The stress distribution defined by (6.24) and (6.25) is sketched in Fig. 6.20 (c). Finally, we use the equilibrium requirement that the stress distribution of Fig. 6.20 (c) should be equivalent to the applied twisting moment M_t.

$$M_t = \int_A r\tau_{\theta z} dA$$

$$= \int_0^{r_Y} r\left(\tau_Y \frac{r}{r_Y}\right) 2\pi r\, dr + \int_{r_Y}^{r_o} r\tau_Y\, 2\pi r\, dr$$

$$= \frac{2\pi}{3}\tau_Y r_0^3 \left(1 - \frac{1}{4}\frac{r_Y^3}{r_0^3}\right) \tag{6.26}$$

9 ▷ This result can be put into a more useful final form by introducing the yield-point twisting moment from (6.20) and the twisting angle from (6.23)

$$M_t = \frac{4}{3}T_Y\left(1 - \frac{1}{4}\frac{\phi_r^3}{\phi^3}\right) \tag{6.27}$$

cf. This nonlinear relationship is valid when $\phi > \phi_Y$

10 ▷ The limit or fully plastic twisting moment T_L (Fig. 6.21)

when $\phi \to \infty$

$T_L \to \frac{4}{3}T_Y$

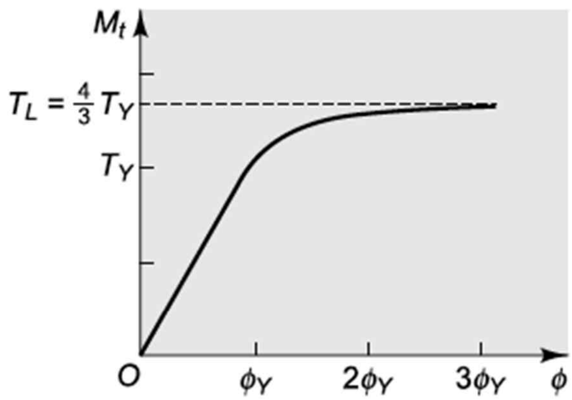


Fig. 6.21 *Twisting-moment–twisting-angle relationship for solid circular shaft made of material with stress-strain curve of Fig. 6.19*

6.11 Residual Stresses

▶ from Fig. 6.22;

If we assume that the material of the shaft unloads elastically after it has been strained plastically, then if at any stage the twisting moment were to be decreased, the twisting moment-twisting angle curve would trace out a straight line parallel to the original elastic relation of (6 8), as sketched in Fig. 6.22.

→ The justification for this lies in the fact that the geometric and equilibrium requirements for torsion remain unchanged while the stress-increment strain-increment relation is now elastic for the entire shaft.

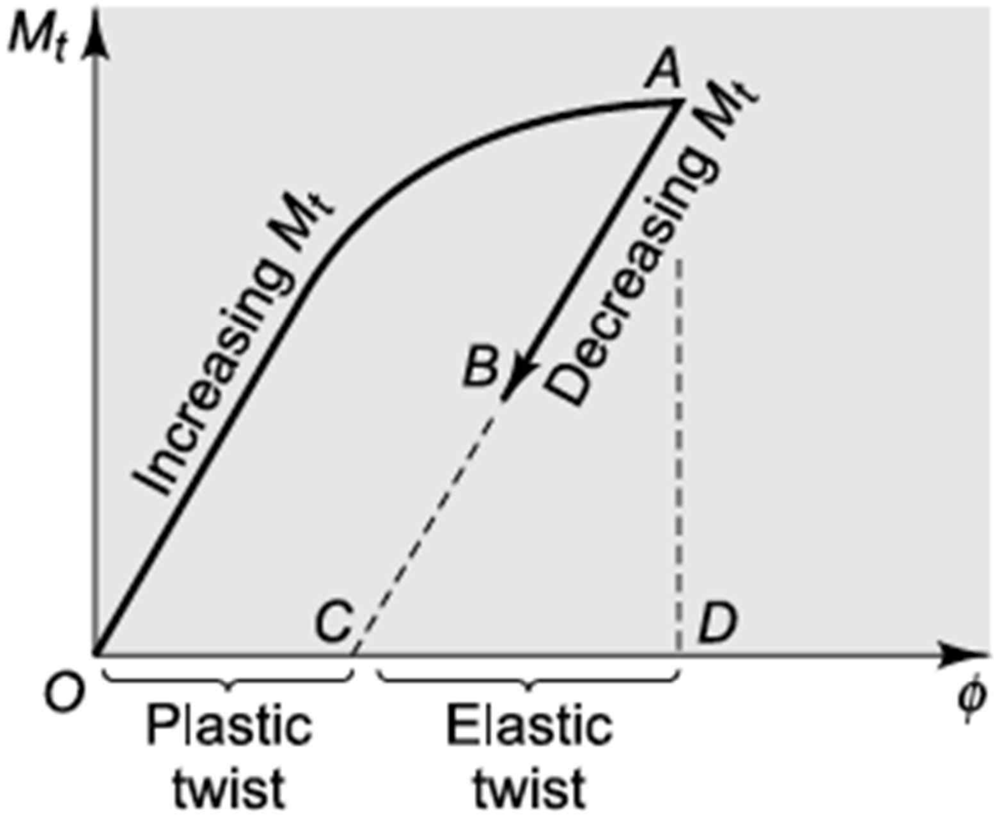


Fig. 6.22 *Unloading a plastically deformed shaft*

▶ <u>Residual stress</u>

Although there is no external load on the shaft in this condition, there is a distribution of self-balancing internal stresses in the shaft. These internal stresses which are "locked in" the material by the plastic deformation are called residual stresses.

→ The distribution of residual stresses can be found by using superposition.

▶ <u>Calculation of residual stress</u>

i) When parts (a) and (b) of Fig. 6.23 are superposed, we end up with no external twisting moment but with a distribution of residual stresses, as shown in Fig. 6.23 (c).
ii) The outer part of the shaft carries shearing stresses of the opposite sense to that imposed by the original application of the load, while the inner part carries stresses of the same sense as those originally imposed.
iii) Under some circumstances the reversed stresses obtained in this manner might be larger than the yield stress in the opposite direction. In this case simple linear superposition would not be applicable (see Prob. 6.41).

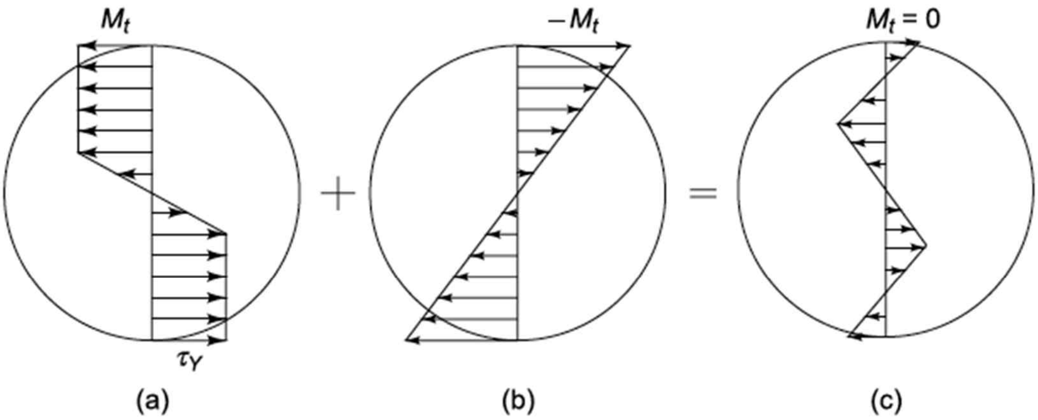


Fig. 6.23 *Residual shear-stress distribution in a shaft which has been twisted into the plastic region and unloaded*