Ch. 7
Stresses due to Bending

7.1 Introduction

→ Our aim in this chapter is to determine the distributions of stresses which have the shear force V and the bending moment M_b as their resultant.

▶ Beam

When a slender member is subjected to transverse loading, we say it acts as a beam.

▶ Pure bending

When there is no shear force, and a constant bending moment is transmitted, we say it is a state of pure bending $\left(\because -V = \frac{dM}{dx}\right)$

cf. Our method of approach will be similar to that followed in the investigation of torsion in Chap. 6, and to a certain extent our results will be similar.

cf. In this chapter we shall also obtain an exact solution within the theory of elasticity of the special case of a beam subjected to pure bending. For more general cases we shall obtain approximate distributions of stresses on the basis of equilibrium considerations.

7.2 Geometry of deformation of a symmetrical beam subjected to pure bending

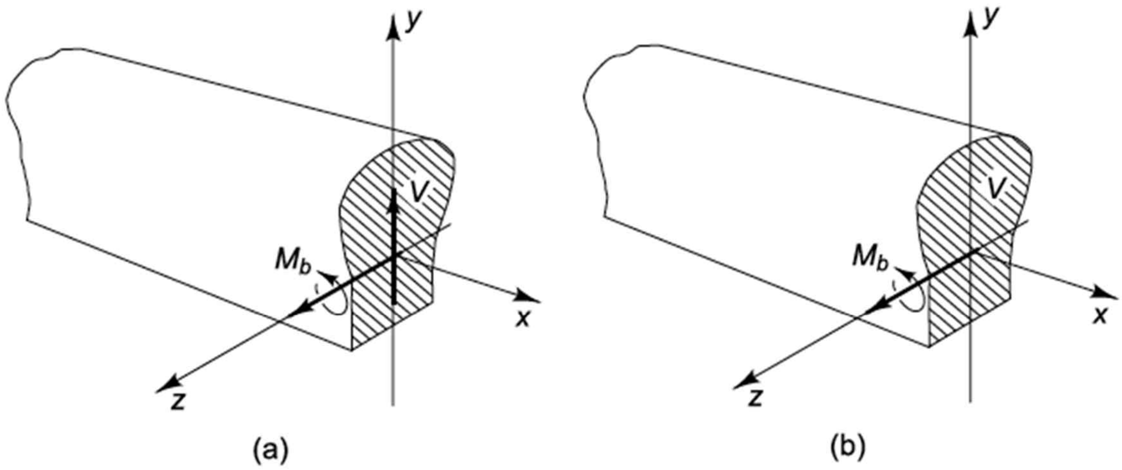

Fig. 7.2 *Symmetrical beam loaded in its plane of symmetry. (a) In general, both shear force and bending moment are transmitted; (b) in pure bending there is no shear force, and a constant bending moment is transmitted*

▶ Assumptions (See Fig. 7.2)

i) We consider an originally straight beam which is uniform along its length, whose cross sections is symmetrical.

ii) Its material properties are constant along the length of the beam.

iii) It is subjected to pure bending.

∴ The deformation pattern can be fixed by symmetry arguments alone.

cf. The result derived from these assumptions is valid to any types of beams whose materials are linear or nonlinear, elastic or plastic.

▶ Curvature

→ The curvature of a plane curve is defined as the rate of the slope angle change of the curve with respect to distance along the curve.

∴ for $\Delta s \rightarrow 0$ (see fig. 7.3)

$$\kappa = \frac{d\phi}{ds} = \lim_{\Delta s \to 0} \frac{\Delta\phi}{\Delta s} = \lim_{\Delta s \to 0} \frac{1}{\overline{O'B}} = \frac{1}{\rho} \qquad (7.1)$$

where $\rho = OB$ is *the radius of curvature* at point B.

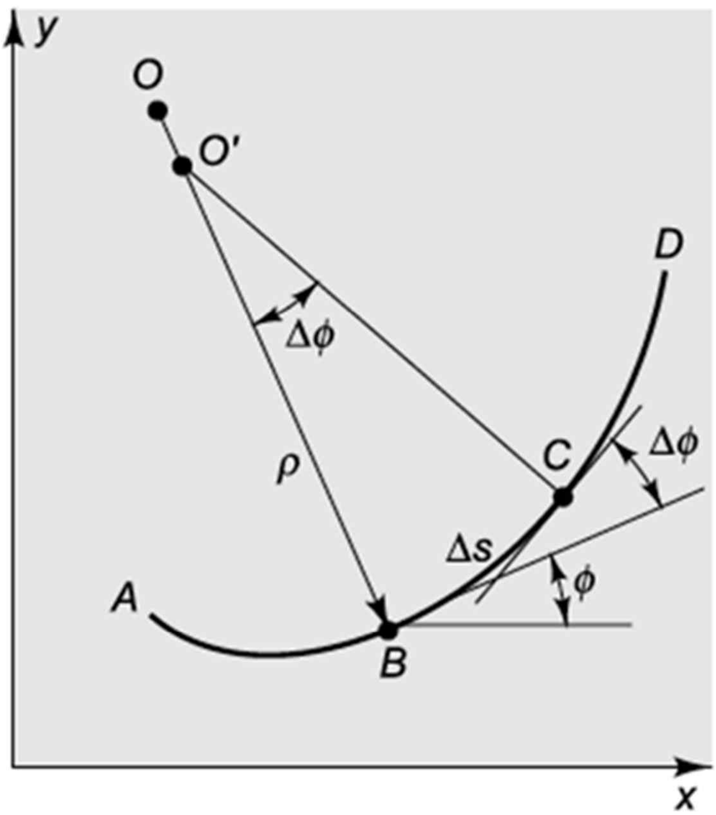

Fig. 7.3 *The line AD has curvature dϕ/ds = 1/ρ at point B, where ρ = OB is the radius if curvature at point B*

▶ Deformation behavior under pure bending

i) The surface A_1D_1, B_1E_1, C_1F_1 must be plane surfaces perpendicular to the plane of symmetry.

∴ In pure bending in a plane of symmetry plane cross sections remain plane.

ii) The fact that each element deforms identically means that the initially parallel plane sections now must have a common intersection, as illustrated by point O in Fig. 7.4b, and that the beam bends into the arc of a circle centered on this intersection.

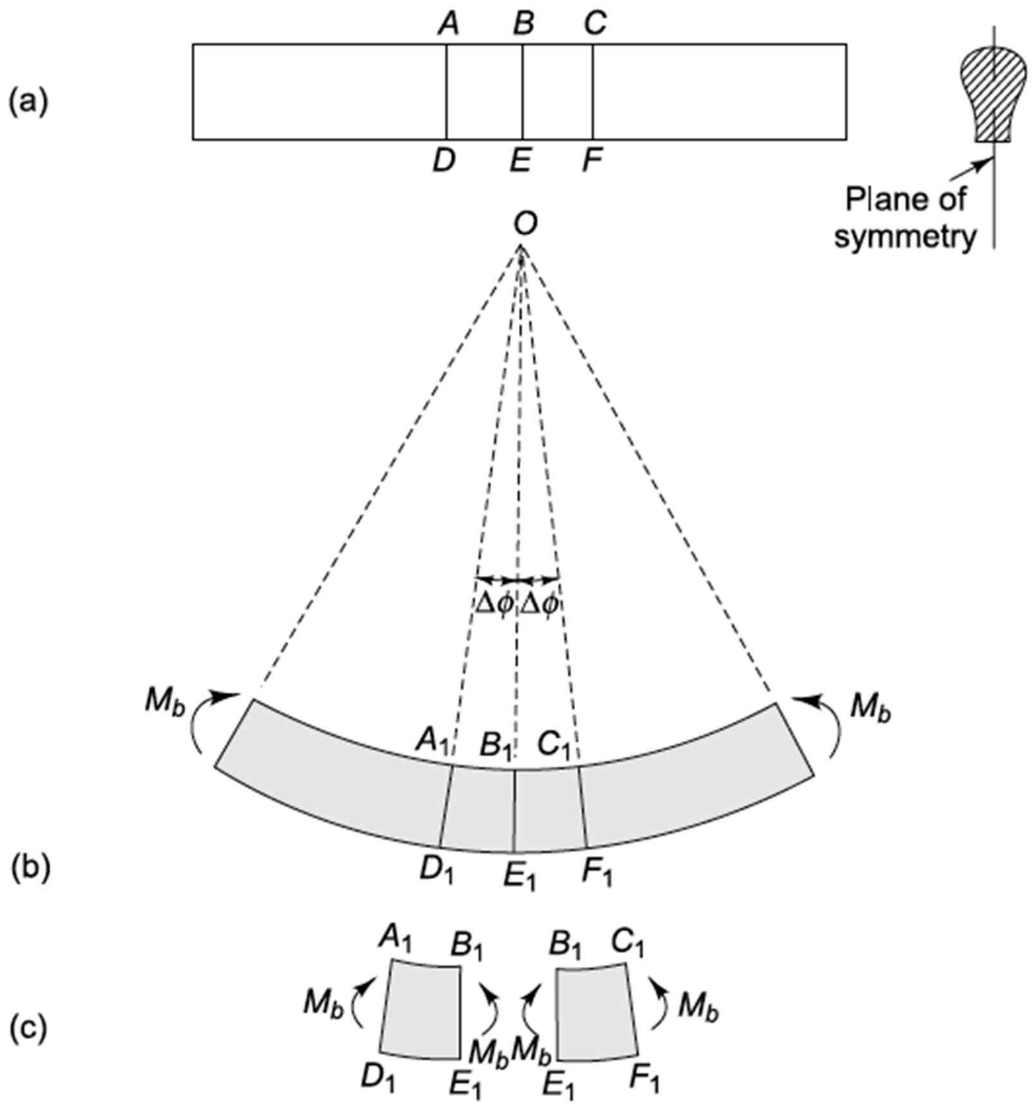

Fig. 7.4 *Overall deformation of a symmetrical beam subjected to pure bending in its plane of symmetry*

▶ Neutral Axis

Neutral axis is one line in the plane of symmetry which has not changed in length.

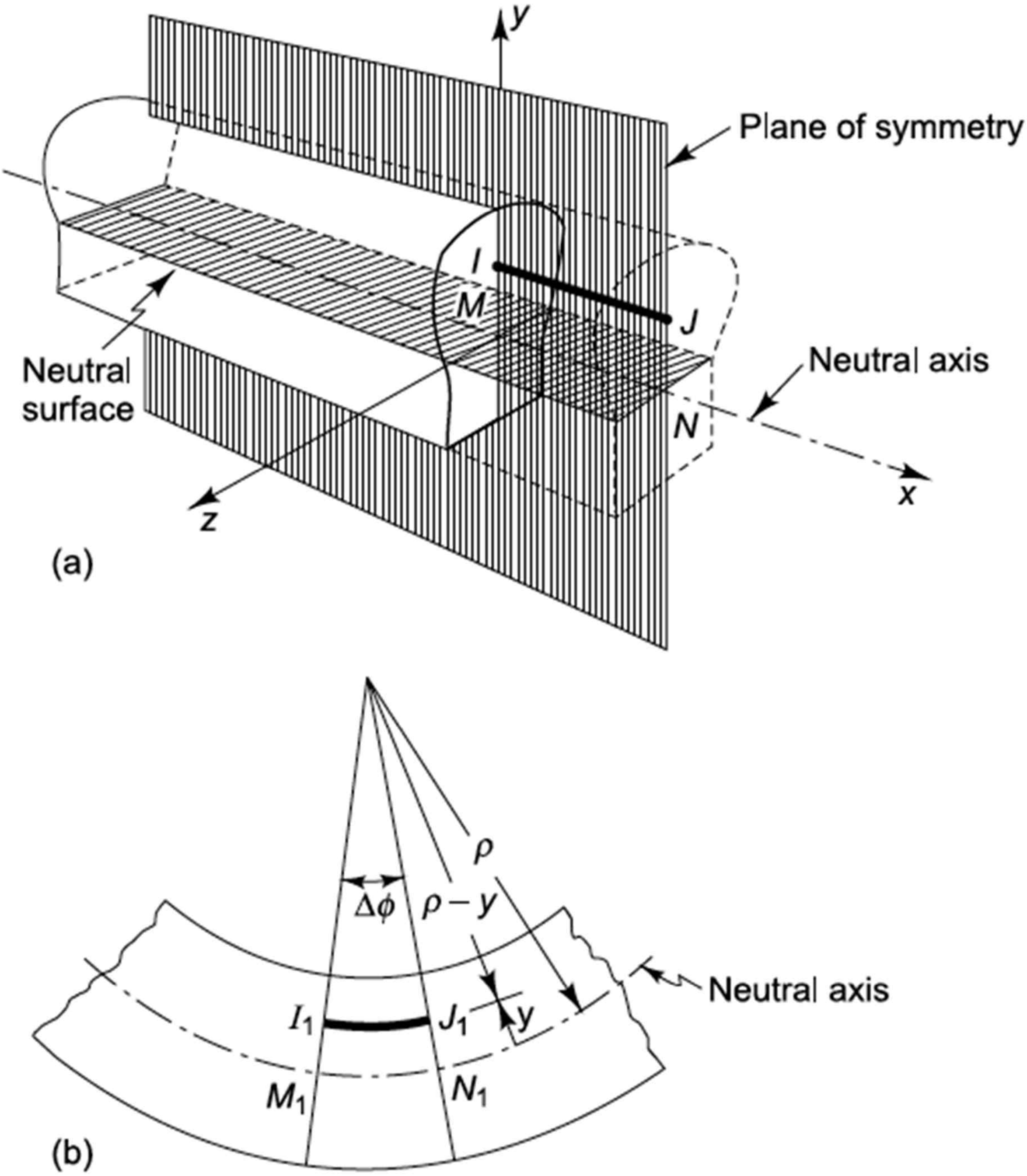

Fig. 7.5 *(a) Undeformed beam; (b) trace of deformed beam in the plane of symmetry*

▶ Distribution of strain (See Fig. 7.5)

$$\epsilon_x = \frac{I_1J_1 - IJ}{IJ} = \frac{I_1J_1 - M_1N_1}{M_1N_1} \quad (\because \ IJ = M_1N_1) \tag{7.2}$$

where $M_1N_1 = \rho\Delta\phi, \quad I_1J_1 = (\rho - y)\Delta\phi$ (7.3)

$$\therefore \quad \epsilon_x = -\frac{y}{\rho} = -\frac{d\phi}{ds}y = -\kappa y \tag{7.4}$$

▷ Brief on Eq.(7.4)

i) Longitudinal strain of the beam ϵ_x is proportional to curvature k (= bending deformation rate) and varies linearly with the distance from the neutral surface y.

ii) The derivation of (7.4) applies strictly only to the plane of symmetry, but we shall assume that (7.4) describes the longitudinal strain at all points in the cross section of the beam.

iii) This equation is irrelevant to the stress-strain relation of material.

▷ Other strain components of strain

$$\gamma_{xy} = \gamma_{xz} = 0 \tag{7.5}$$

cf. We can make no quantitative statements about the strains ϵ_x, ϵ_y *and* γ_{yz} beyond the remark that they must be symmetrical with respect to the *xy* plane.

7.3 Stresses obtained from stress-strain relations

→ In this section we shall restrict ourselves to beams made of linear isotropic elastic material, i.e., to materials which follow Hooke's law.

▶ Strain components

$$\epsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] = -\frac{y}{\rho}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = 0 \qquad (7.6)$$

$$\gamma_{xz} = \frac{\tau_{xz}}{G} = 0$$

→ In pure bending, $\tau_{xy} = \tau_{xz} = 0$

7.4 Equilibrium requirements

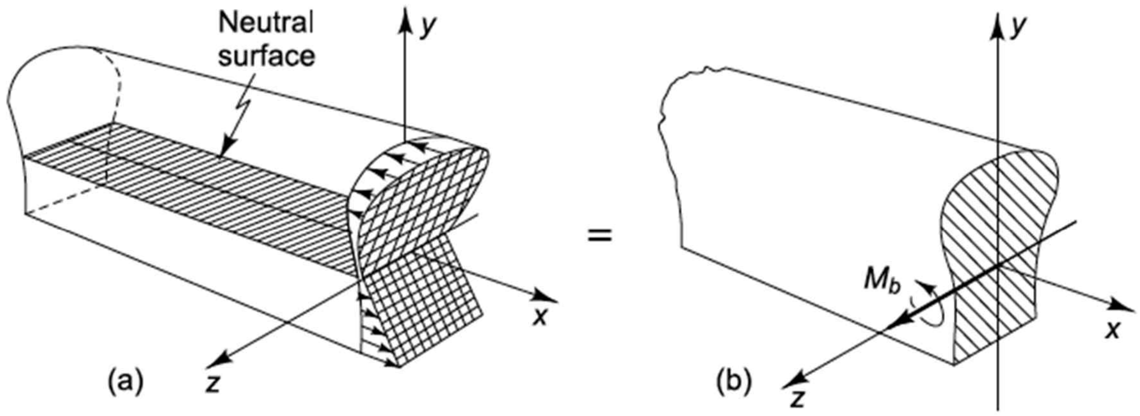

Fig. 7.6 *The resultant of the stress distribution in pure bending must be the bending moment M_b*

▶ Considering equilibrium (fig. 7.6)

$$\sum F_x = \int_A \sigma_x \, dA = 0$$

$$\sum M_y = \int_A z\sigma_x \, dA = 0 \qquad (7.7)$$

$$\sum M_z = -\int_A y\sigma_x \, dA = M_b$$

cf. We make the fundamental assumption that the deformation of the cross section is sufficiently small so that we can use the undeformed coordinates to locate points in the deformed cross section.

7.5 Stress and deformation in symmetrical elastic beams subjected to pure bending

▶ Aim of this section

→ We shall find the solution satisfying strain requirements, eqs (7.6), (7.7)

▶ Assumption

→ Considering that there is no normal or shear stress on the external surface of Δx and that the beam is slender, we can assume as follow.

$$\sigma_y = \sigma_z = \tau_{yz} = 0 \tag{7.8}$$

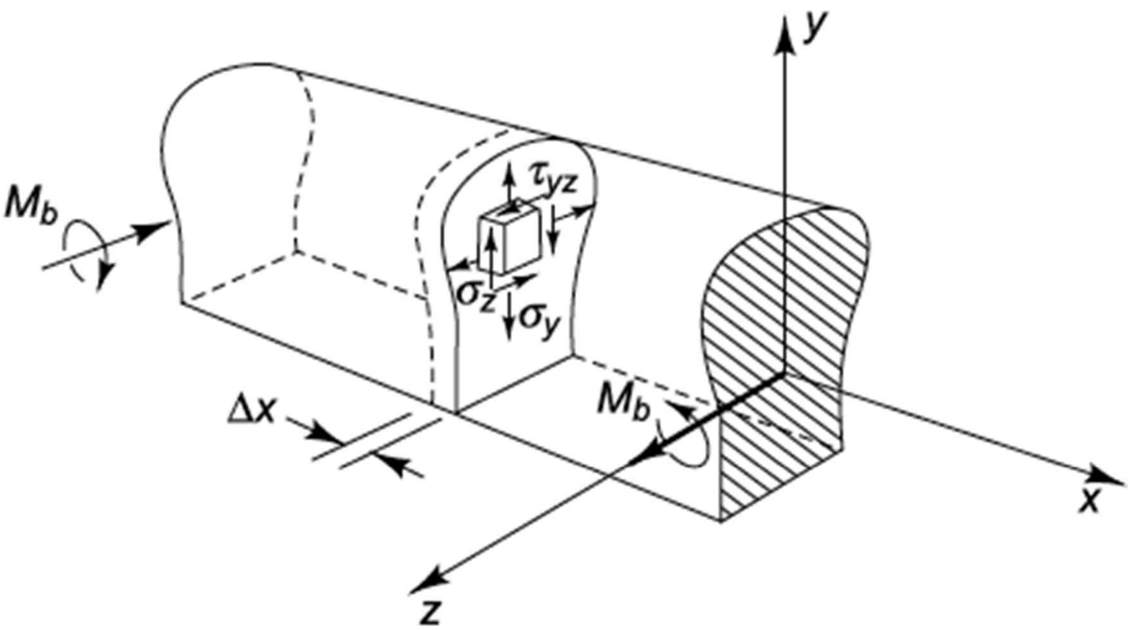

Fig. 7.8 The transverse stresses σ_y, σ_z, and τ_{yz} are assumed to be zero

▶ Analysis of stresses

Given above assumption, at the beam under pure bending which is following Hooke's low, the only stress component is

$$\sigma_x = -E\frac{y}{\rho} = -\kappa E y = -E\frac{d\phi}{ds}y \qquad (7.9)$$

▷ Equilibrium

i) $\sum F_x = \int_A \sigma_x \, dA = -\frac{E}{\rho}\int_A y \, dA = 0$ (7.10)

→ ∴ Since $\int_A \sigma_x \, dA = 0$, the neutral surface must pass through the centroid of the cross-sectional area.

cf. In case of a composite or nonlinear beam, it's possible to apply $\sum F_x = 0$ but the neutral surface doesn't pass through the centroid.

ii) $\sum M_y = \int_A z\sigma_x \, dA = -\frac{E}{\rho}\int_A yz \, dA = 0$ (7.11)

→ As the cross section is symmetrical with respect to *xy* plane, $\int_A yz \, dA = 0$

iii) $\sum M_z = -\int_A y\sigma_x \, dA = \frac{E}{\rho}\int_A y^2 \, dA = M_b$ (7.12)

where $I_z = \int_A y^2 \, dA$ (7.13)

→ ∴ Eq. (7.12) is;

$$\kappa = \frac{1}{\rho} = \frac{d\phi}{ds} = \frac{M_b}{EI_{zz}} \qquad (7.14)$$

cf. Similar with $\theta = \frac{d\phi}{dz} = \frac{M_t}{GI_z}$ (6.7)

$$\therefore \frac{M_b}{EI_z} = \kappa = -\frac{\epsilon_x}{y} \quad \rightarrow \quad \epsilon_x = -\frac{M_b y}{EI_z} \qquad (7.15)$$

$$\therefore \sigma_x = E\epsilon_x = -\frac{M_b y}{I_z} \qquad (7.16)$$

cf. Similar with $\tau_{\theta z} = \frac{M_t r}{I_z}$ (6.9)

▶ Analysis of strains

$$\epsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)] = \frac{1}{E}\left[0 - \nu\left(0 - \frac{M_b y}{I_z}\right)\right]$$
$$= \frac{\nu M_b y}{E I_z} = -\nu\epsilon_x$$
$$\epsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] = \frac{1}{E}\left[0 - \nu\left(-\frac{M_b y}{I_z} + 0\right)\right] \qquad (7.17)$$
$$= \frac{\nu M_b y}{E I_z} = -\nu\epsilon_x$$
$$\gamma_{yz} = \frac{\tau_{yz}}{G} = 0$$

▷ Brief on lateral strain

i) Since the axial normal strain is compressive at the top of the beam and tensile at the bottom, the top of the cross section expands while the bottom of the cross section contracts.

ii) The trace of the neutral surface on the cross section has become an arc with curvature $-\nu(1/\rho)$.

∴ The deformed neutral surface is a surface of double curvature ($1/\rho$ and $-\nu/\rho$). A further result of the anticlastic curvature is that the neutral axis is the only line in the deformed neutral surface whose curvature is in a plane parallel to the original plane of symmetry of the beam

→ This transverse curvature of the beam is called anticlastic curvature.

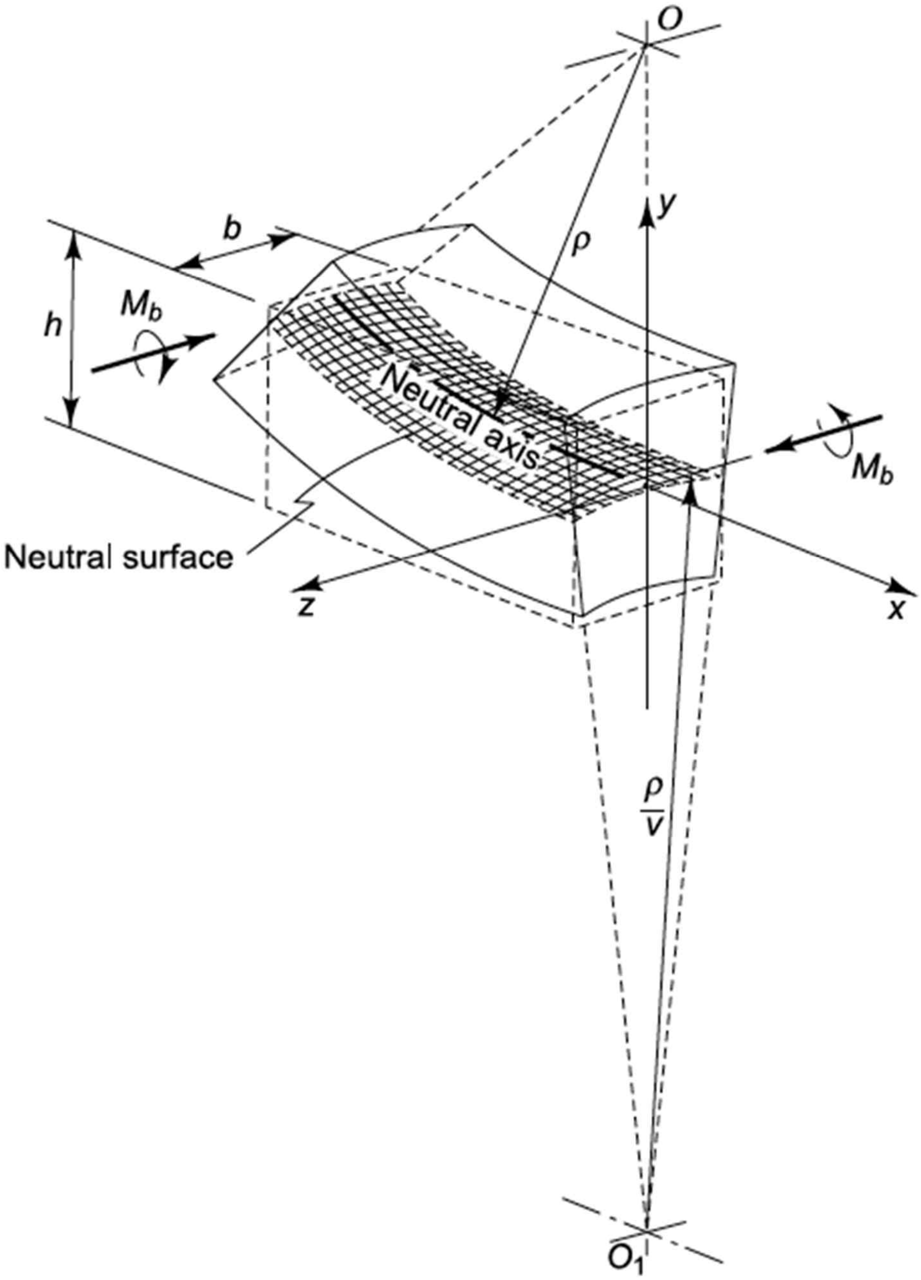

Fig. 7.9 *Deformed shape of an originally rectangular beam subjected to pure bending in a plane of symmetry*

▷ Validity of the assumption

i) The strains (7.5), (7.15), and (7.17) are geometrically compatible; the stresses (7.6), (7.8), and (7.16) satisfy the differential equations of equilibrium; and at every point the stresses and strains satisfy Hooke's law.

ii) Our solution is still very accurate in the central portion of the beam in accord with St. Venant's principle and only becomes appreciably in error near the ends. (The length of these transition regions at the ends is of the order of the depth of the beam cross section.)

cf. The analysis of the pure bending of curved beams is reasonably accurate for the non-uniform bending of curved beams

▶ Section modulus, S

$$S = \frac{M_{max}}{\sigma_{allow}} \quad \text{(a)}$$

or $S_1 = \frac{I_z}{y_{max}}$, $S_2 = \frac{I_z}{y_{min}}$

cf. $\sigma_{max} = -\frac{M_b}{S_2}$, $\sigma_{min} = -\frac{M_b}{S_1}$

→ It's convenience to define a required section modulus when we select the beam.

i) The cross section of the beam must be used when the S is larger than the value that obtained from eq. (a).

ii) It is desirable to select the cross section that has satisfactory section modulus and the smallest cross sectional area.

iii) On the rectangular cross section, the greater height h is, the larger S is.

iv) The square cross section beam is more efficient than the circular cross section beam with respect to the same area.

v) To design the beam economically, material should be placed in location that is away from the neutral axis as possible. (But, in an excessive case, there is a danger of buckling.)

▶Example 7.1

A steel beam 25mm wide and 75mm deep is pinned to supports at points A and B, as shown in Fig. 7.11a, where the support B is on rollers and free to move horizontally. When the ends of the beam are loaded with 5kN loads, find the maximum bending stress at the mid-span of the beam and also the angle $\Delta\phi_0$ subtended by the cross sections at A and B in the deformed beam.

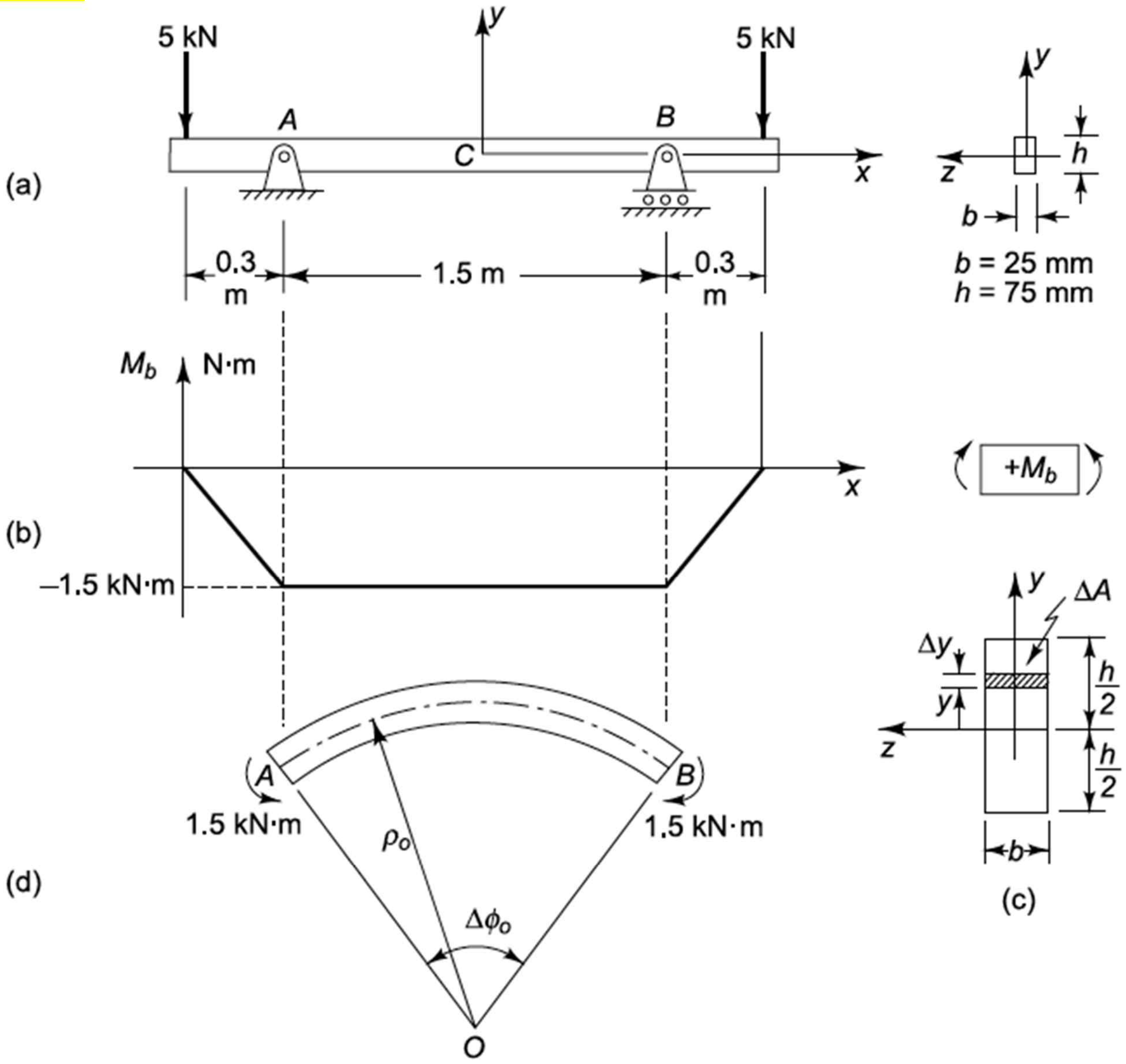

Fig. 7.11 *Example 7.1*

Sol)

from fig. (c)

$$I_z = \int_{-h/2}^{h/2} y^2 b\, dy = \frac{bh^3}{12} = 8.789 \times 10^5\ mm^4$$

$$\therefore \quad \sigma_{max} = -\frac{M_b(h/2)}{I} = -\frac{(-1500)(37.5\times10^{-3})}{8.789\,(10)^{-7}} = 64.0\ MN/m^2$$

$$\kappa = \frac{d\phi}{ds} = \frac{M_b}{EI_z}$$

$$\therefore \quad \phi_B - \phi_A = \int_{-L/2}^{L/2} \frac{d\phi}{ds}\, ds = \frac{M_b L}{EI_z} = \frac{-1500\,(1.5)}{E(8.789\times10^{-7})}$$

here, we let $E = 205$ GPa

$$\Delta\phi_0 = \frac{-1500\,(1.5)}{(205\times10^9)(8.789\times10^{-7})} = -0.0125\ rad = 0.7155°$$

now, $\rho_0 = \frac{1}{\kappa} = \frac{EI_z}{M_b} = -120.12\ m$

▶ Example 7.2

Find the maximum tensile and compressive bending stresses in the symmetrical T beam of Fig. 7.12 (a) under the action of a constant bending moment M_b.

Sol)

$$\bar{y} = \frac{\sum_i \bar{y}_i A_i}{\sum_i A_i} = \frac{(3/2)h(2bh)+(h/4)(3bh)}{2bh+3bh} = \frac{3}{4}h \qquad (a)$$

$$(I_{zz})_1 = \frac{b(2h)^3}{12} + 2bh\left(\frac{3}{4}h\right)^2 = \frac{43}{24}bh^3 \qquad (b)$$

$$(I_{zz})_2 = \frac{6b(h/2)^3}{12} + 3bh\left(\frac{h}{2}\right)^2 = \frac{13}{16}bh^3 \qquad (c)$$

Then, for the entire cross section

$$I_{zz} = (I_{zz})_1 + (I_{zz})_2 = 125/48\, bh^3 \qquad (d)$$

$$\therefore \begin{cases} \sigma_{max} = -\dfrac{M_b(-3/4\,h)}{(125/48)bh^3} = \dfrac{36}{125}\dfrac{M_b}{bh^2} \\ \sigma_{min} = -\dfrac{M_b(7/4\,h)}{(125/48)bh^3} = -\dfrac{84}{125}\dfrac{M_b}{bh^2} \end{cases}$$

cf. $|\sigma_{min}| \approx |2.3 \cdot \sigma_{max}|$

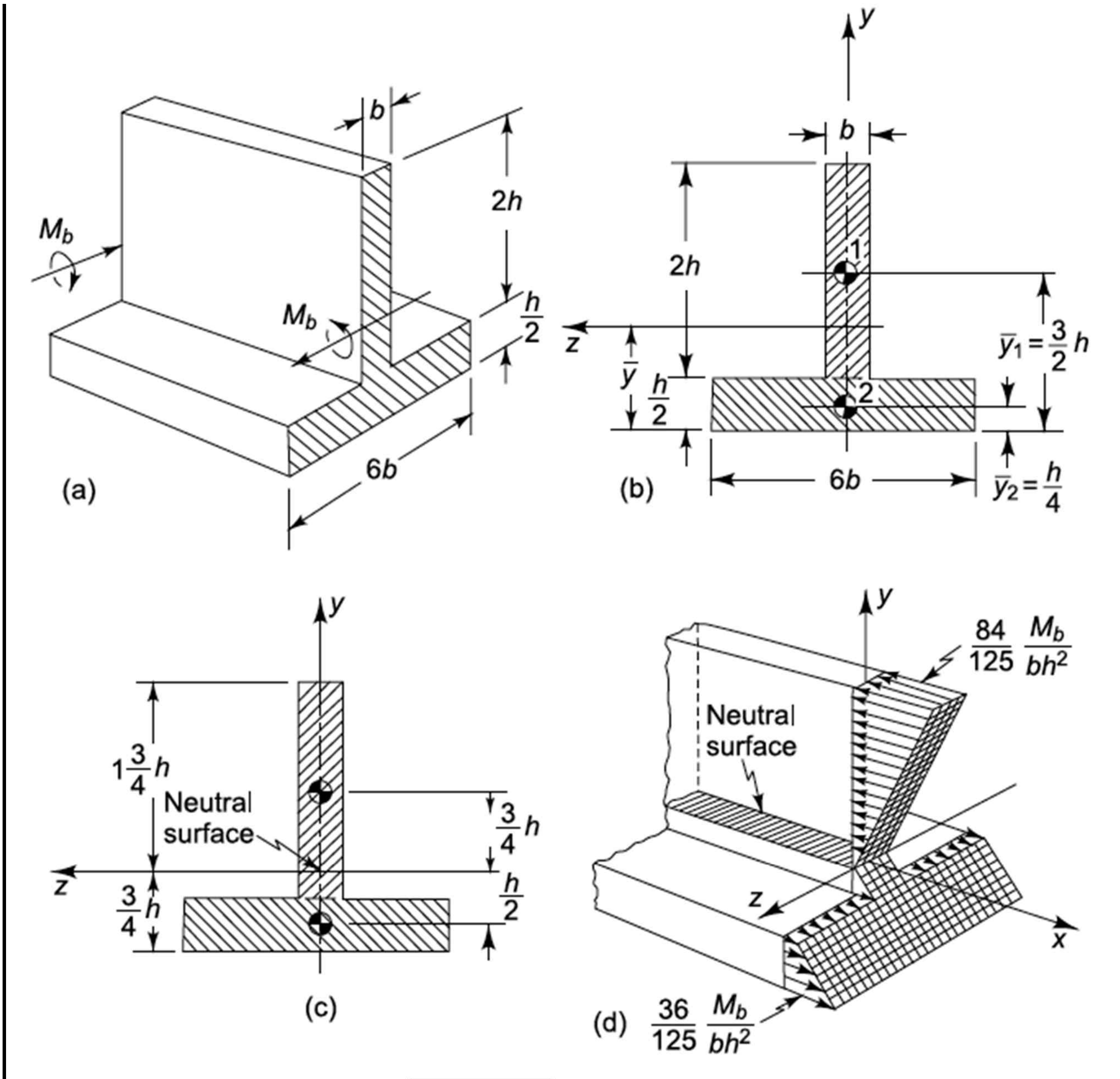

Fig. 7.12 *Example 7.2*