8.6 Energy Method

→ In this section the formulas for strain energy in torsion and bending, developed in Section 6.8 and 7.8, are used to illustrate the application of Castigliano's theorem to more complicated elastic systems.

▶ Strain energy of bending

→ In case of linear elastic beam, small deflection and small slope angle.

$$dU = \frac{M}{2} d\phi, \quad U = \int \frac{M^2}{2EI} dx = \int \frac{EI}{2} \left(\frac{d^2 v}{dx^2}\right)^2 dx$$

cf. Shear strain energy can be neglected in case of common beam which has much bigger length than width. For example L/d >6.

▶ If a slender elastic shaft oriented along the axis of *x* carries a tensile force F(x), a twisting moment $M_t(x)$, and a bending moment $M_b(x)$, then according to (2.11), (6.13), (7.31) the total strain energy in the member is

$$U = \int_L \frac{F^2}{2AE} dx + \int_L \frac{{M_t}^2}{2GJ} dx + \int_L \frac{{M_b}^2}{2EI} dx \qquad (8.8)$$

▶ Castigliano's theorem

$$\delta_i = \frac{\partial U}{\partial P_i} \qquad (8.9)$$

1 ▷ If a deflection δ is desired at a point where there is no load (or in a direction which is not in line with a load),

$$\delta = \left(\frac{\partial U}{\partial Q}\right)_{Q=0} \qquad (8.10)$$

2 ▷ If the elastic energy is expressed in terms of the P_i and the X_i, a set of equations for determining the X_i can be obtained from the condition that there be no in-line deflection at each of the statically indeterminate reaction points

$$\frac{\partial U}{\partial X_i} = 0 \qquad (8.11)$$

▶ Example 8.10

Solve the example 8.2 only using energy method.

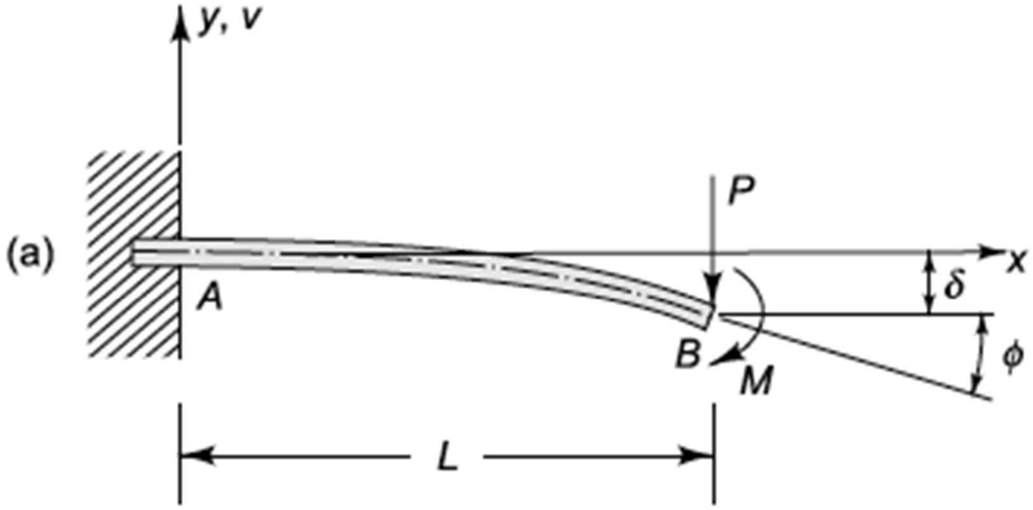


Fig. 8.5 *Example 8.2. Cantilever beam with force and moment load*

$$M_b = -P(L-x) - M$$

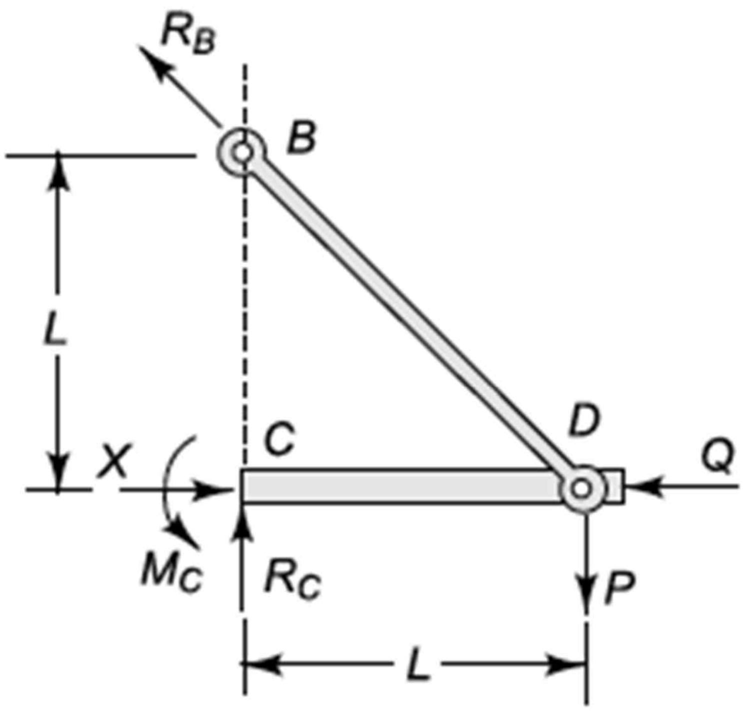
$$\therefore U = \int_0^L \frac{[P(L-x)+M]^2}{2EI} dx$$

$$\therefore \delta = \frac{\partial U}{\partial P} = \int_0^L \frac{P(L-x)^2 + M(L-x)}{EI} dx = \frac{PL^3}{3EI} + \frac{ML^2}{2EI}$$

$$\phi = \frac{\partial U}{\partial M} = \int_0^L \frac{P(L-x)+M}{EI} dx = \frac{PL^2}{2EI} + \frac{M\,L}{EI}$$

▶ Example 8.11

Solve the example 8.8 only using energy method.



(a)

The total strain energy in this case 1s due to longitudinal and bending loading in *CD* and to longitudinal loading in *BD*,

$$U = \int_0^L \frac{X^2}{2A_{CD}E}dx + \int_0^L \frac{(P-X+Q)^2(L-X)^2}{2EI}dx + \int_0^{\sqrt{2}L} \frac{\left[\sqrt{2}(X-Q)\right]^2}{2A_{BD}E}dx \qquad \text{(a)}$$

The vertical and horizontal deflections at point *D* are

$$\delta_V = \left(\frac{\partial U}{\partial P}\right)_{Q=0} = \int_0^L \frac{(P-X)(L-X)^2}{EI}dx = \frac{(P-X)L^3}{3EI} \qquad \text{(b)}$$

$$\delta_H = \left(\frac{\partial U}{\partial Q}\right)_{Q=0} = \int_0^L \frac{(P-X)(L-X)^2}{EI}dx - \int_0^{\sqrt{2}L} \frac{2X}{A_{BD}E}dx$$

$$= \frac{(P-X)L^3}{3EI} - \frac{2\sqrt{2}XL}{A_{BD}E} \qquad \text{(c)}$$

in terms of the statically indeterminate reaction *X*. To determine *X* we use the fact that the deflection at point C in Fig. 8.24a is zero; i.e., according to (8.11) we have

$$0 = \left(\frac{\partial U}{\partial X}\right)_{Q=0} = \int_0^L \frac{X}{A_{CD}E}dx - \int_0^L \frac{(P-X)(L-X)^2}{EI}dx + \int_0^{\sqrt{2}L} \frac{2X}{A_{BD}E}dx$$

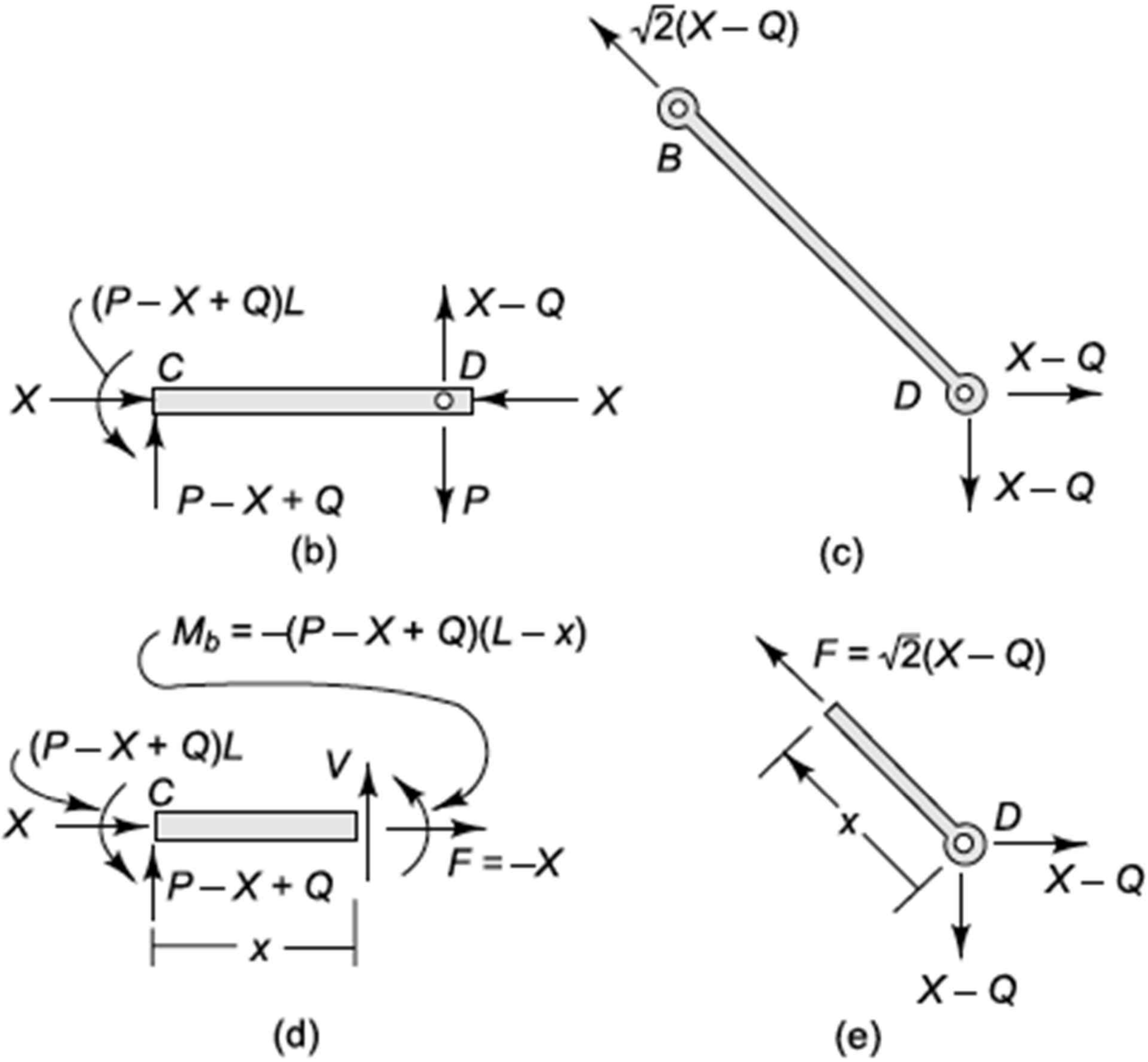
$$= \frac{XL}{A_{CD}E} - \frac{(P-X)L^3}{3EI} + \frac{2\sqrt{2}XL}{A_{BD}E} \qquad \text{(d)}$$

from which we solve for X,

$$\therefore X = \frac{P}{1+3I/(A_{CD}L^2)+6\sqrt{2}I/(A_{BD}L^2)} \tag{e}$$

Note that this result agrees with (e) in Example 8.8 and also that (b) above agrees with (d) in Example 8.8. The horizontal deflection (c) above can be rewritten in the following form by using (d),

$$\delta_H = \frac{XL}{A_{CD}E} \tag{f}$$



▶ Example 8.12

Determine the deflection of the coil spring coiled n times.

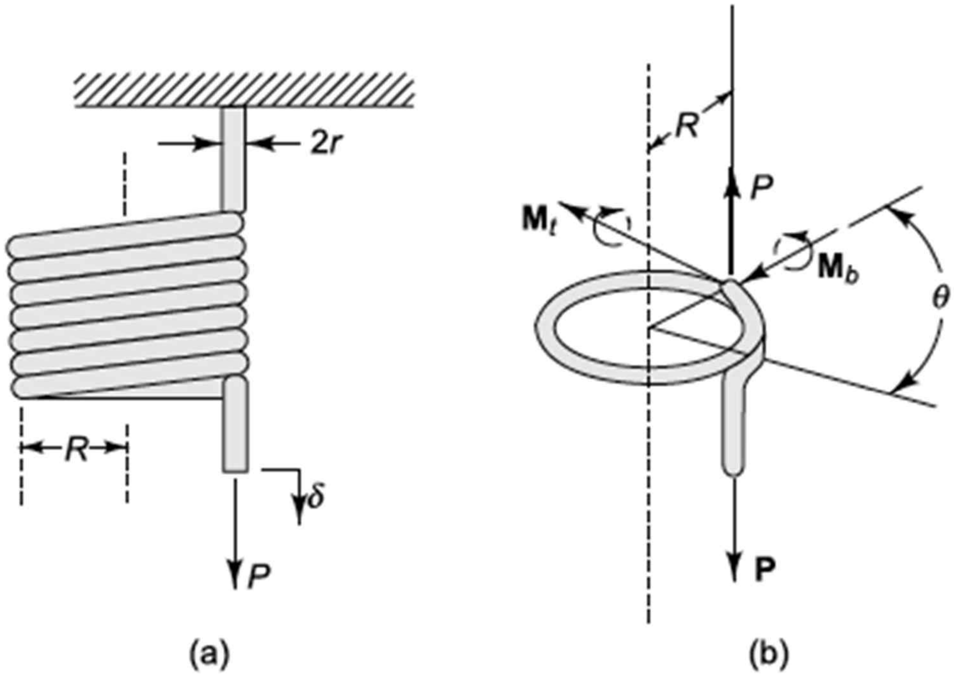


Fig. 8.25 *Example 8.12*

At each section of the spring, the wire carries a transverse shear force P, a twisting moment M_t, and a bending moment M_b, as indicated in Fig 8.25b. By applying the equilibrium requirements to this free body, we find

$$M_t = PR(1-\cos\theta), \quad M_b = PR\sin\theta \tag{a}$$

The total strain energy (8.8) in the wire (of uncoiled length $2n\pi R$) due to the twisting and bending contributions is

$$U = \int_0^{2\pi n} \frac{P^2R^2(1-\cos\theta)^2}{2GJ} R d\theta + \int_0^{2\pi n} \frac{P^2R^2\sin^2\theta}{2EI} R d\theta$$

$$= \frac{P^2R^3}{2GJ} 3\pi n + \frac{P^2R^3}{2EI} \pi n \tag{b}$$

$$(where,\ J = \frac{\pi r^4}{2}, I = \frac{\pi r^4}{4})$$

$$\therefore \delta = \frac{\partial U}{\partial P} = PR^3\pi n\left(\frac{3}{GJ} + \frac{1}{EI}\right) = \frac{4PR^3n}{Gr^4}\left(\frac{3}{2} + \frac{G}{E}\right)$$

$$= \frac{4PR^3n}{Gr^4}\left(\frac{4+3\nu}{2+2\nu}\right) \tag{c}$$

cf. Note that under the same load this spring deflects nearly twice as much as the spring with centered ends m Example 6.4