Field and Wave Electromagnetic

Chapter11

Antennas and Radiating Systems

Introduction (1)

- Potential functions $\overline{A}$ and V
- $\overline{H} = \frac{1}{\mu}\nabla \times \overline{A}$

  $\overline{E} = -\nabla V - j\omega\overline{A}$ for time harmonic cases
- The potential functions $\overline{A}$ and V are solutions of non-homogeneous wave equations.
- For harmonic time dependence, the phasor retarded potentials.

$$\overline{A} = \frac{\mu}{4\pi}\int_{v'} \frac{\overline{J}e^{-jkR}}{R} dv'$$

$$V = \frac{1}{4\pi\varepsilon}\int_{v'} \frac{\rho e^{-jkR}}{R} dv' \qquad k = \omega\sqrt{\mu\varepsilon} = \frac{2\pi}{\lambda}$$

Introduction (2)

cf)
$$\begin{cases} \nabla \bullet \overline{A} + \mu\varepsilon \dfrac{\partial V}{\partial t} = 0 & \rightarrow \text{Lorentz Gauge} \\ i.e\ \ \nabla \bullet \overline{A} + j\omega\mu\varepsilon V = 0 & \end{cases}$$

$$\begin{cases} \nabla \bullet \overline{J} = -\dfrac{\partial \rho}{\partial t} & \rightarrow \text{continuity equation} \\ i.e\ \ \nabla \bullet \overline{J} = -j\omega\rho & \end{cases}$$

$\therefore$ No need to evaluate the both integrals

Introduction (3)

- $\overline{E}$ *and* $\overline{H}$

$$\overline{E} = \frac{1}{j\omega\varepsilon} \nabla \times \overline{H}$$

- Three steps to determine electromagnetic fields from a current distribution.
  1. Determine $\overline{A}$ from $\overline{J}$
  2. Find $\overline{H}$ from $\overline{A}$
  3. Find $\overline{E}$ from $\overline{H}$

Radiation Fields of Elemental Dipoles

* Elemental electric dipoles

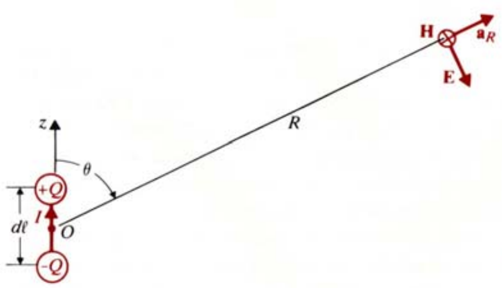


FIGURE 11–1
A Hertzian dipole.

Elemental oscillating electric dipole.

$\Rightarrow$ Short conducting wire of length dl terminated in two small conductive spheres Q disks (capacitive loading) assume the current in the wire to be uniform and to vary sinusoidally with time $i(t) = \cos \omega t = \Re e[Ie^{j\omega t}]$

The Elemental Electric Dipole (1)

- The current vanishes at the ends of the wire,
  charge must be deposited there.

$$i(t) = \pm \frac{dq(t)}{dt} \qquad q(t) = \mathrm{Re}[Qe^{j\omega t}]$$

$$I = \pm j\omega Q$$

$$or \quad Q = \pm \frac{I}{j\omega}$$

$\Rightarrow$ + sign : for the charge on the upper end

– sign : for the charge on the lower end

$\Rightarrow$ The pair of equal and opposite charges separated by a short distance effectively form an electric dipole.

$$\bar{p} = \hat{z}Qdl \quad (C \cdot m)$$

$\Rightarrow$ This kind of oscillating dipole is called a Hertzian dipole

The Elemental Electric Dipole (2)

- Electromagnetic fields of a Hertzian dipole

$$\overline{A} = \hat{z}\frac{\mu_0 Idl}{4\pi}\left(\frac{e^{-j\beta R}}{R}\right)$$

$$\text{where } \beta = k_0 = \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

$$\hat{z} = \hat{R}\cos\theta - \hat{\theta}\sin\theta$$

The Elemental Electric Dipole (3)

$$\overline{A} = \hat{R}A_R + \hat{\theta}A_\theta + \hat{\phi}A_\phi$$

$$\left[\begin{array}{l} A_R = A_z\cos\theta = \dfrac{\mu_0 Idl}{4\pi}\left(\dfrac{e^{-j\beta R}}{R}\right)\cos\theta \\ A_\theta = -A_z\sin\theta = -\dfrac{\mu_0 Idl}{4\pi}\left(\dfrac{e^{-j\beta R}}{R}\right)\sin\theta \\ A_\phi = 0 \end{array}\right.$$

$$\overline{H} = \frac{1}{\mu_0}\nabla\times\overline{A} = \hat{\phi}\frac{1}{\mu_0 R}\left[\frac{\partial}{\partial R}(RA_\theta) - \frac{\partial A_R}{\partial\theta}\right]$$

$$= -\hat{\phi}\frac{Idl}{4\pi}\beta^2\sin\theta\left[\frac{1}{j\beta R} + \frac{1}{(j\beta R)^2}\right]e^{-j\beta R}$$

The Elemental Electric Dipole (4)

$$\overline{E} = \frac{1}{j\omega\varepsilon_0}\nabla\times\overline{H}$$

$$= \frac{1}{j\omega\varepsilon_0}[\hat{R}\frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(H_\phi\sin\theta)-\hat{\theta}\frac{1}{R}\frac{\partial}{\partial R}(RH_\phi)]$$

$$E_R = -\frac{Idl}{4\pi}\eta_0\beta^2 2\cos\theta[\frac{1}{(j\beta R)^2}+\frac{1}{(j\beta R)^3}]e^{-j\beta R}$$

$$E_\theta = -\frac{Idl}{4\pi}\eta_0\beta^2\sin\theta[\frac{1}{j\beta R}+\frac{1}{(j\beta R)^2}+\frac{1}{(j\beta R)^3}]e^{-j\beta R}$$

$$E_\phi = 0$$

where $\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi$

The Elemental Electric Dipole - Near Field (1)

- Near field

In the near zone, $\beta R = \frac{2\pi R}{\lambda} \ll 1,$

Consider only leading term.

$$\underline{\underline{H_\phi \triangleq \frac{Idl}{4\pi R^2}}}\sin\theta, \text{ where } e^{-j\beta R} = 1 - j\beta R - \frac{(\beta R)^2}{2} + \cdots \simeq 1$$

Magnetic field intensity due to a current element
Idl is obtained using the Biot-Savart law in magnetostatics

$$cf)\ \ d\overline{B} = \frac{\mu_0 I}{4\pi}(\frac{dl' \times \hat{R}}{R^2}) \quad , \quad \overline{A} = \frac{\mu_0 I}{4\pi}\oint_{C'}\frac{dl'}{R}$$

The Elemental Electric Dipole - Near Field (2)

$$E = \hat{R}E_R + \hat{\theta}E_\theta$$

$$E_R = \frac{p}{4\pi\varepsilon_0 R^3} 2\cos\theta$$

$$E_\theta = \frac{p}{4\pi\varepsilon_0 R^3}\sin\theta$$

Electric field due to an elemental electric dipole of a moment ρ in the e-direction

$$cf)\ \ \overline{p} = \hat{z}Qdl\ ,\ Q = \pm\frac{I}{j\omega}\ ,\ \overline{E} = \frac{p}{4\pi\varepsilon_0 R^3}(\hat{R}2\cos\theta + \hat{\theta}\sin\theta)$$

Note : the near field of an oscillating time-varying dipole are the quasi-static field

The Elemental Electric Dipole - Far Field (1)

- Far field

In the far zone, $\beta R = \dfrac{2\pi \mathrm{R}}{\lambda} \gg 1,$

Far zone leading terms of $\overline{\mathrm{E}}$ and $\overline{\mathrm{H}}$ fields are

$$H_\phi = j\frac{Idl}{4\pi}\left(\frac{e^{-j\beta R}}{R}\right)\beta\sin\theta$$

$$E_\theta = j\frac{Idl}{4\pi}\left(\frac{e^{-j\beta R}}{R}\right)\eta_0\beta\sin\theta\ ,\ (E_R = 0)$$

The Elemental Electric Dipole - Far Field (2)

note 1. E_θ and H_ϕ are in space quardrature and in time phase.

2. $\dfrac{E_\theta}{\mathrm{H}_\phi} = \eta_0$: constant equal to the intrinsic impedance of the medium.

3. The same properties as those of a plane wave (at very large distances from the dipole a spherical wavefront closely resembles a plane wavefront)

4. The magnitude of the far-zone fields varies inversely with the distance from the source.

5. The phase is a periodic function of R.

$$\lambda = \frac{2\pi}{\beta} = \frac{c}{f} \qquad (R \gg \frac{\lambda}{2\pi})$$

The Elemental Magnetic Dipole (1)

* Elemental magnetic dipole

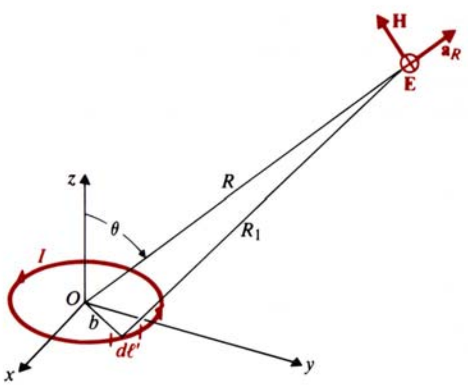


FIGURE 11–2
A magnetic dipole.

- Small filamentary loop of radius b.
- Uniform time harmonic current

$$i(t) = I \cos \omega t$$

$$\bar{m} = \hat{z} I \pi b^2 = \hat{z} m :$$

vector phasor magnetic moment

The Elemental Magnetic Dipole (2)

$$\bar{A} = \frac{\mu_0 I}{4\pi} \oint \frac{d\bar{l}'}{R_1} (e^{-j\beta R_1})$$

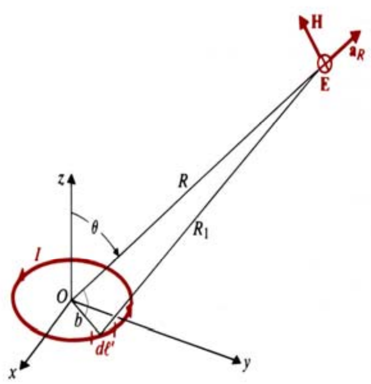
$$e^{-j\beta R_1} = e^{-j\beta R} e^{-j\beta (R_1 - R)}$$

$$\simeq e^{-j\beta R}[1 - j\beta(R_1 - R)]$$

$$\therefore\ \bar{A} = \frac{\mu_0 I}{4\pi} e^{-j\beta R}\left[(1 + j\beta R)\oint \frac{d\bar{l}'}{R_1} - j\beta \underbrace{\cancel{\oint d\bar{l}'}}_{0}\right]$$

$$cf)\ d\bar{l}' = (-\hat{x}\sin\phi' + \hat{y}\cos\phi')bd\phi'$$

The Elemental Magnetic Dipole (3)



For every $Id\bar{l}'$, there is another symmetrically located differential current element on the other side of the y-axis $\Rightarrow$ an equal amount of contribution to $\bar{A}$ in the $-\hat{x}$.

But an equal amount of contribution to $\bar{A}$ in the opposite direction of $\hat{y}$

The Elemental Magnetic Dipole (4)

$$\bar{A} = \hat{\phi}\frac{\mu_0 Ib}{2\pi}(1+j\beta R)e^{-j\beta R}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin\phi'}{R_1}d\phi'$$

$$where\ R_1^2 = R^2 + b^2 - 2bR\cos\psi$$

$$R\cos\psi = R\sin\theta\sin\phi'$$

$$\therefore \frac{1}{R_1} = \frac{1}{R}\left(1+\frac{b^2}{R^2}-\frac{2b}{R}\sin\theta\sin\phi'\right)^{-\frac{1}{2}} \quad (\mathrm{R}^2 \gg b^2)$$

$$\simeq \frac{1}{R}\left(1-\frac{2b}{R}\sin\theta\sin\phi'\right)^{-\frac{1}{2}}$$

$$\simeq \frac{1}{R}\left(1+\frac{b}{R}\sin\theta\sin\phi'\right)$$

The Elemental Magnetic Dipole (5)

$$\bar{A} = \hat{\phi}\frac{\mu_0 Ib^2}{4R^2}(1+j\beta R)e^{-j\beta R}\sin\theta$$

$$= \hat{\phi}\frac{\mu_0 m}{4\pi R^2}(1+j\beta R)e^{-j\beta R}\sin\theta$$

$$\therefore E_\phi = \frac{j\omega\mu_0 m}{4\pi}\beta^2\sin\theta\left[\frac{1}{j\beta R}+\frac{1}{(j\beta R)^2}\right]e^{-j\beta R}$$

$$H_R = -\frac{j\omega\mu_0 m}{4\pi\eta_0}\beta^2 2\cos\theta\left[\frac{1}{(j\beta R)^2}+\frac{1}{(j\beta R)^3}\right]e^{-j\beta R}$$

$$H_\theta = -\frac{j\omega\mu_0 m}{4\pi\eta_0}\beta^2\sin\theta\left[\frac{1}{j\beta R}+\frac{1}{(j\beta R)^2}+\frac{1}{(j\beta R)^3}\right]e^{-j\beta R}$$

The Elemental Magnetic Dipole (6)

- Far field

$$E_\phi = \frac{\omega\mu_0 m}{4\pi}\left(\frac{e^{-j\beta R}}{R}\right)\beta\sin\theta$$

$$H_\theta = -\frac{\omega\mu_0 m}{4\pi\eta_0}\left(\frac{e^{-j\beta R}}{R}\right)\beta\sin\theta$$

The Elemental Magnetic Dipole (7)

H.W 11-1, 11-2, 11-3, 11-4, 11-5

$cf)$ let $(\bar{E}_e, \bar{H}_e)$: Electric and magnetic field of the electric dipole.

$(\bar{E}_m, \bar{H}_m)$: Electric and magnetic field of the magnetic dipole.

$$\bar{E}_e = \eta_0 \bar{H}_m, \quad \bar{H}_e = -\frac{\bar{E}_m}{\eta_0} \qquad \left(\frac{E_\phi}{H_\theta} = \eta_0\right)$$

$$if \quad Idl = j\beta m \qquad where \quad \beta = \frac{\omega\mu_0}{\eta_0} = \omega\sqrt{\mu_0\varepsilon_0}$$

i.e Hertzian electric dipole and elemental magnetic dipole are dual devices.

Antenna Patterns and Antenna Parameters (1)

* Antenna patterns and antenna parameters

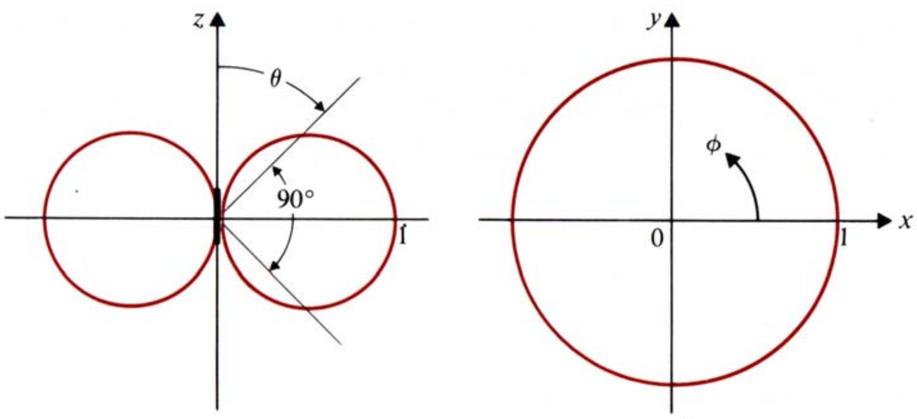
- Antenna pattern (radiation pattern) : the relative far-zone field strength versus direction at a fixed distance from an antenna.

- E-plane pattern = the magnitude of the normalized field strength (with respect to the peak value) versus θ for a constant ϕ

- H-plane pattern = the magnitude of the normalized field strength versus ϕ for $\theta = \frac{\pi}{2}$

Antenna Patterns and Antenna Parameters (2)

❖ e.g. Hertzian dipole



(a) *E*-plane pattern. (b) *H*-plane pattern.

FIGURE 11–3
Radiation patterns of a Hertzian dipole.

Antenna Patterns and Antenna Parameters (3)

- Antenna parameters
  (1) Width of main beam = beamwidth (3dB beamwidth)
  (2) Sidelobe levels $\Rightarrow$ unwanted radiation (first sidelobe : -40dB)
  (3) Directivity
    - Directive gain in terms of radiation intensity.
    - Radiation intensity = the time-average power per unit solid angle
      : (W/sr) cf) sr : steradian (solid angle)

$\therefore$ Radiation intensity $U = R^2 \mathcal{P}_{av}$ (W/sr)

- Total time-average power radiated

$$P_r = \oint \mathcal{P}_{av} \bullet d\overline{s} = \oint U d\Omega \quad \text{(W)}$$

$$d\Omega = \text{differential solid angle} = \sin\theta d\theta d\phi$$

Antenna Patterns and Antenna Parameters (4)

- Directive gain

$G_D(\theta,\phi)$ = the ratio of the radiation intensity in the direction (θ,ϕ) to the average radiation intensity

$$= \frac{U(\theta,\phi)}{P_r / 4\pi} = \frac{4\pi U(\theta,\phi)}{\oint U d\Omega}$$

- Directivity = the maximum directive gain of an antenna

$$D = \frac{U_{\max}}{U_{av}} = \frac{4\pi U_{\max}}{P_r} \quad \text{(Dimensionless)}$$

$$= \frac{4\pi \left|E_{\max}\right|^2}{\int_0^{2\pi}\int_0^{\pi} \left|E(\theta,\phi)\right|^2 \sin\theta d\theta d\phi}$$

Antenna Patterns and Antenna Parameters (5)

$-Ex.$: Hertzian dipole

$$\mathcal{P}_{av} = \frac{1}{2}\Re e\left|\bar{E}\times\bar{H}^*\right| = \frac{1}{2}\left|E_\theta\right|\left|H_\phi\right| = \frac{(Idl)^2}{32\pi^2R^2}\eta_0\beta^2\sin^2\theta$$

$$\mathrm{U} = R^2\mathcal{P}_{av} = \frac{(Idl)^2}{32\pi^2}\eta_0\beta^2\sin^2\theta$$

$$\therefore\ G_D(\theta,\phi) = \frac{4\pi U(\theta,\phi)}{\oint U d\Omega} = \frac{4\pi\sin^2\theta}{\int_0^{2\pi}\int_0^{\pi}(\sin^2\theta)\sin\theta\, d\theta\, d\phi} = \frac{3}{2}\sin^2\theta$$

$$D = G_D(\frac{\pi}{2},\phi) = 1.5 \rightarrow 1.76dB$$

Antenna Patterns and Antenna Parameters (6)

- Power gain $\mathrm{G_p}$ = the ratio of its maximum radiation intensity to the radiation intensity of a lossless isotropic source with the same input power $\mathrm{P_i}$

total input power $P_i = P_r + P_l$ (P_l : ohmic loss power)

$$G_p = \frac{4\pi U_{\max}}{P_i}$$

- Radiation efficiency $\eta_r = \frac{G_p}{D} = \frac{P_r}{P_i}$

- Radiation resistance = the value of a hypothetical resistance

$$P_r = \frac{1}{2}I_m{}^2R_r \qquad \therefore\ R_r = \frac{2P_r}{I_m{}^2}$$

Thin Linear Antennas (1)

* Thin linear antennas

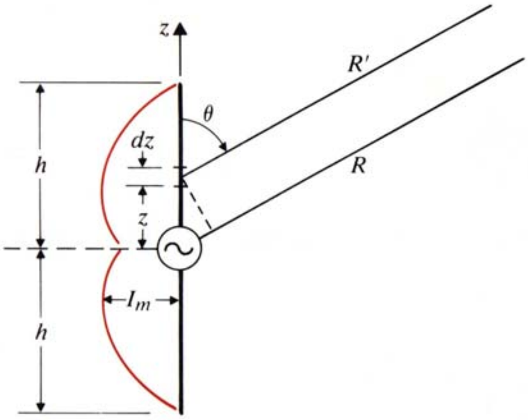


FIGURE 11–5
A center-fed linear dipole with sinusoidal current distribution.

$$I(z) = I_m \sin\beta(h-|z|),$$
$$= \begin{cases} I_m \sin\beta(h-z), & z>0 \\ I_m \sin\beta(h+z), & z<0. \end{cases}$$

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Thin Linear Antennas (2)

- The far-field contribution from the differential current element *Idz* is

$$dE_\theta = \eta_0 dH_\phi = j\frac{Idz}{4\pi}\left(\frac{e^{-j\beta R'}}{R'}\right)\eta_0\beta\sin\theta.$$

$$R' = (R^2 + z^2 - 2Rz\cos\theta)^{\frac{1}{2}} \cong R - z\cos\theta$$

$$\frac{1}{R'} \cong \frac{1}{R}$$

$$E_\theta = \eta_0 H_\phi = j\frac{I_m\eta_0\beta\sin\theta}{4\pi R}e^{-j\beta R}\int_{-h}^{h}\sin\beta(h-|z|)e^{j\beta z\cos\theta}dz$$

$\sin\beta(h-|z|)$: even function

$$e^{j\beta z\cos\theta} = \underbrace{\cos(\beta z\cos\theta)}_{\text{even function}} + j\,\underbrace{\sin(\beta z\cos\theta)}_{\text{Odd function}}$$

Thin Linear Antennas (3)

$$E_\theta = \eta_0 H_\phi = j\frac{I_m \eta_0 \beta \sin\theta}{2\pi R} e^{-j\beta R}\int_0^h \sin\beta(h-z)\cos(\beta z\cos\theta)dz$$

$$= \frac{j60 I_m}{R} e^{-j\beta R} F(\theta),$$

$$\text{where} \quad F(\theta) = \frac{\cos(\beta h\cos\theta) - \cos\beta h}{\sin\theta} \longrightarrow$$ H.W

$$cf)\ \int e^{Ax}\sin(Bx+C)dx = ?$$

$$u' = e^{Ax},\quad u = \frac{1}{A}e^{Ax},\quad v = \sin(Bx+C),\quad v' = B\cos(Bx+C)$$

$$I = \int e^{Ax}\sin(Bx+C)dx$$

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Thin Linear Antennas (4)

$$= \frac{1}{A}e^{Ax}\sin(Bx+C) - \frac{B}{A}\int e^{Ax}\cos(Bx+C)dx$$

$$= \frac{1}{A}e^{Ax}\sin(Bx+C) - \frac{B}{A^2}[e^{Ax}\cos(Bx+C) + B\int e^{Ax}\sin(Bx+C)dx$$

$$= \frac{1}{A}e^{Ax}\sin(Bx+C) - \frac{B}{A^2}e^{Ax}\cos(Bx+C) - \frac{B^2}{A^2}I$$

$$\therefore\ I = \frac{e^{Ax}}{A^2+B^2}[A\sin(Bx+C) - B\cos(Bx+C)]$$

$$let\ \ A = jk\cos\theta,\quad B = k,\quad C = -kh$$

Thin Linear Antennas (5)

- Half wave dipole

$$F(\theta) = \frac{\cos(\beta \frac{\lambda}{4}\cos\theta) - \cos(\frac{2\pi}{\lambda}\frac{\lambda}{4})}{\sin\theta} = \frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}$$

$$\therefore\ E_\theta = \frac{j60I_m}{R} e^{-j\beta R}\left\{\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\right\}$$

$$H_\phi = \frac{jI_m}{2\pi R} e^{-j\beta R}\left\{\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\right\}$$

Thin Linear Antennas (6)

$$\mathsf{P}_{av} = \frac{1}{2}E_\theta H_\phi^* = \frac{15I_m^2}{\pi R^2}\left\{\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\right\}^2$$

$$P_r = \int_0^{2\pi}\int_0^{\pi} \mathsf{P}_{av} R^2 \sin\theta d\theta d\phi = 30I_m^2 \int_0^{\pi} \frac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin\theta} d\theta$$

$$\int_0^{\pi} \frac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin\theta} d\theta = 1.218 \Rightarrow$$ obtained by numerical solution

using Simpson's law

Thin Linear Antennas (7)

$$\therefore \quad R_r = \frac{2P_r}{I_m^2} = 73.1\Omega$$

$$U_{\max} = R^2 \mathrm{P}_{av}(90^0) = \frac{15}{\pi} I_m^2$$

$$D = \frac{4\pi U_{\max}}{P_r} = \frac{60}{36.54} = 1.64$$

Thin Linear Antennas (8)

- Effective antenna length

Assume a center-feed linear dipole

and a general phasor current distribution $I(z)$

$$E_\theta = \eta_0 H_\phi = \frac{j30}{R}\beta e^{-j\beta R}\{\sin\theta \int_{-h}^{h} I(z) e^{j\beta z\cos\theta} dz\}$$

$I(0)$: the input current at the feed point of the antenna

$$\Rightarrow E_\theta = \eta_0 H_\phi = \frac{j30 I(0)}{R}\beta e^{-j\beta R} l_e(\theta)$$

where $l_e(\theta) = \dfrac{\sin\theta}{\mathrm{I}(0)} \displaystyle\int_{-h}^{h} I(z) e^{j\beta z\cos\theta} dz$

$\Rightarrow$ the effective length of the transmitting antenna

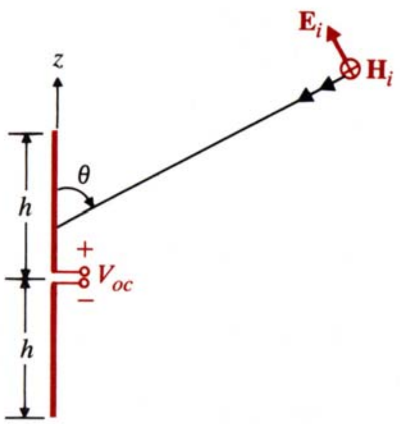
Thin Linear Antennas (9)

$$l_e(\theta = \frac{\pi}{2}) = \frac{1}{I(0)} \int_{-h}^{h} I(z)dz$$

$\Rightarrow$ The length of an equivalent linear antenna with a uniform current $I(0)$ such that it radiates the same far-zone field in the $\theta = \frac{\pi}{2}$ plane

Thin Linear Antennas (10)

Ex 11-6) Assume a sinusoidal current distribution on a center-fed, thin straight half-wave dipole. Find its effective lenth. What is its maximum value?



$$l_e(\theta) = \sin\theta \int_{-\frac{\lambda}{4}}^{\frac{\lambda}{4}} \sin\beta(\frac{\lambda}{4} - |z|) e^{j\beta z\cos\theta} dz$$

$$l_e(\theta) = \frac{2}{\beta}\left[\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\right]$$

Thin Linear Antennas (11)

maximum value of $l_e(\theta) = l_e(\frac{\pi}{2}) = \frac{2}{\beta} = \frac{\lambda}{\pi} < \frac{\lambda}{2}$

note

$$l_e = \frac{1}{I(0)} \int_{-h}^{h} I(z)dz$$

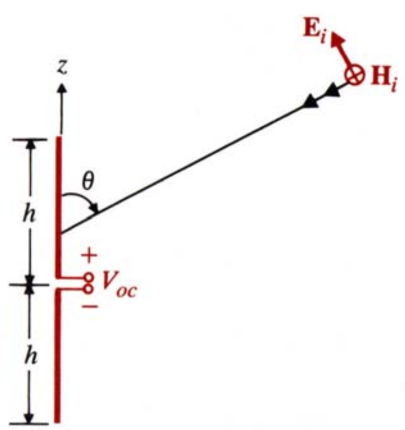
$\Rightarrow$ Valid only for relatively short antennas having a current maximum at the feed point

Thin Linear Antennas (12)

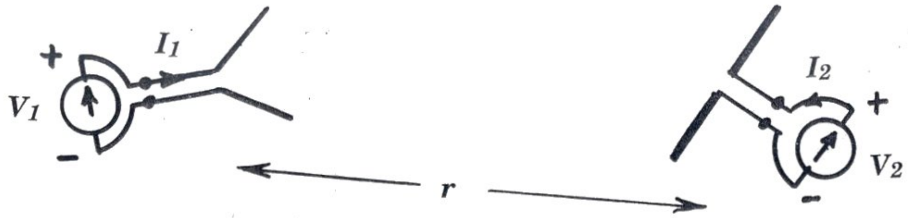
$$l_e(\theta) = -\frac{V_{oc}}{E_i}$$

$\Rightarrow$ Negative sign is to conform with the convention that the electric potential increases in a direction opposite to that of the electric field

$\Rightarrow$ The effective length of an antenna for receiving is the same as that for transmitting.



Receiving Antennas and Reciprocity



For a linear two-port

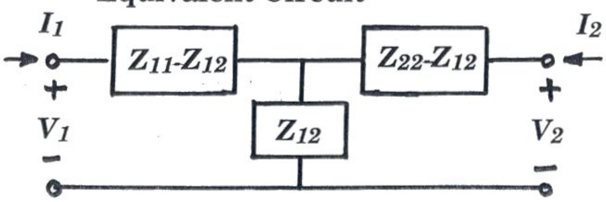
$$V_1 = Z_{11}I_1 + Z_{12}I_2$$

$$V_2 = Z_{21}I_1 + Z_{22}I_2$$

Reciprocity

$$Z_{12} = Z_{21}$$

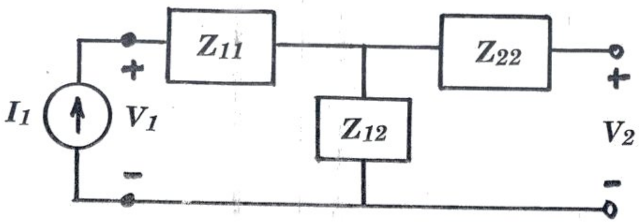
Equivalent Circuit



If $I_2 = 0$, $V_2 = Z_{12}I_1 \sim 1/r$

For r large,
$|Z_{12}| << |Z_{11}|, |Z_{22}|$

Circuit relation for Radiation into Free Space



Open Circuit

$V_1 = Z_{11}I_1$ and $V_2 = V_{oc} = Z_{12}I_1$

$$P_T = (1/2)Re(V_1I_1^*) = (1/2)Re(Z_{11}|I_1|^2) = (1/2)R_{r1}|I_1|^2$$

where R_{r1} = radiation resistance of antenna 1

Therefore: $Z_{11} = R_{r1} + jX_1$

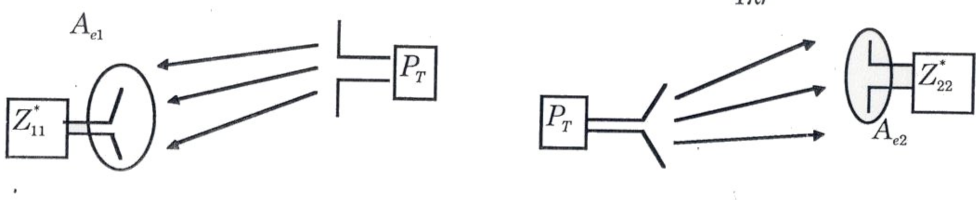
Similarly: $Z_{22} = R_{r2} + jX_2$

where R_{r2} = radiation resistance of antenna 2

Effective Area of Receiving Antenna

Effective Area = A_e

$$P_R = \vec{P} \bullet \vec{a}_r A_e = P_T \frac{g(\theta,\phi)}{4\pi r^2} A_e$$



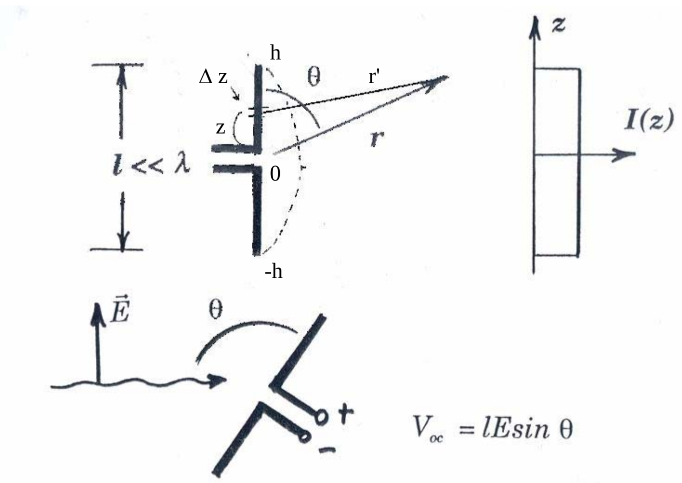
$$PL = \frac{P_R}{P_T} = \frac{g_2 A_{e1}}{4\pi r^2}$$ **and by reciprocity** $$PL = \frac{P_R}{P_T} = \frac{g_1 A_{e2}}{4\pi r^2}$$

$$g_2 A_{e1} = g_1 A_{e2} \quad \textbf{or} \quad \frac{A_{e1}}{g_1} = \frac{A_{e2}}{g_2}$$

For an elemental dipole $\dfrac{A_e}{g} = \dfrac{\lambda^2}{4\pi}$

Therefore for any antennas $PL = g_1 g_2 \left(\dfrac{\lambda}{4\pi r}\right)^2$

Effective Area for Hertzian Dipole



$$\vec{E} = \vec{a}_\theta ZI \frac{e^{-jkr}}{r} f(\theta)$$

$$f(\theta) = \sqrt{3/2}\,\sin\theta$$

$$Z = j\frac{l}{2\lambda}\eta\sqrt{2/3}$$

$$g(\theta) = (3/2)\sin^2\theta$$

cf) Normalization $\int g(\theta,\phi)d\Omega = 4\pi$

$$R_r = \eta\frac{4\pi}{6}\left(\frac{l}{\lambda}\right)^2$$

For matched termination:

$$P_R = \vec{P}\cdot\vec{a}_r A_e = A_e\left(\frac{1}{2\eta}|E|^2\right) \qquad P_R = \frac{|V_{OC}|^2}{8R_r} = \frac{(lE\sin\theta)^2}{8\eta(l/\lambda)^2(4\pi/6)}$$

$$A_e = \frac{3}{8\pi}(\lambda\sin\theta)^2 = g(\theta)\frac{\lambda^2}{4\pi}$$