Mechanics of Composite Materials

CHAPTER 3.
Ply Elasticity

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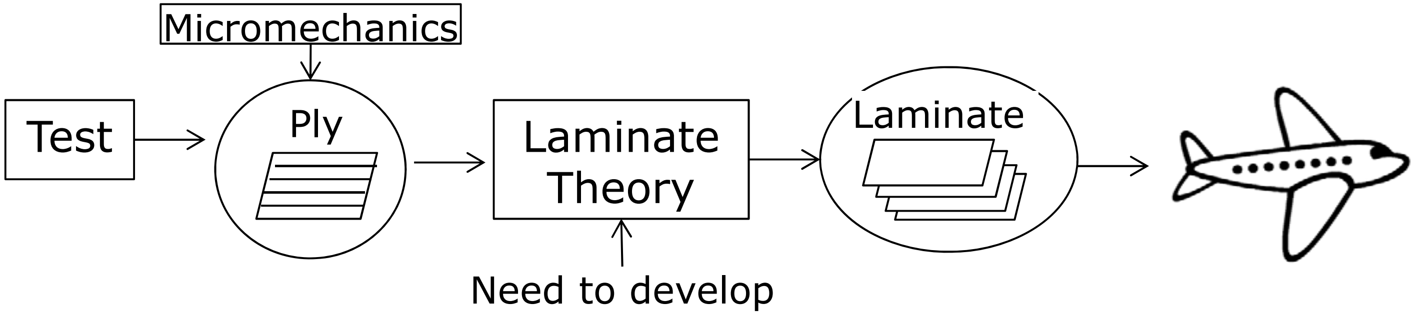
3. Ply Elasticity

- **Look at 3-D and 2-D anisotropic elasticity**

  - See Jones, Chap. 2 at Appendix A



Micromechanics

Test

Ply

Laminate Theory

Laminate

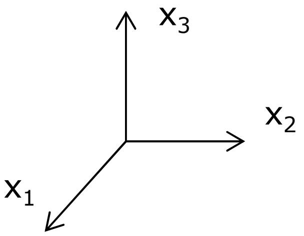
Need to develop

- **Make a brief review**

  1. Jones Book
  2. Bisplinghoff, Mar and Pian, “Statics of Deformable Bodies” -> (tensor notation)
  3. Herrmann, “Applied Anisotropic Elasticity”

- **Notation**

  Right hand coord. System, x_m



3. Ply Elasticity

Components of stress, σ_{mn}

"Stress tensor" , 2 subscripts -> 2nd order

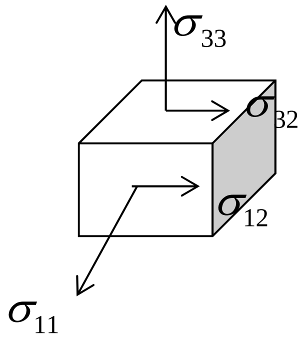
6 independent components

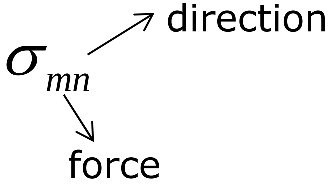
$$\sigma_{11} \quad \sigma_{22} \quad \sigma_{33}$$

Extension

$$\sigma_{12} = \sigma_{21} \quad \sigma_{23} = \sigma_{32} \quad \sigma_{31} = \sigma_{13}$$

Shear





Components of strain, ε_{mn}

"Strain tensor" , 2 subscripts -> 2nd order

6 independent components

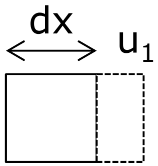
$$\varepsilon_{11} \quad \varepsilon_{22} \quad \varepsilon_{33}$$

Extension

$$\varepsilon_{12} = \varepsilon_{21} \quad \varepsilon_{23} = \varepsilon_{32} \quad \varepsilon_{31} = \varepsilon_{13}$$

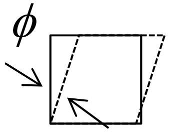
Shear

Extension



$$\varepsilon_{11} \approx \frac{u_1}{dx_1}$$

Shear



$$\varepsilon_{12} \approx \frac{\phi}{2}$$

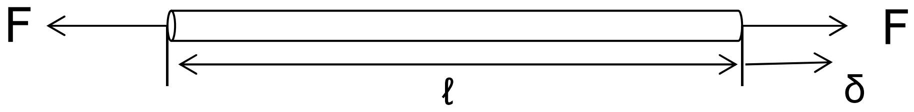
Note : Stress tensor symmetric : $\sigma_{mn} = \sigma_{nm}$ by equilibrium

Strain tensor symmetric : $\varepsilon_{mn} = \varepsilon_{nm}$ by Geometrical consideration

3. Ply Elasticity

Stress – Strain Relations

Hooke's Law, F = kδ : linear relation for rod



Can rewrite as $\sigma = E\varepsilon$

where σ = stress = F/A

ϵ = strain = δ/ℓ

E = modulus of elasticity

Extending to 3-D stress, have "Generalized Hooke's Law"

$$\sigma_{mn} = E_{mnpq}\varepsilon_{pq}$$

E_{mnpq} -> "Elasticity tensor" (3×3×3×3 = 81 components)

3. Ply Elasticity

❖ Recall Tensor Notation Rules

Latin subscripts (*m*, *n*, *p*, *q*, *r*, …) -> 1, 2, 3

Greek subscripts (α, β, γ , …) -> 1, 2

1. Subscripts that appear only once in a term are either 1, 2, or 3

$$\sigma_i = f(x_i) \quad \text{->} \quad \begin{cases} \sigma_1 = f(x_1) \\ \sigma_2 = f(x_2) \\ \sigma_3 = f(x_3) \end{cases}$$

2. Subscripts repeated in a term are “dummy” subscripts -> Sum on them

$$E_{ij}\varepsilon_j = E_{i1}\varepsilon_1 + E_{i2}\varepsilon_2 + E_{i3}\varepsilon_3 = \sum_{j=1}^{3} E_{ij}\varepsilon_j$$

3. No subscript can appear more than twice in a term

$$f_i C_{ij} D_i \quad (\text{x})$$

So, in general stress strain $\sigma_{mn} = E_{mnpq}\varepsilon_{pq}$ 9 equations are represented.

3. Ply Elasticity

$$\sigma_{11} = E_{11pq}\varepsilon_{pq}$$
$$\sigma_{12} = E_{12pq}\varepsilon_{pq}$$
$$\sigma_{13} = E_{13pq}\varepsilon_{pq}$$
$$\sigma_{21} = E_{21pq}\varepsilon_{pq}$$
$$\vdots$$
$$\sigma_{31} = E_{31pq}\varepsilon_{pq}$$
$$\vdots$$
$$\sigma_{33} = E_{33pq}\varepsilon_{pq}$$

Look at 1st equations and sum over P

$$\sigma_{11} = E_{11}\varepsilon_{1q} + E_{11}\varepsilon_{2q} + E_{11}\varepsilon_{3q}$$

Sum over q

$$\begin{aligned}\sigma_{11} = {} & E_{1111}\varepsilon_{11} + E_{1112}\varepsilon_{12} + E_{1113}\varepsilon_{13} \\ & + E_{1121}\varepsilon_{21} + E_{1122}\varepsilon_{22} + E_{1123}\varepsilon_{23} \\ & + E_{1131}\varepsilon_{31} + E_{1132}\varepsilon_{32} + E_{1133}\varepsilon_{33}\end{aligned}$$

9 Eqn.s -> 9 terms -> 81E_{mnpq}'s. Symmetries reduce number of independent E_{mnpq}

$$\sigma_{mn} = \sigma_{nm} \qquad\qquad \varepsilon_{pq} = \varepsilon_{qp}$$
$$\downarrow \qquad\qquad \downarrow$$
$$E_{mnpq} = E_{nmpq} \qquad\qquad E_{mnpq} = E_{nmqp}$$

(Equilibrium consideration)

Overall symmetry (Energy considerations)

$$E_{mnpq} = E_{pqmn}$$

3. Ply Elasticity

So at most 21 independent constants

E_{1111}	E_{1122}		E_{1212}	E_{1213}		E_{1112}	E_{2212}	E_{3312}
E_{2222}	E_{2233}		E_{1313}	E_{1323}		E_{1113}	E_{2213}	E_{3314}
E_{3333}	E_{3311}		E_{2323}	E_{2312}		E_{1123}	E_{2223}	E_{3323}
Extension-Extension			Shear-Shear			Coupling	Shear-Extension	

Material with all 21 independent constants is <u>Anisotropic</u>

Have used Tensor Notation

To here mary books use a "contracted" notation (Jones, Tsai, etc)

Also called "Engineering" Notation

3 major difference

① Subscript changes

Tensor		Contracted		Physical
11	⟶	1	⟶	Extens. in 1
22	⟶	2	⟶	Extens. in 2
33	⟶	3	⟶	Extens. in 3
23	⟶	4	⟶	Rotate about 1
31	⟶	5	⟶	〃 2
12	⟶	6	⟶	〃 3

3. Ply Elasticity

② Shear strain changes

Tensor shear strain is ½ of Engineering shear strain.

We change the notation from ε to γ

Engineering		Tensor	Contracted
γ_{12}	=	$\varepsilon_{12}+\varepsilon_{21}$	ε_6
γ_{13}	=	$\varepsilon_{13}+\varepsilon_{31}$	ε_5
γ_{23}	=	$\varepsilon_{23}+\varepsilon_{32}$	ε_4

③ Elasticity constants represented by C_{ij} instead E_{mnpq} (Still 21 components)

Tensor		Engineering
E_{mnpq}	$\rightarrow$	C_{ij}
$m\,n$	$\rightarrow$	i
$p\,q$	$\rightarrow$	j

The "Generalized Hooke's Law" is

$$\sigma_{mn} = E_{mnpq}\,\varepsilon_{pq}$$ (Tensor notation)

$$\sigma_i = C_{ij}\varepsilon_j$$ (Engineering notation)

Still use summation convention

$$\sigma_i = \sum_j C_{ij}\varepsilon_j = C_{ij}\varepsilon_j$$

j = 1, 2, 3, … 6

3. Ply Elasticity

$C_{ij} = C_{ji}$ ← Symmetry of the Elasticity constants still applies

$E_{mnpq} = E_{pqmn}$ other symm.s in $E_{mnpq} = E_{nmpq}$, etc.

all automatically included in Engineering Notation.

Be careful of 2 in shear strain.

$$2\,\varepsilon_{mn} = \gamma_{mn}$$

- **Can see usefulness of Engineering notation by writing Tensor notation in matrix form**

$$=\underset{\sim}{\sigma} \longleftarrow \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix} = \underbrace{\begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & 2E_{1123} & 2E_{1131} & 2E_{1112} \\ \vdots & & & \vdots & & \\ E_{3311} & & & \vdots & & \\ E_{2311} & & & 2E_{2323} & & \vdots \\ \vdots & & & \vdots & & \\ E_{1211} & & & 2E_{1223} & & \end{bmatrix}}_{\longrightarrow\, = \underset{\sim}{E}} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{31} \\ \varepsilon_{12} \end{Bmatrix} \longrightarrow = \underset{\sim}{\varepsilon}$$

E matrix not symmetric, inconvenient; contracted(Engineering) more convenient

$$\underset{\sim}{\sigma} = \underset{\sim}{C}\ \underset{\sim}{\varepsilon}\ , \qquad \sigma_i = C_{ij}\ \varepsilon_j$$

Note $\boxed{\varepsilon_4} = 2\ \varepsilon_{23}$ etc.

γ_{23} Tensor

$$\sigma_{11} = \cdots\ \cdots\ \boxed{2E_{1123}\varepsilon_{23}} + \cdots$$

$$\sigma_1 = \cdots\ \cdots\ C_{14}\varepsilon_4 + \cdots$$

C is symmetric, $C_{12} = E_{1123}$

3. Ply Elasticity

COMPLIANCE

Just we have $\sigma_{mn} = E_{mnpq}\,\varepsilon_{pq}$

Also have inverse $\varepsilon_{mn} = S_{mnpq}\,\sigma_{pq}$

S_{mnpq} → Compliance tensor

$$\underset{\sim}{\sigma} = \underset{\sim}{E}\;\underset{\sim}{\varepsilon}$$

$$\underset{\sim}{\varepsilon} = \underbrace{E^{-1}}_{=\underset{\sim}{S}}\;\underset{\sim}{\sigma}$$

Same symmetries as E_{mnpq}

Writing out Tensor Relations as matrices

$$\underbrace{\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \\ \sigma_{12} \end{Bmatrix}}_{=\underset{\sim}{\sigma}} = \underbrace{\begin{bmatrix} E_{1111} & E_{1122} & \cdots & 2E_{1112} \\ \vdots & & & \\ \vdots & & & \\ E_{1211} & & & 2E_{1212} \end{bmatrix}}_{=\underset{\sim}{E}} \underbrace{\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \vdots \\ \varepsilon_{12} \end{Bmatrix}}_{=\underset{\sim}{\varepsilon}}$$

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \\ \varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{1111} & S_{1122} & \cdots & 2S_{1112} \\ \vdots & & & \\ \vdots & & & \\ S_{1211} & & & 2S_{1212} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \vdots \\ \sigma_{12} \end{Bmatrix}$$

6x1 6x6 6x1

3. Ply Elasticity

Wish to write in Engineering Notation

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & & & & & \\ \vdots & & & & & \\ \vdots & & & & & \\ S_{61} & & & & & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \\ \vdots \\ \sigma_6 \end{Bmatrix}$$

Symmetric : relate S_{ij} to S_{mnpq}

For Elasticity matrix, use

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \\ \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & & & & & \\ \vdots & & & & & \\ \vdots & & & & & \\ C_{61} & & & & & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \\ \vdots \\ \varepsilon_6 \end{Bmatrix}$$

$$C_{ij}^{-1} = S_{ij}$$

Symmetric

All symmetric matrices in Engineering Notation

- **Problem Set #1**

Weight Figure of Merit $= \dfrac{\sigma_{ULT}}{(Specific \quad Gravity)}$

Cost Figure of Merit $= \dfrac{\sigma_{ULT}}{(S.G) \quad (\$/\mathrm{lb})}$

3. Ply Elasticity

Stress-Strain relations (Engineering Notation)

$$\underset{\sim}{\sigma} = \underset{\sim}{C}\ \underset{\sim}{\varepsilon} \qquad \underset{\sim}{\varepsilon} = \underset{\sim}{S}\ \underset{\sim}{\sigma}$$

where

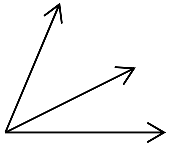
$$\sigma_6 = \sigma_{12,} etc \leftarrow \qquad \underset{\sim}{\sigma} = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_6 \end{bmatrix} \qquad \underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_6 \end{bmatrix} \rightarrow \varepsilon_6 = \gamma_{12} = 2\varepsilon_{12}$$

$$\left.\begin{array}{l} \underset{\sim}{C} : \text{elasticity} \\ \underset{\sim}{S} : \text{compliance} \end{array}\right\} \quad \begin{array}{l} \text{6x6 Symm. matrices} \\ \text{21 Independent constants} \end{array}$$

$$\underset{\sim}{S} = \underset{\sim}{C}^{-1}$$

❖ Type of Materials

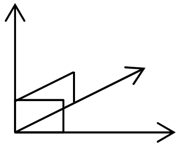
- Fully Anisotropic → 21 constants



Vary along non-orthogonal axis.

Different stiffness along each direction (Same crystals)

- Monoclinic Material → 13 constants



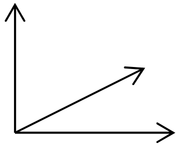
1 axis ⊥ other two

Different stiffness along each direction

(Some crystals, some composites)

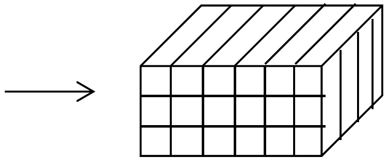
3. Ply Elasticity

- Orthotropic Material → 9 constants



3 axis ⊥ to each other
Different stiffness along each direction

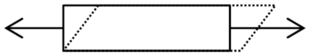
Important
Practical
Case



Crystals.
Composites

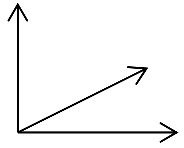
$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & & & \\ & S_{22} & S_{23} & & 0 & \\ & & S_{33} & & & \\ & \text{Symm.} & & S_{44} & 0 & \\ & & & & S_{55} & \\ & & & & & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}$$

But no shear



3. Ply Elasticity

- Transversely isotropic Material → 5 constants

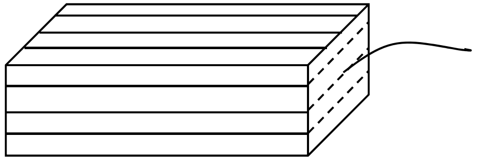


3 perpendicular axis

x_1 is stiffer than $x_2 = x_3$

same stiffness any direction in x_2 , x_3 plane

Wood or composite ⟶



Like orthotropic, but additionally

$S_{33} = S_{22}$

$S_{13} = S_{12}$

$S_{55} = S_{66}$

$S_{44} = 2(S_{22} - S_{23}) \longleftarrow G = \frac{E}{2(1+\nu)}$

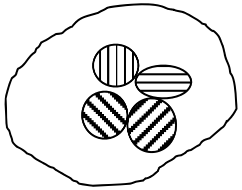
$$S = \left[\begin{array}{ccc|ccc} S_{11} & S_{12} & S_{12} & & & \\ & S_{22} & S_{23} & & 0 & \\ & & S_{22} & & & \\ \hline & & & 2(S_{22} - S_{23}) & & \\ & \text{Symm.} & & & S_{55} & \\ & & & & & S_{55} \end{array}\right]$$

5 constants

3. Ply Elasticity

- Isotropic Material → 2 constants

Same properties in all directions
Most metals, Resin



Many crystals randomly oriented
Polycrystalline material

Same as Transversely Isotropic, but additionally

$S_{22} = S_{11}$
$S_{13} = S_{12}$
$S_{55} = S_{44} = 2(S_{11} - S_{12})$
$S_{66} = S_{66} = 2(S_{11} - S_{12})$

Only 2 constants S_{11} and S_{12}
S_{mn}'s traditionally expressed in terms of Modulus of Elasticity E and Poisson's Ratio ν

$$S_{11} = \frac{1}{E}, \quad S_{12} = -\frac{\nu}{E}$$

$$with \quad these$$

$$\varepsilon_1 = \frac{1}{E}[\sigma_1 - \nu\sigma_2 - \nu\sigma_3]$$

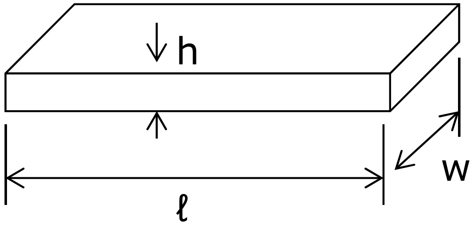
$$\vdots$$

$$\varepsilon_4 = \frac{2(1+\nu)}{E}\sigma_4$$

3. Ply Elasticity

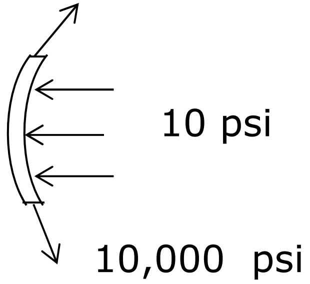
❖ 2-Dim. Plane stress approximations

Many structures are thin (plate)



h << ℓ , w

Also, not heavily loaded through thickness



$$\underset{\sigma_{33}}{\underset{\downarrow}{\sigma_3}},\ \underset{\sigma_{23}}{\underset{\downarrow}{\sigma_4}},\ \underset{\sigma_{31}}{\underset{\downarrow}{\sigma_5}} << \sigma_1,\ \sigma_2,\ \underset{\sigma_{12}}{\underset{\downarrow}{\sigma_6}}$$

Assume $\sigma_3 = \sigma_4 = \sigma_5 = 0$ in stress-strain

only deal with $\sigma_1, \sigma_2, \sigma_3$

In 3-D $\underset{\sim}{\sigma} = \underset{\sim}{C}\,\underset{\sim}{\varepsilon}$ 6x6 matrix

3x1 In 2-D $\underset{\sim}{\sigma} = \underset{\sim}{Q}\,\underset{\sim}{\varepsilon}$ 3x3 matrix

For Transversely isotropic mat'l

$$\sigma_1 = C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + C_{13}\varepsilon_3 + 0 + 0 + 0$$

$$\sigma_2 = etc.$$

Solve for ε_3 and put into others

$$0 = \sigma_3 = C_{12}\varepsilon_1 + C_{13}\varepsilon_2 + C_{22}\varepsilon_3 + 0 + 0 + 0$$

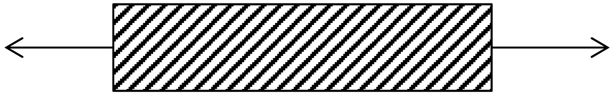
3. Ply Elasticity

$$\sigma_6 = C_{66}\varepsilon_6$$

$$\Rightarrow \left\{ \begin{array}{lllll} \sigma_1 & = & Q_{11}\varepsilon_1 & + & Q_{12}\varepsilon_2 \\ \sigma_2 & = & Q_{21}\varepsilon_1 & + & Q_{22}\varepsilon_2 \\ \sigma_6 & = & Q_{66}\varepsilon_6 & & \end{array} \right\}$$

Transversely isotropic

In general, for fibers not along the axis looks



$$\begin{array}{lllllll} \sigma_1 & = & Q_{11}\varepsilon_1 & + & Q_{12}\varepsilon_2 & + & Q_{16}\varepsilon_6 \\ \sigma_2 & = & \cdots & & & & \\ \sigma_6 & = & Q_{61}\varepsilon_1 & + & Q_{62}\varepsilon_2 & + & Q_{66}\varepsilon_6 \end{array}$$

[Q] 3x3, matrix symm.

Homework Prob. → Relation between 3-D and 2-D

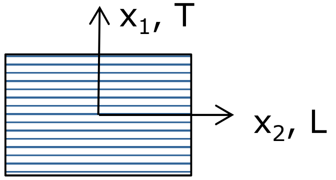
3. Ply Elasticity

❖ Properties of Single Ply

Ply → flat → plane stress

2-D stress-strain Eqns are $\underset{\sim}{\sigma} = \underset{\sim}{Q} \; \underset{\sim}{\varepsilon}$

3x3 matrix (matrix of 6 constants)



L : longitudinal

T : Transverse

On this set of axes - orthotropic

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix}$$

4 constants

Also cloth (0/90) weave works this way, but some funny products may not.

From Strength of Materials, we are familiar with Engineering Constants $E_L \quad E_T \quad \nu_{LT} \quad G_{LT}$

Those are obtained from experimental tests.

Formal definitions from

$$\varepsilon_1 = \frac{1}{E_L}(\sigma_1 - \nu_{LT}\sigma_2)$$

$$\varepsilon_2 = \frac{1}{E_T}(\sigma_2 - \nu_{TL}\sigma_1)$$

$$\varepsilon_6 = \frac{1}{G_{LT}}\sigma_6$$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} \dfrac{E_L}{1-\nu_{LT}\nu_{TL}} & \dfrac{\nu_{LT}E_T}{1-\nu_{LT}\nu_{TL}} & 0 \\ \dfrac{\nu_{LT}E_T}{1-\nu_{LT}\nu_{TL}} & \dfrac{E_T}{1-\nu_{LT}\nu_{TL}} & 0 \\ 0 & 0 & G_{LT} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix}$$

3. Ply Elasticity

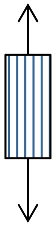
Questions

a) How to find Engineering Constants ?

b) How to relate them to Elastic Constants Q_{ij}

a) Tests for Engineering constants

① Longitudinal Tests

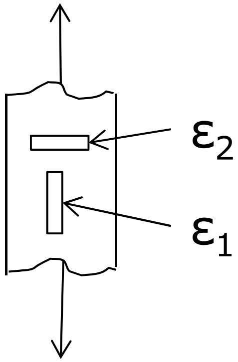


Apply known P (dead weight, calibrated machine) long, narrow specimen

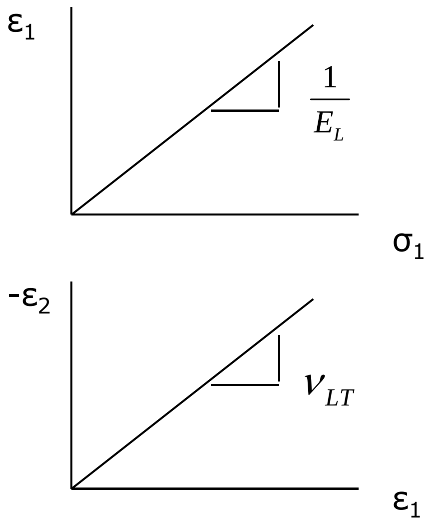
Know σ_1 = P/A (except near ends, reinforce there)

$\sigma_2 = 0, \quad \sigma_6 = 0$

Measure ε_1, ε_2 (ε_6 ?) with strain gages



Plot



$$\varepsilon_1 = \frac{\sigma_1}{E_L}$$

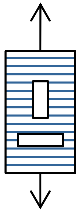
$$\varepsilon_2 = -\nu_{LT}\varepsilon_1$$

$$\varepsilon_6 = 0$$

From this test → 2 constants

3. Ply Elasticity

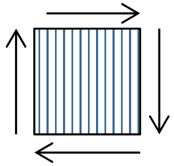
② Transverse Tension



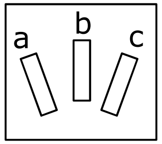
Same deal, apply known P

From this test, got $E_T \ \nu_{TL}$

③ Shear Tests



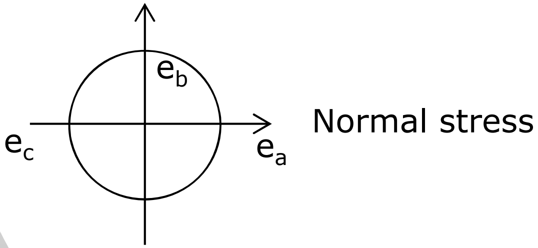
Apply known shear σ_6 (not too easy)

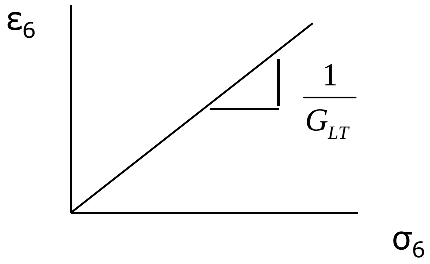


Measure shear strain with a rosette

$$\varepsilon_6 = e_c - e_a$$

Mohr's circle

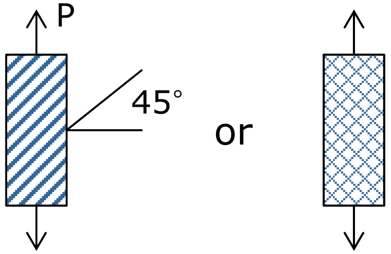




Shear Tests difficult to perform

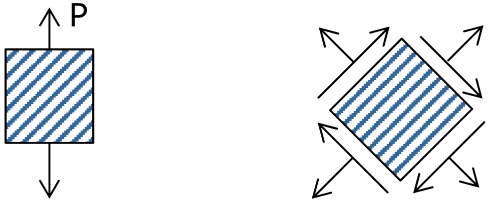
3. Ply Elasticity

Easier to test 45° Ply in tension



$[\pm 45°]_s$ laminate

This gives a mixed state of stress in axis system of the material



But can untangle to get G

So, we have

$$\varepsilon_1 = \frac{1}{E_L}\sigma_1 - \frac{\nu_{TL}}{E_T}\sigma_2$$

$$\varepsilon_2 = -\frac{\nu_{LT}}{E_L}\sigma_1 + \frac{1}{E_T}\sigma_2$$

$$\varepsilon_6 = \frac{1}{G_{LT}}\sigma_6$$

Because of symmetry

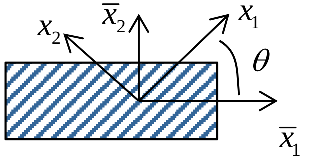
$$\frac{\nu_{TL}}{E_T} = \frac{\nu_{LT}}{E_L}$$

ν_{LT} : Major Poisson's Ratio ~ 0.3

ν_{TL} : $E_T/E_L \cdot \nu_{LT}$ ~ 0.02

3. Ply Elasticity

❖ Rotation of Plies



$x_1 \quad x_2$ → ply axes

$\overline{x}_1 \quad \overline{x}_2$ → laminate axes (also x, y)

Ply at angle θ from lamina axis $\overline{x}_1$

($+\theta$ → x_1 going towards x_2)

In x_1 , x_2 (Ply axes) → 2-D orthotropic Material

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} \quad \text{or} \quad \underset{\sim}{\sigma} = \underset{\sim}{Q} \, \underset{\sim}{\varepsilon}$$

Q's from $E_L \quad E_T \quad \nu_{LT} \quad G_{LT}$

To find stress-strain in $\overline{x}_1$, $\overline{x}_2$ (laminate axes)

First relate stresses in 2 axis systems.

$$\sigma_{mn} = l_{m\overline{p}} \; l_{n\overline{q}} \; \overline{\sigma}_{pq}$$

Standard transform Law

Stress in $\overline{x}_1 \; \overline{x}_2$

Stress tensor
in x_1, x_2

Direction cosine
= cos(angle x_m and $\overline{x}_p$)

3. Ply Elasticity

Table of cosines

	x_1	x_2
$\bar{x}_1$	cos θ	cos(90+θ) = -sin θ
$\bar{x}_2$	cos(90-θ) = sin θ	cos θ

$$\begin{aligned}\sigma_{11} &= l_{1\bar{1}}\, l_{1\bar{1}}\bar{\sigma}_{11} + l_{1\bar{2}} l_{1\bar{2}}\bar{\sigma}_{22} + l_{1\bar{1}} l_{1\bar{2}}\bar{\sigma}_{12} + l_{1\bar{2}} l_{1\bar{1}}\bar{\sigma}_{21} \\ &= \cos^2\theta\bar{\sigma}_{11} + \sin^2\theta\bar{\sigma}_{22} + \cos\theta\sin\theta\bar{\sigma}_{12} + \cos\theta\sin\theta\bar{\sigma}_{21} \\ \sigma_{22} &= etc. \\ \sigma_{12} &= etc.\end{aligned}$$

So obtain

$$\begin{Bmatrix}\sigma_1 \\ \sigma_2 \\ \sigma_6\end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & (c^2 - s^2)\end{bmatrix}\begin{Bmatrix}\bar{\sigma}_1 \\ \bar{\sigma}_2 \\ \bar{\sigma}_6\end{Bmatrix}$$

Ply $\qquad T_\sigma \qquad$ Laminate

or $\underset{\sim}{\sigma} = T_\sigma \bar{\sigma}$

where $c = \cos\theta \qquad s = \sin\theta$

3. Ply Elasticity

Also for strain

$$\varepsilon_{mn} = l_{m\bar{p}}\, l_{n\bar{q}}\, \bar{\varepsilon}_{pq}$$

← Standard transform law

Tensor strain in x_1, x_2

Tensor strain in $\bar{x}_1$ $\bar{x}_2$

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{1}{2}\varepsilon_6 \end{Bmatrix} = T_\sigma \begin{Bmatrix} \bar{\varepsilon}_1 \\ \bar{\varepsilon}_2 \\ \frac{1}{2}\bar{\varepsilon}_6 \end{Bmatrix}$$

Recall

$$\varepsilon_{12} = \frac{1}{2}\gamma_{12} = \frac{1}{2}\varepsilon_6$$

Absorb the ½ into $\underset{\sim}{T}_\sigma$ gives

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & cs \\ s^2 & c^2 & -cs \\ -2cs & 2cs & (c^2-s^2) \end{bmatrix} \begin{Bmatrix} \bar{\varepsilon}_1 \\ \bar{\varepsilon}_2 \\ \bar{\varepsilon}_6 \end{Bmatrix}$$

Ply T$_\varepsilon$ Laminate

or $\underset{\sim}{\varepsilon} = \underset{\sim}{T}_\varepsilon \underset{\sim}{\bar{\varepsilon}}$

Placing into Ply axes stress strain

$$\underset{\sim}{\sigma} = Q\,\varepsilon$$

$$T_\sigma \bar{\sigma} = Q\, T_\varepsilon \bar{\varepsilon}$$

$$\bar{\sigma} = \underbrace{T_\sigma^{-1} Q\, T_\varepsilon}_{\bar{Q}} \bar{\varepsilon}$$

$$or \quad \bar{\sigma} = \bar{Q}\,\bar{\varepsilon}$$

← Stress-strain Relation In laminate $\bar{x}_1$ $\bar{x}_2$ axes

3. Ply Elasticity

Now Note Inverses

$$T_\sigma^{-1} = T_\sigma(-\theta) = T_\varepsilon^{\,T}$$

$$T_\varepsilon^{-1} = T_\varepsilon(-\theta) = T_\sigma^{\,T}$$

So rotated Q matrix is

$$\boxed{\bar{Q} = T_\varepsilon^{\,T} Q T_\varepsilon}$$

Q fully populated now

Also in Jones Notation,

Laminated Axes $\bar{x}_1, \bar{x}_2$ → x , y

Laminated stress $\bar{\sigma}_i \ -> \ \sigma_x, \sigma_y, \tau_{xy}$

Laminated strain $\bar{\varepsilon}_i \ -> \ \varepsilon_x, \varepsilon_y, \gamma_{xy}$

Final Laminated stress – strain Eqns

$$\begin{Bmatrix} \bar{\sigma}_x \\ \bar{\sigma}_y \\ \bar{\tau}_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{13} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \bar{\varepsilon}_x \\ \bar{\varepsilon}_y \\ \bar{\gamma}_{xy} \end{Bmatrix}$$

Multiplies, out matrices

$$\bar{Q}_{11} = Q_{11}\cos^4\theta + 2(Q_{12} + 2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\sin^4\theta$$

$$\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})\sin^2\theta\cos^2\theta + Q_{12}(\sin^4\theta + \cos^4\theta)$$

$$\bar{Q}_{22} = etc.$$

$$\bar{Q}_{66} = etc.$$

3. Ply Elasticity

Similarly can transform compliances.

from $\varepsilon = S\sigma$

obtain $\bar{\varepsilon} = \bar{S}\bar{\sigma}$

where $\bar{S} = T_{\sigma}^{T} S T_{\sigma}$

Alternate ways of rotating

$$\bar{E}_{mnpq} = l_{m\bar{r}} l_{n\bar{s}} l_{p\bar{t}} l_{q\bar{u}} E_{rstu}$$

Also, can mathematically reduce $\bar{Q}_{ij}$ by $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$, etc.

can then express

$$\bar{Q}_{ij} = A_{ij} + B_{ij}\cos 2\theta + C_{ij}\cos 4\theta + D\sin 2\theta + E\sin 4\theta$$

A, B, C, D, E → depend only on 4 invariants