Mechanics of Composite Materials

CHAPTER 4.
Laminate Theory

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4. Laminate Theory

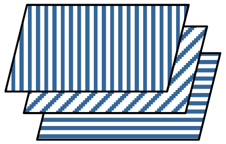
Can now manipulate orthotropic plies in plane stress

$$\underset{\sim}{\sigma} = \underset{\sim}{Q}\underset{\sim}{\varepsilon}, \quad \overline{\underset{\sim}{\sigma}} = \overline{\underset{\sim}{Q}}\,\overline{\underset{\sim}{\varepsilon}}$$

where $\overline{\underset{\sim}{Q}} = f(\underset{\sim}{Q}, \theta)$

Similarly, have $\underset{\sim}{\varepsilon} = \underset{\sim}{S}\,\underset{\sim}{\sigma}$, etc.

But, composites are actually used as laminates



- Many plies (lamina) are arranged at many θ
- Carry load, provide stiffness, strength, etc.

Note, other laminates → electronics - circuit boards

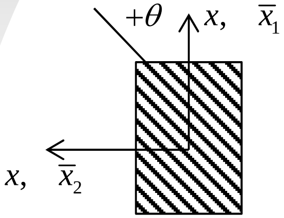
Capacitors, active materials (piezoelectrics), thermal barrier

- coats for engine combustors, etc.

4. Laminate Theory

❖ Laminate Notation

Need keep track of ply orientation
Use a compact notation



θ : ply angle

Note : Usually 0° direction corresponds to principal loading direction

Laminates specified as $[\ \pm 30\ /\ 0_2\ ]_s$ → Top: +30, −30, 0, 0 | 0, 0, −30, +30 :Bottom

repeat   symm.

$[\ \pm 30\ /\ 0_3\ ]_T$ → +30, −30, 0, 0, 0

total

$[\ 0\ /\ 30\ ]_{2S}$ → 0, 30, 0, 30, 30, 0, 30, 0

repeat group

Typical Laminates may bear

Cross Ply - $[\ 0_2\ /\ 90_2\ ]_T$ → 0, 0, 90, 90

Angle ply - $[\pm\theta\ ]_s$

Quasi – Isotropic - $[\ 0\ /\ \pm 45\ /\ 90\ ]_s$, $[\ 0\ /\ \pm 60\ ]_s$

❖ In-plane stress strain & stiffness

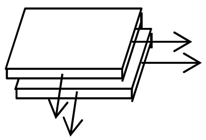
(Symmetric laminates – no bending)

Basic Assumptions :

1. Plies are all glued together
2. Plies are in plane-stress

$$\sigma_{33} = \sigma_{23} = \sigma_{13} = 0$$

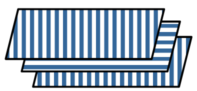
- Strain



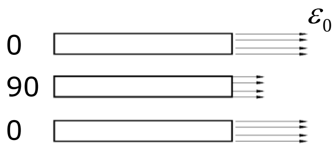
Because of gluing, ε_{ij}'s all same

$$\underset{\text{laminate}}{\underset{\sim}{\varepsilon}^{0}} = \underset{\text{each lamina}}{\underset{\sim}{\varepsilon}} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix}$$

- Stress



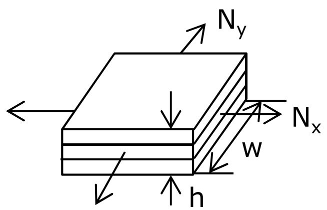
Is stress in 0° and 90° same ?
Not same.



Q_{ij} different

To find stresses, look at average force in plies.

Define N ← force/unit width of laminate



$$N_x = \frac{load \quad in \quad x}{w}$$

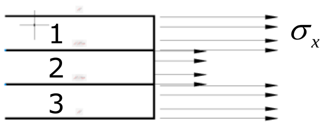
Total load – P (lb)

$$N = \frac{P}{w} \quad (lb/in)$$

4. Laminate Theory

Average stress $(\sigma_x)_A = N_x / h$, h = laminate thickness

$$N_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x dz$$



In discrete plies, $N_x = \sum_{k=1}^{n} \sigma_x^{(k)} t_k$

n = number of plies, k = 1, 2, 3, --- top to down

Similarly $N_y = \sum_{k=1}^{n} \sigma_y^{(k)} t_k$, $N_{xy} = \sum_{k=1}^{n} \sigma_{xy}^{(k)} t_k$

will then have

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \varepsilon_{xy}^0 \end{Bmatrix}$$

$$N_x = \sum_{k=1}^{n} [\bar{Q}_{11}^{(k)} \varepsilon_x + \bar{Q}_{11}^{(k)} \varepsilon_y + \bar{Q}_{11}^{(k)} \gamma_{xy}] t_k$$

$$= \underbrace{[\sum_{k=1}^{n} \bar{Q}_{11}^{(k)} t_k]}_{= A_{11}} \varepsilon_x^0 + \underbrace{[\sum_{k=1}^{n} \bar{Q}_{12}^{(k)} t_k]}_{= A_{12}} \varepsilon_y^0 + \underbrace{[\sum_{k=1}^{n} \bar{Q}_{16}^{(k)} t_k]}_{= A_{16}} \gamma_{xy}^0$$

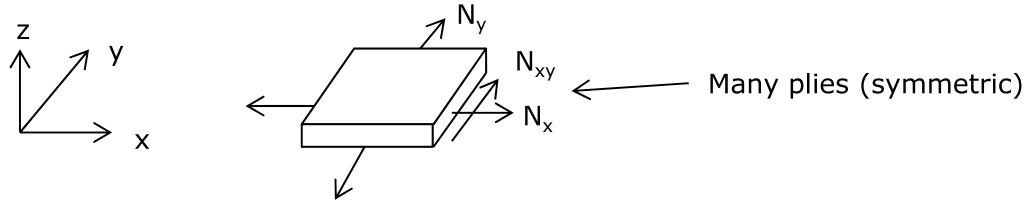
Similarly for N_y and N_{xy}

So finally

$$\underbrace{\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix}}_{= N} = \underbrace{\begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix}}_{= A} \underbrace{\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix}}_{= \varepsilon^0}$$

4. Laminate Theory

Given a symmetric laminate



Have formed a relation $\underset{\sim}{N} = \underset{\sim}{A}\ \underset{\sim}{\varepsilon}^0$

Where,

$$\underset{\sim}{N} = \begin{Bmatrix} N_x \\ N_x \\ N_{xy} \end{Bmatrix}$$ = Force (lbs/in)

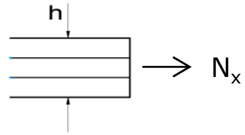
$$\underset{\sim}{\varepsilon}^0 = \begin{Bmatrix} \varepsilon_x{}^0 \\ \varepsilon_x{}^0 \\ \gamma_{xy}{}^0 \end{Bmatrix}$$ = Midplane strains (in/in)
(laminate axes)

$$\underset{\sim}{A} = \begin{Bmatrix} A_{11} & A_{12} & A_{16} \\ A_{21} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{Bmatrix}$$ = Extensional stiffness (lb/in)

and $A_{ij} = \sum_{k=1}^{N} \overline{Q}_{ij}{}^{(k)} t_k$

Could also write as equivalent moduli

$$\underset{\sim}{\bar{\sigma}} = \underset{\sim}{E}^{eq} \; \underset{\sim}{\varepsilon}^{0}$$



Averaged stress = N/h

So $\dfrac{1}{h}\underset{\sim}{N} = \dfrac{1}{h}\underset{\sim}{A}\,\underset{\sim}{\varepsilon}^{0}$

$\underset{\sim}{\bar{\sigma}}$ $\quad\underset{\sim}{E}^{eq}$ ← Lbs/in^2 : modulus

$$\underset{\sim}{E}^{eq} = \begin{bmatrix} E_{11}^{eq} & E_{12}^{eq} & E_{16}^{eq} \\ E_{12}^{eq} & E_{22}^{eq} & E_{26}^{eq} \\ E_{16}^{eq} & E_{26}^{eq} & E_{66}^{eq} \end{bmatrix}$$

← Like $\underset{\sim}{Q}$ matrix for the laminate

These are not the Engineering constants for the laminate.

Also, have Inverse Relations

$$\underset{\sim}{\varepsilon}^{0} = \underset{\sim}{a}\;\underset{\sim}{N} \quad \text{where} \quad \underset{\sim}{a} = A^{-1}$$

This only applies for symmetric laminates (no bending)

Deal later with unsymm. laminates

4. Laminate Theory

❖ Properties of $\underset{\sim}{A}$ matrix

$$\underset{\sim}{A} = \sum_{k=1}^{N} \bar{Q}^{(k)} t_k$$

$$\left.\begin{aligned} \bar{Q}_{11} &= c^4 Q_{11} + s^4 Q_{22} + \cdots \\ \bar{Q}_{11} &= c^2 s^2 (Q_{11} + \cdots \\ \bar{Q}_{11} &= \\ &\vdots \end{aligned}\right\}$$

See Handout, also Jones, p.51

· Remark on $\underset{\sim}{A}$

1. Thickness (area) weight stiffness $\bar{Q}_{ij}$

2. Independent of stacking order

3. Balanced laminates

" a - θ for every $+\theta$ "

$\bar{Q}_{11}$, $\bar{Q}_{12}$, $\bar{Q}_{22}$, $\bar{Q}_{66}$ not sensitive to sign
(c^4, s^4, c^2, s^2, . . .)

$\bar{Q}_{16}$, $\bar{Q}_{26}$ are affected (c^3s, cs^3, …) $A_{16} = A_{21} = 0$

∴ Balanced laminates are orthotropic.

4. Quasi – isotropic laminate

$[0 / \pm 60]_s$, $[0 / \pm 45 / 90]_s$ ← primary

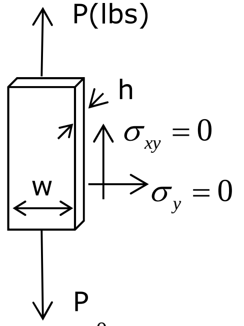
$[0 / \pm 30 / \pm 60 / 90]_s$ ← Built up from $[0 / \pm 60]_s$

$A_{22} = A_{11}$, $A_{66} = f(A_{11}, A_{22})$

∴ Quasi – isotropic have "isotropic" stiffness

4. Laminate Theory

· Example of using $\underset{\sim}{A}$ matrix



Test laminate

$$N_x = P/w, \qquad \bar{\sigma}_x = \frac{N_x}{h} = \frac{P}{hw}$$

$$N_y = 0$$

$$N_{xy} = 0$$

$$\varepsilon^0 = \underset{\sim}{A}^{-1} N = \underset{\sim}{a} \ \underset{\sim}{N}$$

$$\varepsilon_x{}^0 = a_{11} N_x + a_{12} N_y + a_{16} N_{xy} = a_{11}\frac{P}{w} = a_{11} h \bar{\sigma}_x$$

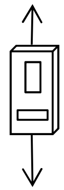
$$\varepsilon_y{}^0 = a_{12}\frac{P}{w} = a_{12} h \bar{\sigma}_x$$

$$\gamma_{xy}{}^0 = a_{16} h \bar{\sigma}_x = 0$$

❖ Laminate Engineering Constants

Constants we get from mechanical tests on laminates (as for plies)

①



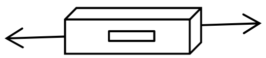
laminate

Measure $\varepsilon_x{}^0, \quad \varepsilon_y{}^0$

$$\bar{\sigma}_x = \frac{P}{wh}, \qquad \bar{\sigma}_x = \bar{E}_L \ \varepsilon_x{}^0$$ ← Engineering stiffness or Modulus

$$\bar{\nu}_{LT} = -\varepsilon_y{}^0 / \varepsilon_x{}^0$$ ← Engineering Poisson's Ratio

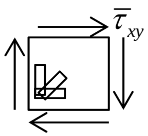
②



Measure $\varepsilon_y{}^0$

$$\bar{\sigma}_x = \frac{P}{wh}, \qquad \bar{\sigma}_y = \bar{E}_T \ \varepsilon_y{}^0$$ ← Engineering Transverse stiffness $\bar{E}_T$

③

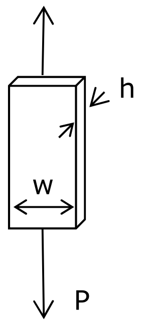


Measure $\gamma_{xy}{}^0$

$$\bar{\tau}_{xy} = \bar{G}_{LT} \quad \gamma_{xy}{}^0$$ ← Engineering shear stiffness $\bar{G}_{LT}$

❖ Obtaining Laminate Engineering Constants

For Test ①



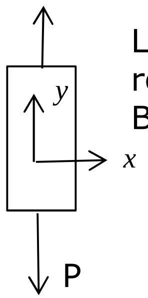
$$N_x = P/w = h\bar{\sigma}_x$$
$$N_y = 0$$
$$N_{xy} = 0$$
$$\varepsilon^0 = \underset{\sim}{A}^{-1}N = \underset{\sim}{a}\ \underset{\sim}{N}$$
$$\varepsilon_x^{\ 0} = a_{11}N_x = a_{11}h\bar{\sigma}_x$$
$$\varepsilon_y^{\ 0} = a_{12}\frac{P}{w} = a_{12}h\bar{\sigma}_x = \frac{a_{12}}{a_{11}}\varepsilon_x^{\ 0}$$
$$\varepsilon_{xy}^{\ 0} = 0$$ For balanced laminate

(If it weren't, $\varepsilon_{xy}^{\ 0} = a_{16}N_x = \dfrac{a_{16}}{a_{11}}\varepsilon_x^{\ 0}$)

Note
$$\bar{E}_x = \frac{\bar{\sigma}_x}{\varepsilon_x^{\ 0}} = \frac{1}{a_{11}h}$$
$$\bar{\nu}_{xy} = -\frac{\varepsilon_y^{\ 0}}{\varepsilon_x^{\ 0}} = -\frac{a_{12}}{a_{11}}$$

If not balanced, would also have $\eta_{xy,x}$ (Lekhnitski coefficient)

For Test②

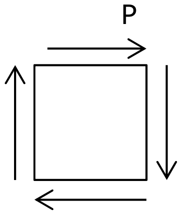


Laminate
rotated 90°
Before cutting

$$N_y = P/w = h\bar{\sigma}_y$$
$$N_x = 0, \quad N_{xy} = 0$$
$$\varepsilon_x^{\ 0} = a_{12}N_y = a_{12}h\bar{\sigma}_y$$
$$\varepsilon_y^{\ 0} = a_{22}N_y = a_{22}h\bar{\sigma}_y$$
$$\varepsilon_{xy}^{\ 0} = 0$$ if balanced
$$\bar{E}_y = \frac{\bar{\sigma}_y}{\varepsilon_y^{\ 0}} = \frac{1}{a_{22}h}$$

4. Laminate Theory

For Test ③



$$N_{xy} = P/w = h\bar{\sigma}_{xy}$$
$$N_x = 0, \quad N_y = 0$$
$$\varepsilon_x{}^0 = a_{16}N_{xy} = 0 \qquad \varepsilon_y{}^0 = a_{26}N_{xy} = 0$$
$$\varepsilon_{xy}{}^0 = a_{66}N_{xy} = a_{66}h\bar{\sigma}_{xy} \qquad \bar{G}_{xy} = \sigma_{xy}/\varepsilon_{xy}{}^0 = 1/a_{66}h$$

For balanced laminate,

$$\underset{\sim}{A} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix}, \quad \underset{\sim}{a} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{12} & a_{22} & 0 \\ 0 & 0 & a_{66} \end{bmatrix}$$

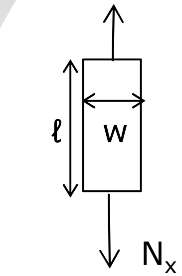
$$a_{66} = \frac{1}{A_{66}}$$
$$\bar{G}_{xy} = \frac{1}{a_{66}h} = \frac{A_{66}}{h} = E_{66}{}^{eq}$$

But $a_{11} \neq \dfrac{1}{A_{11}}$, So $\underbrace{\bar{E}_x}_{\text{Engineering}} \neq E_{11}{}^{eq} \rightarrow$ Equivalent

When does difference come up ?

4. Laminate Theory

For w << ℓ case



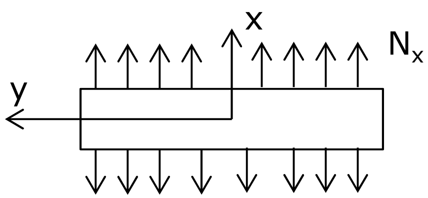
Unconstrained

$$N_y = N_{xy} = 0$$

$$\varepsilon_x^{\ 0} = \bar{\sigma}_x / \bar{E}_x$$

$\bar{E}_x$ → Engineering stiffness

For w >> ℓ case



Can't say $N_y = N_{xy} = 0$

$\varepsilon_y^{\ 0} = \varepsilon_{xy}^{\ 0} = 0$ is a better approximation

Then would have

$$\underset{\sim}{N} = \underset{\sim}{A}\,\underset{\sim}{\varepsilon}^{0}$$

$$N_x = A_{11}\underset{\sim}{\varepsilon}^{0} + 0 + 0$$

$$N_y = A_{12}\underset{\sim}{\varepsilon}^{0}$$

$$N_{xy} = A_{16}\underset{\sim}{\varepsilon}^{0}$$

$$N_x = \bar{\sigma}_x h = A_{11}\varepsilon_x^{\ 0}$$

$$\frac{\bar{\sigma}_x}{\varepsilon_x^{\ 0}} = \frac{A_{11}}{h} = E_{11}^{\ eq}$$

$E_{11}^{\ eq}$ → Equivalent stiffness

❖ **Effect of Boundaries**

Note also this effect in isotropic materials.

current stiffness there → $E^{eq} = E/(1-\nu^2)$

Because plies are constrained by neighbors, usually more convenient to work with $\underset{\sim}{A}$ (or E^{eq}) rather then $\bar{E}_x$, $\bar{E}_y$, etc.

4. Laminate Theory

❖ Ply Stresses

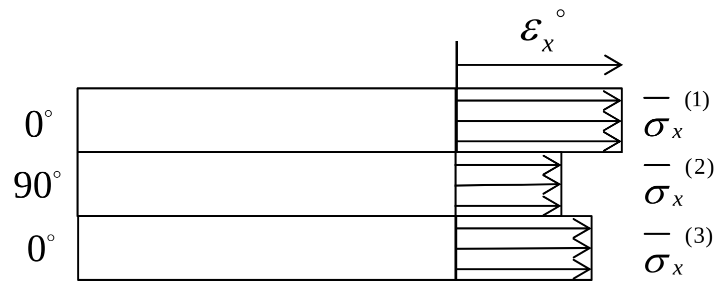
- Given a laminate description, can find laminate engineering constants for input to plate and shell problems.

- Given loads P_x, P_y, P_{xy}, can find N_x, N_y, N_{xy} and then one gets laminate strains from

$$\varepsilon^{\circ} = \underset{\sim}{a} \cdot \underset{\sim}{N}$$

- Average laminate stresses $\overline{\underset{\sim}{\sigma}} = \underset{\sim}{N} / h$

- Now, want to look at individual stress in k^{th} ply. (to predict failure)



Ply strains $\bar{\varepsilon}_x^{(k)}$ all the same

Ply stresses $\bar{\sigma}_x^{(k)}$ all different

(Bar indicates laminate coordinate system)

- How do we calculate $\underset{\sim}{\sigma}^{(k)}$

4. Laminate Theory

- Note: no bar – want stress in ply coordinate system
- Two paths for getting $\underset{\sim}{\sigma}^{(k)}$

Path #1

Know

$$\underset{\sim}{\overline{\varepsilon}}^{(k)} = \underset{\sim}{\varepsilon}^{\circ}$$

laminate strain

ply strain, ply k, laminate coord.

Also know

$$\underset{\sim}{\overline{\sigma}}^{(k)} = \overline{Q}^{(k)}\overline{\varepsilon}^{(k)}$$

ply strain, laminate coord.

ply stress, ply k, laminate coord.

Therefore, can calculate stresses in ply

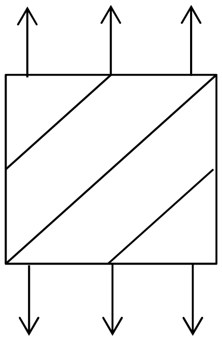
$\bar{\sigma}_x^{(k)}$, etc.

$\longrightarrow N_x$

$$N_x = \sum_{K=1}^{N} \bar{\sigma}_x^{(k)} t_k$$

(can check out N_x)

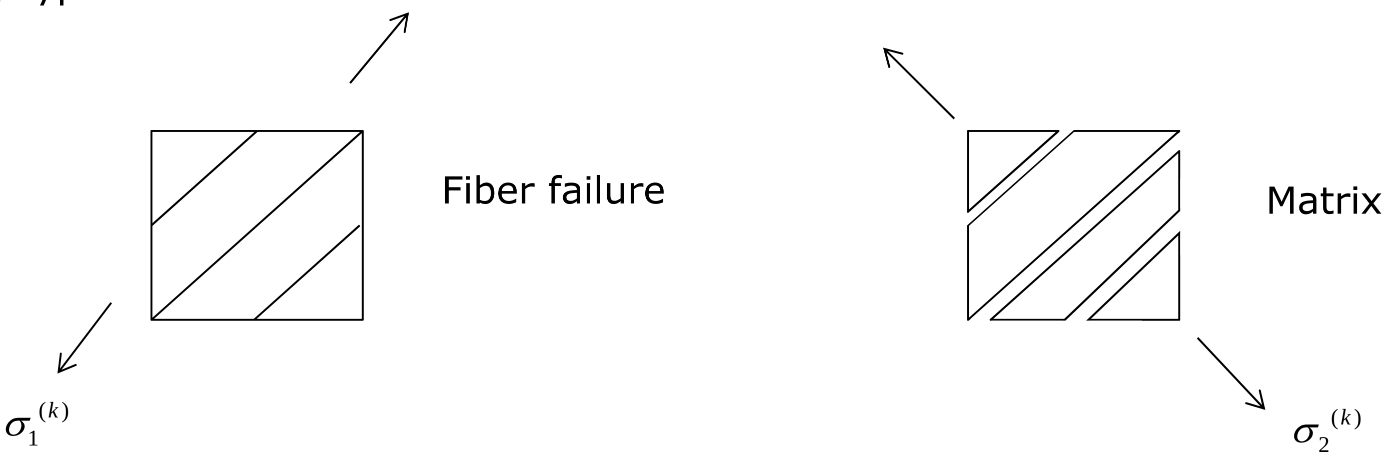
Unfortunately, $\bar{\sigma}_x^{(k)}$ isn't very useful.



$\theta = 45^\circ$

$\bar{\sigma}_x^{(k)}$

Type of failure

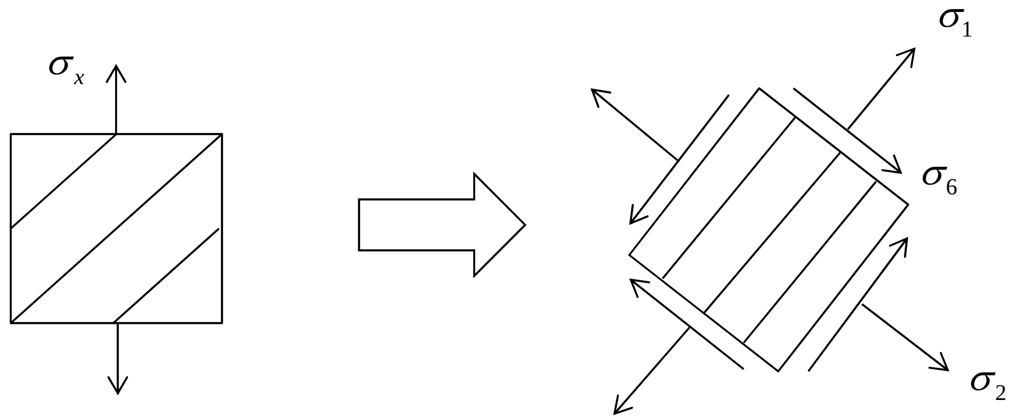


Given $\overline{\underset{\sim}{\sigma}}^{(k)}$, get $\underset{\sim}{\sigma}^{(k)}$ by transformation

$$\underset{\sim}{\sigma}^{(k)} = \underset{\sim}{T}_{\sigma}^{(k)} \overline{\underset{\sim}{\sigma}}^{(k)}$$

ply

laminate



4. Laminate Theory

Also, get ply strain in ply coordinates.

$$\underset{\sim}{\varepsilon}^{(k)} = \underset{\sim}{S}^{(k)} \underset{\sim}{\sigma}^{(k)}$$

So, system we have

$$P \longrightarrow \underset{\sim}{N} \xrightarrow{\underset{\sim}{a}} \underset{\sim}{\varepsilon}^{\circ} \xrightarrow{\text{same}} \underset{\sim}{\bar{\varepsilon}}^{(k)} \xrightarrow{\bar{Q}^{k}} \underset{\sim}{\bar{\sigma}}^{(k)} \xrightarrow{\underset{\sim}{T}_{\sigma}} \underset{\sim}{\sigma}^{k} \xrightarrow{\underset{\sim}{S}} \underset{\sim}{\varepsilon}^{(k)}$$

Path #2

Given $\underset{\sim}{\bar{\varepsilon}}^{(k)}$, go directly to $\underset{\sim}{\varepsilon}^{(k)}$ by transformation

$$\underset{\sim}{\varepsilon}^{(k)} = \underset{\sim}{T}_{\varepsilon}^{(k)} \underset{\sim}{\bar{\varepsilon}}^{(k)}$$

Ply coords. laminate coordinates

Then from $\varepsilon^{(k)}$, get $\underset{\sim}{\sigma}^{(k)}$ from ply stress – strain equations.

$$\underset{\sim}{\sigma}^{(k)} = Q^{(k)} \underset{\sim}{\varepsilon}^{(k)}$$

4. Laminate Theory

So, have another system

$$P \longrightarrow \underset{\sim}{N} \xrightarrow{\underset{\sim}{a}} \underset{\sim}{\varepsilon}^{\circ} \xrightarrow{\text{same}} \underset{\sim}{\bar{\varepsilon}}^{(k)} \xrightarrow{T_{\varepsilon}^{(k)}} \underset{\sim}{\varepsilon}^{(k)} \xrightarrow{\bar{Q}^{(k)}} \underset{\sim}{\sigma}^{(k)}$$

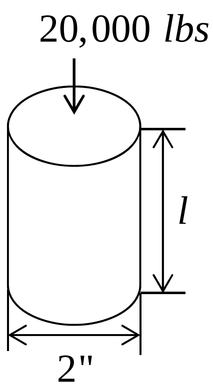
So, summarizing what we know so far, we have arrived at,

- In-Plane Classical Laminate Plate Theory (CLPT)

$$\begin{tikzcd}
\boxed{\text{Loads}} \arrow[d] & & \boxed{\text{Constraints}} \arrow[d] & \\
N \arrow[r] & A \arrow[r] & \varepsilon^{\circ} \arrow[d] & \text{Laminate} \\
\bar{\sigma} \arrow[d, "\text{transform}"'] & \bar{Q} \arrow[l] \arrow[u, "\text{sum}"'] & \bar{\varepsilon} \arrow[l, "\#1"'] \arrow[d] & \text{Ply in laminate coordinate system} \\
\sigma \arrow[r, shift right] & Q \arrow[l, shift right] \arrow[r, shift right] \arrow[u, "\text{transform}"] & \varepsilon \arrow[l, shift right, "\#2"'] & \text{Ply (ply coordinates system)} \\
& \boxed{\text{Laminate properties}} \arrow[u] & &
\end{tikzcd}$$

- Example of In-Plane CLPT

Use previous system to solve a practical problem.



Tubular compression member

Assume l short (no buckling)

Material T300/934 Gr/Ep

Ply Engineering Properties

$$E_L = 20\ Msi,\ E_T = 1.4\ Msi,\ \nu_{LT} = 0.29,\ G_{LT} = 0.7\ Msi$$

Ply thicknesses: $0.005''(5\ mils)$

Lay up : $[0/\pm 45/90]_S$

Referring to ground scheme, already have Ply Eng'g Consts (obtained by micromechanics and test)

4. Laminate Theory

Step #1: Find $\underset{\sim}{Q}$

$$\nu_{LT} = \frac{E_r}{E_L}\nu_{LT} = 0.020$$

$$Q_{11} = \frac{E_L}{1-\nu_{LT}\nu_{TL}} = \frac{20}{1-0.29(0.02)} = 20.12\ Msi \longleftarrow$$ not much different from E_L

$$Q_{12} = \frac{\nu_{LT}E_T}{1-\nu_{LT}\nu_{TL}} = 0.408\ Msi$$

$$Q_{22} = \frac{E_T}{1-\nu_{LT}\nu_{TL}} = 1.41\ Msi$$ approximately E_T

$$Q_{16} = 0,\ Q_{26} = 0,$$

$$Q_{66} = G_{LT} = 0.7\ Msi$$

$$\underset{\sim}{Q} = \begin{bmatrix} 20.12 & 0.408 & 0 \\ 0.408 & 1.41 & 0 \\ 0 & 0 & 0.7 \end{bmatrix} (Msi)$$

4. Laminate Theory

Step #2: Compute $\overline{Q}$ for each ply

0° plies: Trivial $\longrightarrow$ $\overline{\underset{\sim}{Q}}_0 = \underset{\sim}{Q}$

90° plies: Easy $\longrightarrow$ Reverse 1, 2

$$\overline{\underset{\sim}{Q}}_{90} = \begin{bmatrix} 1.41 & 0.408 & 0 \\ 0.408 & 20.12 & 0 \\ 0 & 0 & 0.7 \end{bmatrix} (Msi)$$

45° Plies: Harder → Use transform formulas

$$\overline{Q}_{11} = c^4 Q_{11} + s^2 Q_{22} + 2c^2 s^2 (Q_{12} + 2Q_{66})$$

Note: $\begin{cases} \cos\theta = \sin\theta = 0.707 \\ \cos^2\theta = \sin^2\theta = 0.500 \end{cases}$

$$\overline{Q}_{11} = 0.25(20.12) + 0.25(1.41) + 0.500(0.408 + 2[0.7])$$

$$= \ 6.29\ Msi$$

4. Laminate Theory

Similarly

$$\overline{Q}_{12} = 4.89\ Msi$$

$$\overline{Q}_{16} = \qquad , \overline{Q}_{26} = \qquad , \overline{Q}_{66} =$$

$$\overline{Q}_{45} = \begin{bmatrix} 6.29 & 4.87 & 4.68 \\ 4.87 & 6.29 & 4.68 \\ 4.68 & 4.68 & 5.18 \end{bmatrix} (Msi)$$

-45° Plies: Easy, same as +45 except $\overline{Q}_{16}$ and $\overline{Q}_{66}$ change signs

Note: $\sin(-\theta) = -\sin(\theta) \longrightarrow$ Only $s,\ s^3$

$\cos(-\theta) = \cos(\theta)$

$$\overline{Q}_{-45} = \begin{bmatrix} 6.29 & 4.89 & 4.68 \\ 4.89 & 6.29 & -4.68 \\ 4.68 & -4.68 & 5.18 \end{bmatrix} (Msi)$$

4. Laminate Theory

Step #3: Assemble $\underset{\sim}{A}$ matrix

$$\underset{\sim}{A} = \sum_{k=1}^{N} Q^{(k)} t_k$$

Note: - Thickness all the same

- order doesn't matter here

- Symmetric $\underset{\sim}{\overline{Q}}$ → symmetric $\underset{\sim}{A}$

- Also, $\overline{Q}_{11}^{-45^\circ} = \overline{Q}_{11}^{45^\circ}$ (etc. for 12, 22, 66)

but, $\overline{Q}_{16}^{-45^\circ} = -\overline{Q}_{16}^{45^\circ}$ (and for 26),

so, 16 and 26 term cancel

So summing,

$$A_{11} = t(2Q_{11}^{\circ} + 2Q_{11}^{90} + 4Q_{11}^{45})$$
$$= 0.005(2(20.12\times10^6) + 2(1.41\times10^6) + 4(6.29\times10^6))$$
$$= 0.341\times10^6 \ lb/in$$

$$A_{12} = t(2Q_{12}^{\circ} + 2Q_{12}^{90} + 4Q_{12}^{45})$$
$$= 0.005(2(0.408\times10^6) + 2(0.408\times10^6) + 4(4.89\times10^6))$$
$$= 0.106\times10^6 \ lb/in$$

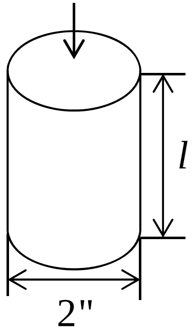
$$A_{22} = 0.341\times10^6 \ lb/in$$

$$A_{66} = 0.118\times10^6 \ lb/in$$

$$A_{16} = 0, \ A_{26} = 0$$

$$\underset{\sim}{A} = \begin{bmatrix} 0.341 & 0.106 & 0 \\ 0.106 & 0.341 & 0 \\ 0 & 0 & 0.118 \end{bmatrix} \times 10^6 \ lb/in$$

Step #4: Establish Loading



Loading assume thin, load distributes evenly

$$N_x = \frac{P}{circumference} = \frac{P}{2\pi r} = \frac{20{,}000}{2\pi(1)} = -3.183\ lb/in$$

Assume unrestrained, $N_x = 0,\ N_{xy} = 0$

Step #5: Calculate Laminate Strain

$$\underset{\sim}{\varepsilon}^{\circ} = \underset{\sim}{a}\underset{\sim}{N}, \quad \underset{\sim}{a} = A^{-1}$$

Can invert 3x3 matrix, on else,

$$\underset{\sim}{a} = \begin{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{bmatrix}^{-1} & 0 \\ 0 \quad\quad 0 & \dfrac{1}{A_{66}} \end{bmatrix} = \begin{bmatrix} \dfrac{A_{22}}{A_{11}A_{22} - A_{12}^{2}} & \dfrac{-A_{22}}{A_{11}A_{22} - A_{12}^{2}} & 0 \\ \dfrac{-A_{22}}{A_{11}A_{22} - A_{12}^{2}} & \dfrac{A_{22}}{A_{11}A_{22} - A_{12}^{2}} & 0 \\ 0 & 0 & \dfrac{1}{A_{66}} \end{bmatrix} = \begin{bmatrix} 3.25 & -1.01 & 0 \\ -1.01 & 3.25 & 0 \\ 0 & 0 & 8.47 \end{bmatrix} \times 10^{6}\ in/lb$$

$$\varepsilon_x^{\circ} = a_{11} N_x = 3.25 \times 10^{-6} (-3183) = -0.0103$$

$$\varepsilon_y^{\circ} = a_{12} N_x = -1.01 \times 10^{-6} (-3183) = +0.0032$$

$$\varepsilon_{xy}^{\circ} = a_{16} N_x = 0$$

Step #6: Calculate Ply Strains in Laminate coordinates

Jones Notation

$$\overline{\varepsilon}^{(k)} = \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix}$$

All $\underset{\sim}{\overline{\varepsilon}}^{(k)}$ equal to $\underset{\sim}{\varepsilon}^{\circ}$

Step #7: Calculate Ply Stresses in Laminate Coordinates System

Jones Notation

$$\overline{\sigma}^{(k)} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

$$\overline{\sigma}^{(k)} = \underset{\sim}{\overline{Q}}^{(k)} \overline{\varepsilon}^{(k)}$$

4. Laminate Theory

$$\begin{Bmatrix} \bar{\sigma}_x \\ \bar{\sigma}_y \\ \bar{\sigma}_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x{}^{\circ} \\ \varepsilon_y{}^{\circ} \\ \gamma_{xy}{}^{\circ} \end{Bmatrix}$$

0° ply: $\bar{\sigma}_x = \overline{Q_{11}}\varepsilon_x{}^{\circ} + \overline{Q_{12}}\varepsilon_y{}^{\circ}$

$= 20.12 \times 10^6(\text{-}0.0103) + 0.408 \times 10^6(+0.0032)$

$= \text{-}206\ (Ksi)$ (high)

$\bar{\sigma}_y = \overline{Q_{12}}\varepsilon_x{}^{\circ} + \overline{Q_{12}}\varepsilon_y{}^{\circ}, \qquad \bar{\sigma}_{xy} = 0$

$= 0.300\ (Ksi)$ (low)

$+90^{\circ}$ ply: $\bar{\sigma}_x = \overline{Q_{11}}^{90}\varepsilon_x{}^{\circ} + \overline{Q_{12}}^{90}\varepsilon_y{}^{\circ}$

$= \text{-}13\ (Ksi)$

$\bar{\sigma}_y = 60\ Ksi$

$\bar{\sigma}_{xy} = 0$

$+45^\circ$ ply:

$$\bar{\sigma}_x = \overline{Q_{11}}\varepsilon_x{}^\circ + \overline{Q_{12}}\varepsilon_y{}^\circ + \cancel{\overline{Q_{16}}\gamma_{xy}{}^\circ} = -49\ (Ksi)$$

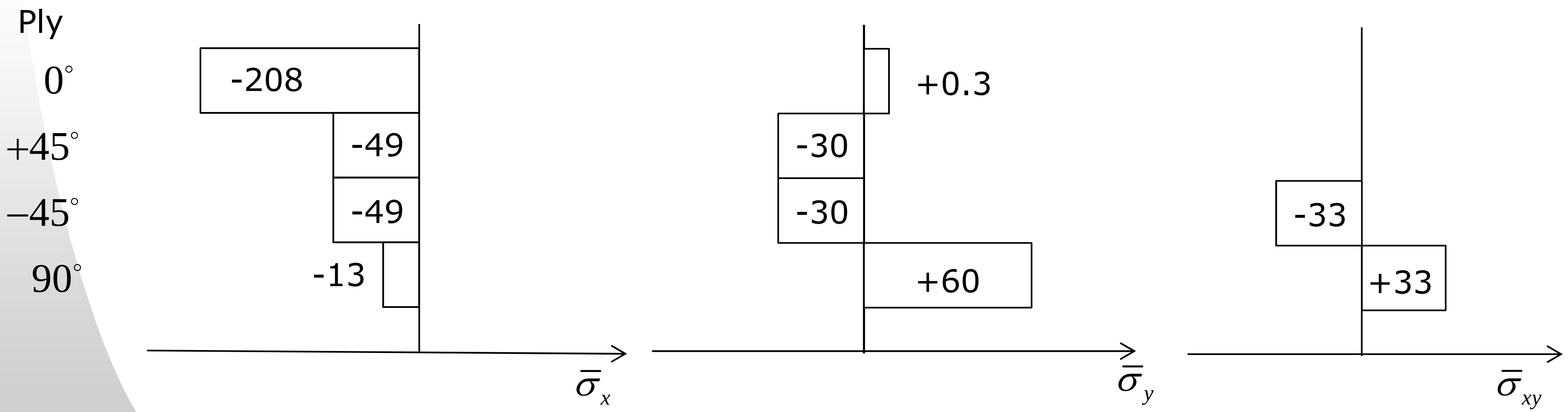
$$\bar{\sigma}_y = \overline{Q_{12}}\varepsilon_x{}^\circ + \overline{Q_{22}}\varepsilon_y{}^\circ + \cancel{\overline{Q_{26}}\gamma_{xy}{}^\circ} = -30\ (Ksi)$$

$$\bar{\sigma}_x = \overline{Q_{16}}\varepsilon_x{}^\circ + \overline{Q_{26}}\varepsilon_y{}^\circ + \cancel{\overline{Q_{66}}\gamma_{xy}{}^\circ} = -33\ (Ksi)$$

-45° ply: same as $+45^\circ$, but

$$\bar{\sigma}_{xy}(-45^\circ) = -\bar{\sigma}_{xy}(+45^\circ)$$

Plotting stresses



Step #8: Calculate Ply Stress $\underset{\sim}{\sigma}$ in ply coordinates

$$\underset{\sim}{\sigma} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}$$ ← Jones Notation

$$\underset{\sim}{\sigma} = T_\sigma \underset{\sim}{\bar{\sigma}} \quad \rightarrow \quad \underset{\sim}{\sigma} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -cs & cs & (c^2 - s^2) \end{bmatrix} \begin{Bmatrix} \bar{\sigma}_x \\ \bar{\sigma}_y \\ \bar{\sigma}_{xy} \end{Bmatrix}$$

0° ply:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} -206 \\ 30 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -206 \\ 30 \\ 0 \end{Bmatrix} Ksi$$

90° ply:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} -13 \\ 60 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 60 \\ -13 \\ 0 \end{Bmatrix} Ksi$$

4. Laminate Theory

$+45^\circ$ ply:
$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_6\end{Bmatrix} = \begin{bmatrix}0.5 & 0.5 & 0\\ -0.5 & -0.5 & -1\\ -0.5 & 0.5 & 0\end{bmatrix}\begin{Bmatrix}-49\\ 30\\ 33\end{Bmatrix} = \begin{Bmatrix}-73\\ -6.5\\ -9.5\end{Bmatrix} Ksi$$

-45° ply: Same signs as $+45^\circ$ by σ_6 change sign

Summary of stress (*Ksi*)

ply, θ	σ_1	σ_2	σ_6
0°	-206	0.3	0
$+45^\circ$	-73	-6.5	9.5
-45°	-73	-6.5	-9.5
$+90^\circ$	60	-13	0

Note

Compare to strength of unidirectional material

Compress ultimate (1-dir) = 160 *Ksi*

Compress ultimate (2-dir) = 25 *Ksi*

Shear ultimate = 10 *Ksi*

Fiber failure in 0° ply, reinforce strut

Look also at ply-axis strains

Step #9: Calculate Ply Strains $\underset{\sim}{\varepsilon}$ in ply coordinates.

$$\underset{\sim}{\varepsilon} = \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} \qquad \leftarrow \text{Jones notation}$$

$$\underset{\sim}{\varepsilon} = \underset{\sim}{S}\underset{\sim}{\sigma} \quad \rightarrow \quad \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}$$

$$S_{11} = \frac{1}{E_L} = \frac{1}{20 \times 10^6} = 0.050 \times 10^{-6}$$

$$S_{12} = -\frac{\nu_{LT}}{E_L} = -\frac{0.29}{20 \times 10^6} = -0.0145 \times 10^{-6}$$

$$S_{22} = \frac{1}{E_T} = \frac{1}{1.4 \times 10^6} = 0.7143 \times 10^{-6}$$

$$S_{66} = \frac{1}{G_{LT}} = \frac{1}{0.7 \times 10^6} = 1.429 \times 10^{-6}$$

$$S_{16} = S_{26} = 0$$

x10^{-6} x10^{-3}

0° ply:
$$\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_6\end{Bmatrix}=\begin{bmatrix}0.0500 & -0.0145 & 0\\ -0.0145 & 0.7143 & 0\\ 0 & 0 & 1.429\end{bmatrix}\begin{Bmatrix}0.206\\ 0.3\\ 0\end{Bmatrix}=\begin{Bmatrix}-0.0103\\ 0.0032\\ 0\end{Bmatrix}$$

$+90^\circ$ ply:
$$\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_6\end{Bmatrix}=\begin{bmatrix}0.0500 & -0.0145 & 0\\ -0.0145 & 0.7143 & 0\\ 0 & 0 & 1.429\end{bmatrix}\begin{Bmatrix}60\\ -13\\ 0\end{Bmatrix}=\begin{Bmatrix}-0.0032\\ -0.0103\\ 0\end{Bmatrix}$$

$$\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_6\end{Bmatrix}=\begin{bmatrix}0.0500 & -0.0145 & 0\\ -0.0145 & 0.7143 & 0\\ 0 & 0 & 1.429\end{bmatrix}\begin{Bmatrix}-73\\ -6.5\\ 9.5\end{Bmatrix}=\begin{Bmatrix}-0.0036\\ -0.0036\\ 0.0136\end{Bmatrix}$$

-45°ply - same as +45° by ε_6 change sign

Summary of strains

ply, θ	ε_1	ε_2	ε_6
0°	-0.0103	0.0032	0
$+45^\circ$	-0.0036	-0.0036	0.0136
-45°	-0.0036	-0.0036	-0.0136
$+90^\circ$	-0.0032	-0.0103	0

Sometimes use a max. strain criteria instead of max. stress $(\varepsilon_1 = 7000\ \mu\varepsilon)$

Also can do ply stress analysis by Path #2

Steps 1 ~ 6 same as before

Step #7A Calculate ply strain $\underset{\sim}{\varepsilon}$ in ply coords.

$$\underset{\sim}{\varepsilon} = \underset{\sim}{T_\varepsilon}\,\underset{\sim}{\bar{\varepsilon}} \rightarrow \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & sc \\ s^2 & c^2 & -sc \\ -2cs & 2cs & (c^2 - s^2) \end{bmatrix} \begin{Bmatrix} \bar{\varepsilon}_x \\ \bar{\varepsilon}_y \\ \bar{\varepsilon}_{xy} \end{Bmatrix}$$

4. Laminate Theory

0° ply:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} -0.0103 \\ 0.0032 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -0.0103 \\ 0.0032 \\ 0 \end{Bmatrix}$$

$+90^\circ$ ply:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} -0.0103 \\ 0.0032 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0.0032 \\ -0.0103 \\ 0 \end{Bmatrix}$$

etc. 45° and -45° plus

Step #8A: Calculate ply stress $\underset{\sim}{\sigma}$ in ply coords.

$$\underset{\sim}{\sigma} = \underset{\sim}{Q}\underset{\sim}{\varepsilon} \quad \rightarrow \quad \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix}$$

0° ply:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} 20.12 & 0.408 & 0 \\ 0.408 & 1.41 & 0 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{Bmatrix} -0.0103 \\ -0.0032 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -206 \\ -3 \\ 0 \end{Bmatrix} \; Ksi$$

x10^{-6}

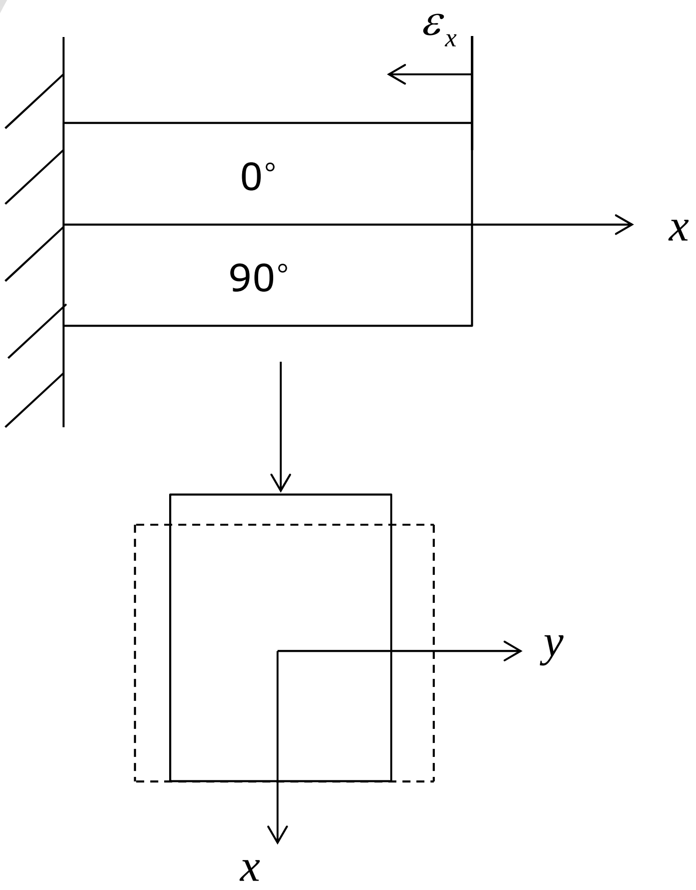
$+90^\circ$ ply:

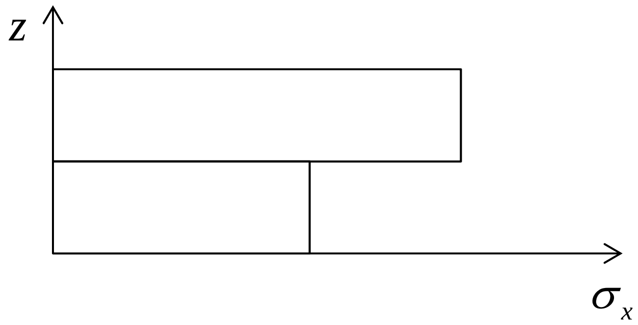
$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} & & \\ & same & \\ & & \end{bmatrix} \begin{Bmatrix} -0.0032 \\ -0.0103 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 60 \\ -13 \\ 0 \end{Bmatrix} \; Ksi$$

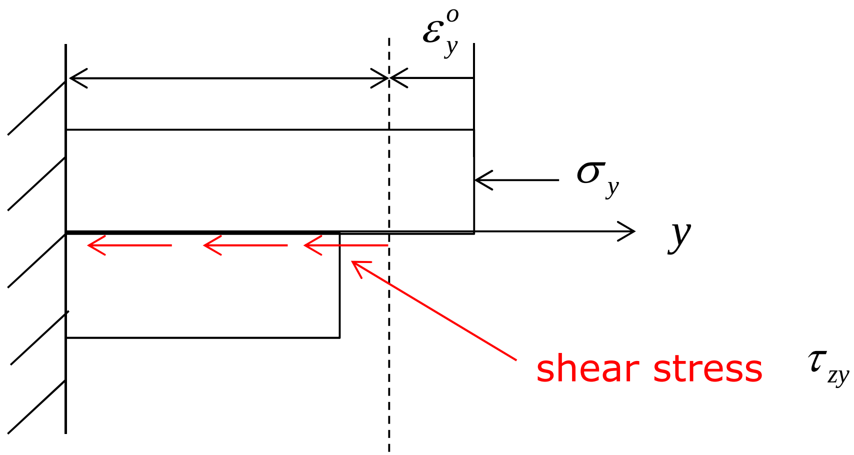
same $+45^\circ$ and -45°

Same results from Path #1 and Path #2 $\rightarrow$ easier (but no $\bar{\underset{\sim}{\sigma}}$)

4. Laminate Theory







$$\tau_{zy} dydx = \sigma_y h(\varepsilon_y^F - \varepsilon_y^\circ) dydx$$

$$\tau_{zy} = \sigma_y h(\varepsilon_y^F - \varepsilon_y^\circ)$$