Mechanics of Composite Materials

CHAPTER 6.
Bending of Laminated Plate

SangJoon Shin

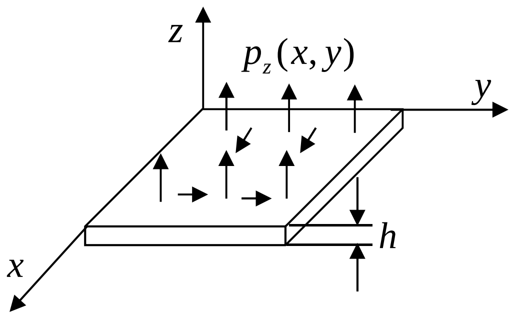
School of Mechanical and Aerospace Engineering

Seoul National University

6. Bending of Laminated Plate

See Jones, Chap 4.

- Consider plate under loading $p_z(\mathrm{x},\mathrm{y})$, p_x, p_y



$h << l_x, l_y$

- Notation............coords x,y,z
  deflections u,v,w
  Engineering Strain $\varepsilon_x, \varepsilon_y, \varepsilon_z$
  $\gamma_{yz}, \gamma_{zx}, \gamma_{xy}$
  Stresses $\sigma_x, \sigma_y, \sigma_z$
  $\tau_{yz}, \tau_{zx}, \tau_{xy}$

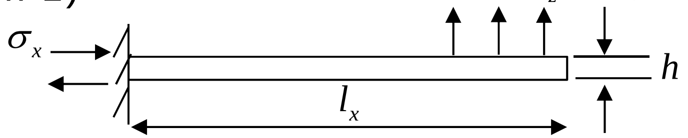
Three Basic Assumptions

1. $\sigma_z << \sigma_x, \sigma_y \longrightarrow$ hence $\sigma_z = 0$
2. Plane sections remain plane and $\perp$ after deformation (Bernoulli-Euler, Navier, Lave)
3. Use Stress Resultants (averages) instead of stresses themselves

All these assumptions are a consequence of plate, being thin. $h << l_x, l_y$

6. Bending of Laminated Plate

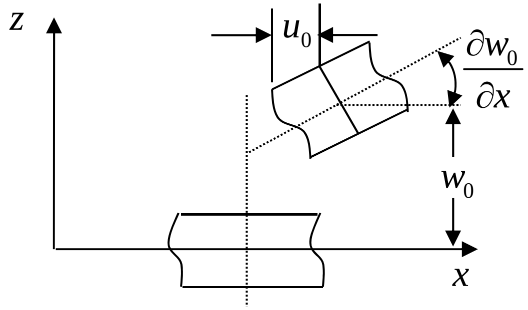
Assumption 1)



because l/h large, much leverage action, gives $\sigma_x >> \sigma_z$

Neglect σ_z in analysis

Assumption 2 - Geometry



For small angles

$$\sin\frac{\partial w}{\partial x} \cong \frac{\partial w}{\partial x}$$

$$\cos\frac{\partial w}{\partial x} \cong 1$$

$$u(x,y,z) = u_0 - z\frac{\partial w}{\partial x}$$

$$w(x,y,z) = w_0$$

Looking along y axis

$$v(x,y,z) = v_0 - z\frac{\partial v_0}{\partial y}$$

u_0, v_0, w_0 mid plane displacement (function of x, y only)

6. Bending of Laminated Plate

Assumption 1) $\partial_z = 0$

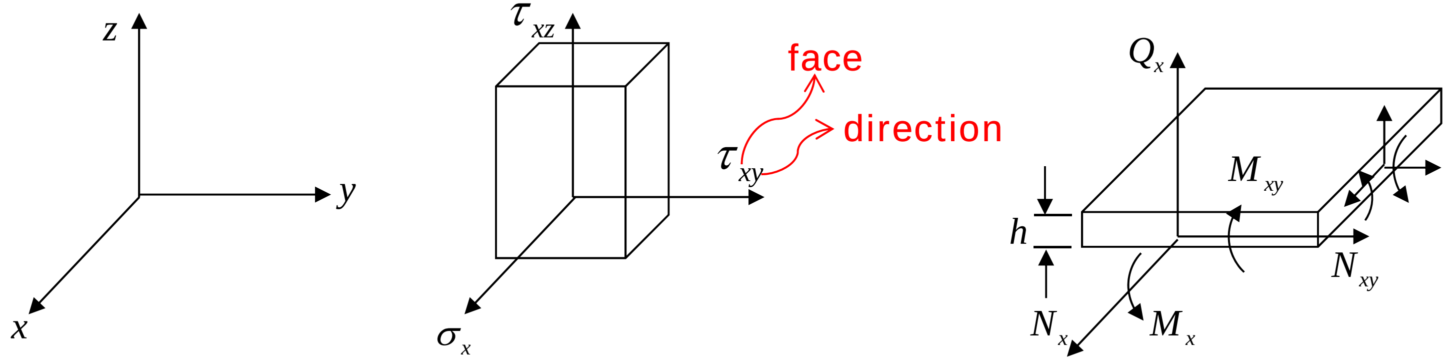
2) B.E.N Hypothesis

$$u = u_0 - z\frac{\partial w_0}{\partial x}$$

$$v = v_0 - z\frac{\partial w_0}{\partial y}$$

$$w = w_0$$

Assumption 3) Stress Resultants



6. Bending of Laminated Plate

Define following 8 Stress Resultants

$$\left.\begin{aligned} N_x &= \int_{-h/2}^{h/2} \sigma_x dz \\ N_y &= \int_{-h/2}^{h/2} \sigma_y dz \\ N_{xy} &= \int \tau_{xy} dz \end{aligned}\right\}$$ Mid Plane Forces (lbs/in)

$$\left.\begin{aligned} M_x &= \int z\sigma_x dz \\ M_y &= \int z\sigma_y dz \end{aligned}\right\}$$ Bending Moments (lb-in/in)

$$\left. M_{xy} = \int z\tau_{xy} dz \right\}$$ Twisting Moments (lb-in/in)

$$\left.\begin{aligned} Q_x &= \int \tau_{xy} dz \\ Q_y &= \int \tau_{yz} dz \end{aligned}\right\}$$ Transvers Shear Forces (lbs/in)

M_x, Q_x, etc Slightly different conventions than Beam theory

Introduce Assumption 1),2),3) into Equations of Elasticity to obtain Plate Theory

6. Bending of Laminated Plate

From Strain-Displacement Equations and Assumption 2)

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w}{\partial x^2}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = \frac{\partial v_0}{\partial y} - z\frac{\partial^2 w}{\partial y^2}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2z\frac{\partial^2 w}{\partial x \partial y}$$

Write generally as

$$\underset{\sim}{\varepsilon} = \underset{\sim}{\varepsilon}^0 + z\underset{\sim}{\kappa}$$

Where,

$$\varepsilon^0 = \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial y} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial u_0}{\partial x} \end{Bmatrix} \longleftarrow \text{mid-plane strains} = \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix}$$

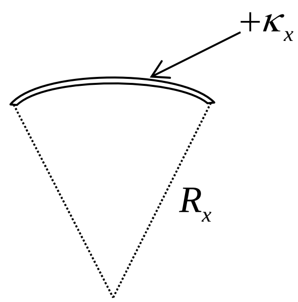
6. Bending of Laminated Plate

$$\underset{\sim}{\kappa} = \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_z \end{Bmatrix} = \begin{Bmatrix} -\dfrac{\partial^2 w}{\partial x^2} \\ -\dfrac{\partial^2 w}{\partial y^2} \\ -2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}$$

"curvature strains"

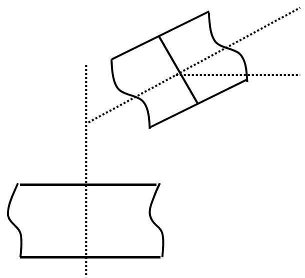
factor of 2

κ_x is curvature $= \dfrac{1}{R_x}$ $\longrightarrow$



For kinematics,

$$\left.\begin{matrix} \gamma_{xz} = 0 \\ \gamma_{yz} = 0 \end{matrix}\right\} \longrightarrow$$ BEN Hypothesis



"No" shear

6. Bending of Laminated Plate

But τ_{xz} and τ_{yz} are present (from equilibrium)

From Stress-Strain Equations and Assumption 1) and 3)

Stress Strain $\longrightarrow$ $\underset{\sim}{\bar{\sigma}} = \underset{\sim}{\bar{Q}}\underset{\sim}{\bar{\varepsilon}} \leftarrow$ laminate axis system

2-D plane stresses

($\sigma_z = 0$)

With assumption 3)

$$\underset{\sim}{N} = \int_{-h/2}^{h/2} \underset{\sim}{\bar{\sigma}} dz = \int \underline{\tilde{Q}} \underset{\sim}{\bar{\varepsilon}} dz$$

$$= \int \underset{\sim}{\bar{Q}} (\underset{\sim}{\varepsilon}^0 + z\underset{\sim}{\kappa}) dz$$

$$\underset{\sim}{M} = \int z \underset{\sim}{\bar{\sigma}} dz = \int z \underset{\sim}{Q} (\underset{\sim}{\varepsilon}^0 + z\underset{\sim}{\kappa}) dz$$

$$\underset{\sim}{N} = \underbrace{\left(\int \underset{\sim}{\bar{Q}} dz \right)}_{=\underset{\sim}{A}} \underset{\sim}{\varepsilon}^0 + \underbrace{\left(\int \underset{\sim}{\bar{Q}} z dz \right)}_{=\underset{\sim}{B}} \kappa$$

$$\underset{\sim}{M} = \underbrace{\left(\int \underset{\sim}{\bar{Q}} z dz \right)}_{=\underset{\sim}{B}} \underset{\sim}{\varepsilon}^0 + \underbrace{\left(\int \underset{\sim}{\bar{Q}} z^2 dz \right)}_{=\underset{\sim}{D}} \kappa$$

6. Bending of Laminated Plate

In matrix form

$$\begin{Bmatrix} \underset{\sim}{N} \\ \underset{\sim}{M} \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \underset{\sim}{\varepsilon}^0 \\ \underset{\sim}{\kappa} \end{Bmatrix}$$

expanded stress resultants - strain relation

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{21} & A_{22} & A_{26} & B_{21} & B_{22} & B_{26} \\ & & A_{66} & B_{61} & B_{62} & B_{66} \\ \hline & & & D_{11} & D_{12} & D_{16} \\ & B^T & & D_{21} & D_{22} & D_{26} \\ & & & D_{61} & D_{62} & D_{66} \end{array}\right] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}$$

symm.

coupling between ε^0 and κ present

Stress-strain equations for laminate become

$$\begin{Bmatrix} \underset{\sim}{N} \\ \underset{\sim}{M} \end{Bmatrix} = \begin{bmatrix} \underset{\sim}{A} & \underset{\sim}{B} \\ \underset{\sim}{B} & \underset{\sim}{D} \end{bmatrix} \begin{Bmatrix} \underset{\sim}{\varepsilon}^0 \\ \underset{\sim}{\kappa} \end{Bmatrix}$$

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{21} & A_{22} & A_{26} & B_{21} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{61} & B_{62} & B_{66} \\ \hline B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{21} & B_{22} & B_{26} & D_{21} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{61} & D_{62} & D_{66} \end{array}\right] \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}$$

$\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{D} \longleftarrow$ All symmetric matrices

$\underset{\sim}{B}$:Couples extension $\underset{\sim}{\varepsilon}^0$ and bending $\underset{\sim}{\kappa}$

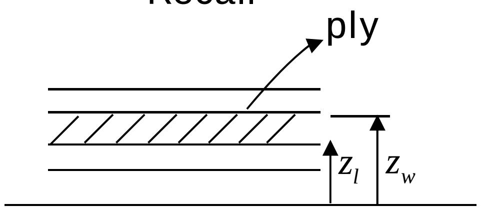
Learn your A, B, D's !

6. Bending of Laminated Plate

Look at $\underset{\sim}{D}$ matrix

Integral form not useful computationally evaluate.

Recall



z_{uk} : upper surface location

z_{lk} : lower surface location

$$\underset{\sim}{D} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \underset{\sim}{\bar{Q}} z^2 dz = \sum_{k=1}^{N} \int_{z_{lk}}^{z_{uk}} \bar{Q}^k z^2 dz$$

constant within a ply

sum on plies

$$= \sum_{k=1}^{N} \underset{\sim}{\bar{Q}}^k \int_{z_{lk}}^{z_{uk}} z^2 dz$$

$$= \frac{1}{3}\left(z_{uk}^3 - z_{lk}^3\right)$$

$$\underset{\sim}{D} = \frac{1}{3} \sum_{k=1}^{N} \underset{\sim}{\bar{Q}}^k \left(z_{uk}^3 - z_{lk}^3\right)$$

Similarly,

$$\underset{\sim}{B} = \frac{1}{2} \sum_{k=1}^{N} \underset{\sim}{\bar{Q}}^k \left(z_{uk}^2 - z_{lk}^2\right)$$

$$\underset{\sim}{A} = \sum_{k=1}^{N} \underset{\sim}{\bar{Q}}^k \left(z_{uk} - z_{lk}\right)$$

6. Bending of Laminated Plate

Example for T300/934, [0/±45/90], laminate (t=0.005" ply)

θ	z_u	z_l	$\frac{z_u^3 - z_l^3}{3}$	$\frac{z_u^2 - z_l^2}{2}$	$z_u - z_l$	$\bar{Q}_{11}$	$\bar{Q}_{22}$	$\bar{Q}_{66}$
	(mils=*10^{-3})		(10^{-9})	(10^{-6})	(10^{-3})	(Msi=10^{6})		
0	20	15	1541	89	5	20.1	1.4	0
45	25	10	792	63	5	6.3	6.3	4.7
-45	10	5	292	38	5	6.3	6.3	-4.7
90	5	0	42	12.5	5	1.4	20.1	0
90	0	-5	42	-12.5	5	1.4	20.1	0
-45	-5	-10	292	-38	5	6.3	6.3	-4.7
45	-10	-15	792	-63	5	6.3	6.3	4.7
0	-15	-20	1541	-89	5	1.4	1.4	0

Let's look at $\underset{\sim}{B}$ terms

Note $\frac{1}{2}\left(z_u^2 - z_l^2\right)$ has a (+) for every $+Q_{11}$ and (-) for every $+Q_{11}$ symm.

$$\underset{\sim}{B} = \sum_{k=1}^{N} \bar{Q}^k \frac{1}{2}\left(z_{uk}^2 - z_{lk}^2\right)$$

sym.

+ upper z

- lower z

$B = 0$ for a symmetric laminate

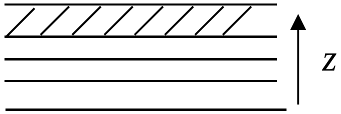
For Anti-symmetric laminate,

$$[0/90/0/90]_T \quad \text{or} \quad [45/45/-45/-15]_T \quad B \neq 0$$

Look at D_{11} term

$$D_{11} > D_{22}$$

Because 0° ply is at large z than for D_{22} , plies out further have more influence (larger z)



$$D_{16} \to Q_{16}^{+45} = -Q_{16}^{-45}$$

but z further out

Although $A_{16} = 0, \quad D_{16} \neq 0$

coupling between bending κ_x and twist κ_{xy}

For this $[0/\pm 45/90]_s$ layup, we get

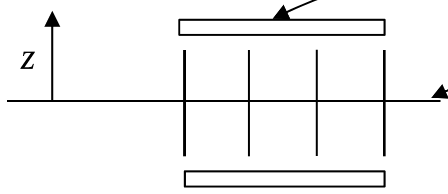
$$\underset{\sim}{A} = \begin{bmatrix} .341 & .106 & 0 \\ .106 & .341 & 0 \\ 0 & 0 & .118 \end{bmatrix} \times 10^6 \text{ (lbs/in)}$$

Note $A_{16}, A_{26} = 0$

$$\underset{\sim}{D} = \begin{bmatrix} 75.8 & 11.9 & 4.68 \\ 11.9 & 19.6 & 4.68 \\ 4.68 & 4.68 & 13.4 \end{bmatrix}$$

small D_{16}, D_{26} present, $D_{22} \neq D_{11}$

Other notes: sandwich construction



Honeycomb boosts value of z, $\frac{1}{3}\left(z_u^3 - z_l^3\right)$ increases greatly.

Can model honeycomb as a thick ply with trivial properties.

❖ Some Laminate Nomenclature

Stiffness Matrix, $\begin{bmatrix} \underset{\sim}{A} & \underset{\sim}{B} \\ \underset{\sim}{B} & \underset{\sim}{D} \end{bmatrix}$

a) Symmetric Cross-ply Laminate (0° and 90°'s only)

$[0/90_2/0_3]_s \rightarrow \underset{\sim}{B} = 0 \qquad A_{16} = A_{26} = 0 \quad D_{16} = D_{26} = 0$

b) Symmetric Balanced Laminate ($-\theta$ for every θ)

$[0/+45/-45]_s \rightarrow \underset{\sim}{B} = 0 \qquad A_{16} = A_{26} = 0$

c) Symmetric Angle-ply Laminate

$[0/+45/-60]_s \rightarrow \underset{\sim}{B} = 0$

d) Anti-symmetric Cross-ply Laminate

$[0_2/90/0/90_2]_T \rightarrow \underset{\sim}{B}$ present, $\quad A_{16} = A_{26} = 0 \quad D_{16} = D_{26} = 0$

e) Anti-symmetric Angle-ply Laminate

$\left[+45/+60/0/90/-60/-45\right]_{\tau} \rightarrow \underset{\sim}{B}$ present, $A_{16} = A_{26} = 0$

$D_{16} = D_{26} = 0$

f) General Laminate

$\left[+45/+60/0\right]_{\tau} \rightarrow$ all may be present

❖ Stiffness in Bending

Same basic procedure

Will look at symmetric laminate

1) Calculate $\underset{\sim}{D} = \frac{1}{3}\sum_{k=1}^{N} \underset{\sim}{\bar{Q}}^{k}\left(z_{uk}^{3} - z_{lk}^{3}\right)$

$\underset{\sim}{M} = \underset{\sim}{D}\underset{\sim}{\kappa}$

2) $\therefore \underset{\sim}{\kappa} = \underset{\sim}{d}\underset{\sim}{M}$, $\underset{\sim}{d} = \underset{\sim}{D}^{-1}$

3) $\underset{\sim}{\bar{\varepsilon}} = \underset{\sim}{\varepsilon}^{0} + \underset{\sim}{\kappa}z$

=0 if symmetric and $\underset{\sim}{N} = 0$

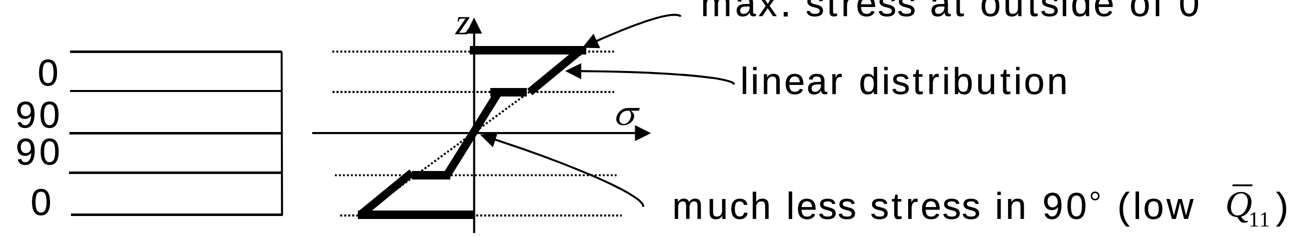
4) $\bar{\sigma}^{k} = \underset{\sim}{\bar{Q}}^{k}\underset{\sim}{\bar{\varepsilon}} = \underset{\sim}{\bar{Q}}^{k}\underset{\sim}{\kappa}z$

depend on ply (k) and position z

6. Bending of Laminated Plate

Simple laminate $[0/90]_s$

Simple load, $M_x = 1$ (in-lb/in)



Usually compute stress either

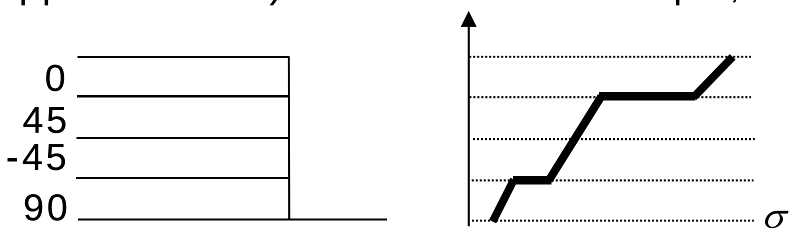
a) at upper and lower surface (more accurate)

b) at mid-plane of ply (approximate)

$$\bar{\sigma}^k = \underset{\sim}{\bar{Q}}^k \underset{\sim}{\bar{\varepsilon}} \approx \underset{\sim}{\bar{Q}}^k \underset{\sim}{\kappa} \underset{\sim}{z}_h$$

mid-line of ply

Approximate b) bad for above example, but often O.K



even 8 plies, not bad.

More plies, and sandwiches. Approximately good

6. Bending of Laminated Plate

Note…. Looking at

$$\underset{\sim}{\bar{\sigma}} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

laminate coordinate

5) If we want ply coordinates, Rotate as before.

$$\underset{\sim}{\sigma}^k = \mathrm{T}^k_{\underset{\sim}{\sigma}} \underset{\sim}{\bar{\sigma}}^k \qquad \underset{\sim}{\sigma} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}$$

ply coordinate

❖ Example Calculation

$[0/\pm45]_s$ T300/934

$$\underset{\sim}{D} = \begin{bmatrix} 36 & 3.9 & 2.3 \\ 3.9 & 6.4 & 2.3 \\ 2.3 & 2.3 & 4.6 \end{bmatrix}$$ lb-in

$$\underset{\sim}{d} = \begin{bmatrix} 0.03 & -0.015 & -0.007 \\ -0.015 & 0.2 & -0.09 \\ -0.07 & -0.09 & 0.27 \end{bmatrix}$$

$$\underset{\sim}{d} = \underset{\sim}{D}^{-1}$$

$$\underset{\sim}{B} = 0$$

6. Bending of Laminated Plate

Apply gentle curvature, $R_x = 100", \kappa_x = 0.01$

Also keep $\kappa_y = 0, \kappa_{xy} = 0$ (Cylindrical Bending)

$$\underset{\sim}{M} = \underset{\sim}{D}\underset{\sim}{\kappa} = \underset{\sim}{D}\begin{Bmatrix} 0.01 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0.36 \\ 0.04 \\ 0.02 \end{Bmatrix} \text{ lb-in/in}$$

M_x cause κ_x

M_y, M_{xy} needed to retrain bending in y direction and twisting

$$\underset{\sim}{\bar{\varepsilon}} = \underset{\sim}{\varepsilon}^0 + z\underset{\sim}{\kappa}$$

=0 since $\underset{\sim}{N} = 0$

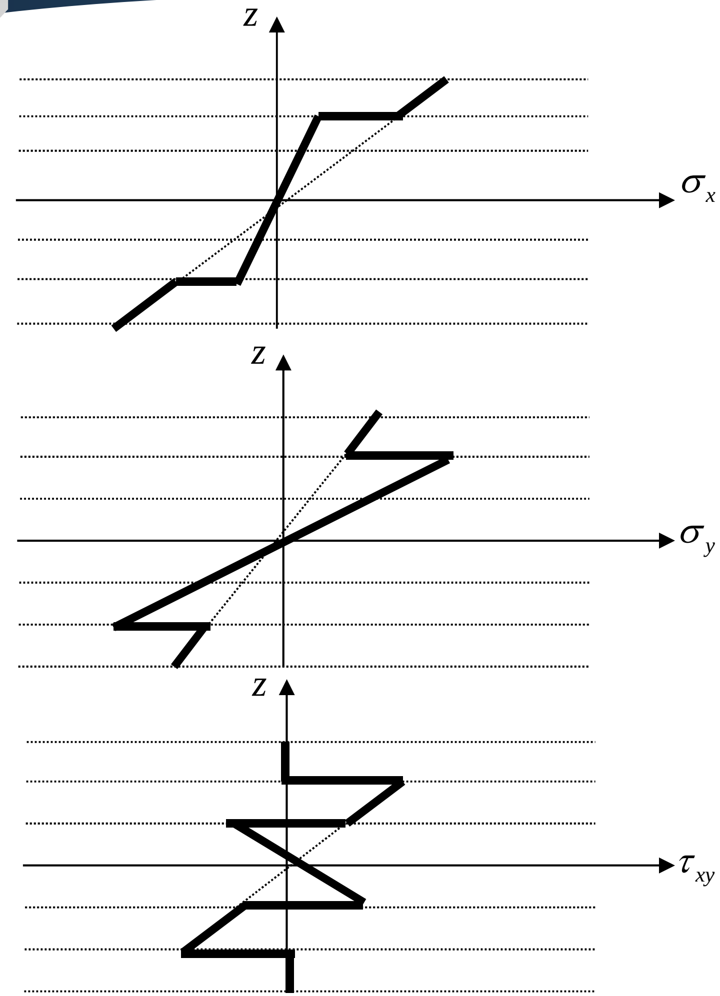
$$\varepsilon_x = 0.01z, \varepsilon_y = 0, \varepsilon_z = 0$$

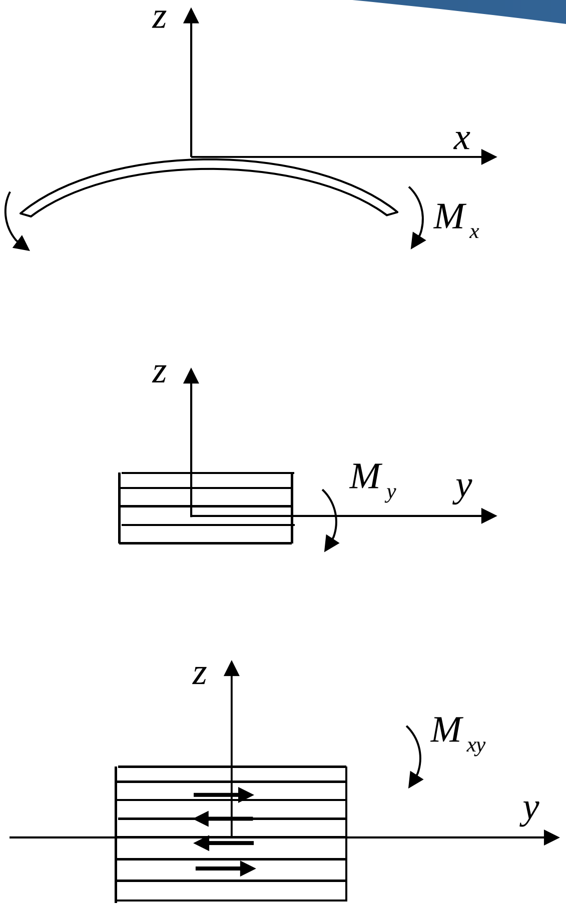
So, $\underset{\sim}{\bar{\sigma}} = \underset{\sim}{\bar{Q}}\underset{\sim}{\bar{\varepsilon}}$

$$\sigma_x = \bar{Q}_{11}\varepsilon_x, \sigma_y = \bar{Q}_{12}\varepsilon_x, \tau_{xy} = \bar{Q}_{16}\varepsilon_x$$

	ply	z	ε_x	σ_x	σ_y	τ_{xy}
		(in)	$(\times 10^{-6})$	(psi)	(psi)	(psi)
1	Top	0.015	150	3018	61	0
	Bottom	0.10	100	2012	41	0
2	Top	0.10	100	628	488	467
	Bottom	0.005	50	314	244	233
3	Top	0.005	50	314	244	-233
	Bottom	0	0	0	0	0

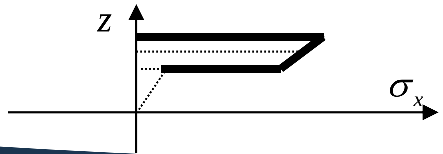
6. Bending of Laminated Plate





oppose the twisting

One last point – for honeycomb



6. Bending of Laminated Plate

❖ General Laminate

Have noted that for a laminate that is not symmetric,

$$\underset{\sim}{N} = \underset{\sim}{A}\underset{\sim}{\varepsilon}^0 + \underset{\sim}{B}\underset{\sim}{\kappa}$$

$$\underset{\sim}{M} = \underset{\sim}{B}\underset{\sim}{\varepsilon}^0 + \underset{\sim}{D}\underset{\sim}{\kappa}$$

where,

$$\underset{\sim}{A} = \sum \underset{\sim}{\bar{Q}}^k \left(z_{uk} - z_{lk} \right)$$

$$\underset{\sim}{B} = \frac{1}{2} \sum \underset{\sim}{\bar{Q}}^k \left(z_{uk}^2 - z_{lk}^2 \right)$$

$$\underset{\sim}{D} = \frac{1}{3} \sum \underset{\sim}{\bar{Q}}^k \left(z_{uk}^3 - z_{lk}^3 \right)$$

or,

$$\begin{bmatrix} \underset{\sim}{N} \\ \underset{\sim}{M} \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} \underset{\sim}{\varepsilon}^0 \\ \underset{\sim}{\kappa} \end{bmatrix}$$

6*6

inverting 6x6 gives,

$$\begin{bmatrix} \underset{\sim}{\varepsilon}^0 \\ \underset{\sim}{\kappa} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{a}^* & \underset{\sim}{b}^* \\ \underset{\sim}{b}^* & \underset{\sim}{d}^* \end{bmatrix} \begin{bmatrix} \underset{\sim}{N} \\ \underset{\sim}{M} \end{bmatrix}$$

In general

$\underset{\sim}{a}^* \neq \underset{\sim}{a} = \underset{\sim}{A}^{-1}$ $\quad \underset{\sim}{d}^* \neq \underset{\sim}{d} = \underset{\sim}{D}^{-1}$ $\quad \underset{\sim}{b}^*$ may not be symmetric coupling between stretching and bending

"Engineering" constants not well defined for general laminate (E_x etc.)

Look at $[0/90]_T$ T300/934

$$A=\begin{bmatrix}108 & 408 & 0\\ 408 & 108 & 0\\ 0 & 0 & 7\end{bmatrix}\times 10^{3}\ \text{lb/in}$$

$$B=\begin{bmatrix}234 & 0 & 0\\ 0 & -234 & 0\\ 0 & 0 & 0\end{bmatrix}\ \text{lb/in}$$

$$\underset{\sim}{D}=\begin{bmatrix}90 & 0.034 & 0\\ 0.034 & 90 & 0\\ 0 & 0 & 0.058\end{bmatrix}\ \text{lb-in}$$

$$\underset{\sim}{a}^{*}=\begin{bmatrix}21.5 & -0.82 & 0\\ -0.82 & 21.5 & 0\\ 0 & 0 & 14\end{bmatrix}\times 10^{-6}\ \text{in/lb}$$

$$\underset{\sim}{b}^{*}=\begin{bmatrix}-5.6 & 0 & 0\\ 0 & 5.6 & 0\\ 0 & 0 & 0\end{bmatrix}\times 10^{-3}\ \text{1/lb}$$

6. Bending of Laminated Plate

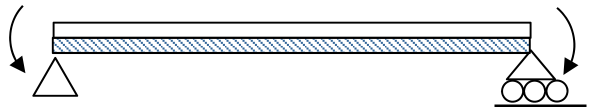
$$\underset{\sim}{d}^* = \begin{bmatrix} 2.58 & -0.98 & 0 \\ -0.98 & 2.58 & 0 \\ 0 & 0 & 17.1 \end{bmatrix} \text{ 1/in-lb}$$

If had inverted A^{-1}, would get

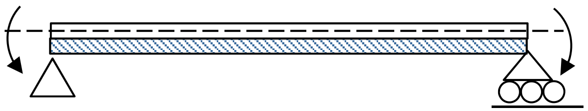
$$\underset{\sim}{a} = \underset{\sim}{A}^{-1} = \begin{bmatrix} 9.35 & -0.36 & 0 \\ -0.36 & 9.35 & 0 \\ 0 & 0 & 143 \end{bmatrix} \times 10^{-6}$$

$$\underset{\sim}{d} = \underset{\sim}{D}^{-1} = \begin{bmatrix} 1.12 & -0.04 & 0 \\ -0.04 & 1.12 & 0 \\ 0 & 0 & 17.1 \end{bmatrix} \longleftarrow \text{ way different}$$

$\underset{\sim}{d}$ is different because $\underset{\sim}{d}$ forces laminate to bend about mid plane (become N.A)



Actually



this results in $\underset{\sim}{d}^*$
Bends about true N.A.

Apply $M_x = 1$ in-lb/in

$$\begin{Bmatrix} \varepsilon \\ \kappa \end{Bmatrix} = \begin{bmatrix} \underset{\sim}{a}^{*} & \underset{\sim}{b}^{*} \\ \underset{\sim}{b}^{*T} & \underset{\sim}{d}^{*} \end{bmatrix} \begin{Bmatrix} N \\ M \end{Bmatrix} \qquad \begin{Bmatrix} \varepsilon_x \\ \vdots \\ \kappa_x \\ \vdots \end{Bmatrix} = \left[\begin{array}{cccc|cc} \cdots & \cdots & \cdots & \cdots & 0.0056 & \cdots \\ \cdots & \cdots & \cdots & \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & 0 & \cdots \\ \hline \cdots & \cdots & \cdots & \cdots & 2.58 & \cdots \\ \cdots & \cdots & \cdots & \cdots & -0.98 & \cdots \\ \cdots & \cdots & \cdots & \cdots & 0 & \cdots \end{array}\right] \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix}$$

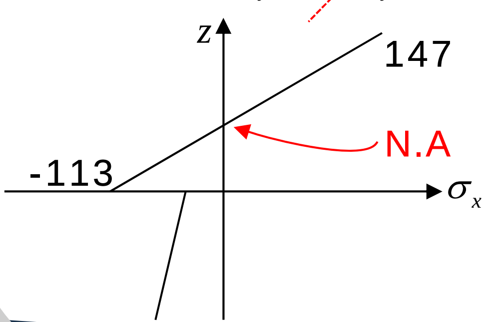
$$\underset{\sim}{\varepsilon} = \begin{Bmatrix} -0.0056 \\ 0 \\ 0 \end{Bmatrix} \qquad \underset{\sim}{\kappa} = \begin{Bmatrix} 2.58 \\ -0.098 \\ 0 \end{Bmatrix}$$

$$\bar{\varepsilon}_x = \varepsilon_x^0 + \kappa_x z = -0.0056 + 2.58z$$

$$\bar{\varepsilon}_y = 0 - 0.098z$$

negligible

$$\sigma_x = \bar{Q}_{11}\bar{\varepsilon}_x + \bar{Q}_{12}\bar{\varepsilon}_y + \bar{Q}_{16}\bar{\gamma}_{xy}$$



ply		$\bar{\varepsilon}_x$	$\bar{Q}_{11}$	$\bar{\sigma}_x$
0	Top	0.073	20.1	147
	Bot	-0.0056		-113
90	Top	-0.0056	1.41	-7.9
	Bot	-0.013		-18

Neutral axis is not z=0

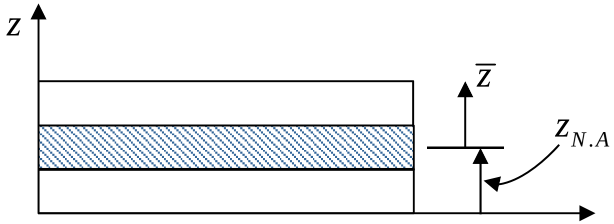
❖ Concept of Neutral Axis

Consider only κ direction

Find $z_{N.A.}$ to make B_{11} term equal to zero

$$N_{11} = A_{11}\varepsilon_x + \cancel{B_{11}}\kappa_x$$
$$M_{11} = \cancel{B_{11}}\varepsilon_x + D_{11}\kappa_x$$

make these uncoupled



$$\varepsilon_x = \varepsilon^0 + \bar{z}\kappa_x$$

About $z_{N.A.}$, $\bar{B}_{11} = \int \bar{Q}_{11}^k \bar{z} dz = \int \bar{Q}^k (z - z_{NA}) dz$

$$= \underbrace{\int Q_{11}^k z dz}_{B_{11}} - z_{NA} \underbrace{\int \bar{Q}_{11}^k dz}_{A_{11}}$$

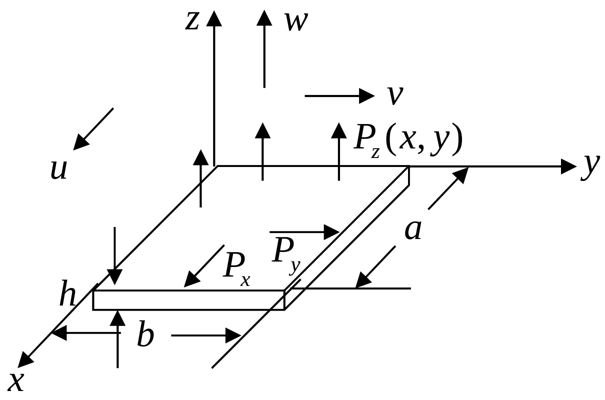
For $\bar{B}_{11} = 0$ require $z_{N.A} = \dfrac{B_{11}}{A_{11}}$ $\longleftarrow$ location of N.A (modulus weighted C.G)

For y direction, $z_{N.A} = \dfrac{B_{22}}{A_{22}}$ $\longleftarrow$ not generally the same

6. Bending of Laminated Plate

❖ Basic Equations of Plate Theory

Plate under loading p_x, p_y, p_z (lb/in^2) h<<a, b



3 Basic Assumptions

1) $\sigma_z = 0$

2) B.E.N hypothesis

3) Use stress resultants

Placing these assumptions into the Theory of Elasticity,
the Strain-Displacement Equations become

$$\bar{\underset{\sim}{\varepsilon}} = \underset{\sim}{\varepsilon}^0 + z\underset{\sim}{\kappa}$$

where,

$$\underset{\sim}{\varepsilon}^0 = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial y} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{Bmatrix} \longleftarrow \text{mid plane strains}$$

$$\underset{\sim}{\kappa} = \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = \begin{Bmatrix} -\dfrac{\partial^2 w_0}{\partial x^2} \\ -\dfrac{\partial^2 w_0}{\partial y} \\ -2\dfrac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix} \longleftarrow \text{curvature strains}$$

6. Bending of Laminated Plate

Similarly, the stress-strain Equations become

$$\begin{Bmatrix} \underset{\sim}{N} \\ \underset{\sim}{M} \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \underset{\sim}{\varepsilon}^0 \\ \underset{\sim}{\kappa} \end{Bmatrix}$$

where

$$\underset{\sim}{N} = \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \int_{-h/2}^{h/2} \underset{\sim}{\bar{\sigma}} dz$$ → force resultants

$$\underset{\sim}{M} = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \int_{-h/2}^{h/2} z \underset{\sim}{\bar{\sigma}} dz$$ → moment resultants

To complete the plate Theory formulation, go to the Equation of Equilibrium and Apply Assumption 3)

Equations of Equilibrium

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + f_x = 0 \qquad \text{(E -1)}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + f_y = 0 \qquad \text{(E -2)}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z = 0 \qquad \text{(E -3)}$$

Integrating $\int_{-h/2}^{h/2} dz,$ Equation (E -1) gives

$$\underbrace{\int_{-h/2}^{h/2} \frac{\partial \sigma_x}{\partial x} dz}_{=\frac{\partial N_x}{\partial x}} + \underbrace{\int_{-h/2}^{h/2} \frac{\partial \tau_{yz}}{\partial y} dz}_{=\frac{\partial N_{xy}}{\partial y}} + \underbrace{\int_{-h/2}^{h/2} \frac{\partial \tau_{zx}}{\partial z} dz}_{=\left|\tau_{zx}\right|_{-h/2}^{h/2}} + \int f_x dz = 0$$

$$\underbrace{\left|\tau_{zx}\right|_{-h/2}^{h/2} + \int f_x dz}_{= p_x (\text{lb/ in}^2)}$$

Total applied load (surface + gravity)

Similarly for (E -2), so obtain

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + p_x = 0 \qquad \text{(E -4a)}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} + p_y = 0 \qquad \text{(E -4b)}$$

Integrating $\int_{-h/2}^{h/2} dz,$ Equation (E -3) gives

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \underbrace{\left|\sigma_z\right|_{-h/2}^{h/2} + \int f_z dz}_{= p_z \text{ (surface + gravity)}} = 0$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p_z = 0 \qquad \text{(E -4c)}$$

6. Bending of Laminated Plate

Multiplying (E -1) by z, then taking $\int dz$

$$\underbrace{\int z\frac{\partial\sigma_x}{\partial x}dz}_{=\frac{\partial M_x}{\partial x}} + \underbrace{\int z\frac{\partial\tau_{yx}}{\partial y}dz}_{=\frac{\partial M_{xy}}{\partial y}} + \underbrace{\int z\frac{\partial\tau_{zx}}{\partial z}dz}_{\left|z\tau_{zx}\right|_{-h/2}^{h/2} - \int\tau_{zx}dz} + \underbrace{\int zf_x dz}_{=\left|f_x\frac{z^2}{2}\right|_{-h/2}^{h/2}} = 0$$

$$\left|z\tau_{zx}\right|_{-h/2}^{h/2} = 0, \qquad \int\tau_{zx}dz = Q_x$$

f_x = const. through plate

Similarly for (E -2), so obtain

$$\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_x=0 \qquad \text{(E -4d)}$$

$$\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}-Q_y=0 \qquad \text{(E -4e)}$$

Reduction of Plate Equations

Have 17 Equations, and 17 unknowns

6 strain-Displacement	$N_x.............., M_x........ \rightarrow 6$
6 Stress-strain	$Q_x, Q_y........ \rightarrow 2$
5 Equilibrium	$\varepsilon_x.............., \kappa_x........ \rightarrow 6$
	$u_0, v_0, w_0 \rightarrow 3$
	17

6. Bending of Laminated Plate

To reduce, first place strain-displacement into stress-strain

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} \underset{\sim}{A} & \underset{\sim}{B} \\ \underset{\sim}{B} & \underset{\sim}{D} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \\ -\dfrac{\partial^2 w}{\partial x^2} \\ -\dfrac{\partial^2 w}{\partial y^2} \\ -2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}$$

For example

$$N_x = A_{11}\frac{\partial u}{\partial x} + A_{12}\frac{\partial v}{\partial y} + A_{16}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - B_{11}\frac{\partial^2 w}{\partial x^2} - B_{12}\frac{\partial^2 w}{\partial y^2} - 2B_{16}\frac{\partial^2 w}{\partial x \partial y}$$

$$N_y = etc.$$

Note: Have replaced mid-plane displacement u_0, v_0, w_0 by u, v, w for writing convenience

6. Bending of Laminated Plate

Next, the 5 equilibrium equations can be reduced to 3 by placing Q_x and Q_y from (E-4d) and (E-4e) into (E-4c). This gives,

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = -p_x$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = -p_y$$

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -p_z$$

Finally placing N_x, N_y.....into 3 Equilibrium Equations gives 3 Equations

$$A_{11}\frac{\partial^2 u}{\partial x^2} + \cdots\cdots - B_{11}\frac{\partial^2 w}{\partial x^2} = -p_x$$

$$A_{16}\frac{\partial^2 u}{\partial x^2} + \cdots\cdots - B_{16}\frac{\partial^2 w}{\partial x^2} = -p_y$$

$$D_{11}\frac{\partial^4 w}{\partial x^4} + \left(2D_{12} + 4D_{66}\right)\frac{\partial^4 w}{\partial x^2 \partial y^2}\cdots\cdots - B_{11}\frac{\partial^3 u}{\partial y^3} + \cdots = p_z$$

3 basic Equations in u, v, w

6. Bending of Laminated Plate

See Jonse, chap. 5

" Classical Laminated Plate Theory" (CLPT)

For general unsymmetric plates, must treat u, v, w equations together.

For mid-plane symmetric plate $\longrightarrow B_{ij} = 0$

$$A_{11}\frac{\partial^2 u}{\partial x^2} + \cdots\cdots = -p_x$$

$$A_{16}\frac{\partial^2 u}{\partial x^2} + \cdots\cdots = -p_y$$

"stretching" u, v

$$D_{11}\frac{\partial^4 w}{\partial x^4} + \left(2D_{12} + 4D_{66}\right)\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4} + 4D_{16}\frac{\partial^4 w}{\partial x^3 \partial y} + 4D_{26}\frac{\partial^4 w}{\partial x \partial y^3} = p_z$$

"bending" w

Solve bending equations separately for w

For special orthotropic lay-ups $\longrightarrow D_{16} = D_{26} = 0$

For isotropic plate

$$D_{11} = D_{22} = \left(D_{12} + 2D_{66}\right) = D \longrightarrow D\nabla^4 w = p_z$$

Recall: $A_{66} = \frac{1}{2}\left(A_{11} - A_{12}\right)$

$$\frac{Eh^3}{12(1-\nu^2)}$$

6. Bending of Laminated Plate

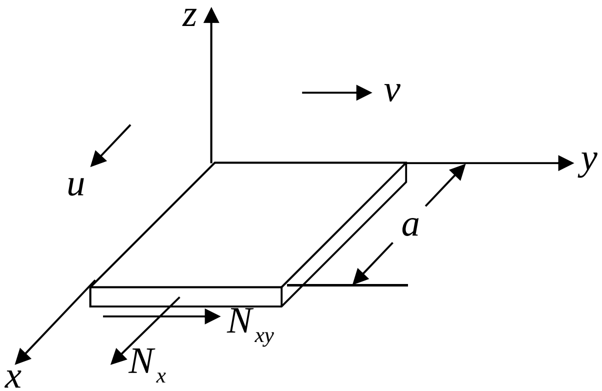
Look briefly at boundary condition

For plate stretching along: Have 2 B.C's at each end =0

force displacement

At $x=a$, either N_x or u prescribed

either N_{xy} or v prescribed



For forces can express in terms of u,v

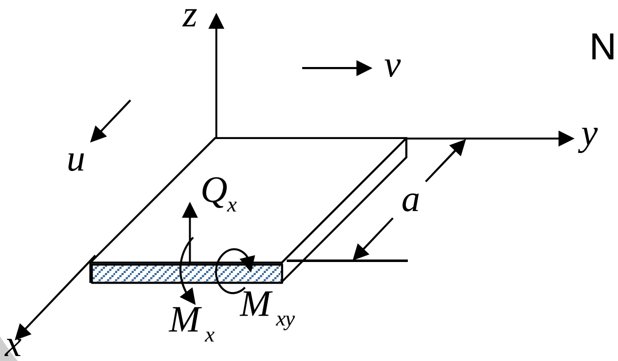
$$\{N_x\} = \underset{\sim}{A}\begin{Bmatrix} \frac{\partial u}{\partial x} \\ \vdots \end{Bmatrix} \longrightarrow N_x = A_{11}\frac{\partial u}{\partial x} + A_{12}\frac{\partial v}{\partial y} + \cdots$$

$$N_{xy} = A_{16}\frac{\partial u}{\partial x}\cdots$$

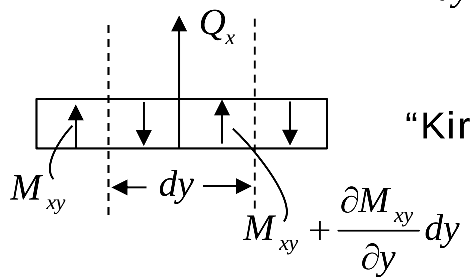
Any combinations N_x, N_{xy} $\quad u, N_{xy}$

N_x, v $\quad u, v$

For plate bending alone, have 2 B.C's at each end



Note: vertical force $V_x = Q_x + \frac{\partial M_{xy}}{\partial y}$



"Kirchhoff" condition

6. Bending of Laminated Plate

At $x=a$, either V_x or w prescribed

either M_x or $\frac{\partial w}{\partial x}$ prescribed

For moments, express in terms of w by

$$\begin{Bmatrix} M_x \\ \vdots \end{Bmatrix} = \underset{\sim}{D} \begin{Bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ \vdots \end{Bmatrix} \longrightarrow \begin{aligned} M_x &= -D_{11}\frac{\partial^2 w}{\partial x^2} - D_{12}\frac{\partial^2 w}{\partial y^2} + \cdots \\ V_x &= Q_x + \frac{\partial M_{xy}}{\partial y} = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} + \frac{\partial M_{xy}}{\partial y} = D_{11}\frac{\partial^3 w}{\partial x^3} + \cdots \end{aligned}$$

Again combinations w, M_x .... simply supported

$w, \frac{\partial w}{\partial x}$ .... clamped

V_x, M_x .... free

$V_x, \frac{\partial w}{\partial x}$

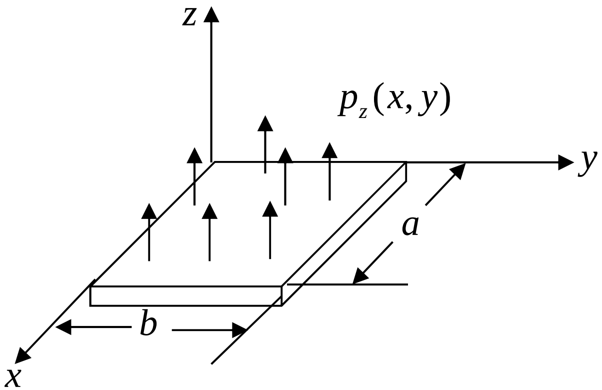
For complete equations u,v,w together

Go to $\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} \underset{\sim}{A} & \underset{\sim}{B} \\ \underset{\sim}{B} & \underset{\sim}{D} \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \vdots \\ -\frac{\partial^2 w}{\partial x^2} \\ \vdots \end{Bmatrix}$ ← all coupled together

❖ Bending of a Loaded Rectangular Plate

Consider specially orthotropic (cross-ply, symmetric)

Hence $D_{16}, D_{26} = 0$



uniform plate, all edges simply supported

Governing D.E is

$$D_{11}\frac{\partial^4 w}{\partial x^4} + \left(2D_{12} + 4D_{66}\right)\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4} = p_z \quad (1)$$

B.C's are at $x = 0, \mathrm{a} \rightarrow \begin{cases} w = 0 \\ M_x = -D_{11}\dfrac{\partial^2 w}{\partial x^2} - D_{12}\dfrac{\partial^2 w}{\partial y^2} = 0 \end{cases}$

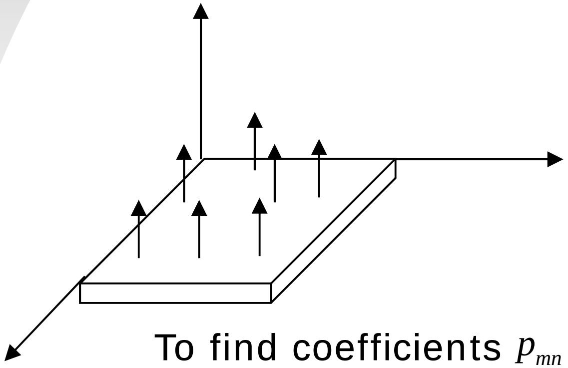
$$y = 0, \mathrm{b} \rightarrow \begin{cases} w = 0 \\ M_y = -D_{12}\dfrac{\partial^2 w}{\partial x^2} - D_{22}\dfrac{\partial^2 w}{\partial y^2} = 0 \end{cases}$$

Assume a solution, $w = \sum_{m=1}\sum_{n=1} a_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$ ……………………………………. (2)

6. Bending of Laminated Plate

Each term satisfies B.C's on all edges.

Try to satisfy the D.E. as well. (Navier solution)



$$D_{11}\frac{\partial^4 w}{\partial x^4}+\left(2D_{12}+4D_{66}\right)\frac{\partial^4 w}{\partial x^2\partial y^2}+D_{22}\frac{\partial^4 w}{\partial y^4}=p_z$$

$$w=\sum_{m=1}\sum_{n=1}a_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

First, expand $p_z(\mathrm{x},\mathrm{y})$ in a Fourier series.

To find coefficients p_{mn}, multiply p_z by $\sin\frac{k\pi x}{a}\sin\frac{l\pi y}{b}$, and integrate $\iint dxdy$

$$\iint p_z(\mathrm{x},\mathrm{y})\sin\frac{k\pi x}{a}\sin\frac{l\pi y}{b}dxdy$$

$$=\iint\sum_m\sum_n p_{mn}p_z\sin\frac{k\pi x}{a}\sin\frac{m\pi x}{a}\sin\frac{l\pi y}{b}\sin\frac{n\pi y}{b}dxdy$$

Then

$$\int_0^a\sin\frac{k\pi x}{a}\sin\frac{m\pi x}{a}dx=\begin{cases}\frac{a}{2} & if \quad m=k\\ 0 & if \quad m\neq k\end{cases}$$

Similarly for $\int_0^b\cdots dy$

$$\iint p_z\sin\frac{k\pi x}{a}\sin\frac{l\pi y}{b}dxdy=p_{kl}\frac{ab}{4}$$ or $$p_{mn}=\frac{4}{ab}\int_0^b\int_0^a p_z(\mathrm{x},\mathrm{y})\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy$$

$$p_z(x,y)=\sum_m\sum_m p_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

6. Bending of Laminated Plate

Now, placing $p_z(\mathrm{x,y})$ and w into differential equation

$$\sum_m \sum_n \left\{ a_{mn} \left[D_{11}\left(\frac{m\pi}{a}\right)^4 + (2D_{12}+4D_{66})\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + D_{22}\left(\frac{n\pi}{b}\right)^4 \right] - p_{mn} \right\} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = 0$$

This D.E. is satisfied if { }=0. i.e.

$$a_{mn} = \frac{p_{mn}}{\left[D_{11}\frac{\partial^4 w}{\partial x^4} + (2D_{12}+4D_{66})\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + D_{22}\left(\frac{n\pi}{b}\right)^4 \right]}$$

Solution is then

$$w = \sum_m \sum_n a_{mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

"Navier solution for s.s. plate"

$$M_x = -D_{11}\frac{\partial^2 w}{\partial x^2} - D_{12}\frac{\partial^2 w}{\partial y^2} = \sum_m \sum_n M_{x,mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$M_y = \sum_m \sum_n M_{y,mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$M_{xy} = \sum_m \sum_n M_{xy,mn} \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

Take many terms, converges fast on w $(1/m^4)$

converges slower on M $(1/m^2)$

Then find ply stresses $\underset{\sim}{\bar{\sigma}}$ from $\underset{\sim}{M}$ and laminate stacking, as before.

Apply to uniform load $p_z(x, y) = p_0 \; \left(lbs / in^2\right)$

$$p_{mn} = \frac{4p_0}{ab}\int\int \sin\frac{m\pi x}{a}\sin\frac{n\pi x}{b}dxdy$$

$$= \frac{4p_0}{ab}\left[-\frac{\cos\frac{m\pi x}{a}}{\frac{m\pi}{a}}\right]_0^a \left[-\frac{\cos\frac{b\pi x}{b}}{\frac{n\pi}{b}}\right]_0^b = \frac{16p_0}{\pi^2 mn}$$

$m, n = 1, 3, 5, 7, \cdots$

$$w = \frac{16p_0}{\pi^6}\sum_m\sum_n \frac{\sin\frac{m\pi x}{a}\sin\frac{n\pi x}{b}}{mn\left[D_{11}\left(\frac{m\pi}{a}\right)^4 + \left(2D_{12} + 4D_{66}\right)\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + D_{22}\left(\frac{n\pi}{b}\right)^4\right]}$$

$M_x, M_y, M_{xy} = etc.$

See Jones book Sec. 5.3

For mid-plane symmetric plate, $D_{16}, D_{26} \neq 0$, Above exact solution not possible .

$$M_x = -D_{11}\frac{\partial^2 w}{\partial x^2} - D_{12}\frac{\partial^2 w}{\partial y^2} - D_{16}\frac{\partial^2 w}{\partial x \partial y}$$

causes trouble

Simple sine solution doesn't satisfy B.C

6. Bending of Laminated Plate

Must obtain approximate solutions by

a) Energy method $\longrightarrow$ Rayleigh-Ritz

b) Numerical methods $\longrightarrow$ F.E.M

Jones book Sec.5.3 gives some results for angle-ply symmetric and anti-symmetric laminates.

❖ Vibration of a Plate – Jones Sec.5.5

Here $p_z = -\rho h \ddot{w}$

ρ : density

$\rho h : mass / in^2$

For specially orthotropic (symmetric, $D_{16} = D_{26} = 0$)

Differential equation is

$$D_{11}\frac{\partial^4 w}{\partial x^4} + \left(2D_{12} + 4D_{66}\right)\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4} = -\rho h \ddot{w}$$

For s.s. boundary condition, and assume harmonic motion, a relation is

$$w = \bar{w} e^{i\omega t} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

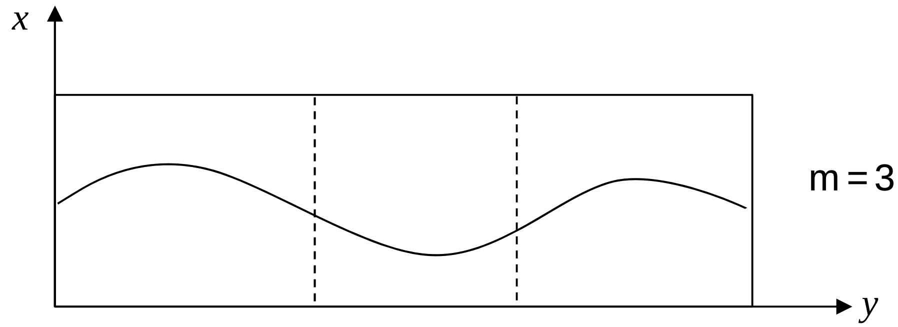
satisfies B.C's

6. Bending of Laminated Plate

placing into diff. eqn. gives

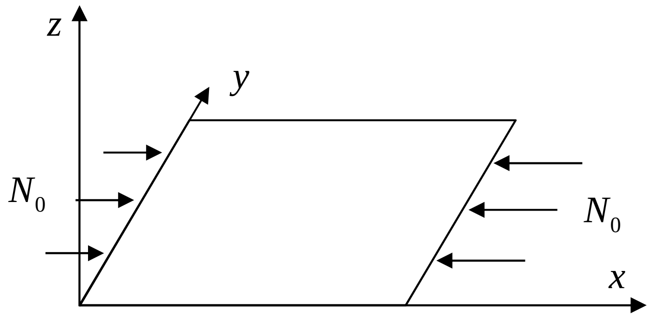
$$\omega^2 = \frac{\pi^4}{\rho h}\left[D_{11}\left(\frac{m}{a}\right)^4 + \left(2D_{12}+4D_{66}\right)\left(\frac{m}{a}\right)^2\left(\frac{n}{b}\right)^2 + D_{22}\left(\frac{n}{b}\right)^4 \right]$$

frequency for a given vibration mode m, n



m=3, n=1

Also buckling of a plate – See Jones Sec.5.4



Here, $p_z = 0$, add $-N_0 \dfrac{\partial^2 w}{\partial x^2}$ to R.H.S

For s.s. boundary condition

$$N_0 = \pi^2\left[D_{11}\left(\frac{m}{a}\right)^2 + \left(2D_{12}+4D_{66}\right)\left(\frac{n}{b}\right)^2 + D_{22}\left(\frac{n}{b}\right)^4\left(\frac{m}{a}\right)^2 \right]$$

Find *m*, *n* to give minimum N_0

$n = 1$

m depends on *a*/*b* ratio

See Jones Sec.5.4