CHAPTER 7.
Thermal Stresses and Deformation

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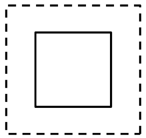
7. Thermal stresses and deformation

Will look at

- free expansion of a ply
- constraint and thermal stress
- rotation of plies
- laminate and effective properties
- stresses and deformation

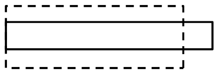
Consider a body changing temperature



$$\varepsilon = \alpha \Delta T = \alpha (\mathrm{T} - \mathrm{T}_0)$$

coefficient of thermal expansion, *CTE*

ref. temperature



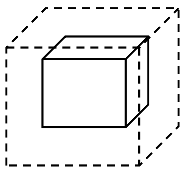
Fiber : Anisotropic CTE

$$\alpha_L \simeq -.5 \mu\varepsilon / {}^o F$$

$$\alpha_T \ \text{small positive} \simeq 2 \sim 3 \mu\varepsilon / {}^o F$$

7. Thermal stresses and deformation

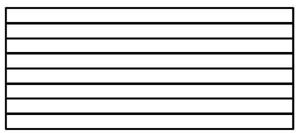
Matrix isotropic



$\alpha \simeq 20$ to $30 \mu\varepsilon / {}^{o}F$

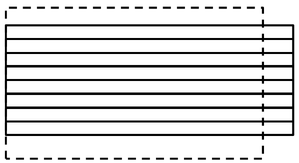
ply? Did micromechanics

$\rightarrow$ ply equivalent properties



Also microstresses between fiber and matrix some will ignore these here

Ply Properties (G_r / E_P material)



$$\alpha_L \simeq -1.0 \;\; to \;\; +5 \mu\varepsilon / {}^{o}F$$

$$\alpha_T \simeq 16 \mu\varepsilon / {}^{o}F$$

7. Thermal stresses and deformation

Consider In-plane Thermal strains

$$\underset{\sim}{\varepsilon}^T = \underset{\sim}{\alpha}\,\Delta T$$

$$\underset{\sim}{\varepsilon}^T = \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} \qquad \underset{\sim}{\alpha} = \begin{Bmatrix} \alpha_L \\ \alpha_T \\ 0 \end{Bmatrix}$$

No stress!!

Before had $\underset{\sim}{\sigma} = \underset{\sim}{Q}\,\underset{\sim}{\varepsilon}$

Now need new constitutive law

To modify, note

$$\underset{\sim}{\varepsilon} = \underset{\sim}{\varepsilon}^M + \underset{\sim}{\varepsilon}^T$$

total strain "real" $\simeq \frac{\Delta l}{l}$ — mechanical — thermal

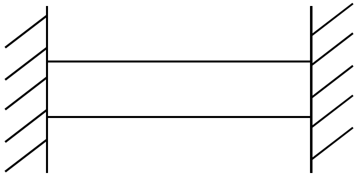
$\underset{\sim}{\varepsilon}^M = \underset{\sim}{S}\,\underset{\sim}{\sigma}$ mechanical stress- strain

$$\underset{\sim}{\varepsilon} = \underset{\sim}{S}\,\underset{\sim}{\sigma} + \underset{\sim}{\varepsilon}^T = \underset{\sim}{S}\,\underset{\sim}{\sigma} + \underset{\sim}{\alpha}\Delta T$$

or $\underset{\sim}{\sigma} = \underset{\sim}{Q}\{\underset{\sim}{\varepsilon} - \underset{\sim}{\alpha}\,\Delta T\}$

[Thermoelastic stress –strain laminate]

What if constrained ?



$$\underset{\sim}{\varepsilon} = 0$$

$$\underset{\sim}{\varepsilon} = 0 = \underset{\sim}{S}\,\underset{\sim}{\sigma} + \underset{\sim}{\alpha}\,\Delta T$$

$$\underset{\sim}{S}\underset{\sim}{\sigma} = -\underset{\sim}{\alpha}\,\Delta T$$

$$\underset{\sim}{\sigma} = -\underset{\sim}{Q}\,\underset{\sim}{\alpha}\,\Delta T$$

In 1-Dim, $\bar{\sigma}_x = -E\underset{\sim}{\alpha}\,\Delta T$

The $\underline{\sigma}$ obtained is called "Thermal Stress"

Actually, this is a mis-usage of the terminology.

Thermal strain O.K

7. Thermal stresses and deformation

Thermal stresses caused by mechanical forces due to constraints,
Also one defines “equivalent thermal stress”

$$\underset{\sim}{\sigma}^{T} = +\underset{\sim}{Q}\,\underset{\sim}{\alpha}\,\Delta T$$

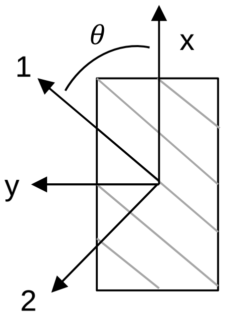
This is a fictitious but is computationally useful.

$$\bar{\underset{\sim}{\sigma}} = \underset{\sim}{\sigma}^{m} + \underset{\sim}{\sigma}^{T}$$

$$\underset{\sim}{\varepsilon} = \underset{\sim}{S}\underset{\sim}{\sigma} = \underset{\sim}{S}\underset{\sim}{\sigma}^{m} + \underbrace{\underset{\sim}{S}\underset{\sim}{Q}}_{I}\underset{\sim}{\alpha}\Delta T$$

↖ Allows one to use old constitutive law with “fictitious” thermal stress

❖ Ply at Arbitrary Angle



$$\bar{\underset{\sim}{\varepsilon}} = \underset{\sim\varepsilon}{T}^{-1} \underset{\sim}{\varepsilon}$$: strain transformation

(lamin) (ply)

$$\bar{\underset{\sim}{\alpha}} \Delta T = \underset{\sim\varepsilon}{T}^{-1} \underset{\sim}{\alpha} \Delta T$$

$$\boxed{\bar{\underset{\sim}{\alpha}} = \underset{\sim\varepsilon}{T}^{-1} \underset{\sim}{\alpha}}$$ ← CTE in laminate axes

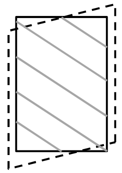
where,

$$\underset{\sim\varepsilon}{T}^{-1} = \underset{\sim\sigma}{T}^{T} = \begin{bmatrix} c^2 & s^2 & -cs \\ s^2 & c^2 & cs \\ 2cs & -2cs & (c^2 - s^2) \end{bmatrix}$$

In general,

$$\bar{\underset{\sim}{\alpha}} = \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}$$

Can get shear



← α_{xy}

7. Thermal stresses and deformation

❖ Laminate Thermal Properties

Have for a single ply

$$\underset{\sim}{\varepsilon} = \underset{\sim}{\varepsilon}^m + \underset{\sim}{\varepsilon}^T = \underset{\sim}{S}\underset{\sim}{\sigma} + \underset{\sim}{\alpha}\Delta T \qquad \text{(ply words)}$$

$$\underset{\sim}{\bar{\varepsilon}} = \underset{\sim}{\varepsilon}^0 + \underset{\sim}{\kappa}\, z = \underset{\sim}{\bar{S}}\underset{\sim}{\bar{\sigma}} + \underset{\sim}{\bar{\alpha}}\Delta T \quad \text{(laminate coordinate)}$$

For laminate, want force and moment resultants,

$$\underset{\sim}{N} = \int \underset{\sim}{\bar{\sigma}}\, dz \quad , \underset{\sim}{M} = \int \underset{\sim}{\bar{\sigma}}\, z\, dz$$

Rewriting stress-strain, get

$$\underset{\sim}{\bar{\sigma}} = \underset{\sim}{\bar{Q}}(\underset{\sim}{\varepsilon}^0 + \underset{\sim}{\kappa}\, z - \underset{\sim}{\bar{\alpha}}\,\Delta T)$$

$$\underset{\sim}{N} = \int \underset{\sim}{\bar{\sigma}} dz = \underbrace{(\int \underset{\sim}{\bar{Q}}\, dz)}_{\underline{A}}\underset{\sim}{\varepsilon}^0 + \underbrace{(\int \underset{\sim}{\bar{Q}} z\, dz)}_{\underline{B}}\underset{\sim}{\kappa} - \underbrace{\int \underset{\sim}{\bar{Q}}\, \underset{\sim}{\bar{\alpha}}\, \Delta T dz}_{\underline{N}^T}$$

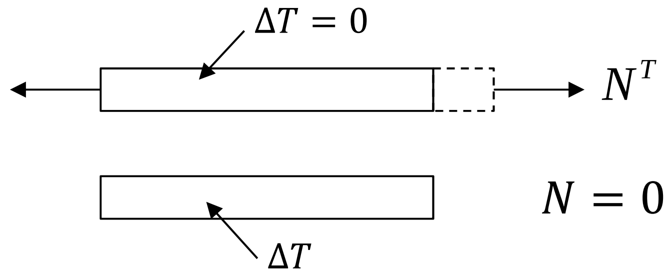
Last is what we call “Thermal Force”

$$\underset{\sim}{N}^T = \int \underset{\sim}{\bar{Q}}\, \underset{\sim}{\bar{\alpha}}\, \Delta T dz$$

$$\begin{pmatrix} fake \\ useful\ quantity \end{pmatrix}$$

7. Thermal stresses and deformation

$\underset{\sim}{N}^T$ is not a physical load , it is a convenience



N^T is mechanical load necessary to provide same deformation in laminate as ΔT with no *N*.

For “thermal stresses”

$$\underset{\sim}{\bar{\sigma}} = \underset{\sim}{\bar{Q}}(\underset{\sim}{\varepsilon}^0 + \underset{\sim}{\kappa} z - \underset{\sim}{\bar{\alpha}} \Delta T)$$

$$\underset{\sim}{N} = \int \underset{\sim}{\bar{\sigma}} dz = \underset{\sim}{A}\underset{\sim}{\varepsilon}^0 + \underset{\sim}{B}\underset{\sim}{\kappa} - N^T$$

Likewise,

$$\underset{\sim}{M} = \int \underset{\sim}{\bar{\sigma}} z dz = \underset{\sim}{B}\underset{\sim}{\varepsilon}^0 + \underset{\sim}{D}\underset{\sim}{\kappa} - \underbrace{\int \underset{\sim}{\bar{Q}}\, \underset{\sim}{\bar{\alpha}}\, \Delta T z dz}_{\underset{\sim}{M}^T}$$

7. Thermal stresses and deformation

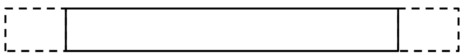
Combining,

$$\begin{Bmatrix} \underset{\sim}{N} + \underset{\sim}{N}^T \\ \underset{\sim}{M} + \underset{\sim}{M}^T \end{Bmatrix} = \begin{bmatrix} \underset{\sim}{A} & \underset{\sim}{B} \\ \underset{\sim}{B} & \underset{\sim}{D} \end{bmatrix} \begin{Bmatrix} \underset{\sim}{\varepsilon}^0 \\ \underset{\sim}{\kappa} \end{Bmatrix}$$

or

$$\begin{Bmatrix} \underset{\sim}{\varepsilon}^0 \\ \underset{\sim}{\kappa} \end{Bmatrix} = \begin{bmatrix} \underset{\sim}{a} & \underset{\sim}{b} \\ \underset{\sim}{b} & \underset{\sim}{d} \end{bmatrix} \begin{Bmatrix} \underset{\sim}{N} + \underset{\sim}{N}^T \\ \underset{\sim}{M} + \underset{\sim}{M}^T \end{Bmatrix}$$

If laminate is unloaded – free thermal deformation



One step up from single ply case

Plies may have stresses, but $\underset{\sim}{N} = \underset{\sim}{M} = 0$

$$\begin{Bmatrix} \underset{\sim}{\varepsilon}^0 \\ \underset{\sim}{\kappa} \end{Bmatrix} = \begin{bmatrix} \underset{\sim}{a} & \underset{\sim}{b} \\ \underset{\sim}{b}^T & \underset{\sim}{d} \end{bmatrix} \begin{Bmatrix} \underset{\sim}{N}^T \\ \underset{\sim}{M}^T \end{Bmatrix}$$

If ΔT constant with z

$$= \begin{bmatrix} \underset{\sim}{a} & \underset{\sim}{b} \\ \underset{\sim}{b} & \underset{\sim}{d} \end{bmatrix} \begin{Bmatrix} \int \overline{\underset{\sim}{Q}}\, \overline{\underset{\sim}{\alpha}}\, dz \\ \int \overline{\underset{\sim}{Q}}\, \overline{\underset{\sim}{\alpha}}\, z\, dz \end{Bmatrix} \Delta T$$

7. Thermal stresses and deformation

In symmetric case, $\underset{\sim}{b} = 0,\ \underset{\sim}{M}^{T} = 0$

$$\begin{Bmatrix} \varepsilon_x{}^0 \\ \varepsilon_y{}^0 \\ \varepsilon_z{}^0 \end{Bmatrix} = \underbrace{\underline{a} \int \underset{\sim}{\bar{Q}}\, \underset{\sim}{\bar{\alpha}}\, dz\, \Delta T}_{\begin{Bmatrix} \delta_x \\ \delta_y \\ \delta_z \end{Bmatrix}}$$

$\begin{Bmatrix} \delta_x \\ \delta_y \\ \delta_z \end{Bmatrix}$ ← Engineering CTE of laminate

single ply case → $\underset{\sim}{\varepsilon}^{0} = \underset{\sim}{\bar{\alpha}}\, \Delta T$

This is stiffness – weighted rotated average CTE of each ply

- order doesn`t matter. (like $\underset{\sim}{A}$)

NOTE

$\alpha_L = -.5 \qquad \alpha_T = 16\,\mu\varepsilon / {}^{o}F$

$\theta_{11} = 20 \qquad \theta_{22} = 1.4 \qquad (AS4/3501-6)$

Can play off α and ply angle θ to get zero CTE`S

$E_L >> E_T$ helps. (scissor`s effect with θ)

7. Thermal stresses and deformation

❖ Bending

If ΔT constant and laminate symmetric

$$\underset{\sim}{M}^T = \int \underset{\sim}{\bar{Q}}\, \underset{\sim}{\bar{\alpha}}\, \Delta T z dz$$ → no bending

If ΔT gradient and laminate symmetric, $\underset{\sim}{M}^T \neq 0$

laminate bends / twists

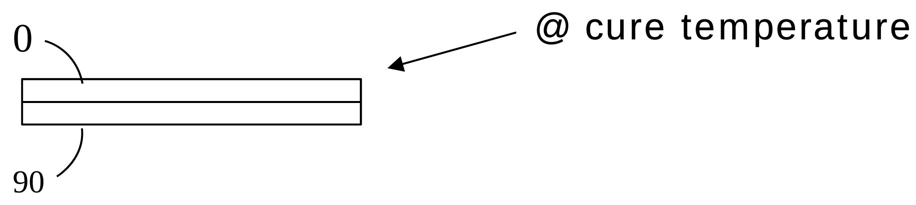
If laminate unsymmetric, $\underset{\sim}{b}$ and $\underset{\sim}{M}^T \neq 0$,

laminate bends / twists. (some exceptions)

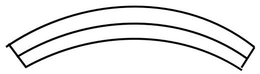
Note on ΔT

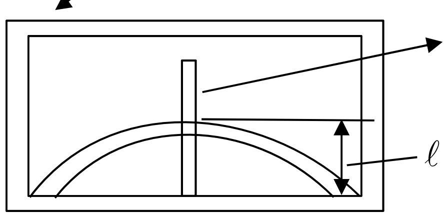
$\Delta T = T - T_0$, What is T_0 ?

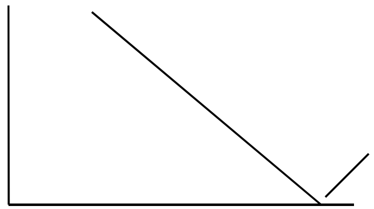
An experiment – $[0/90]_T$



7. Thermal stresses and deformation

@ room temperature →  $\Delta T \neq 0$





T_o: usually the cure temperature

NOTE : To calculate N^T and M^T for $\Delta T =$ const.

$$\underset{\sim}{N}^T = \int \underset{\sim}{\bar{Q}}\,\underset{\sim}{\bar{\alpha}}\,\Delta T dz = \Delta T \sum^{k} \underset{\sim}{\bar{Q}}^k \underset{\sim}{\bar{\alpha}}^k (z_{uk} - z_{lk})$$

$$\underset{\sim}{M}^T = \int \underset{\sim}{\bar{Q}}\,\underset{\sim}{\bar{\alpha}}\,\Delta T z dz = \Delta T \frac{1}{2} \sum^{k} \underset{\sim}{\bar{Q}}^k \underset{\sim}{\bar{\alpha}}^k (z_{uk}{}^2 - z_{lk}{}^2)$$

Same as $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{D}$ matrices

Always use z_{k-1} and $\underset{\sim}{A}\underset{\sim}{B}$ for each ply, rather than z_k and z_{lk}

(confusing, z direction
Signs, Jones book)

7. Thermal stresses and deformation

❖ **Thermal stresses in Plies**

Have

$$\begin{Bmatrix} \underset{\sim}{\varepsilon}^0 \\ \underset{\sim}{\kappa} \end{Bmatrix} = \begin{bmatrix} a & b \\ b^T & d \end{bmatrix} \begin{Bmatrix} \underset{\sim}{N} + \underset{\sim}{N}^T \\ \underset{\sim}{M} + \underset{\sim}{M}^T \end{Bmatrix}$$

What happens at ply level?

Total strains are just

$$\underset{\sim}{\bar{\varepsilon}} = \underset{\sim}{\varepsilon}^0 + z\underset{\sim}{\kappa} \qquad \text{(Laminate coordinate)}$$

Just transform to get ply coordinate

$$\underset{\sim}{\varepsilon} = \underset{\sim}{T}_{\varepsilon} \underset{\sim}{\bar{\varepsilon}} \qquad \text{(ply coordinate)}$$

Mechanical strain (these cause stress in material)

$$\underset{\sim}{\bar{\varepsilon}}^m = \underset{\sim}{\bar{\varepsilon}} - \underset{\sim}{\bar{\alpha}} \Delta T$$

total thermal

What are stresses?

Recall, $\underset{\sim}{\bar{\sigma}} = \underset{\sim}{\bar{Q}} \underset{\sim}{\bar{\varepsilon}}^m = \underset{\sim}{\bar{Q}} \{ \underset{\sim}{\bar{\varepsilon}} - \underset{\sim}{\bar{\alpha}} \Delta T \}$ (Laminate coordinate)

Also, $\underset{\sim}{\sigma} = \underset{\sim}{T}_{\sigma} \underset{\sim}{\bar{\sigma}}$ (ply coordinate)

7. Thermal stresses and deformation

Example → $[0/90]_s$ T300/934 material

$$\underset{\sim}{Q} = \begin{bmatrix} 20.1 & .4 & 0 \\ .4 & 1.4 & 0 \\ 0 & 0 & .7 \end{bmatrix} \text{ Msi}, \quad \underset{\sim}{\alpha} = \begin{bmatrix} +.05 \\ 16.0 \\ 0 \end{bmatrix} \mu\varepsilon / {}^{o}F$$

$T_0 = 350^{o}F$ (stress – free)

$T = 70^{o}F$

$\Delta T = -280^{o}F$

Laminate coordinates

$$\underset{\sim}{\bar{Q}} = \underset{\sim}{T}_{\varepsilon}^{T} \underset{\sim}{Q} \underset{\sim}{T}_{\varepsilon} \quad \text{(as before)}$$

$$\underset{\sim}{\bar{\alpha}} = \underset{\sim}{T}_{\varepsilon}^{-1} \underset{\sim}{\alpha} = \begin{bmatrix} c^2 & s^2 & -cs \\ s^2 & c^2 & cs \\ 2cs & -2cs & (c^2 - s^2) \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{Bmatrix}$$

$$\underset{\sim}{\bar{n}} = \underset{\sim}{\bar{Q}} \underset{\sim}{\alpha} = \begin{Bmatrix} n_x \\ n_y \\ n_{xy} \end{Bmatrix}$$

7. Thermal stresses and deformation

$$\underset{\sim}{N}^T = \int \underset{\sim}{\bar{Q}}\,\underset{\sim}{\bar{\alpha}}\,\Delta T dz = \Delta T \sum \underset{\sim}{\bar{n}}^k (z_{uk} - z_{lk})$$

$$\underset{\sim}{M}^T = \int \underset{\sim}{\bar{Q}}\,\underset{\sim}{\bar{\alpha}}\,\Delta T z dz = \Delta T \frac{1}{2} \sum^{k} \underset{\sim}{\bar{n}}^k ({z_{uk}}^2 - {z_{lk}}^2)$$

(for symmetric laminate, $\underset{\sim}{M}^T = 0$)

For 0° ply,

$$\underset{\sim}{\bar{\alpha}} = \begin{Bmatrix} c^2\alpha_1 + s^2\alpha_2 \\ s^2\alpha_1 + c^2\alpha_2 \\ 2cs(\alpha_1 - \alpha_2) \end{Bmatrix} = \begin{Bmatrix} +.05 \\ 16.0 \\ 0 \end{Bmatrix} \mu\varepsilon / {}^oF$$

$$\underset{\sim}{\bar{n}} = \begin{bmatrix} 20.1 & .4 & 0 \\ .4 & 1.4 & 0 \\ 0 & 0 & .7 \end{bmatrix} \begin{Bmatrix} +.05 \\ 16.0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 7.41 \\ 22.4 \\ 0 \end{Bmatrix} lbs / in^2\,{}^\circ F$$

$\times 10^6$ $\times 10^6$

For 90° ply

$$\underset{\sim}{\bar{\alpha}} = \left\{ \qquad\qquad \right\} = \begin{Bmatrix} 16.0 \\ +.05 \\ 0 \end{Bmatrix} \mu\varepsilon / {}^{o}F$$

$$\underset{\sim}{\bar{n}} = \begin{bmatrix} 1.4 & .4 & 0 \\ .4 & 20.1 & 0 \\ 0 & 0 & .7 \end{bmatrix} \begin{Bmatrix} 16.0 \\ .05 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 22.4 \\ 7.41 \\ 0 \end{Bmatrix}$$

Ply	z_{ku}	z_{kl}	$z_{ku}-z_{kl}$	$\bar{n}_x$	$\bar{n}_y$	$\bar{n}_{xy}$
0°	.010	.005	.005	7.41	22.4	0
90°	.005	0	.005	22.4	7.41	0

Sym.

$$\underset{\sim}{N}^T = \begin{Bmatrix} N_x^{\ T} \\ N_y^{\ T} \\ N_{xy}^{\ T} \end{Bmatrix} = \Delta T \sum^k \underset{\sim}{\overline{n}}^k (z_{uk} - z_{lk}) = (-280) \begin{Bmatrix} .298 \\ .298 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -84 \\ -84 \\ 0 \end{Bmatrix} lbs/\text{in}$$

$$\underset{\sim}{M}^T = \begin{Bmatrix} M_x^{\ T} \\ M_y^{\ T} \\ M_{xy}^{\ T} \end{Bmatrix} = \Delta T \frac{1}{2} \sum^k \underset{\sim}{\overline{n}}^k (z_{uk}^{\ 2} - z_{lk}^{\ 2}) = 0$$ (symmetric)

$$\underset{\sim}{A} = \begin{bmatrix} .215 & .0082 & 0 \\ .0082 & .215 & 0 \\ 0 & 0 & .014 \end{bmatrix} \times 10^6$$

$$\underset{\sim}{\varepsilon}^0 = \underset{\sim}{\alpha} \underset{\sim}{N}^T = \begin{bmatrix} 4.65 & -.177 & 0 \\ -.177 & 4.65 & 0 \\ 0 & 0 & 71.4 \end{bmatrix} \begin{Bmatrix} -84 \\ -84 \\ 0 \end{Bmatrix} \times 10^6 = \begin{Bmatrix} -377 \\ -377 \\ 0 \end{Bmatrix} \mu\varepsilon$$

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7. Thermal stresses and deformation

$$\underset{\sim}{\bar{\alpha}}_{\text{average}} = \underset{\sim}{a}\int \underset{\sim}{\bar{Q}}\,\underset{\sim}{\bar{\alpha}}\,dz = \varepsilon^{\circ} / \Delta T = \begin{Bmatrix} 1.3 \\ 1.3 \\ 0 \end{Bmatrix} \mu\varepsilon / {}^{o}F$$

0°Ply

$$\underset{\sim}{\varepsilon}^{m} = \underset{\sim}{\varepsilon}^{0} - \underset{\sim}{\bar{\alpha}}\,\Delta T = \begin{Bmatrix} -377 \\ -377 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0.5 \\ 16 \\ 0 \end{Bmatrix}(-280) = \begin{Bmatrix} -363 \\ 4103 \\ 0 \end{Bmatrix} \mu\varepsilon$$

$$\underset{\sim}{\bar{\sigma}} = \underset{\sim}{\bar{Q}}\,\underset{\sim}{\varepsilon}^{m} = \begin{bmatrix} 20.1 & .4 & 0 \\ .4 & 1.4 & 0 \\ 0 & 0 & .7 \end{bmatrix} \begin{Bmatrix} -363 \\ -4103 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -5600 \\ +5600 \\ 0 \end{Bmatrix} \ell bs / \text{in}^{2}$$

$\times 10^{6}$ $\times 10^{-6}$

Similarly obtain 90° Ply

In ply coordinate, $\underset{\sim}{\sigma}^{\circ} = \underset{\sim}{T}_{\sigma}\underset{\sim}{\bar{\sigma}}^{\circ} = \underset{\sim}{\bar{\sigma}}^{\circ}$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{Bmatrix} -5.6 \\ +5.6 \\ 0 \end{Bmatrix} Ksi$$

7. Thermal stresses and deformation

Recall allowables,

σ_1	σ_2	σ_6
$+190$	$+6$	$+10$
-160	-25	

Residual stresses close to allowable Y_t here.

In progressive failure analysis , should include this

$$\underset{\sim}{N}^{ToT} = \lambda \underset{\sim}{N}^{0} + \underset{\sim}{N}^{T}$$

Include this

A little complicates

See Jones Sec.4 failure with ΔT

❖ summary

Thermal strains $\underset{\sim}{\alpha}\Delta T$ cause residual stresses due to cool down,

$\Delta T = -280\,^{\circ}F$

For symmetric laminates, $\underset{\sim}{\kappa} = 0$ → no accompanying warping

For unsymmetric laminate, $\underset{\sim}{\kappa} \neq 0$ →

Warping $\begin{cases} \kappa_x, \kappa_y & = \text{bending} \\ \kappa_{xy} & = \text{twisting} \end{cases}$

$$= \frac{\partial}{\partial x}\frac{\partial w}{\partial y} \quad = \propto$$

One unsymmetric laminate that doesn`t warp

$[\theta / (\theta - 90)_2 / \theta]_A$

$i.e.)\ [\theta / (\theta - 90)_2 / \theta / -\theta / -(\theta - 90)_2 / -\theta]_t$

(Also give extension–twist coupling)

❖ Moisture

See Tsai and Hahn, Chap. 8

Matrix absorbs water, and swells

By micromechanics, can calculate ply swelling

(also have microstresses, ignore here)

Hydro → $\varepsilon^h = \beta \Delta M$ ← Moisture change = weight of moisture/dry weight

β ← CME : Coefficient of Moisture Expansion

$$\Delta M = M - M_0$$

$M_0 = 0$ Dry condition

Careful : ΔM sometimes expressed as percent (factor of 100)

$\Delta M \cong .5$ to 2% typical

$$\beta = \begin{Bmatrix} 45 \\ 5500 \\ 0 \end{Bmatrix} \mu\varepsilon / \%$$ For T300/934 , $$\alpha = \begin{Bmatrix} .05 \\ 16 \\ 0 \end{Bmatrix} \mu\varepsilon / {}^\circ F$$

7. Thermal stresses and deformation

Note: typically, $\Delta T \simeq -280,\ \Delta M = 1\%$

$$\varepsilon^T + \varepsilon^h = \alpha\,\Delta T + \beta \Delta T$$

$$= \begin{Bmatrix} -14 \\ -4480 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 45 \\ 5500 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 31 \\ 1.020 \\ 0 \end{Bmatrix} \mu\varepsilon$$

Moisture partly cancels some of strains.

Fortunate Relaxes Stresses.

CLPT works exactly same as before.

$$N^h = \int \bar{Q}\,\bar{\beta}\,\Delta M\, z\, dz$$

$$M^h = \int \bar{Q}\,\bar{\beta}\,\Delta M\, z\, dz$$

Where, $\bar{\beta} = \underset{\sim}{T}_{\varepsilon}^{-1}\,\beta$

rotated β

$$\begin{Bmatrix} \varepsilon^0 \\ c \end{Bmatrix} = \begin{bmatrix} a & b \\ b^T & d \end{bmatrix} \begin{Bmatrix} N + N^T + N^h \\ M + M^T + M^h \end{Bmatrix}$$

$$\bar{\sigma} = \bar{Q}(\varepsilon^0 + z\kappa - \bar{\alpha}\Delta T - \bar{\beta}\Delta M)$$

D_0 CLPT as before

❖ Moisture Absorption

Define $m = \dfrac{mass\ of\ water}{mass\ of\ dry\ material}$

$$M = m \times 100(\%)$$

$$\bar{m} = average\ through\ specimen$$

↖ can measure

❖ Fick`s Diffusion

$$q^H = -D\frac{\partial m}{\partial x} (or\ generally,\ q_i^H = -D_{ij}\frac{\partial M}{\partial x_{ij}})$$

$$\frac{\partial m}{\partial t} = -\frac{\partial}{\partial x} q^H = D_{ij}\frac{\partial^2 m}{\partial x_j^2} \qquad (3-Dim.)$$

$$\frac{\partial m}{\partial t} = D\frac{\partial^2 m}{\partial z^2} \qquad (1-Dim.)$$

(like heat conduction)

7. Thermal stresses and deformation

D : diffusion constant $= K^H$

$generally,\ D = D_0 e^{-C/T}$

T 300/1034→ $D_0 = 2.28\ mm^2/\sec$

$C = 5554\ {}^0K$

Moisture can affect cracks, cyclic effects, edge effects.

Equilibrium moisture content is, M_∞

Typically $m_\infty = m_{\infty\,\circ}\phi$ in air

(property of material: $m_{\infty\,\circ}$; Ref. Humidity: ϕ)

$m_\infty = m_{\infty\,w}$ in water $\neq 100^\circ RH$ air

The m_∞ is usually the B.C,

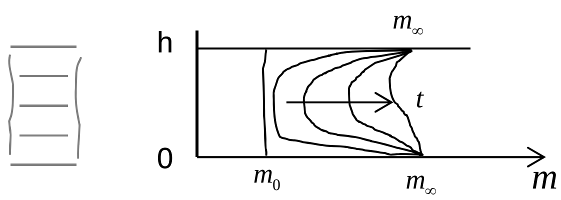
Differential Equation $\frac{\partial m}{\partial t} = D\frac{\partial^2 m}{\partial z^2}$

B.C : z=0 , h → $m = m_\infty$

Initial condition : t=0 → $m = m_0$

Solution

$$m^* = \frac{m - m_0}{m_\infty - m_0} = 1 - \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{1}{2j+1}\sin\frac{(2j+1)nz}{n}\, e^{-\frac{(2j+1)^2 \mathrm{n}^2}{n^2}Dt}$$



Similar to heat conduction

But very long times ($\times 10^5$)

Also interested in average moisture in specimen

$$\bar{m} = \frac{1}{h}\int_0^h m\,dz \leftarrow \text{can measure}$$

can then show

$$G = \frac{\bar{m} - m_0}{m_\infty - m_0} = 1 - \frac{8}{\pi^2}\sum \frac{1}{(2\,\mathrm{j}+1)^2} e^{-\frac{(2j+1)^2\pi^2}{n^2}Dt}$$

A single approximation to above is

$$G \simeq 1 - e^{-7.3\left(\frac{Dt}{n^2}\right)^{.75}}$$

Time t_p to reach 95% final value

$$e^{-7.3(Dt/n^2)^{.75}} = .05$$

or $$7.3\left(\frac{Dt}{n^2}\right)^{.75} = 3$$

$$t_p = \left(\frac{3}{7.3}\right)^{1.93}\frac{h^2}{D} \simeq .3\frac{h^2}{D}$$

Some formulas apply to heat conduction with appropriate constants
In addition to swelling, moisture causes deterioration of material properties.

See Tsai, " Composite Design" 4th Ed. 1988
Chap. 16, 17

7. Thermal stresses and deformation

❖ **summary**

1. Moisture tends to relieve residual thermal stresses obtained from cure (some moisture better than dry)
2. Similarly can do other strains. e.g. → piezoelectric

$$\rightarrow \varepsilon^P = d_T \Delta V$$

(ΔV: voltage; d_T: Coefficient of piezo expansion)

Then $\varepsilon^M = \varepsilon^0 + \kappa z - (\alpha \Delta T + \beta \Delta M + d_T \Delta V)$

(ε^M: mechanical strain; $\varepsilon^0 + \kappa z$: total strain; $\alpha \Delta T$: ε^T; $\beta \Delta M$: ε^m; $d_T \Delta V$: ε^P)

For computing convenience, can sometimes combine

$$\alpha \Delta T + \beta \Delta M + d_T \Delta V \rightarrow \alpha_{equivalent} \Delta T_{Eq}$$

And do analysis with Equivalent $\alpha_{Eq} \Delta T_{Eq}$