CHAPTER 8.
Advanced Topics

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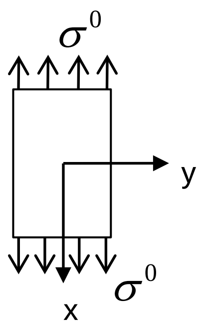
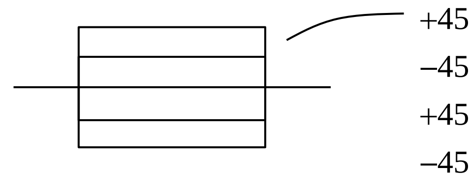
❖ Interlaminar Stresses due to Free Edges

Ref. { Pipes & Pagano, J.composite Mat`l, Oct 1970. p,538
Jones book, Chap.4 p,210

Consider $[\pm 45]_s$ angle-ply laminate under simple tension σ_0





For each ply, have

$$\bar{\underset{\sim}{\sigma}} = \bar{\underset{\sim}{Q}}\ \bar{\underset{\sim}{\varepsilon}}$$

Material Properties

$$E_L = 20.0\ Msi,\quad E_T = 2.1\ Msi,$$

$$G_{LT} = .85\ Msi,\quad \upsilon_{LT} = .21,\quad t_P = 0.005''$$

$$\underset{\sim}{Q} = \begin{bmatrix} 20.1 & .44 & 0 \\ .44 & 2.11 & 0 \\ 0 & 0 & .855 \end{bmatrix} \times 10^6\ (\text{lbs/ in}^2)$$

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Using transformation $\bar{Q} = T_\varepsilon^{\,T} Q T_\varepsilon$, get

$$\bar{\underset{\sim}{Q}}^{45} = \begin{bmatrix} 6.63 & 4.92 & 4.50 \\ 4.92 & 6.63 & 4.50 \\ 4.50 & 4.50 & 5.33 \end{bmatrix} \times 10^6$$

$\bar{\underset{\sim}{Q}}^{-45} \rightarrow$ Same with $\bar{Q}_{16} = \bar{Q}_{26} = -4.50 \times 10^6$

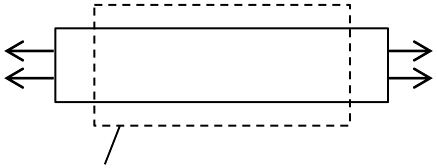
Laminate stiffness $\underset{\sim}{A} = \sum \bar{Q}^K (z_{uk} - z_{lk})$

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} .1325 & .0983 & 0 \\ .0983 & .1325 & 0 \\ 0 & 0 & .1066 \end{bmatrix} \begin{Bmatrix} \varepsilon_x^{\,o} \\ \varepsilon_y^{\,o} \\ \varepsilon_{xy} \end{Bmatrix} \times 10^6$$

For problem, $N_x = \sigma_0 h, \quad N_y = N_{XY} = 0$

Inverting gives

$$\begin{Bmatrix} \varepsilon_x^{\,o} \\ \varepsilon_y^{\,o} \\ \varepsilon_{xy} \end{Bmatrix} = \begin{bmatrix} 16.8 & -12.4 & 0 \\ -12.4 & 16.8 & 0 \\ 0 & 0 & 9.38 \end{bmatrix} \begin{Bmatrix} \sigma_0 h \\ 0 \\ 0 \end{Bmatrix} \times 10^6 = \begin{Bmatrix} 16.8 \\ -12.4 \\ 0 \end{Bmatrix} \sigma_0 h \times 10^6$$



$$\upsilon_{Laminate} = \frac{-\varepsilon_y{}^0}{\varepsilon_x{}^0} = .74$$

Stresses in Top ply $+45^\circ$ are

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} 6.63 & 4.92 & 4.50 \\ 4.92 & 6.63 & 4.50 \\ 4.50 & 4.50 & 5.33 \end{bmatrix} \begin{Bmatrix} 16.8 \\ -12.4 \\ 0 \end{Bmatrix} \sigma_0 h = \begin{Bmatrix} 50.3 \\ 0 \\ 19.8 \end{Bmatrix} \sigma_0 h$$

since $h = 4(.005) = .020$

$$= \begin{Bmatrix} 1.00\sigma_0 \\ 0 \\ 396\sigma_0 \end{Bmatrix}$$

← Note big shear stress

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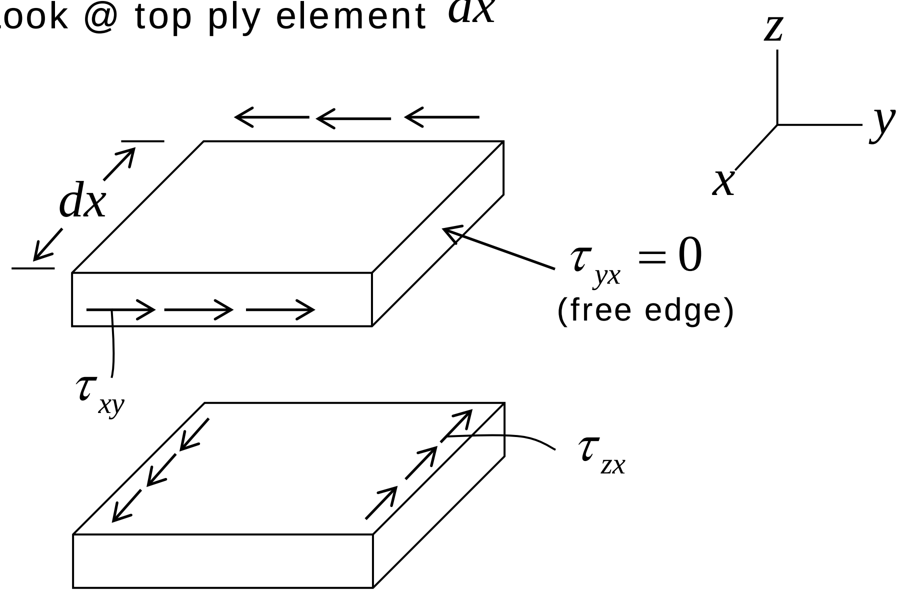
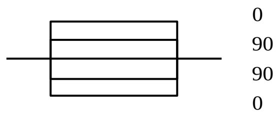
Look @ top ply element dx



To balance τ_{xy} , $\sum M_z = 0$, must have τ_{zx} develop on the interface

Similarly consider $[0/90]_s$ cross-ply laminate under tension σ_0



Would obtain

$$\begin{cases} \sigma_x = 1.811\sigma_0 \\ \sigma_y = .032\sigma_0 \\ \tau_{xy} = 0 \end{cases}$$

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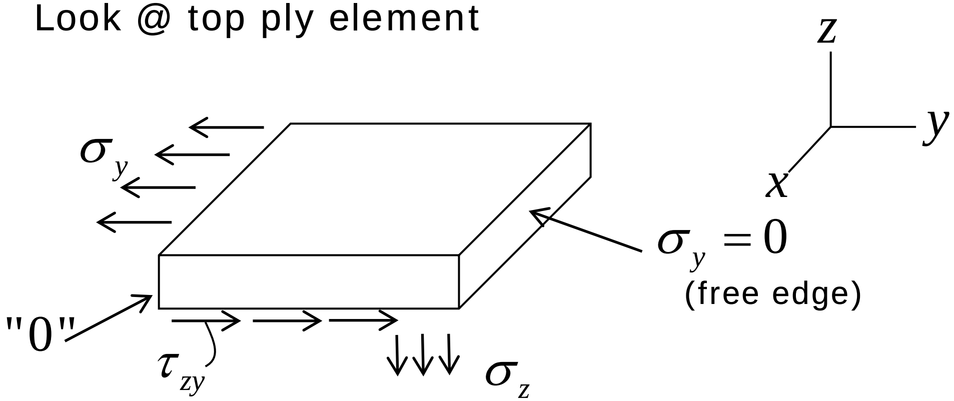
Look @ top ply element



For $\sum F_y = 0,$ must have τ_{zy} develop

For $\sum M_{x_0} = 0,$ must have σ_z develop, but $\int \sigma_z dy = 0$

$\tau_{xy,} \tau_{zy,} \sigma_z =$ "Interlaminar stresses", Develop on z face

$\sigma_{x,} \sigma_{y,} \tau_{xy} =$ "In-plane stresses"

Note : For $[\pm\theta]_s \Rightarrow$ only τ_{zx} develops

For $[0/90]_s \rightarrow$ only τ_{zy} , σ_z

For general combination, all 3 interlaminar stresses present

Must use 3-D Elasticity to solve complete problem.

❖ 3-Dim, Solution

Ref. Pipes & Pagano, J. Composite Mat`l, Oct 1970. p,583~

consider $[\pm 45]_s$ laminate under tension σ^0

For each ply,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{zy} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & & & c_{16} \\ c_{12} & c_{22} & c_{23} & & & c_{26} \\ c_{13} & c_{23} & c_{33} & & & c_{36} \\ & & & c_{44} & & \\ & & & & c_{55} & \\ c_{16} & c_{26} & c_{36} & & & c_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{zy} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix}$$

Rotated stiffness matrix about z-axis
(depends on 9 constraints only →orthotropic)

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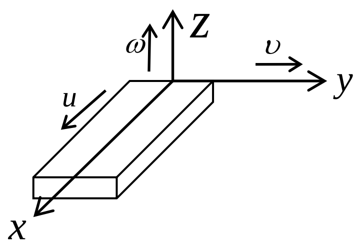
Strain-Displacement

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{zy} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Displacement pattern is



$$u = kx + U(y, z)$$

$$v = V(y, x)$$

$$w = W(y, z)$$

From symmetry, no x dependence of stresses

Stresses found as,

$$\sigma_x = c_{11}\varepsilon_x + c_{12}\varepsilon_y + c_{13}\varepsilon_z + c_{16}\gamma_{xy}$$

$$= c_{11}\frac{\partial u}{\partial x} + c_{12}\frac{\partial v}{\partial y} + c_{16}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$= c_{11}K + c_{12}V_{,y} + c_{13}W_{,z} + c_{16}U_{,y}$$

Similarly,

$$\sigma_y = c_{12}K + c_{22}V_{,y} + c_{23}W_{,z} + c_{26}U_{,y}$$
$$\tau_{yz} = c_{44}(V_{,z} + W_{,y}) + c_{45}(W_{,x} + U_{,z})$$

Placing into Equilibrium Eqns

$$\cancel{\frac{\partial \gamma_x}{\partial x}} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \cancel{p_x} = 0$$
$$\cancel{\frac{\partial \tau_{xy}}{\partial x}} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \cancel{p_y} = 0$$

no body force

Equations reduce to

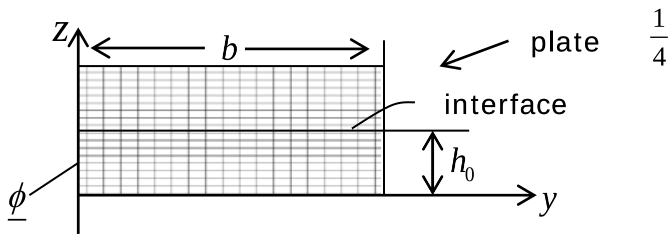
$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{12} & L_{22} & L_{23} \\ L_{13} & L_{23} & L_{33} \end{bmatrix} \begin{Bmatrix} U \\ V \\ W \end{Bmatrix} = 0$$

symmetric

Where,
$$L_{11} = c_{66}\frac{\partial^2}{2\partial y^2} + c_{55}\frac{\partial^2}{2\partial z^2}$$
$$L_{12} = c_{26}\frac{\partial^2}{2\partial y^2} + c_{45}\frac{\partial^2}{2\partial z^2}$$
$$L_{13} = (c_{36} + c_{45})\frac{\partial^2}{\partial y \partial x}$$

6th order set of Differential Equations

Solve by Finite Difference (like CFD)



Used up to 400 points

Boundary conditions

on Top face $\rightarrow \tau_{ZX} = 0,\ \ \tau_{ZY} = 0,\ \ \sigma_Z = 0$

on interface $\rightarrow \tau_{ZX},\ \ \tau_{ZY},\ \ \sigma_Z, \mathrm{u}, v, w$ continuous

at sideface $\rightarrow \tau_{yx} = 0,\ \ \sigma_y = 0,\ \ \tau_{yz} = 0$

@ z=0 $\rightarrow symmetry,\ \ \frac{\partial u}{\partial z} = 0,\ \ \frac{\partial v}{\partial z} = 0,\ \ w = 0$

@ y=0 $\rightarrow symmetry,\ \ u = 0,\ \ v = 0,\ \ \frac{\partial w}{\partial y} = 0$

Numerical solution by computer

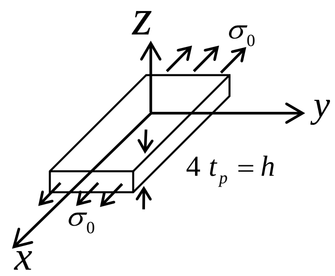
$$\text{Input} = K = \frac{\partial u}{\partial x} = \varepsilon_{x_0}$$

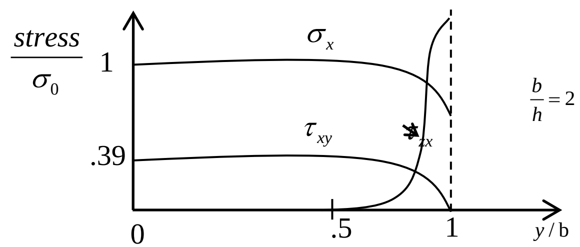
Enters from B.C.`s

$\sigma_y = 0$ @ sideface

$\sigma_z = 0$ @ top and interface

Results - $[\pm 45]_s$



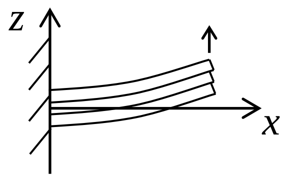


$\left|\frac{y/b}{\Delta y}\right|$ Boundary layer develops

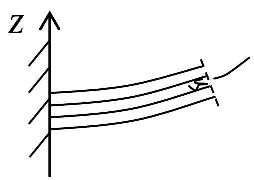
$(\Delta y < h)$

❖ Interlaminar stresses due to Bending

Consider a symmetric laminate in Bending ($\underset{\sim}{B} = 0, \quad \underset{\sim}{\varepsilon}^{\circ} = 0$)

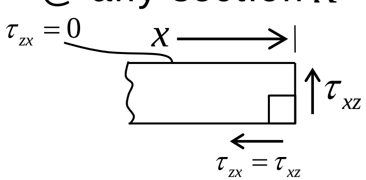


Layers would slide if not glued together



τ_{zx} develops (same as beam theory)

@ any section x



$$Q_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xz} \, dz$$

To obtain τ_{xz}, go to Equilibrium Equation

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + p_x = 0$$

$$\frac{\partial \tau_{zx}}{\partial z} \approx -\frac{\partial \sigma_x}{\partial x} - \frac{\partial \tau_{yx}}{\partial y} - p_x$$

(In red: the $\frac{\partial \tau_{yx}}{\partial y}$ and p_x terms are crossed out; $p_x = 0$)

Main contribution (→ $\frac{\partial \sigma_x}{\partial x}$) Small (→ $\frac{\partial \tau_{yx}}{\partial y}$)

Now, $\underset{\sim}{\bar{\sigma}} = \underset{\sim}{\bar{Q}}\,\underset{\sim}{\kappa}\, z$

$$\sigma_x = z\bar{Q}_{11}\kappa_x = z\bar{Q}_{11} d_{11} M_x$$

Placing into Equilibrium Equation & Integrating,

$$\int_{z_{lk}}^{z_{uk}} \frac{\partial \tau_{zx}}{\partial z} dz = -\int_{z_{lk}}^{z_{uk}} \frac{\partial}{\partial z}(z\bar{Q}_{11} d_{11} M_x) dz$$

$$= -\int z\,\bar{Q}_{11} d_{11} \frac{\partial M_x}{\partial x} dz$$

$= Q_x$

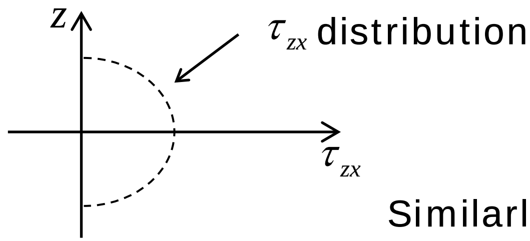
Integrating gives

$$\tau_{zx}(z_{uk}) - \tau_{zx}(z_{lk}) = -d_{11} Q_x \bar{Q}_{11}^k \left.\frac{z^2}{2}\right|_{z_{lk}}^{z_{uk}}$$

$$\tau_{zx}(z_{lk}) = \tau_{zx}(z_{uk}) + d_{11} Q_x \bar{Q}_{11}^k (z_{uk}{}^2 - z_{lk}{}^2)/2$$

Start at tip where $\tau_{zk}(h/2) = 0$

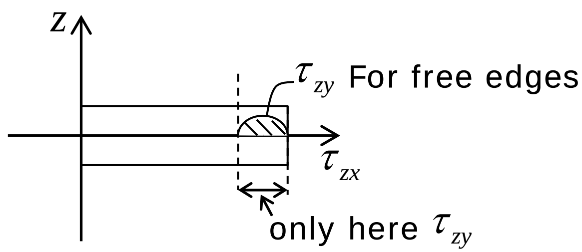
And work down.

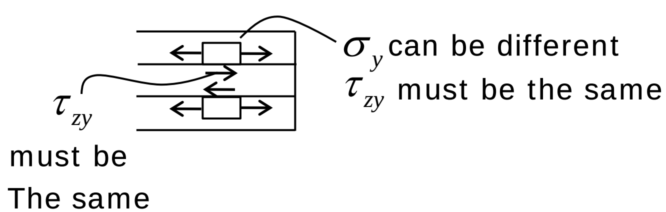


Similarly do for τ_{zy} distribution

Interlaminar stresses due to bending → everywhere in plate

Interlaminar stresses due to free edges → boundary layer near edges





8. Advanced Topics

- Introduction
- Micromechanics
- Ply Elasticity (orthotropic)
- Laminate Theory
- Failure
- Bending of Plate
- Thermal Stresses
- Advanced → Interlaminar stresses

  -Composites generally

  -Deal in design

  -Physical parameters

  -Organization → Computer program

8. Advanced Topics

❖ **Many current problems**

Failure, Fracture, holes
Cracking, Delamination, Fiber Breaking
Impact, Thermal stresses
Environmental Degradation
Buckling of the plates, Large deflections

16.26 Heating Effects (McManus)
16.251 Longevity (Lagace)
16.295 Failure of Composites (Spearing)
16.29 Seminar (current work)
16.230 Plates & shells