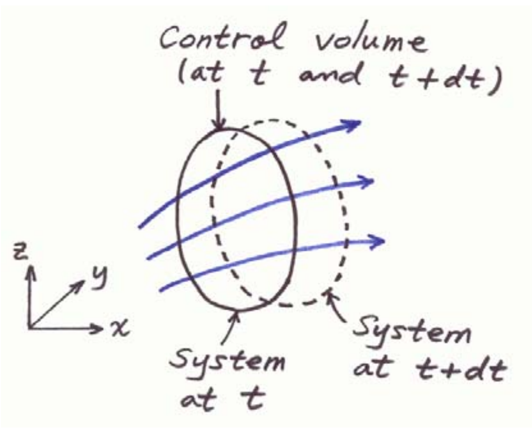
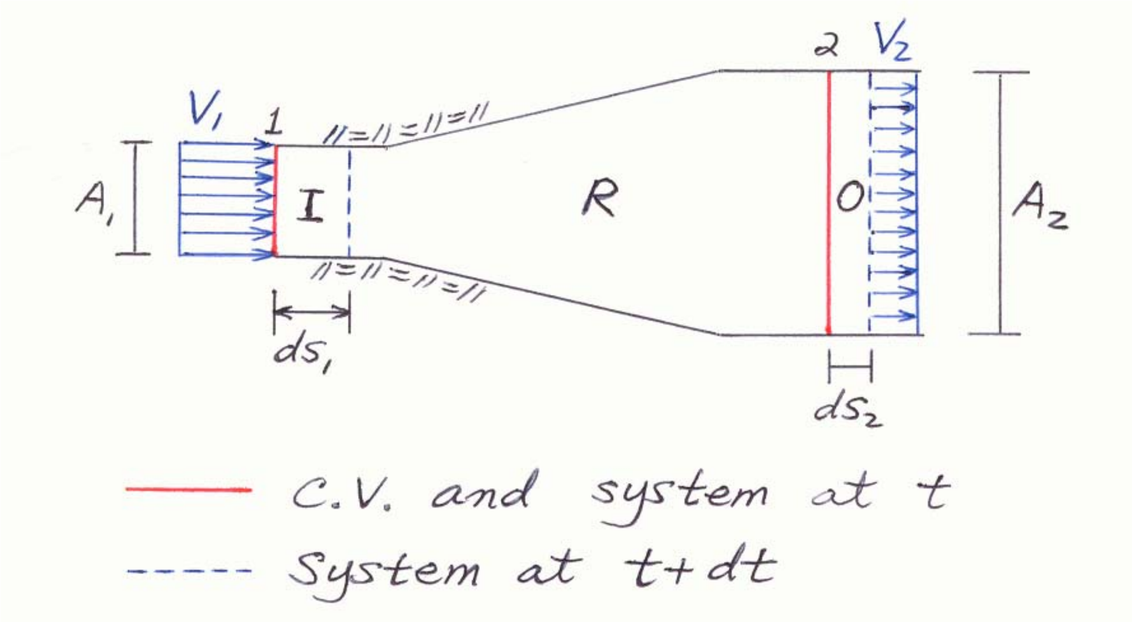
Chap.4 SYSTEMS, CONTROL VOLUMES, CONSERVATION OF MASS, AND THE REYNOLDS TRANSPORT THEOREM

4.1 System versus Control Volume

- System:
  - 폐쇄된 경계에 의해 주변의 물질과 분리되어 그 경계 내에 포함되어 있는 물질을 계속 동일하게 유지하는 물질의 집단
  - 흐름에 의해 위치가 이동되고 경계면의 형상이 바뀜 (Lagrangian view)
  - 유체 입자의 운동이 매우 불규칙한 난류의 경우에는 system의 경계가 유지되기 어려움 → 유체 흐름 해석이 어려움
- Control volume (檢査體積):
  - 흐름 내에 임의로 선정한 가상의 고정된 부피 (또는 공간)
  - 흐름에 상관없이 위치와 형태가 바뀌지 않음 (Eulerian view)
  - Control volume의 경계면 = Control surface
  - Control surface를 통해서 질량, 운동량, 에너지 등이 전달됨



4.2 Conservation of Mass: Continuity Equation–One-Dimensional Steady Flow



Conservation of system mass:

$$(m_I + m_R)_t = (m_R + m_O)_{t+dt}$$

Steady flow $\rightarrow$ $(m_R)_t = (m_R)_{t+dt}$

$$\therefore \quad (m_I)_t = (m_O)_{t+dt}$$

$$\rho_1 A_1 ds_1 = \rho_2 A_2 ds_2$$

$$\rho_1 A_1 \frac{ds_1}{dt} = \rho_2 A_2 \frac{ds_2}{dt}$$

$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 = \dot{m}$ $\leftarrow$ mass flow rate (4.1)

$\boxed{\dot{m} = \rho A V = \text{const.}}$ in 1-D steady flow

Note:

1. Volume flow rate $Q = AV$

For constant-density 1-D steady flow,

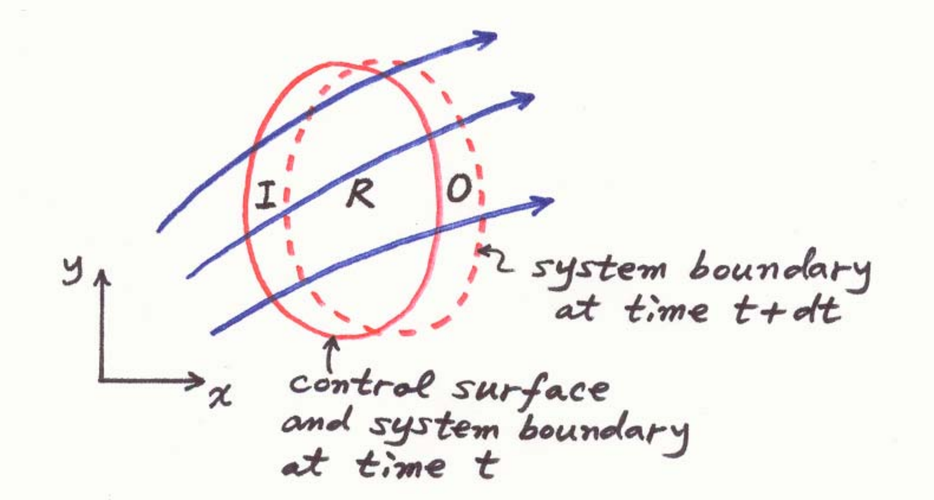
$$Q = \text{const} = A_1V_1 = A_2V_2$$

2. Non-uniform cross-sectional vel. (Fig. 4.2)

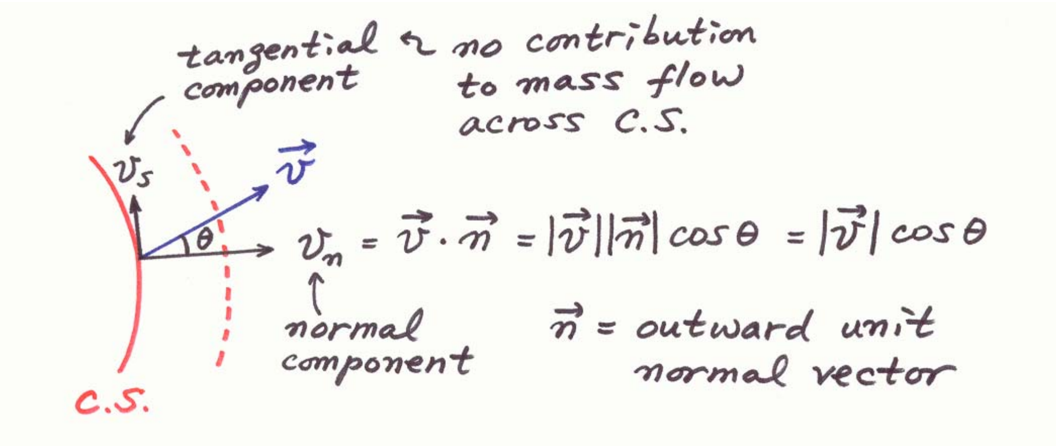
$\rightarrow$ Eq. (4.1) is valid with $V =$ mean velocity

(Read text)

4.3 Conservation of Mass: Continuity Equation–Two-Dimensional Steady Flow



As in 1-D flow, $(m_I)_t = (m_O)_{t+dt}$



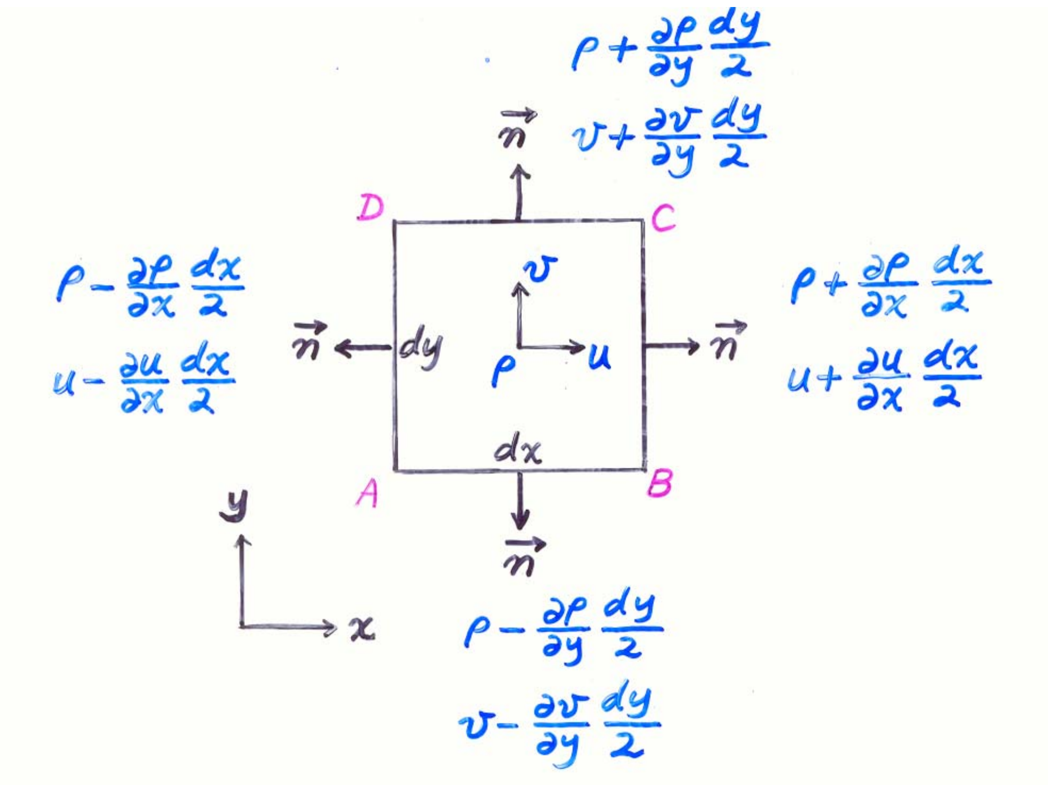
$$(m_O)_{t+dt} = dt\iint_{CS_{out}} \rho\vec{v}\cdot\vec{n}dA$$

$$(m_I)_t = dt\iint_{CS_{in}} \rho\vec{v}\cdot(-\vec{n})dA = -dt\iint_{CS_{in}} \rho\vec{v}\cdot\vec{n}dA$$

$$\therefore \iint_{CS_{out}} \rho\vec{v}\cdot\vec{n}dA + \iint_{CS_{in}} \rho\vec{v}\cdot\vec{n}dA = 0 = \iint_{CS} \rho\vec{v}\cdot\vec{n}dA$$

(Sum of normal mass flux across entire C.S. = 0)

Note: Conservation of system mass
$\rightarrow$ in terms of C.S. (or C.V.)



$$\iint_{ABCDA} \rho\vec{v}\cdot\vec{n}dA = 0 \rightarrow \boxed{\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)=0}$$

Conservation of mass equation for
2-D steady flow in a differential form

if ρ=const in space

$$\boxed{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0}$$

Continuity equation

4.4 Reynolds Transport Theorem

$$\begin{pmatrix} \text{Conservation of mass} \\ \text{Conservation of momentum} \\ \text{Conservation of energy} \\ \text{etc} \end{pmatrix} \text{System} \rightarrow \text{Control volume}$$

$\boxed{\text{Reynolds transport theorem}}$

E : system 전체의 mass, momentum 또는 energy

i : 단위질량당 mass, momentum 또는 energy

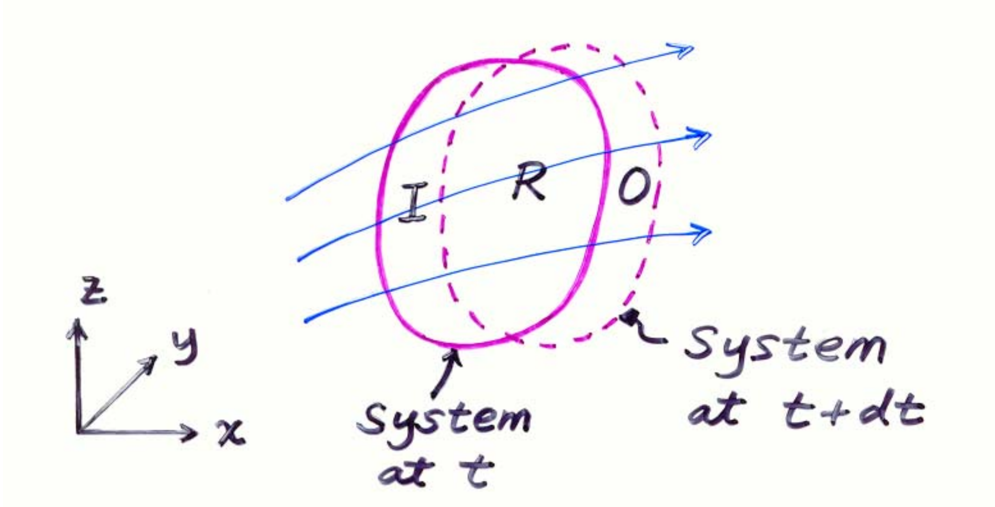
$$E = \iiint_{system} i\,dm = \iiint_{system} i\rho\, dV\!\!\!-$$

$i = 1 \rightarrow$ mass

$i = v \rightarrow$ momentum

$i = gz \rightarrow$ potential energy

$i = \dfrac{v^2}{2} \rightarrow$ kinetic energy



$$E_t = (E_I + E_R)_t \; ; \quad E_{t+dt} = (E_R + E_O)_{t+dt}$$

$$E_{t+dt} - E_t = (E_R + E_O)_{t+dt} - (E_I + E_R)_t$$

Using $(E_O)_{t+dt} = dt \iint_{CS_{out}} i\rho \vec{v} \cdot \vec{n} dA$

$$(E_I)_t = -dt \iint_{CS_{in}} i\rho \vec{v} \cdot \vec{n} dA$$

$$(E_R)_{t+dt} = \left(\iiint_R i\rho d\mathcal{V} \right)_{t+dt}$$

$$(E_R)_t = \left(\iiint_R i\rho d\mathcal{V} \right)_t$$

we get $\approx \iiint_{C.V.} i\rho d\mathcal{V}$ as $dt \to 0$

$$\frac{E_{t+dt} - E_t}{dt} = \frac{1}{dt} \left[\left(\iiint_R i\rho d\mathcal{V} \right)_{t+dt} - \left(\iiint_R i\rho d\mathcal{V} \right)_t \right] + \iint_{CS_{out}} i\rho \vec{v} \cdot \vec{n} dA + \iint_{CS_{in}} i\rho \vec{v} \cdot \vec{n} dA$$

$$\frac{dE}{dt} = \frac{\partial}{\partial t}\left(\iiint_{C.V.} i\rho dV\!\!\!\!-\right) + \iint_{CS_{out}} i\rho\vec{v}\cdot\vec{n}dA + \iint_{CS_{in}} i\rho\vec{v}\cdot\vec{n}dA$$

$$\frac{dE}{dt} \quad = \quad \frac{\partial}{\partial t}\left(\iiint_{C.V.} i\rho dV\!\!\!\!-\right) \quad + \quad \iint_{CS} i\rho\vec{v}\cdot\vec{n}dA$$

Total derivative
(or material derivative)
following fluid particles
(or system)

Control surface를 통해서
들어오고 나가는 flux의 합

Control volume 내에서의
시간에 따른 변화율
(= 0 for steady flow)

좌변 = system 내에서의 변화율

$\downarrow$ ← Reynolds transport theorem

우변 = C.V.과 C.S.로 표시한 변화율

• Application to conservation of mass
($E = m$, $i = 1$)

Conservation of system mass → $\dfrac{dm}{dt} = 0$

$$\frac{\partial}{\partial t}\left(\iiint_{C.V.} \rho d\forall\right) = -\iint_{CS_{out}} \rho \vec{v}\cdot\vec{n} dA - \iint_{CS_{in}} \rho \vec{v}\cdot\vec{n} dA$$

If the flow is steady ($\partial/\partial t = 0$) or the fluid is incompressible and uniform density ($\iiint_{CV} \rho d\forall = \text{const}$), then

$$\frac{\partial}{\partial t}\left(\iiint_{C.V.} \rho d\forall\right) = 0$$

$$\therefore \quad \iint_{CS_{out}} \rho \vec{v}\cdot\vec{n} dA + \iint_{CS_{in}} \rho \vec{v}\cdot\vec{n} dA = 0 \qquad (4.9)$$

Conservation of mass equation for steady flow or incompressible uniform density fluid

For 1-D steady flow,

$$\left.\begin{aligned} \iint_{CS_{in}} \rho \vec{v}\cdot\vec{n} dA &= -\rho_1 V_1 A_1 \\ \iint_{CS_{out}} \rho \vec{v}\cdot\vec{n} dA &= \rho_2 V_2 A_2 \end{aligned}\right\} \rightarrow \quad \rho_1 V_1 A_1 = \rho_2 V_2 A_2$$