Chap.6 The Impulse-Momentum Principle

- So far, we have studied the flow characteristics (velocity, pressure, etc.) in a fluid. In this chapter, we study the force acting on a structure by fluid in a flow. As for the force acting on a structure in quiescent fluid, we studied it in Chapter 2.
- Newton's 2nd law (in a vector form):

$$\sum \vec{F} = m\vec{a}_c = \frac{d}{dt}\left(m\vec{v}_c\right)$$

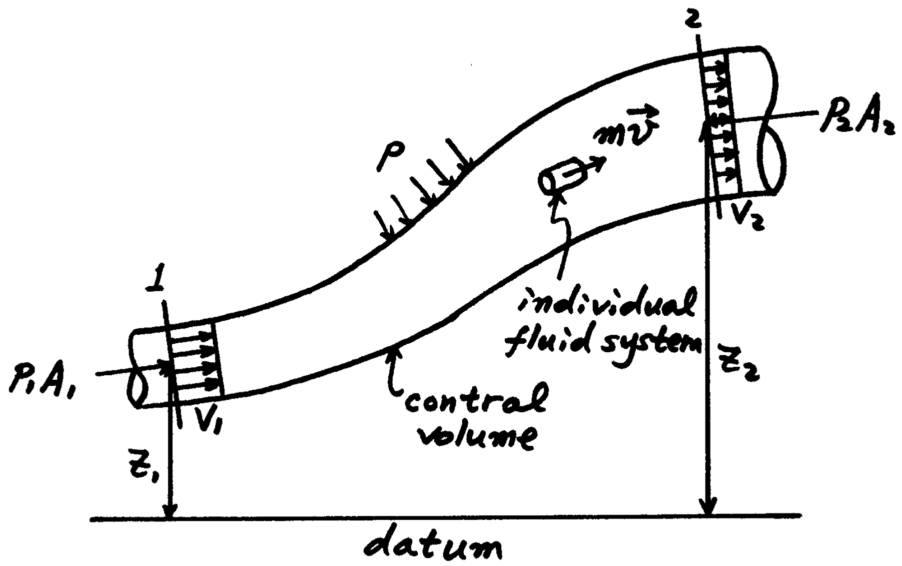
where $\vec{v}_c$ = velocity of the center of mass of the system

$$\underbrace{\left(\sum \vec{F}\right)dt}_{\text{impulse in the time } dt} = d\ \underbrace{\left(m\vec{v}_c\right)}_{\text{linear momentum}}$$

- Linear impulse-momentum equation
  → magnitude and direction of resultant forces exerted on fluid by structures (or on structures by fluid)

- Angular impulse-momentum equation
  → line of action of the resultant force

6.1 Linear Impulse-Momentum Equation



For the individual fluid system,

$$\sum \vec{F} = m\vec{a} = \frac{d}{dt}\left(m\vec{v}\right) = \frac{d}{dt}\left(\rho \vec{v} d\forall\right)$$

Summation over the whole flow system:

$$\begin{aligned}\sum \vec{F}_{\text{ext}} &= \iiint_{\text{system}} \frac{d}{dt}\left(\rho \vec{v} d\forall\right) = \frac{d}{dt}\iiint_{\text{system}} \rho \vec{v} d\forall \\ &= \iint_{\text{CS}_{\text{out}}} \rho \vec{\mathrm{v}}(\vec{v}\cdot\vec{n}dA) + \iint_{\text{CS}_{\text{in}}} \rho \vec{\mathrm{v}}(\vec{v}\cdot\vec{n}dA) \\ &= \rho_2 \vec{V}_2 Q_2 - \rho_1 \vec{V}_1 Q_1 = Q\rho\left(\vec{V}_2 - \vec{V}_1\right)\end{aligned}$$

where $\sum \vec{F}_{\text{ext}}$ = body force + forces on C.S.

(∵ all the internal forces b/w individual fluid systems are cancelled out)

$$\therefore \quad \sum \vec{F}_{\text{ext}} = Q\rho\left(\vec{V}_2 - \vec{V}_1\right)$$

In a 2-D flow (x and z),

$$\left.\begin{aligned}\left(\sum F_{\text{ext}}\right)_x &= Q\rho\left(V_{2_x} - V_{1_x}\right)\\ \left(\sum F_{\text{ext}}\right)_z &= Q\rho\left(V_{2_z} - V_{1_z}\right)\end{aligned}\right\}$$

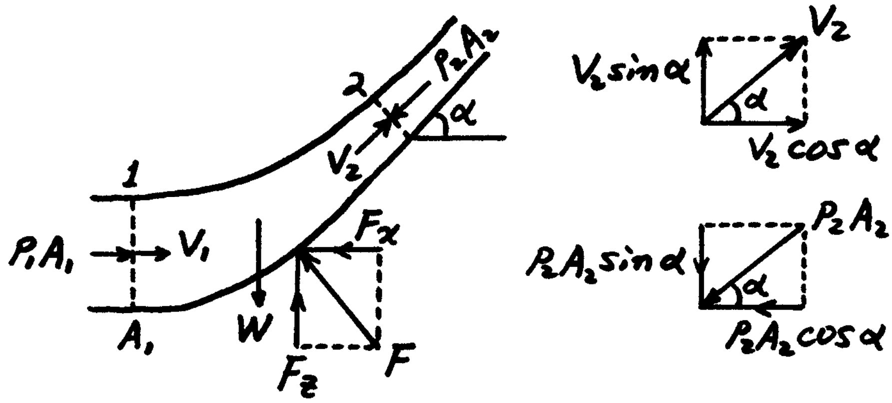
→ Magnitude and direction of $\sum \vec{F}_{\text{ext}}$.
But line of action is not known yet.

If the flow enters and leaves at more than one location,

$$\sum \vec{F}_{\text{ext}} = \left(\sum Q\rho\vec{V}\right)_{\text{out}} - \left(\sum Q\rho\vec{V}\right)_{\text{in}}$$

6.2 Pipe Flow Applications

(1) Force acting on pipe bend



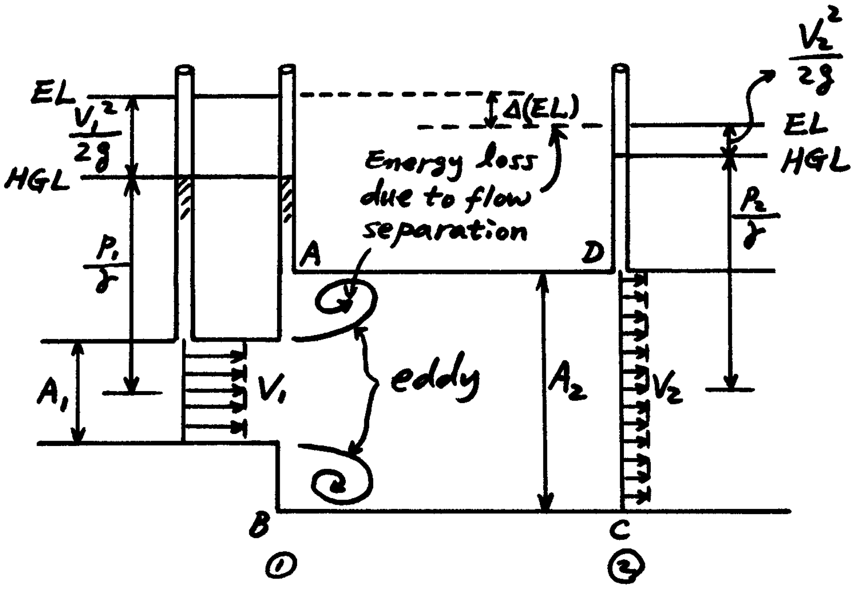
$$x: \quad p_1A_1 - p_2A_2\cos\alpha - F_x = Q\rho(V_2\cos\alpha - V_1)$$

$$F_x = p_1A_1 - p_2A_2\cos\alpha - Q\rho(V_2\cos\alpha - V_1)$$

$$z: \quad -W - p_2A_2\sin\alpha + F_z = Q\rho(V_2\sin\alpha - 0)$$

$$F_z = W + p_2A_2\sin\alpha + Q\rho V_2\sin\alpha$$

(2) Energy loss at abrupt expansion of a passage



Impulse-momentum equation on C.V. ABCD:

$$\sum F_x \approx p_1 A_2 - p_2 A_2 = Q\rho(V_2 - V_1) = A_2 V_2 \rho(V_2 - V_1)$$

Note: p may be different from p_1 in eddy area.

$$\therefore \quad \frac{p_1 - p_2}{\gamma} = \frac{V_2(V_2 - V_1)}{g}$$

Work-energy equation:

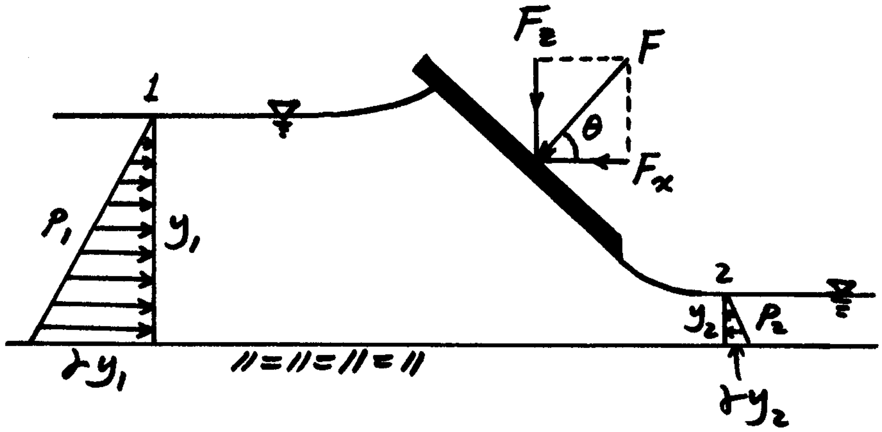
$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} - \Delta(\mathrm{EL}) = \frac{p_2}{\gamma} + \frac{V_2^2}{2g}$$

$$\therefore \Delta(\mathrm{EL}) = \frac{p_1 - p_2}{\gamma} + \frac{V_1^2 - V_2^2}{2g} = \frac{V_2(V_2 - V_1)}{g} + \frac{V_1^2 - V_2^2}{2g}$$

$$= \frac{(V_1 - V_2)^2}{2g}$$: Borda-Carnot head loss

6.3 Open Channel Flow Applications

(1) Force acting on a sluice gate



$$\left(\sum F_{\text{ext}}\right)_x = \frac{\gamma y_1^2}{2} - \frac{\gamma y_2^2}{2} - F_x = Q\rho(V_2 - V_1) = q\rho\left(\frac{q}{y_2} - \frac{q}{y_1}\right)$$

$$\therefore \quad F_x = \frac{\gamma}{2}\left(y_1^2 - y_2^2\right) - q^2\rho\left(\frac{1}{y_2} - \frac{1}{y_1}\right)$$

$\vec{F} \perp$ gate ($\because$ no shear force for ideal fluid)

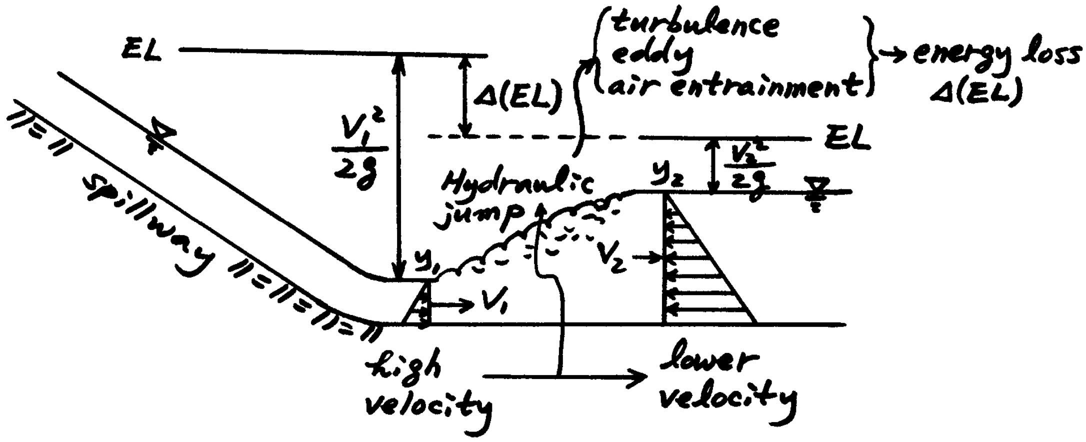
$$\therefore \quad F\cos\theta = F_x \;\rightarrow\; F = \frac{F_x}{\cos\theta}$$

$$F_z = F\sin\theta$$

No need for impulse-momentum equation in z-direction.

See IP 6.2 (p 197) for horizontal force acting on an overflow structure (same principle).

(2) Energy loss due to hydraulic jump



Impulse-momentum equation:

$$\left(\sum F_{\text{ext}}\right)_x = \frac{\gamma y_1^2}{2} - \frac{\gamma y_2^2}{2} = q\rho(V_2 - V_1) \ \leftarrow \text{no structure}$$

no external force by structure

Continuity: $q = V_1 y_1 = V_2 y_2 \ \rightarrow \ V_1 = \frac{q}{y_1}, \quad V_2 = \frac{q}{y_2}$

$$\therefore \frac{\rho g}{2}\left(y_1^2 - y_2^2\right) = q^2 \rho \left(\frac{1}{y_2} - \frac{1}{y_1}\right)$$

$$y_2^2 - y_1^2 = \frac{2q^2}{g}\left(\frac{1}{y_1} - \frac{1}{y_2}\right)$$

$$(y_2 + y_1)(y_2 - y_1) = \frac{2q^2}{g}\frac{y_2 - y_1}{y_1 y_2}$$

$$\left(\frac{y_2}{y_1}\right)^2 + \frac{y_2}{y_1} - \frac{2q^2}{g y_1^3} = 0$$

$$\therefore \frac{y_2}{y_1} = \frac{1}{2}\left(-1+\sqrt{1+\frac{8q^2}{gy_1^3}}\right) = \frac{1}{2}\left(-1+\sqrt{1+\frac{8V_1^2}{gy_1}}\right)$$

(a) $\mathbf{F}_1^2 = \frac{V_1^2}{gy_1} = 1 \rightarrow \frac{y_2}{y_1} = 1$; no hydraulic jump

(b) $\mathbf{F}_1^2 = \frac{V_1^2}{gy_1} > 1 \rightarrow \frac{y_2}{y_1} > 1$; hydraulic jump occurs

(c) $\mathbf{F}_1^2 = \frac{V_1^2}{gy_1} < 1 \rightarrow \frac{y_2}{y_1} < 1$; physically impossible

Note: $\mathbf{F} = \frac{V}{\sqrt{gy}} = \text{Froude number}$

$$\Delta(\text{EL}) = \left(y_1 + \frac{V_1^2}{2g}\right) - \left(y_2 + \frac{V_2^2}{2g}\right) = \left(y_1 + \frac{q^2}{2gy_1^2}\right) - \left(y_2 + \frac{q^2}{2gy_2^2}\right)$$

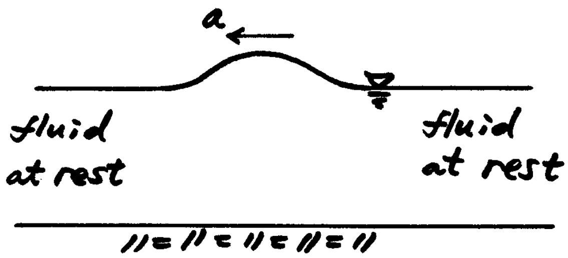
Using $q^2 = \frac{g}{2} y_1 y_2 (y_1 + y_2)$

$\Delta(\text{EL}) = \frac{(y_2 - y_1)^3}{4y_1 y_2} > 0$ for energy loss

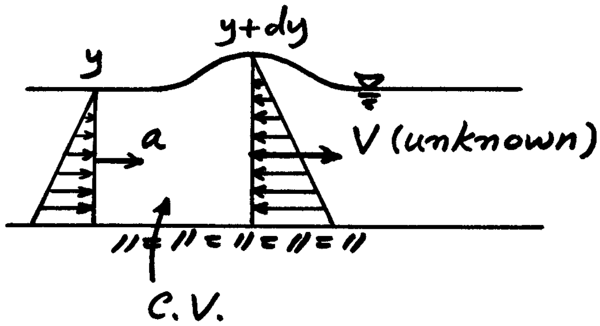
$\therefore y_2 > y_1 \rightarrow \frac{y_2}{y_1} > 1$

(3) Velocity of small gravity waves

→ not fluid velocity but propagating speed of waves



Under the wave, fluid velocity is unknown except that it is constant over depth for a small gravity wave (or long wave).
For a stationary observer, it is an unsteady problem. For an observer moving with speed a, however, it is a steady problem.



Continuity: $ay = V(y+dy) = q$

Impulse−momentum: $\dfrac{\gamma y^2}{2} - \dfrac{\gamma (y+dy)^2}{2} = q\rho (V-a)$

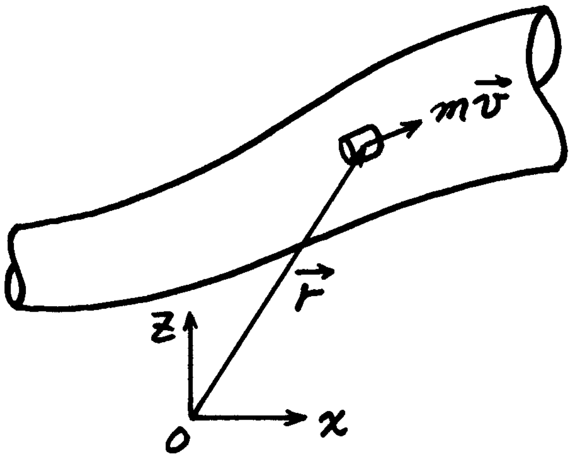
$$\therefore a^2 = g\left(1+\frac{dy}{y}\right)\left(y+\frac{1}{2}dy\right)$$

If $y >> dy$, then $a \approx \sqrt{gy}$ ← a depends only on y.

6.4 Angular Impulse-Momentum Principle

→ line of action of resultant forces

$\left(\sum \vec{F} = m\vec{a}\right)_{\text{moment}} \rightarrow$ moment of momentum
= angular momentum



$$\sum\left(\vec{r}\times\vec{F}\right) = \vec{r}\times m\vec{a} = \frac{d}{dt}\left(\vec{r}\times m\vec{v}\right) = \frac{d}{dt}\left(\vec{r}\times\vec{v}\rho d\mathcal{V}\right)$$

Summation over the whole flow system:

$$\sum\left(\vec{r}\times\vec{F}_{\text{ext}}\right) = \frac{d}{dt}\iiint_{\text{sys}}\left(\vec{r}\times\vec{v}\right)\rho d\mathcal{V}$$

$$= \iint_{\text{CS}_{\text{out}}}\left(\vec{r}\times\vec{v}\right)\rho\left(\vec{v}\cdot\vec{n}dA\right) + \iint_{\text{CS}_{\text{in}}}\left(\vec{r}\times\vec{v}\right)\rho\left(\vec{v}\cdot\vec{n}dA\right)$$

$$= \iint_{\text{CS}_{\text{out}}}\left(\vec{r}\times\vec{v}\right)\rho dQ - \iint_{\text{CS}_{\text{in}}}\left(\vec{r}\times\vec{v}\right)\rho dQ$$

$$= Q\rho\left(\vec{r}_{\text{out}}\times\vec{V}_{\text{out}}\right) - Q\rho\left(\vec{r}_{\text{in}}\times\vec{V}_{\text{in}}\right)$$

$$= Q\rho\left[\left(\vec{r}\times\vec{V}\right)_{\text{out}} - \left(\vec{r}\times\vec{V}\right)_{\text{in}}\right]$$

$$\therefore \sum\left(\vec{r}\times\vec{F}_{\text{ext}}\right)=\sum\overrightarrow{M}_0 = Q\rho\left[\left(\vec{r}\times\vec{V}\right)_{\text{out}} - \left(\vec{r}\times\vec{V}\right)_{\text{in}}\right]$$

Note: $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, $\quad \vec{V} = V_x\vec{i} + V_y\vec{j} + V_z\vec{k}$

$$\vec{r}\times\vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ V_x & V_y & V_z \end{vmatrix} = \left(yV_z - zV_y\right)\vec{i} + \left(zV_x - xV_z\right)\vec{j} + \left(xV_y - yV_x\right)\vec{k}$$

In 2-D problem (x,z),

$$\vec{r}\times\vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & 0 & z \\ V_x & 0 & V_z \end{vmatrix} = \left(zV_x - xV_z\right)\vec{j}$$

Similarly, $\vec{r}\times\vec{F} = \left(zF_x - xF_z\right)\vec{j}$

$$\therefore \sum\left(zF_{ext_x} - xF_{ext_z}\right) = Q\rho\left[\left(z_2V_{2_x} - x_2V_{2_z}\right) - \left(z_1V_{1_x} - x_1V_{1_z}\right)\right]$$

where 1=CS_{in} and 2=CS_{out}.

For the resultant force, use $F_{\text{ext}}\times r$, where F_{ext} is known from linear impulse-momentum equation. For other external forces (i.e. body force and pressure forces at 1 and 2), we know the point of action (i.e. x and z).