

Basic Classical Mechanics

Reading: Atkins, Appendix 3 (pp. 979-984)

17 C, Isaac Newton

→ successful at explaining the motion of everyday objects and planets

→ failed at very small particles (20th century)

1. Main features

- 1) Trajectory (궤적) is completely determined → deterministic
cf) quantum mechanics: probability
- 2) Energy can take any non-negative value → allows the translational, rotational, and vibrational modes of motion to be excited to any energy simply by controlling the forces
cf) Q. M.: discrete
- 3) Measurement of dynamic variables can be made as precisely as we wish → no inherent limitation in accuracy
cf) Q. M.: uncertain

2. Total energy

Total energy of a particle : $E = E_k + V(x)$

E_k : kinetic energy (from the motion)

$V(x)$: potential energy (from the position)

Force

$$F = -(dV(x)/dx)$$

→ The direction of the force is towards decreasing potential energy

$$E_k = (1/2)mv^2 = p^2/2m \text{ (since } \mathbf{p} = m\mathbf{v}, \text{ linear momentum)}$$

Total energy $E = p^2/2m + V(x)$

→ Particle has a definite trajectory, or define position and momentum at each instant

e.g.) $V = 0, v = (dx/dt)$

$$v = (dx/dt) = (2E_k/m)^{1/2}$$

$$\Rightarrow x(t) = x(0) + (2E_k/m)^{1/2}t$$

$$p(t) = mv(t) = m(dx/dt) = (2mE_k)^{1/2}$$

If we know initial position & momentum, we can predict all later positions & momenta exactly

3. Newton's 2nd law of motion

the rate of change of momentum = the force acting on the particle

$$\mathbf{F} = d\mathbf{p}/dt = d(m\mathbf{v})/dt = m(d^2\mathbf{r}/dt^2) = m\mathbf{a}$$

$$\mathbf{r} = \mathbf{r}(x, y, z)$$

e.g.) 1-dimension, $F = m(d^2x/dt^2)$ (d^2x/dt^2) : acceleration of particle

trajectory: $\mathbf{r} = \mathbf{r}(t, \mathbf{v}_0, \mathbf{r}_0)$, $\mathbf{p} = \mathbf{p}(t, \mathbf{v}_0)$

E_k ; $0 \rightarrow E_k = F^2\tau^2/(2m)$ by $t = 0 \rightarrow t = \tau$; any energy value available

4. Rotational motion

angular momentum J

$$\mathbf{J} = I\boldsymbol{\omega}$$

$\boldsymbol{\omega}$: angular velocity (radians/s),

I : moment of inertia ($I = mr^2$)

To accelerate a rotation, it is necessary to apply a torque
torque T , a twisting force

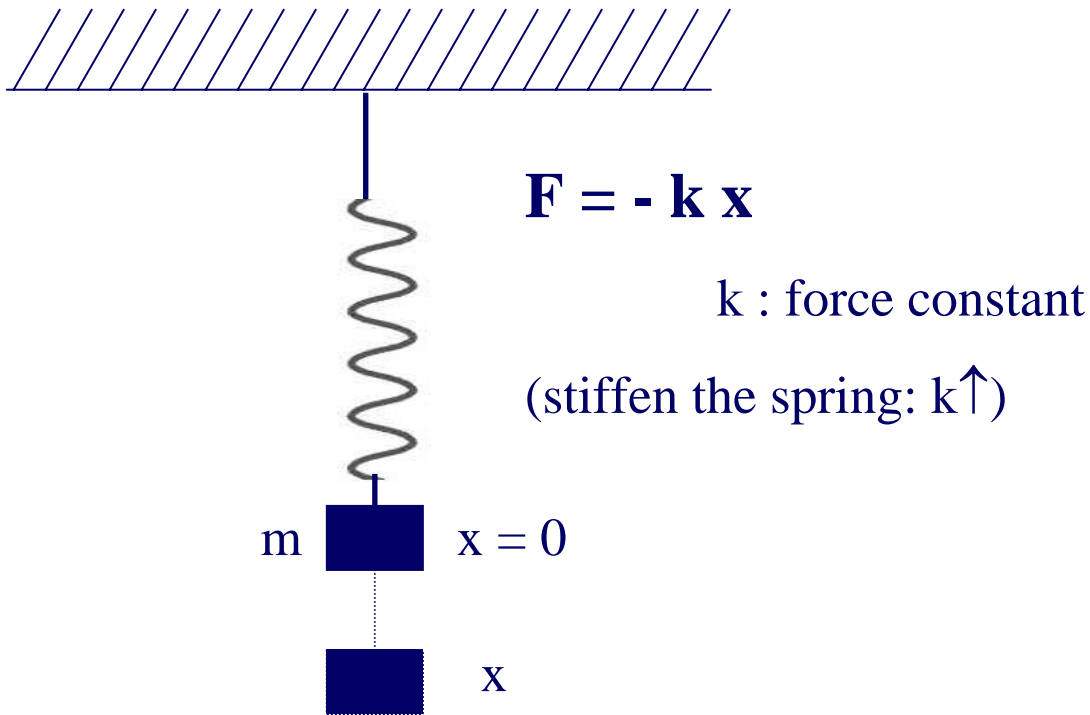
$$T = dJ/dt$$

Rotation energy

$$0 \rightarrow E_k = T^2\tau^2/2I \text{ by time } 0 \rightarrow \tau$$

→ arbitrary rotational energy available

5. Harmonic oscillator → vibration



-: force direction → opposite to displacement

$$F = -kx$$

k: force constant

$$m(d^2x/dt^2) = -kx$$

$$d^2x/dt^2 + \omega^2x = 0$$

where $\omega = \sqrt{k/m}$; angular frequency

$$\rightarrow \omega = 2\pi\nu, \text{ frequency } \nu = \omega/2\pi = (1/2\pi)\sqrt{k/m}$$

$$x(t) = A\sin\omega t \quad (A: \text{amplitude})$$

$$p(t) = m\omega A\cos\omega t \quad (\text{since } p = mv = m(dx/dt))$$

$$E_k = p^2/2m = 1/2m\omega^2A^2\cos^2\omega t = 1/2kA^2\cos^2\omega t$$

$$F = -kx = -dV/dx$$

$$V = 1/2kx^2 = 1/2kA^2\sin^2\omega t$$

$$\text{Total } E = E_k + V = 1/2kA^2\cos^2\omega t + 1/2kA^2\sin^2\omega t = 1/2kA^2$$

→ any vibrational energy available (amplitude (A) governs the energy, independent of frequency)

Classical mechanics: agree with everyday experience → **does not** extend to very small particles, such as individual atoms

Electromagnetic Theory of Radiation

Electromagnetic field

- In classical physics, electromagnetic radiation is understood in terms of electromagnetic field
 - An oscillating electric and magnetic disturbance (even in vacuum) (electric field + magnetic field)
- travel at constant speed: “speed of light” (c)
- wavelength λ (lambda),
frequency ν (nu, Hz) $\lambda \nu = c$
wavenumber ($\tilde{\nu}$) (nu tilde, cm^{-1}):

$$\tilde{\nu} = \nu / c = 1 / \lambda$$

Electromagnetic radiation

Travelling x-direction

Electric field $E(x,t) = E_0 \cos[2\pi \nu t - (2\pi/\lambda)x + \phi]$

Magnetic field $B(x,t) = B_0 \cos[2\pi \nu t - (2\pi/\lambda)x + \phi]$

E_0 & B_0 : the amplitudes of the fields

ϕ : phase of the wave ($-\pi$ to π): two waves $\phi = 0 \rightarrow$ constructive

$\phi = \pi$ or $-\pi \rightarrow$ destructive

E & B: plane polarized

Differentiation

$$\left(\frac{\partial^2}{\partial x^2}\right)\Psi(x,t) = -(4\pi^2/\lambda^2)\Psi(x,t), \quad \left(\frac{\partial^2}{\partial t^2}\right)\Psi(x,t) = -4\pi^2\nu^2\Psi(x,t)$$

$\Psi(x,t)$: either $E(x,t)$ or $B(x,t)$

Intensity of electromagnetic radiation \propto square of the wave amplitude
(energy) E_0^2 or B_0^2

The Failures of Classical Mechanics (Physics)

Reading: Atkins, ch. 8 (p. 244-252)

e.g., black-body radiation

heat capacity of solid

photoelectric effect

atomic spectra etc

1. Black-body radiation

- hot object emit electromagnetic radiation

e.g., iron bar: red \rightarrow yellow \rightarrow blue \rightarrow ...; $T \uparrow \rightarrow$ wavelength $\lambda \downarrow$,
frequency $\nu \uparrow$

The energy distribution in a black-body cavity at several temperatures. Note how the energy density increases in the visible region as the temperature is raised, and how the peak shifts to shorter wavelengths. The total energy density (the area under the curve) increases as the temperature is increased (as T^4).

- black-body: ideal emitter, perfect absorber & perfect emitter
many times absorption & emission → thermal equilibrium at temperature T
→ leaking out through pinhole

$T \uparrow$ color shifts toward the blue

An experimental representation of a black body is a pinhole in an otherwise closed container. The radiation is reflected many times within the container and comes to thermal equilibrium with the walls at a temperature T . Radiation leaking out through the pinhole is characteristic of the radiation within the container.

- **Wien's law** (Wien's displacement law, 1893)

$$T\lambda_{\max} = \text{const.} = 1/5c_2, c_2 = 1.44 \text{ cm}\cdot\text{K}$$

λ_{\max} : maximum distribution wavelength at T

e.g., at 1000 K $\rightarrow \lambda_{\max} \sim 2900 \text{ nm}$

Sunlight peak at $\sim 500 \text{ nm} \rightarrow T = 5800 \text{ K}$

- **Stefan-Boltzman law** (1879)

Total energy density ε , ($\varepsilon = E/V$, radiation energy per unit volume)

$$\varepsilon = aT^4$$

or

excitance M , (radiation power per unit surface, the brightness of the emission)

$$M = \sigma T^4$$

σ (Stefan-Boltzman constant): $5.67 \times 10^{-8} \text{ Wm}^{-2}\text{K}^{-4}$

e.g., 1 cm^2 surface at 1000 K radiate about 6 W (cf. $\text{W} = \text{J/s}$)

- Rayleigh-Jeans law

19 C, Rayleigh → classical approach: electromagnetic field as a collection of a oscillators of all possible frequencies

→ energy distribution calculation from mean energy $\langle E \rangle = kT$ for each oscillator

The electromagnetic vacuum can be regarded as able to support oscillations of the electromagnetic field. When a high-frequency short-wavelength oscillator (a) is excited, that frequency of radiation is present. The presence of low-frequency long-wavelength radiation (b) signifies that an oscillator of the corresponding frequency has been excited.

cf. 온도 T 에서 평균 에너지 = $\frac{1}{2}kT$

Translation: $E = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) = \frac{3}{2}kT$

↑
linear momentum

Rotation: $E = \frac{J^2}{2I} = \frac{1}{2}kT$

Vibration: $E = \frac{p^2}{2m} + \frac{1}{2}kx^2 = kT \Rightarrow$ "1-D oscillator"

Thermal equilibrium에서 population

$$P(E) = Ae^{-E/kT} \quad \text{"Boltzman distribution law"}$$

Rayleigh-Jeans law

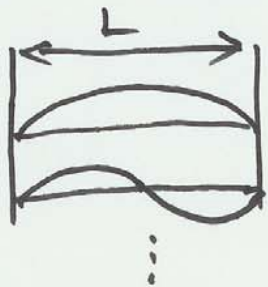
$$d\varepsilon = \rho d\lambda, \quad \rho = (8\pi \langle E \rangle / \lambda^4) d\lambda = (8\pi kT / \lambda^4) d\lambda$$

ε : energy density, ρ : proportionality constant,

k : Boltzmann constant ($1.381 \times 10^{-23} \text{ JK}^{-1}$)

The Rayleigh-Jeans law predicts an infinite energy density at short wavelengths. This prediction is called the ultraviolet catastrophe.

Why $\rho = \frac{8\pi}{\lambda^4} \langle E \rangle$?



$$L = n \left(\frac{\lambda}{2} \right), \quad n = 0, 1, 2, \dots$$

$$\lambda = \frac{2L}{n}, \quad n = \frac{2L}{\lambda}$$

if $L \gg \lambda$, n is large number. ($n(\lambda)$)

$$\lambda \rightarrow \lambda + d\lambda \text{ 사이 } \text{oscillator 수} : d n = - \frac{2L}{\lambda^2} d\lambda$$

1-D 를 3-D 로 확장하면 각기 부피당 $\lambda \rightarrow \lambda + d\lambda$ 사이 oscillator 수는

$$d n = \frac{8\pi}{\lambda^4} d\lambda$$

각 oscillator 의 평균에너지 $\langle E \rangle$ 라 하면

$$d\mathcal{E} = \langle E \rangle d n = \frac{8\pi}{\lambda^4} \langle E \rangle d\lambda$$

|||
 ρ

⇒ quite successful at long λ

⇒ but it fails at lower λ (UV, X-rays...) “Ultraviolet catastrophe”(자외선 파탄)

or $\lambda = c/\nu$, $\lambda\nu = c \rightarrow d\nu = -c(d\lambda/\lambda^2) \rightarrow d\lambda = -\lambda^2 d\nu/c$

$$d\varepsilon = (8\pi\nu^2 kT/c^3) d\nu$$

- classical mechanics: even cool objects should radiate in the visible and UV regions → no darkness even at low T (?)

- **The Planck distribution**

- In 1900, Max Planck propose each oscillator is not continuous → **energy quantization**

(proposing that the energy of each oscillator is limited to discrete values and cannot be varied arbitrarily)

cf) classical mechanics: all possible energies are allowed

$E = h\nu, 2h\nu, 3h\nu, \dots$ (integer multiples of $h\nu$)

$E = nh\nu, n = 0, 1, 2, 3, \dots$

h : Planck constant, 6.626×10^{-34} J·s

$\lambda\nu = c$, λ : wavelength, ν : frequency

- classical mechanics

average energy $\langle E \rangle$

$$\langle E \rangle = \frac{\int_0^{\infty} EP(E) dE}{\int_0^{\infty} P(E) dE}, \quad P(E) = Ae^{-E/kT}$$

put $-\frac{1}{kT} = a$, use $\int_0^{\infty} e^{ax} dx = \frac{e^{ax}}{a}$, $\int x e^{ax} dx = \frac{e^{ax}}{a^2}(ax-1)$

$$\frac{N}{Z}: \int_0^{\infty} P(E) dE = A \int_0^{\infty} e^{+a \cdot E} = \frac{A}{a} e^{+a \cdot E} = -kTA \cdot e^{-E/kT} \Big|_0^{\infty} = kT \cdot A$$

$$\begin{aligned} \frac{N}{Z} \langle E \rangle: \int_0^{\infty} EP(E) dE &= A \int_0^{\infty} E \cdot e^{+aE} \cdot dE = A \cdot (kT)^2 \cdot e^{-E/kT} \left(-\frac{E}{kT} - 1 \right) \Big|_0^{\infty} \\ &= (kT)^2 \cdot A \end{aligned}$$

$$\Rightarrow \langle E \rangle = \frac{(kT)^2 \cdot A}{kT \cdot A} = \underline{\underline{kT}}$$

- Planck

average energy $\langle E \rangle$

$$\langle E \rangle = \frac{\int_0^{\infty} E P(E) dE}{\int_0^{\infty} P(E) dE}, \quad P(E) = A e^{-E/kT}, \quad E = n h \nu, \quad n = 0, 1, 2, 3, \dots$$

$$\langle E \rangle = \frac{A(0 + h\nu e^{-h\nu/kT} + 2h\nu e^{-2h\nu/kT} + \dots)}{A(1 + e^{-h\nu/kT} + e^{-2h\nu/kT} + \dots)}$$

$$\text{if } e^{-h\nu/kT} = d < 1, \quad 1 + d + d^2 + \dots = \frac{1}{1-d}$$

$$\begin{aligned} \text{分子 (numerator): } h\nu (d + 2d^2 + 3d^3 + \dots) &= h\nu \cdot d (1 + 2d + 3d^2 + \dots) \\ &= h\nu \cdot d \cdot \frac{d}{dd} (d + d^2 + d^3 + \dots) = h\nu \cdot d \cdot \frac{d}{dd} \left(\frac{d}{1-d} \right) \\ &= h\nu \cdot d \cdot \frac{1 \cdot (1-d) + d}{(1-d)^2} = \frac{d h \nu}{(1-d)^2} \end{aligned}$$

$$\text{分母 (denominator): } \frac{1}{1-d}$$

$$\langle E \rangle = \frac{h\nu \alpha / (1-\alpha)^2}{1/(1-\alpha)} = h\nu \frac{\alpha}{1-\alpha} = h\nu \frac{e^{-h\nu/kT}}{1 - e^{-h\nu/kT}}$$

$$= h\nu \frac{1}{e^{h\nu/kT} - 1} = \left(h \frac{c}{\lambda} \right) \frac{1}{e^{hc/\lambda kT} - 1}$$

$$\therefore \nu = \frac{c}{\lambda}$$

$$dE = \rho d\lambda = \left(\frac{8\pi \langle E \rangle}{\lambda^4} \right) d\lambda$$

$$\therefore \rho = \frac{8\pi hc}{\lambda^5} \left(\frac{1}{e^{hc/\lambda kT} - 1} \right)$$

The Planck distribution accounts very well for the experimentally determined distribution of radiation. Planck's quantization hypothesis essentially quenches the contributions of high-frequency, short-wavelength oscillators. The distribution coincides with the Rayleigh-Jeans distribution at long wavelengths.

- Short λ ($\lambda \rightarrow 0$) $\Rightarrow \frac{hc}{\lambda kT} \rightarrow \infty \Rightarrow e^{\frac{hc}{\lambda kT}} \gg \frac{1}{\lambda^5}$
 $\xrightarrow{\infty}$ faster than $\xrightarrow{\infty}$

$\therefore \rho = 0$

(R-J law와 다른 것)

~~long~~ - long $\lambda \Rightarrow \frac{hc}{\lambda kT} \ll 1 \Rightarrow e^{\frac{hc}{\lambda kT}} - 1 = \left(1 + \frac{hc}{\lambda kT} + \dots\right) - 1 \simeq \frac{hc}{\lambda kT}$

$\therefore \rho = \frac{8\pi hc}{\lambda^5} \cdot \frac{\lambda kT}{hc} = \frac{8\pi}{\lambda^4} kT \Rightarrow$ "Rayleigh-Jeans law"
 (C.M.와 동일)

- $\frac{d\rho}{d\lambda} = 0 \Rightarrow \lambda_{\max}$ (max. λ) \Rightarrow "Wien's law"

- $\epsilon = \int_0^{\infty} d\epsilon = \int_0^{\infty} \rho d\lambda \Rightarrow$ "Stefan-Boltzman law"

2. Heat capacity

- Dulong & Petit's law (19 C)

monatomic solid: $\langle E \rangle = kT$ for each direction $\rightarrow 3kT$ for 3-D

N atoms: molar internal energy $U_m = 3N_A kT = 3RT$

constant volume heat capacity

$$C_{V,m} = (\partial U_m / \partial T)_V = 3R (= 24.9 \text{ JK}^{-1}\text{mol}^{-1})$$

Deviation at low temperature: $T \rightarrow 0 \Rightarrow C_{V,m} \rightarrow 0$

- Einstein formula (1905)
 - all the atoms oscillate with the same frequency → at low T, few oscillators possess energy to oscillate; T ↑, enough energy for all the oscillators
 - using Planck's hypothesis ($E = nh\nu$)
all $3N$ atomic oscillators → vibrational energy of crystal: $3N\langle E \rangle$

put $\theta_E = \frac{h\nu}{k}$, "Einstein temperature"

$$C_{v,m} = 3R \left(\frac{\theta_E}{T} \right)^2 \frac{e^{\theta_E/T}}{(1 - e^{\theta_E/T})^2} = 3Rf^2, \quad f = \frac{\theta_E}{T} \left(\frac{e^{\theta_E/2T}}{e^{\theta_E/T} - 1} \right)$$

at high T ($T \gg \theta_E$)

$$f = \frac{\theta_E}{T} \left\{ \frac{1 + \theta_E/2T + \dots}{(1 + \frac{\theta_E}{T} + \dots)^{-1}} \right\} \approx 1 \quad \therefore C_{v,m} = 3R \quad \left(\begin{array}{l} \text{classical Mechanics} \\ \frac{F}{E} \\ \text{Dulong \& Petit's law} \end{array} \right)$$

at low T ($T \ll \theta_E$)

$$f \approx \frac{\theta_E}{T} \left(\frac{e^{\theta_E/2T}}{e^{\theta_E/T}} \right) = \frac{\theta_E}{T} e^{-\theta_E/2T}$$

$e^{-\theta_E/2T} \rightarrow 0$ more rapidly than $\frac{1}{T} \rightarrow \infty$

$$\therefore T \rightarrow 0, f \rightarrow 0 \Rightarrow C_{v,m} \rightarrow 0$$

Experimental low-temperature molar heat capacities and the temperature dependence predicted on the basis of Einstein's theory. His equation accounts for the dependence fairly well, but is everywhere too low.

- still poor in experimental data since Einstein assumed all the atoms oscillate with the same frequency

- Debye formula

- consider to oscillate $\nu \rightarrow 0$ to ν_D

- (in real crystal atoms are coupled by the interatomic forces and do not oscillate independently)

Debye's modification of Einstein's calculation gives very good agreement with experiment. For copper, $\theta_D = 2$ corresponds to about 170 K, so the detection of deviations from Dulong and Petit's law had to await advances in low-temperature physics.

- Quantization must be introduced in order to explain thermal properties of solids

$$C_{v,m} = 3Rf, \quad f = 3 \left(\frac{T}{\theta_D} \right)^3 \int_0^{\theta_D/T} \frac{x^4 \cdot e^x}{(e^x - 1)^2} \cdot dx$$

where, $\theta_D = \frac{h\nu_D}{k}$ (Debye Temp.), $x = \frac{h\nu}{kT}$

high T ($T \gg \theta_D$): x is small $\Rightarrow \int_0^{\theta_D/T} \frac{x^4 \cdot e^x}{(e^x - 1)^2} dx = \frac{1}{3} \left(\frac{\theta_D}{T} \right)^3$

$$\therefore C_{v,m} = 3R$$

low T ($T \ll \theta_D$): $\int_0^{\theta_D/T} \frac{x^4 \cdot e^x}{(e^x - 1)^2} dx = \frac{4}{5} \pi^4$

$$\therefore C_{v,m} = 3R \cdot \left(\frac{12}{5} \pi^4 \right) \left(\frac{T}{\theta_D} \right)^3 : \text{"Debye } T^3\text{-law"}$$

\therefore Improved agreement with experiment

3. The photoelectric effect

Philipp Lenard (1862-1947): the light ejects electrons from the metal (**photoelectron**)

Classical viewpoint would assume that the light waves beat on the metal surface like ocean waves and the electrons are distributed like pebbles on a beach. **More intense illumination (i.e. brighter) would deliver more energy to the electrons.** However,.....

1902 Lenard: the electron energy were entirely independent of the light intensity. Further, there was a certain threshold frequency below which no photoelectron were ejected, no matter how bright the light beam.

Albert Einstein (1905) showed that the puzzle of photoelectric effect are easily explained once the illuminating radiation is a collection of particles (photons): qV (kinetic energy of electron) = $h\nu$ (energy of the incoming photon) – P (work to get out of the metal)

In the photoelectric effect, it is found that no electrons are ejected when the incident radiation has a frequency below a value characteristic of the metal and, above that value, the kinetic energy of the photoelectrons varies linearly with the frequency of the incident radiation.

- Einstein (1905-6)

- $\nu < \nu_0$ (threshold ν): no emission, even at strong radiation intensity
 $\nu > \nu_0$: electron emission even at very low intensity

- kinetic energy of ejected electron $\propto \nu$, independent of radiation intensity
if $h\nu > \Phi$ (work function): electron emission

$$\frac{1}{2}m_e v^2 = h\nu - \Phi$$

- Φ : the energy required to remove an electron from the metal to infinity
→ energy depended on the frequency of the incident light $\Rightarrow nh\nu$

The photoelectric effect can be explained if it is supposed that the incident radiation is composed of photons that have energy proportional to the frequency of the radiation. (a) The energy of the photon is insufficient to drive an electron out of the metal. (b) The energy of the photon is more than enough to eject an electron, and the excess energy is carried away as the kinetic energy of the photoelectron (the ejected electron).

4. Atomic & molecular spectra

- spectrum: radiation absorbed or emitted by atoms & molecules

A region of the spectrum of radiation emitted by excited iron atoms consists of radiation at a series of discrete wavelengths (or frequencies).

- radiation is emitted or absorbed at a series of discrete frequencies
- energy of atoms/molecules is confined to discrete values

Spectral lines can be accounted for if we assume that a molecule emits a photon as it changes between discrete energy levels. Note that high-frequency radiation is emitted when the energy change is large.

- Balmer (1885): visible spectrum of atomic hydrogen
 $1/\lambda = R_H(1/2^2 - 1/n^2)$, $n = 3, 4, \dots$; empirical expression
 R_H : Rydberg const ($= 1.09678 \times 10^5 \text{ cm}^{-1}$)

Lyman: UV series, $1/\lambda = R_H(1/1^2 - 1/n^2)$,

Paschen: IR, $1/\lambda = R_H(1/3^2 - 1/n^2)$,

- Why lines? \Rightarrow Bohr (1913): Planck quantum hypothesis + classical mechanics
- Bohr's hypothesis
 - (1) electrons exist in a discrete set of stable, stationary orbits in the atom
 - cf) perfect orbit: different from that in quantum mechanics
 - (2) transition between orbits: $\Delta E = h\nu \rightarrow$ frequency $\nu = \Delta E/h$
 - (3) dynamical equilibrium between proton and electron
 - electrostatic attraction force = centrifugal force
 - (4) angular momentum of orbital: quantized

$$\vec{L} = \vec{r} \times \vec{p} = r_e m v = \frac{n h}{2\pi} \quad n=1, 2, \dots \quad \text{--- (2)}$$

$$\left(\begin{array}{l} \text{Eq. 2} \\ \frac{1}{2} m v^2 = \frac{1}{2} m r^2 \omega^2 = \frac{1}{2} I \omega^2 \\ v = r \omega \\ L = I \omega = m r^2 \cdot \frac{v}{r} = m v r \end{array} \right)$$

From (1)

$$r_e = \frac{1}{4\pi \epsilon_0} \frac{e^2}{m v^2} = \frac{1}{4\pi \epsilon_0} \frac{e^2}{m v^2} \quad \text{--- (3)}$$

$$\text{(2) } \oplus v = \frac{n h}{r_e m \cdot 2\pi} \quad \text{(3) } \ominus \text{ or } \text{or } r_e = \frac{1}{4\pi \epsilon_0} \frac{e^2}{m} \cdot \frac{r_e^2 \cdot m^2 \cdot 4\pi^2}{n^2 h^2} = \frac{n^2 h^2 \epsilon_0}{e^2 m \pi}$$

$n=1 \Rightarrow r_e = a_0 \approx 0.53 \text{ \AA}$, radius of atom "Bohr radius" $\approx \frac{h^2}{4\pi^2}$

$$E = E_k + V = \frac{1}{2} m v^2 - \frac{1}{4\pi \epsilon_0} \frac{e^2}{r_e}$$

$$= \frac{1}{8\pi \epsilon_0} \frac{e^2}{r_e} - \frac{1}{4\pi \epsilon_0} \frac{e^2}{r_e} = - \frac{1}{8\pi \epsilon_0} \frac{e^2}{r_e} \leftarrow$$

(1/2 * 1/4)

$$= - \frac{1}{8\pi \epsilon_0} e^2 \cdot \frac{e^2 m \pi}{n^2 h^2 \epsilon_0} = - \frac{m e^4}{8 \epsilon_0^2 h^2} \cdot \frac{1}{n^2}$$

$m \rightarrow \mu$ (reduced mass = $\frac{m_e m_p}{m_e + m_p}$)

$$V = \frac{\Delta E}{h} = - \left(\frac{\mu e^4}{8 \epsilon_0^2 h^3} \right) \left(\frac{1}{n^2} - \frac{1}{m^2} \right) = - R \left(\frac{1}{n^2} - \frac{1}{m^2} \right)$$

⇒ Energy is quantized

- Only explain hydrogen (one-electron) spectra
 - 1) fail to explain the spectra of atoms more than one electron
 - 2) incorrect to regard the electrons in atoms as discrete particles with precise positions and velocities