CHAPTER 4. Steady Electric Currents

Reading assignments: Cheng Ch.4, Ulaby Ch.3, Hayt Ch 5, Halliday Chs.26-27

1. Electric Currents and Ohm's Law

A. Electric Currents Caused by Moving Charges

1) Convection current by mass transfer (hydrodynamic motion) of net charge in charged medium (or in a vacuum or rarefied gas)
2) Conduction current by charge carrier drift in neutral medium (metal) governed by Ohm's law

- Charge carriers (species)
  - Dielectrics or insulators (quartz, paraffin, glass): polarized charges
  - Metals (Cu, Ag, Fe): free (or valence) electrons in crystal structure
  - Plasma, gas or electrolyte: electrons, ions (cation, anion)
  - Intrinsic (pure) semiconductors (Ge, Si): electrons, holes
  - n-type semiconductors (IV+V donor): electrons > holes
  - p-type semiconductors (IV+III acceptor): holes > electrons

B. Convection Current

Consider charged particles moving with velocity $\boldsymbol{u}$ due to fluid or particle motion.

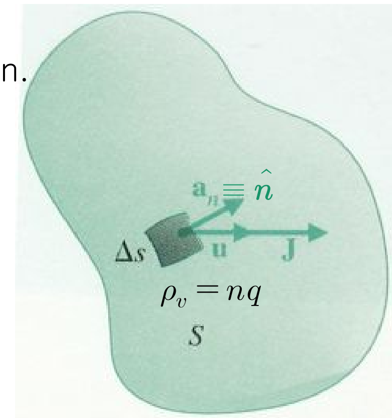


FIGURE 4-1

Electric current passing through Δs:

$$\Delta I = \frac{\Delta Q}{\Delta t} = \frac{nq\boldsymbol{u}\cdot\hat{\boldsymbol{n}}\,\Delta s\,\Delta t}{\Delta t} = \rho_v \boldsymbol{u}\cdot\Delta\boldsymbol{s}$$

Convection current density:

$$\boldsymbol{J} \equiv \frac{\Delta I}{\Delta \boldsymbol{s}} = nq\boldsymbol{u} = \rho_v \boldsymbol{u} \quad (\text{A/m}^2) \qquad (4\text{-}3,\ 4\text{-}6)$$

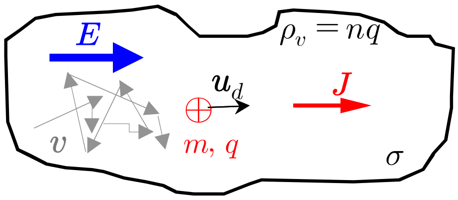
Total current flowing thru S:

$$I = \int_S \boldsymbol{J}\cdot d\boldsymbol{s} \quad (\text{A}) \qquad (4\text{-}5)$$

C. Conduction Current

1) Mobilty

Consider free charge carriers moving with drift velocity $\boldsymbol{u}_d$ due to collisions in the medium in externally applied $\boldsymbol{E}$.



Equation of motion for a single charge carrier:

$$m\frac{d\boldsymbol{u}_d}{dt} = q\boldsymbol{E} - b\boldsymbol{u}_d \qquad \text{(1)}$$

resisting force due to collisions with lattice atoms

where $\boldsymbol{u}_d(t=0) = \mathbf{0}$: initial condition (IC)

Steady-state $(\frac{d}{dt}=0)$ drift velocity from (1):

$$\boldsymbol{u}_d = \mu\boldsymbol{E} \qquad \text{(4-8)}$$

where $\mu \equiv q/b$ is the mobility of the charge carrier.

For a transient period after applying $\boldsymbol{E}$, the solution of (1) with IC is

$$\boldsymbol{u}_d(t) = \mu\boldsymbol{E}(1-e^{-t/\tau}) \qquad \text{(3)}$$

where $\tau \equiv m/b$ is the relaxation time or mean collision time.

Notes) i) (3) $\rightarrow$ (4-8) as $t \rightarrow \infty$.

ii) $\tau \approx 10^{-14}$ s in metals, 10^{-13} in semiconductors

Therefore, the mobility of the charge carrier is

$$\mu = \frac{q\tau}{m} = \frac{q}{m\nu} \quad (\text{m}^2/\text{V·s}) \qquad \text{(4)}$$

collision frequency

Notes) i) $\mu_e^{semicon} \approx 10\,\mu_h^{semicon} \approx 100\,\mu_e^{metal}$

ii) $\mu_e \gg \mu_i$ in plasmas

	Mobility, $m^2 V^{-1} s^{-1}$		Charge density, $C m^{-3}$		
	Electrons μ_e	Holes μ_h	Electrons $\rho_e\ (=\rho_h)$	Conductivity $\mho m^{-1}$	Relative permittivity, dimensionless
Semiconductors (pure):					
Germanium (Ge)	0.39	0.19	4	2.3	12
Silicon (Si)	0.14	0.05	0.002	0.0004	16
Gallium-arsenide (GaAs)	0.68	0.07			11
Indium-antimony (InSb)	8.0	0.40	2×10^3	1.7×10^4	16
Conductors (pure):					
Aluminum (Al)	0.0014		2.5×10^{10}	3.5×10^7	
Copper (Cu)	0.0040		1.4×10^{10}	5.7×10^7	
Silver (Ag)	0.0050		1.2×10^{10}	6.1×10^7	

2) Current density and Ohm's law – Field equation

For a single charge carrier in a linear medium, the current density of the carrier is

$$\boldsymbol{J} \equiv \frac{\triangle I}{\triangle s} = nq\boldsymbol{u_d} = \rho_v \boldsymbol{u_d} = \rho_v \mu \boldsymbol{E}$$

$\therefore\ \boldsymbol{J} = \sigma\,\boldsymbol{E} = \boldsymbol{E}/\eta$: point form of Ohm's law (Constitutive relation) (4-10)

where $\sigma \equiv \rho_v \mu = nq\mu = \dfrac{nq^2}{m\nu}$ (S/m or ℧/m) : conductivity (5)

$\eta \equiv \dfrac{1}{\sigma} = \dfrac{m\nu}{nq^2}$ (V·m/A or Ω·m) : resistivity (6)

Notes)

i) For multi-species (s=e, i, h, ...),

$$\boldsymbol{J} = \sum_{s=e,i,h} \rho_s \boldsymbol{u_{ds}} = \sum_s n_s q_s \boldsymbol{u_{ds}} = \left(\sum_s n_s q_s \mu_s\right)\boldsymbol{E} = \sigma \boldsymbol{E} \qquad \text{(4-8), (4-10)*}$$

where $\sigma = \sum_{s=e,i,h} \rho_s \mu_s = \sum_s n_s q_s \mu_s$ (5)*

$s=e$ in metal; $s=e,h$ in semiconductor; $s=e,i$ in plasma and electrolyte

ii) In general, $J = \sigma(E)\boldsymbol{E}$ (nonlinear) and $J = \overleftrightarrow{\sigma}\cdot\boldsymbol{E}$ (anisotropic)

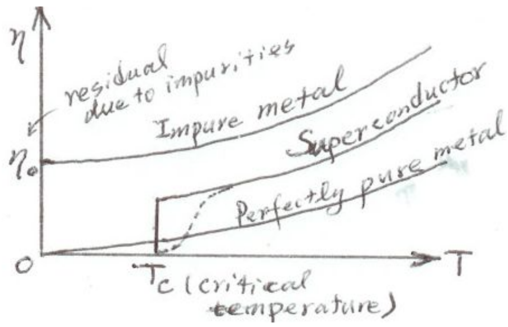
iii) Temperature dependence of η (or σ)

Cause of electric resistance in metals:

Scattering of conduction electrons ① by thermal vibration of lattice atoms, and ② by any impurities or geometric imperfections

$\eta(T) = \eta_o[1+\alpha(T-T_o)]$

$\Rightarrow \quad \alpha = \dfrac{1}{\eta}\dfrac{d\eta}{dT} \sim \dfrac{1}{T}$ (7)



(eg) α at $20^o C$

Cu 0.0068, Au 0.004

Nichrome 0.0004

NaCl solution -0.005

Ge (pure) -0.048

(eg) T_c: NbTi 9.5 K, Nb_3Sn 18.2 K

iv) In unmagnetized plasmas,

① $\eta = \dfrac{m_e \nu_{en}}{ne^2} \propto \dfrac{n_n}{n}$ for weakly-ionized plasmas

② $\eta \equiv \dfrac{m_e \nu_{ei}}{ne^2} = \dfrac{e^2 m_e^{1/2} \ln\Lambda}{16\pi\epsilon_o^2 (kT_e)^{3/2}}$ for fully-ionized plasmas (6)*

collisionless at high T

(eg) $\eta = 5\times 10^{-7}$ Ω·m at $T_e = 10^6 K$

(cf) $\eta_{cu} = 2\times 10^{-8}$, $\eta_{s.s} = 7\times 10^{-7}$

3) Current and Ohm's law – Circuit equation

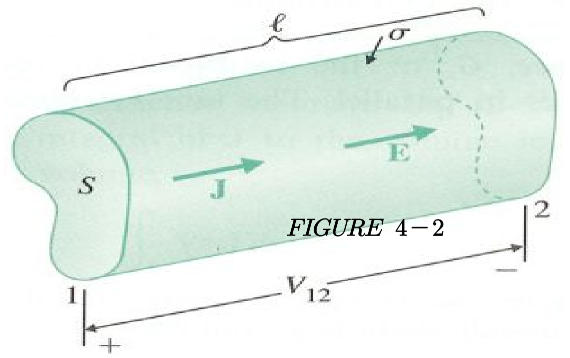
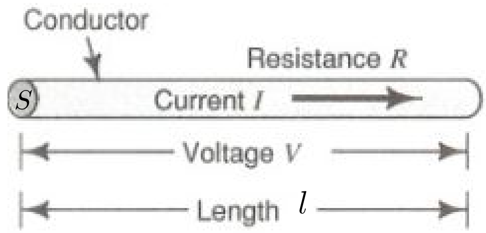
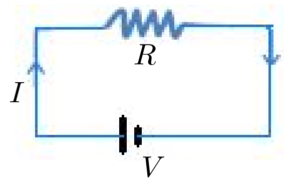


FIGURE 4−2



Equivalent circuit



For uniform fields,

$$I = \int_S \boldsymbol{J} \cdot d\boldsymbol{s} = JS \overset{(4-10)}{=} \sigma ES \overset{V_{12}=El}{=} \left(\frac{\sigma S}{l}\right) V_{12}$$

$\Longrightarrow$ $V_{12} = RI$ or $I = G V_{12}$: Ohm's law (4-15)

where $R = \dfrac{l}{\sigma S} = \eta \dfrac{l}{S}$ (Ω) : Resistance (4-16)

$G \equiv \dfrac{1}{R} = \sigma \dfrac{S}{l}$ (S or ℧) : Conductance (4-17)

2. Equation of Current Continuity and Kirchhoff's Current Law

A. Charge Conservation and Current Continuity Equation

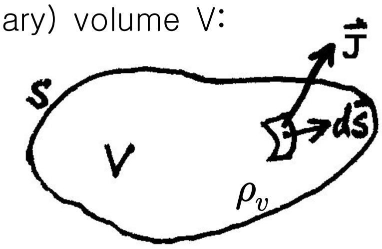
Total charge conservation over a control (stationary) volume V:

Time rate of charge accumulation in V

= Inflow of current thru S

– Outflow of current thru S

= Net inflow of current thru S



$$\Rightarrow \frac{dQ}{dt} = -\oint_S \boldsymbol{J} \cdot d\boldsymbol{s} \Rightarrow \frac{d}{dt}\int_V \rho_v \, dv = -\int_V (\nabla \cdot \boldsymbol{J}) \, dv$$

(divergence theorem)

$$\Rightarrow \int_V \left(\frac{\partial \rho_v}{\partial t} + \nabla \cdot \boldsymbol{J}\right) dv = 0$$

$$\Rightarrow \frac{\partial \rho_v}{\partial t} + \nabla \cdot \boldsymbol{J} = 0$$ (A/m^3): Equation of current continuity (4-20)

(cf) Hydrodynamic equation of continuity (Mass conservation):

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\boldsymbol{u}) = 0 \quad \Longrightarrow \quad \frac{\partial n}{\partial t} + \nabla \cdot \boldsymbol{\Gamma} = 0$$ (7)

$\boldsymbol{\Gamma} \equiv n\boldsymbol{u}$: particle flux

Note) $q \times (7)$ becomes (4-20).

Circuit equation form of (4-20):

$$-\oint_S \boldsymbol{J} \cdot d\boldsymbol{s} = \int_V \frac{\partial \rho}{\partial t} dv$$

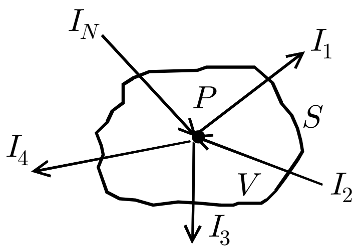
$$\Rightarrow \quad I = \frac{dQ}{dt} \qquad (8)$$

For steady ($\partial / \partial t = 0$) currents,

$$\nabla \cdot \boldsymbol{J} = 0 \qquad (4\text{-}21)$$

Notes) Steady electric currents are solenoidal (no flow source $\rho_v = 0$), and current streamlines close upon themselves.

B. Kirchhoff's Current Law



□$\int_V (4-21)\, dv$ with divergence theorem:

$$\oint_S \boldsymbol{J} \cdot d\boldsymbol{s} = 0 \qquad (4\text{-}22)$$

If conductors form a network inside volume and meet at a junction *P*,

$$\sum_j I_j = 0 \qquad (4\text{-}23)$$

(A) : Kirchhoff's 2nd law (Junction theorem)

Algebraic sum of all currents at a junction is zero

C. Electrostatic Equilibrium in a Conductor

: ∃ ρ_s on the conductor surface,

$\rho_v = 0$ & $\boldsymbol{E} = \boldsymbol{0}$ & $V = const$ inside the conductor

(Proof)

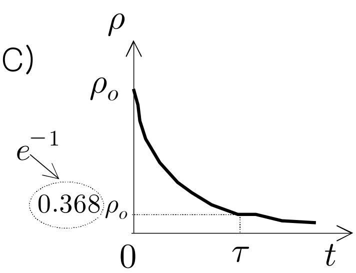
$J = \sigma\, \boldsymbol{E}$ (4-10) and $\nabla \cdot \boldsymbol{E} = \rho_v / \epsilon$ (3-63) in (4-20):

$$\frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\epsilon} \rho_v = 0 \qquad (4\text{-}25)$$

where $\rho_v(\boldsymbol{r}, t)|_{t=0} = \rho_o(\boldsymbol{r})$: initial condition (IC)

Solution:

$$\rho_v(\boldsymbol{r}, t) = \rho_o(\boldsymbol{r})\, e^{-t/\tau} \quad (\text{C/m}^3) \qquad (4\text{-}26)$$



$$\text{where } \tau = \frac{\epsilon}{\sigma} = \eta\epsilon \quad (\text{s}) \qquad (4\text{-}27)$$

: Relaxation time = a measure of how fast the medium reaches an electrostatic equilibrium

(eg) For Cu ($\sigma = 5.8 \times 10^7$, $\epsilon \approx \epsilon_o = 8.85 \times 10_{12}$), $\underline{\tau = 1.53 \times 10^{-19}\, s}$

$\Rightarrow$ Net charge within a conductor is zero.
If present, it must reside on the surface.

3. Governing Equations and BCs For Steady J

A. Governing Equations

For steady currents,

(4-21) $\Rightarrow \quad \nabla \cdot \boldsymbol{J} = 0$ (4-32)

(4-10) Ohm's law $\boldsymbol{J} = \sigma \boldsymbol{E} = \boldsymbol{E}/\eta$ in $\nabla \times \boldsymbol{E} = 0$

$$\Rightarrow \quad \nabla \times \left(\frac{\boldsymbol{J}}{\sigma} \right) = \mathbf{0} \quad \text{or} \quad \nabla \times (\eta \boldsymbol{J}) = \mathbf{0} \qquad (4\text{-}33)$$

Integral form:

$\int_V (4-32)\, dv$ and divergence theorem

$$\Rightarrow \quad \oint_S \boldsymbol{J} \cdot d\boldsymbol{s} = 0 \qquad (4\text{-}32)^*$$

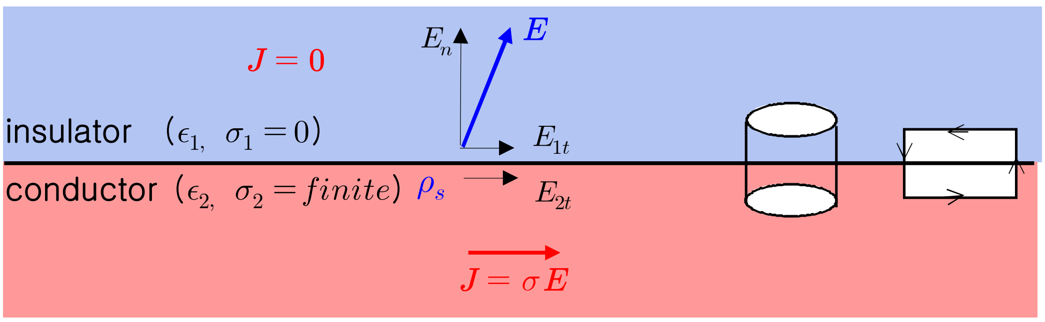
$\oint_S (4-33)\, d\boldsymbol{s}$ and Stokes's theorem

$$\Rightarrow \quad \oint_C \sigma^{-1} \boldsymbol{J} \cdot d\boldsymbol{l} = \oint_C \eta \boldsymbol{J} \cdot d\boldsymbol{l} = 0 \qquad (4\text{-}33)^*$$

Once (4-32) and (4-33) are specified, $\boldsymbol{J}$ is determined according to Helmholtz's theorem.

B. Boundary Conditions

1) Insulator-Conductor Interface



$$\nabla \times \left(\frac{\boldsymbol{J}}{\sigma} \right) = 0 \text{ and } \boldsymbol{J} = \sigma \boldsymbol{E} \quad \Rightarrow \quad J_{2t} = \sigma_2 E_{2t} \qquad (9)$$

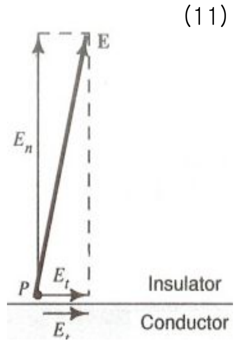
$$\nabla \times \boldsymbol{E} = 0 \quad \Rightarrow \quad E_{1t} = E_{2t} = J_{2t}/\sigma_2 \qquad (10)$$

(cf) $E_{1t} = E_{2t} = 0$ (3-45) for an electrostatics case with no $\boldsymbol{J}$

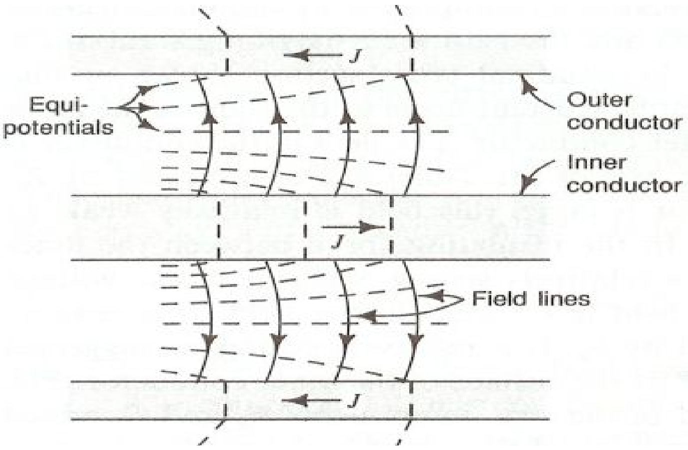
$$\nabla \cdot \boldsymbol{D} = \rho_v \quad \Rightarrow \quad D_{1n} = \rho_s \quad \Rightarrow \quad E_{1n} = \rho_s/\epsilon_1 \qquad (11)$$

Note)

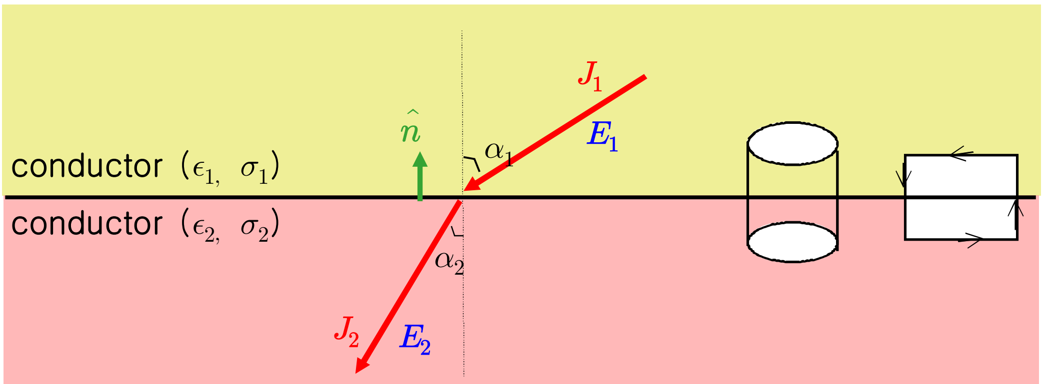
$$E_{1n} \gg E_{1t} \quad \text{since} \quad E_{1t} = E_{2t} = \frac{J_{2t}}{\sigma_2} = \frac{finite}{large} \rightarrow small$$



(eg) Coaxial transmission line



2) Conductor–Conductor Interface



For steady currents,

$$\nabla \cdot \boldsymbol{J} = 0 \quad \Rightarrow \quad J_{1n} = J_{2n} \quad \text{or} \quad \hat{\boldsymbol{n}} \cdot [\boldsymbol{J}_1 - \boldsymbol{J}_2] = 0 \qquad \text{(4-33)}$$

$$\sigma_1 E_{1n} = \sigma_2 E_{2n}$$

$$\nabla \times \left(\frac{\boldsymbol{J}}{\sigma}\right) = 0 \quad \Rightarrow \quad \frac{J_{1t}}{\sigma_1} = \frac{J_{2t}}{\sigma_2} \quad \text{or} \quad \hat{\boldsymbol{n}} \times \left[\frac{\boldsymbol{J}_1}{\sigma_1} - \frac{\boldsymbol{J}_2}{\sigma_2}\right] = 0 \qquad \text{(4-34)}$$

$$\boldsymbol{J} = \sigma \boldsymbol{E} \ \text{or} \ \nabla \times \boldsymbol{E} = 0 \quad \Rightarrow \quad E_{1t} = E_{2t} \quad \text{or} \quad \hat{\boldsymbol{n}} \times [\boldsymbol{E}_1 - \boldsymbol{E}_2] = 0$$

(4-34)*

Note)

(4-33) / (4-34) $\Rightarrow$

$$\frac{\tan\alpha_1}{\tan\alpha_2} = \frac{\sigma_1}{\sigma_2} = \frac{J_{1t}}{J_{2t}} = \frac{E_{2n}}{E_{1n}} \qquad \text{(12)}$$

: similar to Snell's Law of refraction $\left(\frac{\sin\alpha_1}{\sin\alpha_2} = \frac{n_1}{n_2} = \sqrt{\frac{\epsilon_1}{\epsilon_2}}\right)$ (7-118)

(cf) At a dielectric–dielectric interface *(e.g. 3-14)*,

$$\frac{\tan\alpha_1}{\tan\alpha_2} = \frac{\epsilon_1}{\epsilon_2} = \frac{E_{2n}}{E_{1n}} \qquad \text{(3-83)}$$

4. Ohmic Power Dissipation and Joule's Law

Work done by the electric field $\boldsymbol{E}$ on a charge carrier q moving a distance $\triangle \boldsymbol{l}$:

$$\triangle w = q\boldsymbol{E}\cdot \triangle \boldsymbol{l}$$

Power dissipated as ohmic (Joule) heating by collisions of charge carrier with lattice atoms:

$$p = \lim_{\triangle t \to 0} \frac{\triangle w}{\triangle t} = q\boldsymbol{E}\cdot \boldsymbol{u}_d \qquad (4\text{-}28)$$

Total power delivered to all charge carriers in a volume:

$$dP = \sum_{s=e,i,h} p_s = \boldsymbol{E}\cdot \left(\sum_s n_s q_s \boldsymbol{u}_{ds}\right)dv \underset{(4-8)}{=} \boldsymbol{E}\cdot \boldsymbol{J}\,dv$$

∴ Dissipated ohmic power density at a point is

$$\frac{dP}{dv} = \boldsymbol{E}\cdot \boldsymbol{J} \underset{\boldsymbol{J}=\boldsymbol{E}/\eta}{=} \eta J^2 = E^2/\eta \qquad (\mathrm{W/m^3}) \qquad (4\text{-}29)$$

Total electric power converted into heat becomes

$$P = \int_V \boldsymbol{E}\cdot \boldsymbol{J}dv = \int_L Edl \int_S Jds = VI \underset{V=IR}{=} I^2R = V^2/R \quad (\mathrm{W})$$

: Joule's law (4-30), (4-31)

5. Resistance Calculations

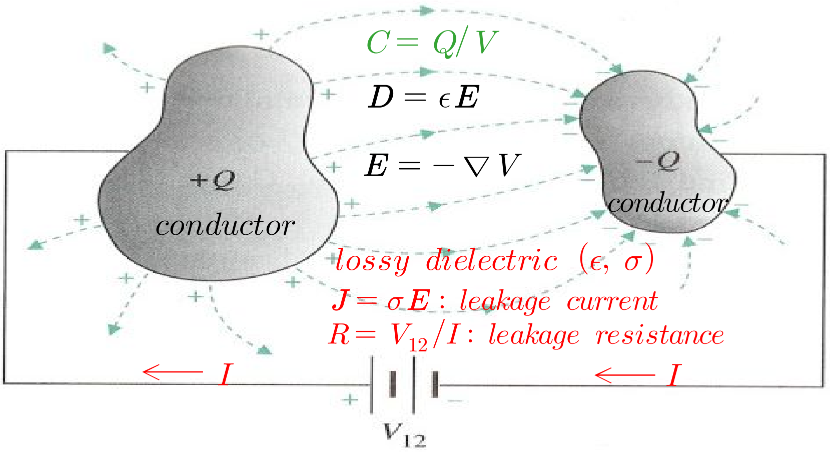


FIGURE 4−3

Capacitance : $C = \dfrac{Q}{V_{12}} = \dfrac{\oint_S \epsilon \boldsymbol{E}\cdot d\boldsymbol{s}}{-\int_L \boldsymbol{E}\cdot d\boldsymbol{l}}$ (4-36)

Resistance : $R = \dfrac{V_{12}}{I} = \dfrac{-\int_L \boldsymbol{E}\cdot d\boldsymbol{l}}{\oint_S \sigma \boldsymbol{E}\cdot d\boldsymbol{s}}$ from $\boldsymbol{E}$ obtained by BVPs (4-37)

$\Rightarrow \quad RC = \dfrac{C}{G} = \dfrac{\epsilon}{\sigma} \quad \Rightarrow \quad R = \dfrac{\epsilon}{\sigma}\left(\dfrac{1}{C}\right)$ from known C (4-38) *(e.g. 4-3)* Find the leakage resistance per unit length : R_1

a) For a lossy coaxial line,

(e.g. 3-16), (3-90) $\Rightarrow$

$$C_1 \equiv \frac{C}{L} = \frac{2\pi\epsilon}{\ln(b/a)}$$

(3-38) $\Rightarrow$

$$R_1 \equiv \frac{\epsilon/\sigma}{C_1} = \frac{1}{2\pi\sigma} ln\left(\frac{b}{a}\right) \quad (\Omega\cdot\text{m})$$

(4-39)

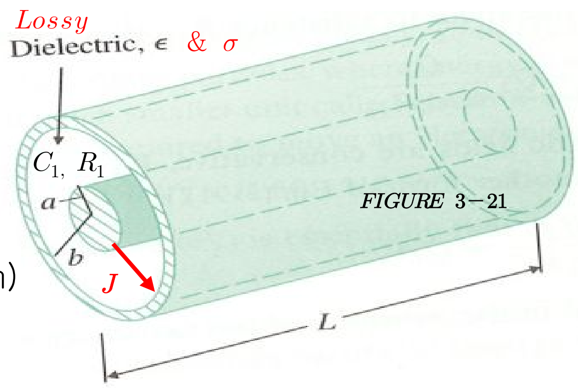


FIGURE 3-21

b) For a two-wire transmission line,

(e.g. 3-25), (3-165) $\Rightarrow$

$$C_1 \equiv \frac{C}{L} = \frac{\pi\epsilon_o}{\cosh^{-1}(D/2a)}$$

(3-38) $\Rightarrow$

$$R_1 \equiv \frac{\epsilon_o/\sigma}{C_1} = \frac{1}{\pi\sigma} cosh^{-1}\left(\frac{D}{2a}\right)$$

$$= \frac{1}{\pi\sigma} ln\left[\frac{D}{2a} + \sqrt{\left(\frac{D}{2a}\right)^2 - 1}\right] \quad (\Omega\cdot\text{m}) \qquad (4\text{-}40)$$

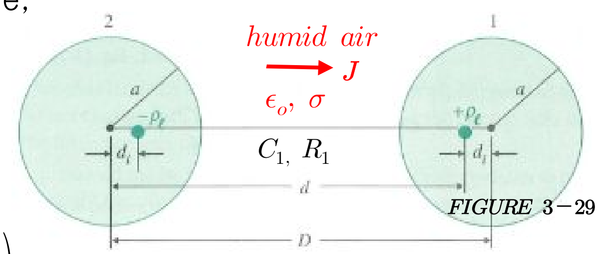


FIGURE 3-29

(e.g. 4-4) A quarter of a circular washer : $R = V/I = ?$

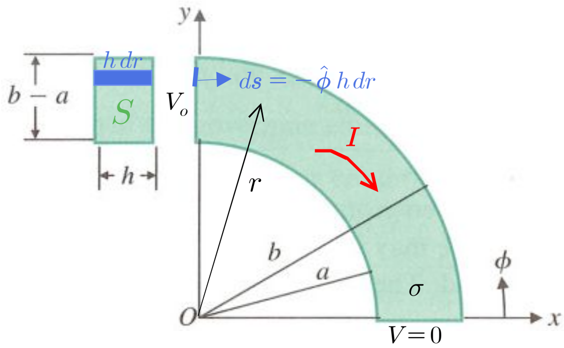
<u>BVP</u>

Laplace's equation:

$$\frac{d^2V}{d\phi^2} = 0, \quad 0 \le \phi \le \pi/2 \qquad ①$$

BCs: $V(\phi)|_{\phi=0,} = 0$ ②

$V(\phi)|_{\phi=\pi/2} = V_o$ ③



General solution : $V(\phi) = c_1\phi + c_2$ ④

②, ③ in ④: $c_2 = 0$, $c_1 = 2V_o/\pi$ ⑤

⑤ in ④: $$V(\phi) = \frac{2V_o}{\pi}\phi \qquad (4\text{-}43)$$

$$\boldsymbol{J} = \sigma \boldsymbol{E}(\phi) = -\sigma\nabla V = -\hat{\phi}\frac{\sigma}{r}\frac{dV}{d\phi} = -\hat{\phi}\frac{2\sigma V_o}{\pi r} \qquad (4\text{-}44)$$

$$I = \int_S \boldsymbol{J}\cdot d\boldsymbol{s} = \int_a^b J_\phi(r)(h\,dr) = \frac{2\sigma V_o}{\pi} h \int_a^b \frac{dr}{r} = \frac{2\sigma h V_o}{\pi} ln\frac{b}{a} \qquad (4\text{-}45)$$

$d\boldsymbol{s} = -\hat{\phi}\,h\,dr$

$$\therefore\ R = \frac{V_o}{I} = \frac{\pi}{2\sigma h \ln(b/a)} \quad (\Omega) \qquad (4\text{-}46)$$

Homework Set 5

1) P.4-3
2) P.4-6
3) P.4-7
4) P.4-11
5) P.4-12