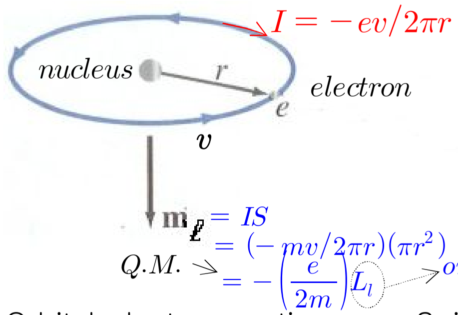
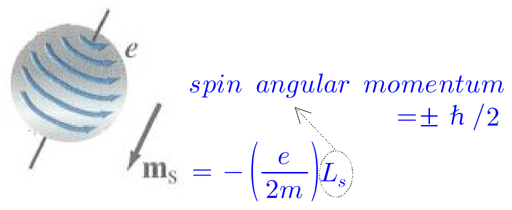
2. Magnetostatics in Magnetic Materials

A. Nature of Magnetic Materials

1) Sources of magnetic moment of an atom (ion) = Atomic currents



Orbital electron motion



Spinning electron (or nucleus)

a) Orbit magnetic dipole moment m_l due to an electron orbit motion

$$m_l = \left(\frac{e}{2m}\right)L_l \approx \left(\frac{e}{2m}\right)\sqrt{\frac{e^2 mr}{4\pi\epsilon_o}} \approx 9.1\times 10^{-24} \quad (\text{A}\cdot\text{m}^2) \qquad (2)$$

b) Spin magnetic dipole moment m_s due to an electron spin motion

$$m_s = \left(\frac{e}{2m}\right)L_s \approx \left(\frac{e}{2m}\right)\left(\pm\frac{\hbar}{2}\right) \approx \pm 9.3\times 10^{-24} \quad (\text{A}\cdot\text{m}^2) \qquad (3)$$

$\exists$ spin-up $\Uparrow$ $(+\frac{\hbar}{2})$ & spin-down $\Downarrow$ $(-\frac{\hbar}{2})$ in pairs

$\Rightarrow$ Only unfilled shell electrons contribution

c) Nuclear spin dipole moment m_n due to the spin motion of a nucleus

$$m_n = \left(\frac{Ze}{2M}\right)L_s \approx \left(\frac{Ze}{2M}\right)\left(\pm\frac{\hbar}{2}\right) \ll m_s \quad : \quad \text{negligible}$$

2) Classification of magnetic materials *(cf)* Appendix B-5

a) Nonmagnetic material $(\mu = \mu_o,\ \mu_r \equiv \mu/\mu_o = 1+\chi_m = 1,\ \chi_m = 0)$

Vacuum (Free space) *relative permeability* *magnetic suceptibility*

b) Weakly magnetic materials $(\mu \approx \mu_o,\ \mu_r \approx 1)$ $\approx$ Nonmagnetic materials

▶ Diamagnetic materials $(\mu_r \lesssim 1,\ -10^{-4} \lesssim \chi_m \lesssim -10^{-6})$

: Bi, Au, Ag, Cu, Pb, Ge, H_2O, H, He, Inert gas, Plasmas, ...

In $B_{ext} = 0$, $m_i = m_l + m_s = 0$ and $m = \sum_i m_i = 0$ due to $m_l = -m_s$

In $B_{ext} \neq 0$, $m_i = m_l + m_s \neq 0$ and $\sum_i m_i = m \upharpoonleft\downharpoonright B_{ext}$ due to $m_l < -m_s$

$$\Rightarrow \quad B \lesssim B_{ext}$$

Experiment : Bi repelled by bar magnet (M. Faraday 1846)

▶ Paramagnetic materials ($\mu_r \gtrsim 1$, $10^{-6} \lesssim \chi_m \lesssim 10^{-4}$)

: Ti, Pd, Al, Mg, W, Air, ...

In $B_{ext} = 0$, $m_i = m_l + m_s \neq 0$ *but* $m = \sum_i m_i = 0$ due to random orientation

In $B_{ext} \neq 0$, $m_i = m_l + m_s \neq 0$ and $\sum_i m_i = m \upuparrows B_{ext}$ due to partial parallel alignment m_i to B_{ext}

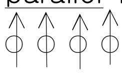
$\Rightarrow \quad B \gtrsim B_{ext}$: Attracted to bar magnet

c) Strongly magnetic materials

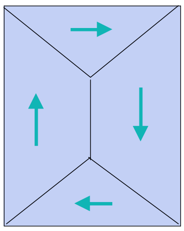
▶ Ferromagnetic materials ($\mu \gg \mu_o$, $\mu_r \gg 1$)

: Fe, Co, Ni, Mumetal (75Ni, 5Cu, 2Cr), Super alloy (5Mo, 79Ni), ...

∃ Strong m ($m_l < m_s$) + Exchange coupling (Interatomic forces cause m to line up in a parallel fashion over a region)



$\Rightarrow$ Domain structure



Single crystal
(anisotropic)

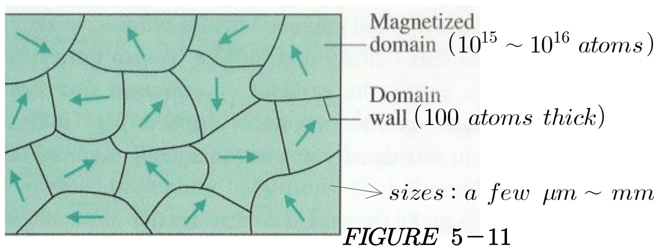
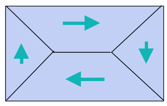


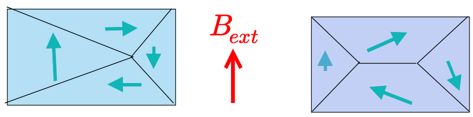
FIGURE 5−11

Polycrystalline material
(isotropic)

In $B_{ext} = 0$, $m = 0$ due to random domain orientations



In $B_{ext} \neq 0$, $m \neq 0$ due to domain-wall motion and domain rotation



$\Rightarrow \quad B \gg B_{ext}$ (induced magnetization)

After removal of B_{ext}, $m \neq 0$ (permanent magnetization) due to residual dipole field (B_r) in Hysteresis loop

Above Currie temperature T_c,

thermal energy > exchange coupling energy

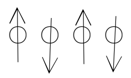
$\Rightarrow$ disorganized domains

$\Rightarrow$ ferromagnetic $\rightarrow$ paramagnetic

(cf) $T_c = 1043\,K$ ($770^\circ C$) for Fe

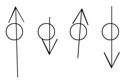
- ▶ Other kinds of magnetic materials
  - Antiferromagnetic (MnO_2, NiO, FeS, $CoCl_2$ .....)

    Even in $B_{ext} \neq 0$, $m = 0$, i.e., no response to B_{ext} due to line up of adjacent atoms in an antiparallel fashion with same magnitudes

  - Ferrimagnetic (Ferrites) (Fe_3O_4, Fe_2O_4, $NiFe_2O_4$, ...)

    Antiparallel alignment, but not equal

    $\Rightarrow$ Large response to $B_{ext} < B < B_{ferro}$

    Low conductivity $\sigma(10^{-4} \sim 1\ S/m) < \sigma_{semicon.} \ll \sigma_{metal}$

    $\rightarrow$ Reduced eddy currents (reduced ohmic losses) in cores of transformers, antennas, ... for ac or rf applications

  - Superparamagnetic (Audio, video & data recording tapes or disks)

    Ferromagnetic particles suspended in dielectric (or plastic) matrix

    $\Rightarrow$ Domain in individual particles

    $\rightarrow$ No penetration of domain wall

    $\rightarrow$ Store large amount of information in magnetic form

Notes)

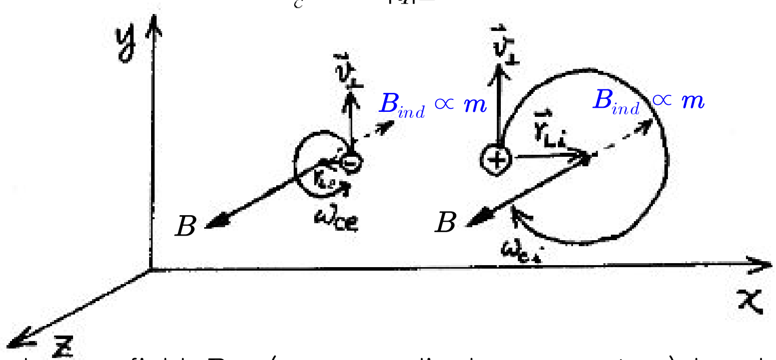
Plasma particle (e, i) motions in $\boldsymbol{B}$ field : $\boldsymbol{F}_m \equiv m\dfrac{d\boldsymbol{v}}{dt} = q\boldsymbol{v} \times \boldsymbol{B}$ (5-4)

i) For $\boldsymbol{v}_\parallel = \boldsymbol{0}$, *Direction of* $\dfrac{d\boldsymbol{v}_\perp}{dt}$ = *Direction of* $q\boldsymbol{v}_\perp \times \boldsymbol{B}$

$\Rightarrow$ *Circular motion* with radius r_L and ang. freq. ω_c around B line

$$\text{where} \quad \omega_c \equiv \frac{|q|B}{m} \; : \; \begin{pmatrix} \textit{Cyclotron} \\ \textit{Larmor} \\ \textit{Gyro} \end{pmatrix} \textit{frequency} \qquad (4)$$

$$r_L \equiv \frac{v_\perp}{\omega_c} = \frac{m v_\perp}{|q|B} \; : \; \textit{Larmor radius} \qquad (5)$$



Induced mag. field $\boldsymbol{B}_{ind}$ ($\propto$ mag. dipole momemt $\boldsymbol{m}$) by plasma particle circular motions opposes the externally applied $\boldsymbol{B}$ ($\boldsymbol{B} \upharpoonleft\downharpoonright \boldsymbol{B}_{ind} \propto \boldsymbol{m}$)

$\Rightarrow$ Plasmas are diamagnetic

ii) For $\boldsymbol{v}_{\parallel} \neq \mathbf{0}$

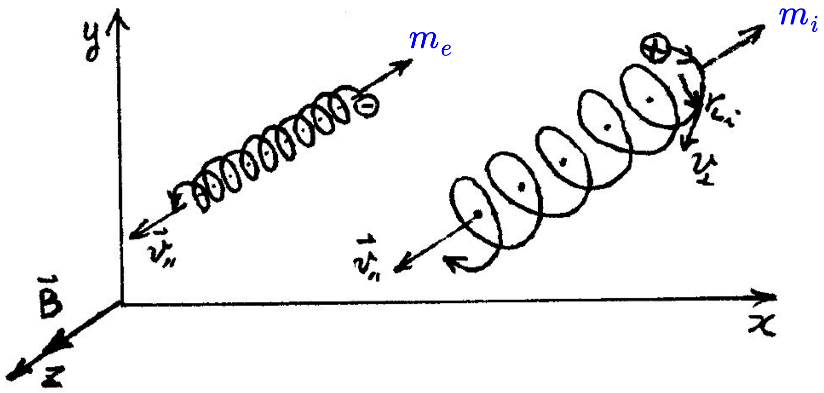
Plasma particle motion in constant $\boldsymbol{B}$

= Translational motion in ∥ direction + Circular motion in ⊥ direction

= *Gyromotion* with radius r_L about *guiding center*

(gyration motion) *Instantaneous center of gyration*

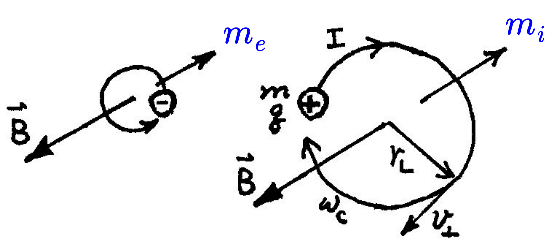
(helical motion)



iii) Magnetic moment $\boldsymbol{m}$ = (ring) current × area = IS

$$v_\perp / r_L \rightarrow \omega_c$$

$$= |q| \frac{\omega_c}{2\pi} \left(\pi r_L^2\right) = \frac{1}{2} |q| r_L v_\perp$$



$$\frac{v_\perp}{\omega_c} = \frac{v_\perp}{|q|B/m}$$

Magnitude of $\boldsymbol{m}$: $m = \frac{1}{2} |q| r_L v_\perp = \frac{\frac{1}{2} m v_\perp^2}{B}$

Direction of $\boldsymbol{m}$ is independent of q

and antiparallel to $\boldsymbol{B}$ *(Diamagnetic).*

$$\boldsymbol{m} = (\tfrac{1}{2})\, q \boldsymbol{r}_L \times \boldsymbol{v}_\perp = -\frac{(\frac{1}{2}) m v_\perp^2}{B} \frac{\boldsymbol{B}}{B} \tag{6}$$

B. Magnetization and Equivalent Current Densities

1) Magnetization

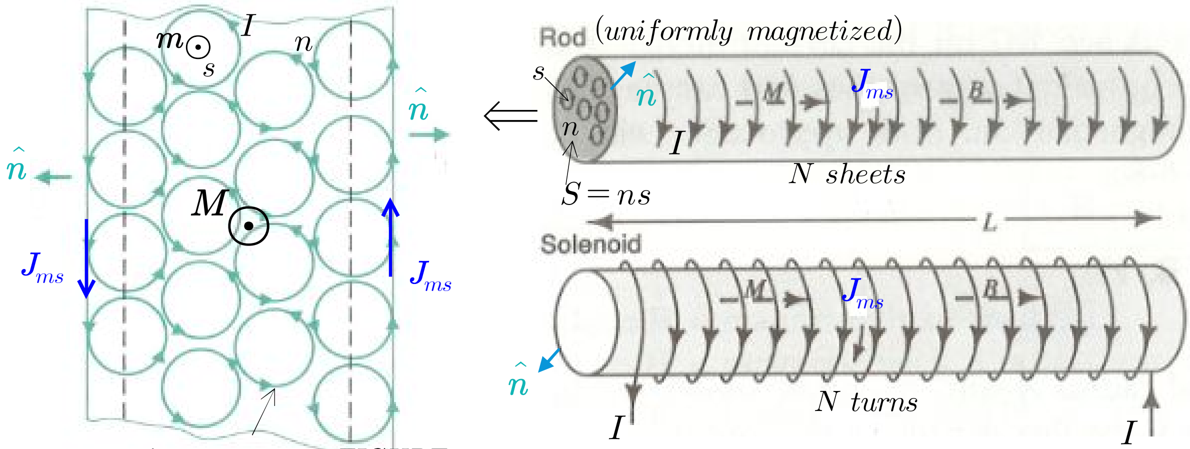


FIGURE 5−9

a) Magnetization vector $\boldsymbol{M}$

= magnetic dipole moments per unit volume

$$\boldsymbol{M} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\Delta v} \boldsymbol{m}_k}{\Delta v} = \frac{d\boldsymbol{m}}{dv} \qquad \text{(A/m)} \qquad \text{(5−48)}\leftrightarrow\text{(3−53)}$$

Vector magnetic potential due to $\boldsymbol{M}$:

$$\text{(5−44)} \Rightarrow \quad d\boldsymbol{A} = \frac{\mu_o \boldsymbol{M} \times \hat{\boldsymbol{R}}}{4\pi R^2} dv' \Rightarrow \boldsymbol{A} = \frac{\mu_o}{4\pi} \int_{V'} \frac{\boldsymbol{M} \times \hat{\boldsymbol{R}}}{R^2} dv' \quad \text{(5−49)}\leftrightarrow\text{(3−56)}$$

b) Magnetization surface (or sheet) current density $\boldsymbol{J}_{ms}$

$$\text{(5−48)} \Rightarrow \quad M = \frac{\Delta m}{\Delta v} = \frac{nNIs}{LS} = \frac{NI}{L} = J_{ms}$$

$$\Rightarrow \quad \boldsymbol{J}_{ms} = \boldsymbol{M} \times \hat{\boldsymbol{n}} \quad \text{or} \quad \boldsymbol{M} = \hat{\boldsymbol{n}} \times \boldsymbol{J}_{ms} \ \text{(A/m)} \qquad \text{(5−50)}\leftrightarrow\text{(3−57)}$$

(cf) For two parallel current sheets,

$$\boldsymbol{B} = \mu_o \boldsymbol{J}_s \times \hat{\boldsymbol{n}} \qquad \text{(2)*}$$

c) Magnetization volume current density $\boldsymbol{J}_{mv}$

Net current remaining within V bounded by the magnetized surface:

$$I_v = \underline{\int_{S'} \boldsymbol{J}_{mv} \cdot d\boldsymbol{s}'} = \int_L \boldsymbol{J}_{ms} \cdot d\boldsymbol{l} = \int_L (\boldsymbol{M} \times \hat{\boldsymbol{n}}) \cdot d\boldsymbol{l}$$

$$= \int_L \boldsymbol{M} \cdot (\hat{\boldsymbol{n}} \times d\boldsymbol{l}) = \underline{\int_{S'} (\nabla \times \boldsymbol{M}) \cdot d\boldsymbol{s}'}$$

$$\Rightarrow \quad \boldsymbol{J}_{mv} = \nabla \times \boldsymbol{M} \quad \text{(A/m}^2\text{)} \ : \ \text{bound currents} \qquad \text{(5−51)}\leftrightarrow\text{(3−59)}$$

d) Magnetic flux densities in magnetized material

Internal flux density B_i due to magnetization : $B_i = \mu_o M$ (5-52)

(5-7) for the external flux B_e due to free current density J :

$$\nabla \times B_e = \mu_o J \qquad (5\text{-}54)$$

For internal flux density B_i , (5-52) & (5-51) in (5-54) :

$\nabla \times B_i = \mu_o \nabla \times M = \mu_o J_{mv}$ due to bound current J_{mv} (5-55)

For total flux density $B = B_e + B_i$, (5-54) + (5-55) :

$$\nabla \times B = \mu_o (J + J_{mv}) = \mu_o J_t \qquad (5\text{-}56)\leftrightarrow(3\text{-}61)$$

(free + bound = total currents)

(e.g. 5-7) Uniformly magnetized cylindrical rod (~ solenoid)

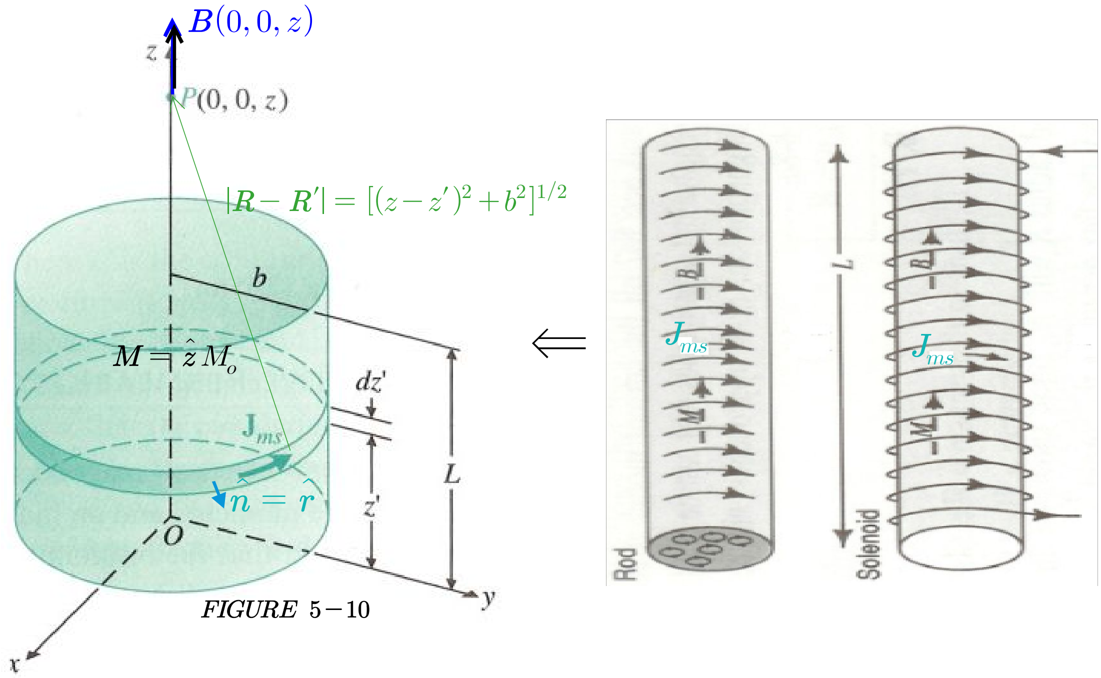


FIGURE 5−10

Uniformly magnetized : $M = \hat{z} M_o$

Volume current density : $J_{mv} = \nabla' \times M = 0$

Surface current density : $J_{ms} = M \times \hat{n} = (\hat{z} M_o) \times \hat{r} = \hat{\phi} M_o$

Surface current on dz' : $J_{ms} dz' = \hat{\phi} M_o dz' = \hat{\phi}\, dI'$

$$(5\text{-}37) \Rightarrow dB(0,0,z) = \hat{z} \frac{\mu_o dI' \, b^2}{2[(z-z')^2 + b^2]^{3/2}} = \hat{z} \frac{\mu_o M_o b^2 dz'}{2[(z-z')^2 + b^2]^{3/2}}$$

$$\Rightarrow B(0,0,z) = \int dB = \hat{z} \int_0^L \frac{\mu_o M_o b^2 dz'}{2[(z-z')^2 + b^2]^{3/2}}$$

$$= \hat{z} \frac{\mu_o M_o}{2} \left[\frac{z}{\sqrt{z^2 + b^2}} - \frac{z - L}{\sqrt{(z-L)^2 + b^2}} \right] \qquad (5\text{-}58)$$

C. Magnetic Field Intensity and Field Equations

1) Magnetic field intensity

In a magnetic material, from (5-56) and (5-51)

$$\frac{1}{\mu_o}\nabla \times \boldsymbol{B} = \boldsymbol{J} + \boldsymbol{J}_{mv} = \boldsymbol{J} + \nabla \times \boldsymbol{M}$$

$$\Rightarrow \quad \nabla \times \left(\frac{1}{\mu_o}\boldsymbol{B} - \boldsymbol{M}\right) = \boldsymbol{J} \qquad \text{(5-59)}$$

The magnetic field intensity is defined as

$$\boldsymbol{H} = \frac{1}{\mu_o}\boldsymbol{B} - \boldsymbol{M} \quad \text{(A/m)} \qquad \text{(5-60)}$$

2) Ampere's law

(5-60) in (5-59) $\Rightarrow$

Differential form of Ampere's law for free current :

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} \quad \text{(A/m}^2\text{) : free current density} \qquad \text{(5-61)}\leftrightarrow\text{(5-7)}$$

$\int_S (5-61) \cdot d\boldsymbol{s}$ using Stokes's theorem $\Rightarrow$

Integral form of Ampere's circuital law in magnetic materials :

$$\oint_C \boldsymbol{H} \cdot d\boldsymbol{l} = \int_S \boldsymbol{J} \cdot d\boldsymbol{s} = I \quad \text{(A) : free current} \qquad \text{(5-63)}\leftrightarrow\text{(5-10)}$$

Circulation of $\boldsymbol{H}$ around any closed path C

= Free current crossing the area bounded by the path

(Right-had rule for the directions)

3) Constitutive relation and magnetic material properties

For a simple (homogeneous, linear and isotropic) magnetic medium,

$$\boldsymbol{M} = \chi_m \boldsymbol{H} \qquad \text{(5-64)}$$

where χ_m is the magnetic susceptibility. Then, (5-60) becomes

$$\boldsymbol{B} = \mu_o(\boldsymbol{H} + \boldsymbol{M}) = \mu_o(1+\chi_m)\boldsymbol{H} = \mu_o\mu_r\boldsymbol{H} = \mu\boldsymbol{H} \qquad \text{(5-65)}\leftrightarrow\text{(3-67)}$$

Relative permeability : $\mu_r = 1 + \chi_m = \mu/\mu_o$ (5-67)↔(3-68)

(cf) Appendix B-5

Notes)

i) $\boldsymbol{M} = \chi_m \boldsymbol{H}$

$\mu_r \lesssim 1$, $\chi_m < 0$: diamagnetic

$\mu_r \gtrsim 1$, $\chi_m > 0$: paramagnetic

$\Rightarrow \quad |\chi_m| \ll 1$

ii) For ferromagnetic materials,

$$\mu = \mu(H), \qquad \chi_m = \chi_m(H) \gg 1 \quad : \quad \text{nonlinear medium}$$

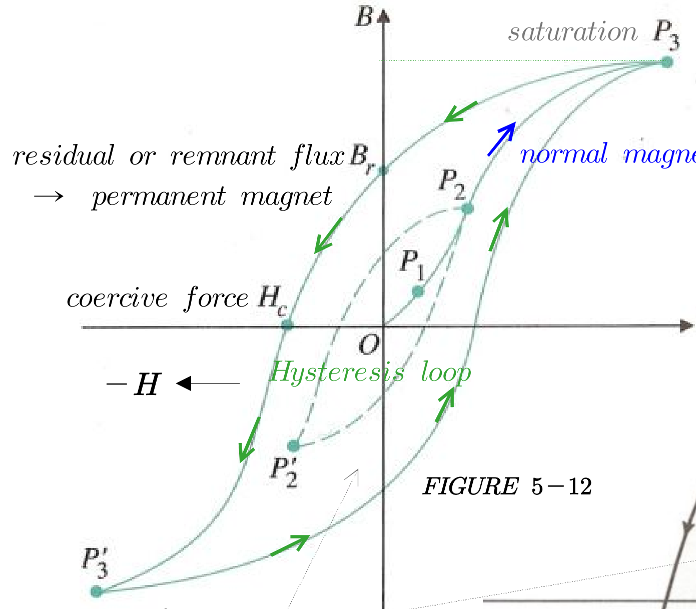
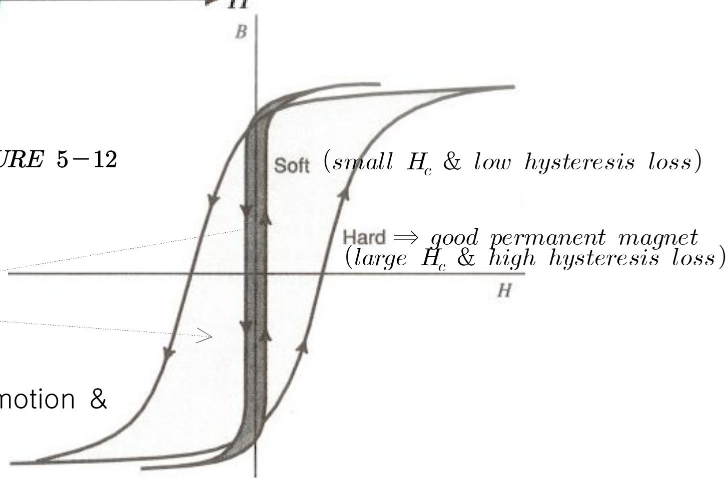


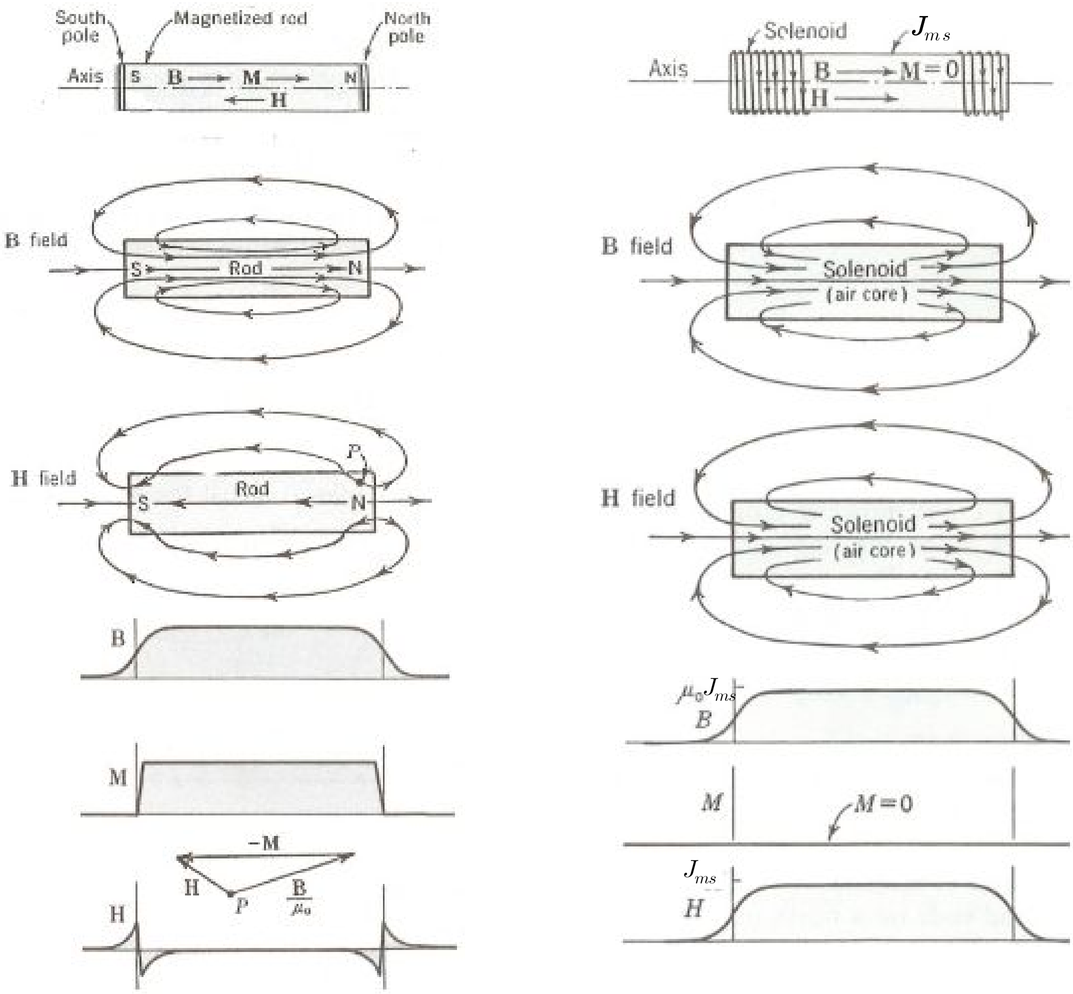
FIGURE 5−12



Loof area ($\oint_{loop} B\,dH$)

= Hysteresis loss of friction heat encountered during domain-wall motion & domain rotation

iii) Comparison of magnetic fields



4) Boundary conditions

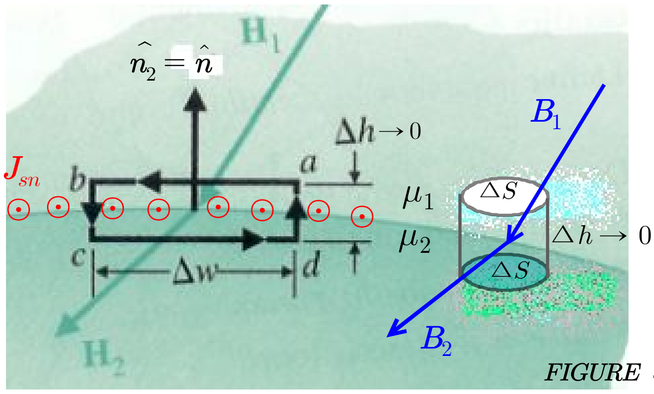


FIGURE 5−13

$\nabla \cdot B = 0$: Gauss's law (5-6)

$\nabla \times H = J$: Ampere's law (5-61)

a) Normal components of B and H

$$\nabla \cdot B = 0 \quad \Rightarrow \quad \oint_S B \cdot ds = 0$$

$$\Rightarrow \oint_{pillbox} B \cdot ds = (B_1 - B_2) \cdot \hat{n} \Delta S$$

$$\Delta h \to 0 \qquad = (B_{1n} - B_{2n}) \Delta S = 0$$

$$\Rightarrow B_{1n} = B_{2n} \text{ (T)} \quad \text{or} \quad \hat{n} \cdot (B_1 - B_2) = 0 \qquad (5\text{-}68)$$

For linear and isotropic materials, $B = \mu H$ in (5-68) :

$$\mu_1 H_{1n} = \mu_2 H_{2n} \quad \text{or} \quad \hat{n} \cdot (\mu_1 H_1 - \mu_2 H_2) = 0 \qquad (5\text{-}69)$$

b) Tangential components of H

$$\nabla \times H = J_{ms} \quad \Rightarrow \oint_C H \cdot dl = \int_S J \cdot ds$$

$$\Delta h \to 0 \quad \Rightarrow \int_{abcda} H \cdot dl = H_1 \cdot \Delta w + H_2 \cdot (-\Delta w) = J_{sn} \Delta w$$

$$\Rightarrow (H_{1t} - H_{2t}) \Delta w = J_{sn} \Delta w$$

$$\Rightarrow H_{1t} - H_{2t} = J_{sn} \text{ (A/m)} \quad \text{or} \quad \hat{n} \times (H_1 - H_2) = J_s \qquad (5\text{-}70)$$

$$\Rightarrow \frac{B_{1t}}{\mu_1} - \frac{B_{2t}}{\mu_2} = J_{sn} \quad \text{or} \quad \hat{n} \times \left(\frac{H_1}{\mu_1} - \frac{H_2}{\mu_2}\right) = J_s \qquad (5\text{-}70)^*$$

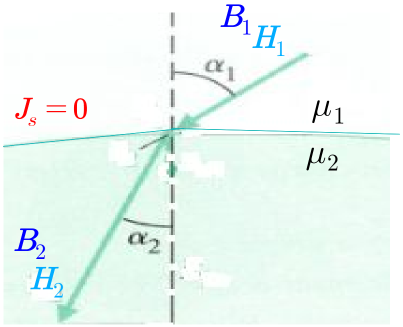
Notes)

i) For an interface with an ideal perfect conductor (σ_1 or $\sigma_2 \to \infty$),

$J_s \neq 0$

ii) For $\sigma_1 = finite$ and $\sigma_2 = finite$, $J_s = 0$

iii) For $J_s = 0$,

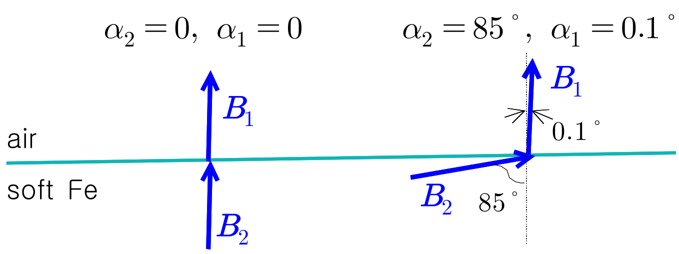


$$\left.\begin{array}{l} B_{1n} = B_{2n} \\ H_{1t} = H_{2t} \end{array}\right)$$

$$\Rightarrow \quad \frac{B_1 \cos\alpha_1}{H_1 \sin\alpha_1} = \frac{B_2 \cos\alpha_2}{H_2 \sin\alpha_2}$$

$$\Rightarrow \quad \frac{\tan\alpha_1}{\tan\alpha_2} = \frac{\mu_1}{\mu_2} = \frac{\mu_{r1}}{\mu_{r2}}$$

For $\mu_{r1} = 1$ (air), $\mu_{r2} = 7000$ (soft Fe),



$\Rightarrow$ When $\mu_2 \gg \mu_1$,

B_1 and H_1 are nearly normal to the boundary

Summary of Boundary Conditions

Electrostatics

$D_{1n} - D_{2n} = \rho_s$	or	$\hat{n} \cdot (D_1 - D_2) = \rho_s$	$\Leftarrow$	$\nabla \cdot D = \rho_v$
$E_{1t} = E_{2t}$	or	$\hat{n} \times (E_1 - E_2) = 0$	$\Leftarrow$	$\nabla \times E = 0$
$J_{1n} = J_{2n}$	or	$\hat{n} \cdot (J_1 - J_2) = 0$	$\Leftarrow$	$\nabla \cdot J = 0$
$E_{1t} = E_{2t} = J_{2t}/\sigma_2$			$\Leftarrow$	$J = \sigma E$

Magnetostatics

$B_{1n} = B_{2n}$	or	$\hat{n} \cdot (B_1 - B_2) = 0$	$\Leftarrow$	$\nabla \cdot B = 0$
$H_{1t} - H_{2t} = J_{sn}$	or	$\hat{n} \times (H_1 - H_2) = J_s$	$\Leftarrow$	$\nabla \times H = J$

$$\frac{\tan\alpha_1}{\tan\alpha_2} = \frac{\epsilon_1}{\epsilon_2} = \frac{\sigma_1}{\sigma_2} = \frac{\mu_1}{\mu_2}$$