Numerical Methods in Rock Engineering
- Introduction to Finite Element Method

(Lecture 2, 3, 4, 5) (15, 22, 29 Sept, 20 Oct 2014)

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Term Paper
Proposal

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Term Paper
Proposal

- Keep it simple – determine what kind of problem you want to tackle that can not be known otherwise.
- Be clear what will the be verification case
- Distinguish between laboratory and numerical investigation – numerical study cannot, in general, produce a new constitutive relation
- If this term paper is *part* of your thesis or project, then make sure *only part of the whole project* will be conducted during class.
- Well begun is half done!

Home Assignment #1

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- Make your own summary
  - Don't just make a *copy and paste*
  - Your views on this summary is the most important

Assignment #2
FEM Exercise

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- 6 Oct 2014 (or 29 Sept 2014)
- Introduction to COMSOL, a general FEM solver
- Exercise
  - Saint Venant Principle
  - Mesh size effect
  - Brazilian Test
  - Uniaxial Test
  - Heat Conduction (Thermal Conductivity measurement)

Numerical Approach in Rock Engineering

Physical variables for THMC problems

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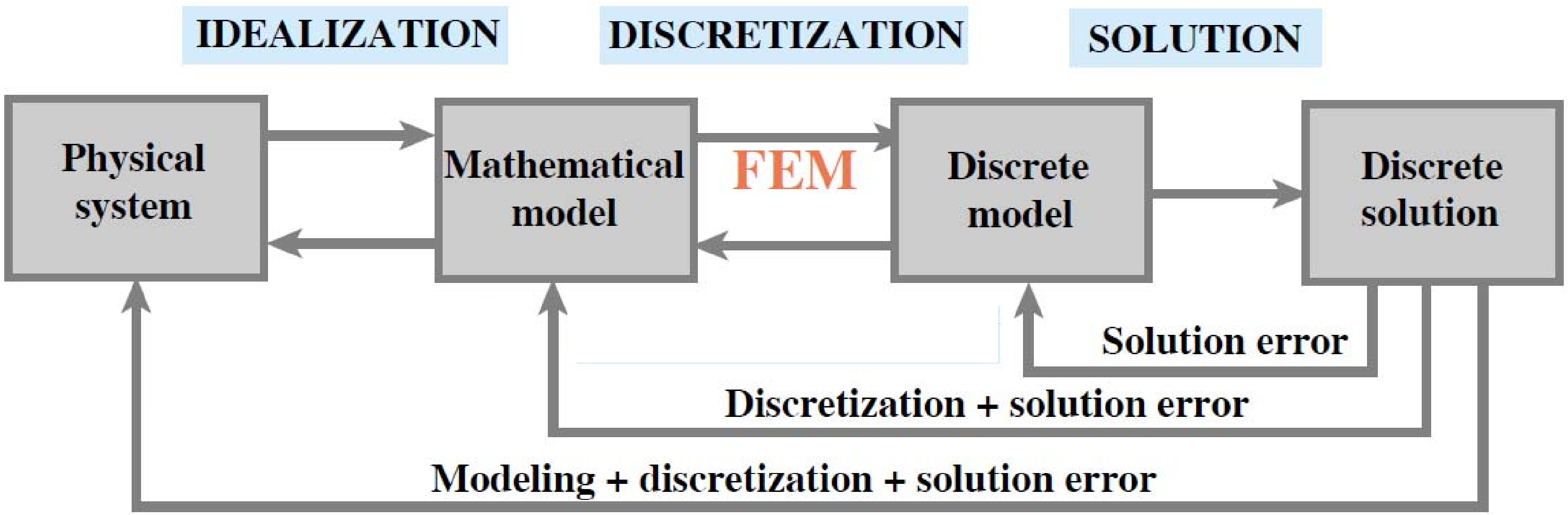
Physical problem	Conservation Principle $\nabla \cdot q = 0$	State Variable u	Flux σ	Material properties k	Source f	Constitutive equation $\sigma = ku'$
Elasticity	Conservation of linear momentum (equilibrium)	Displacement u	Stress σ	Young's modulus & Poisson's ratio	Body forces	Hooke's law
Heat conduction	Conservation of energy	Temperature T	Heat flux Q	Thermal conductivity k	Heat sources	Fourier's law
Porous media flow	Conservation of mass	Hydraulic head h	Flow rate Q	Permeability k	Fluid source	Darcy's law
Mass transport	Conservation of mass	Concentration C	Diffusive flux q	Diffusion coefficient D	Chemical source	Fick's law

Structure of state variables and fluxes are mathematically similar –
a convenient truth!

Mathematical model



Basics of Finite Element Method

- Governing Equations: 1D Boundary Value Problem
  - Elasticity
  - Diffusion equation (Heat conduction & Fluid flow in porous media)
  - 3D expansion
- Finite Elements in One Dimension (Boundary Value Problem)
  - Basics of FEM
  - Weak Formulation and Galerkin's Method
  - Illustrative Example (12 steps)
  - General 1D Boundary Value Problem
- Finite Elements in One Dimension (Mixed Initial-Boundary-Value Problem)
  - Time stepping method
- Finite Elements in two- and three-Dimensions

Elasticity formulation

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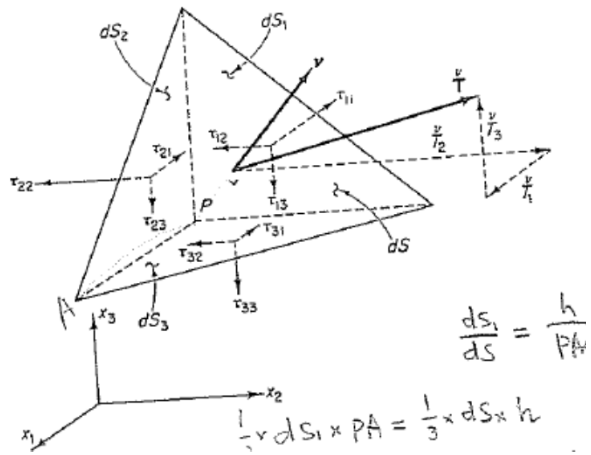
- Elasticity formulation was extensively covered in ‘theory of poroelasticity’.

2D & 3D elasticity
Stress in 3D

- By Cauchy's formula

$$\overset{\nu}{T}_i = \nu_j \sigma_{ji}$$

  - Knowing the stress component, we can write down the stress vector acting on any surface
  - Stress state in a body is characterized completely by stress tensor

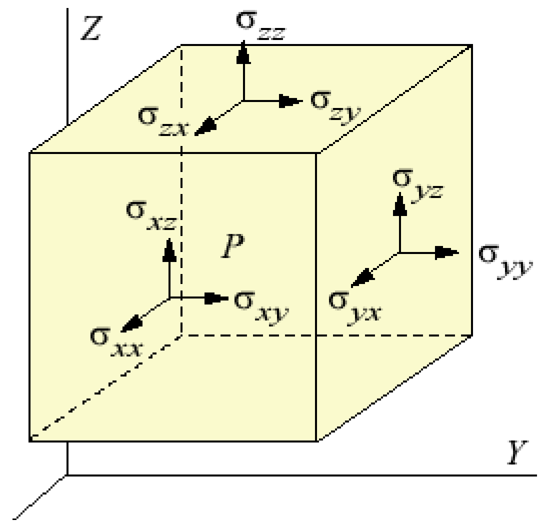


$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

Tensor form

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{pmatrix}$$

matrix



$\sigma_{\xi\eta}$: Stress on the ξ plane along η direction.

Direction of the stress component.

Direction of the surface normal upon which the stress acts.

www.efunda.com

2D & 3D elasticity
Strain – 2D & 3D

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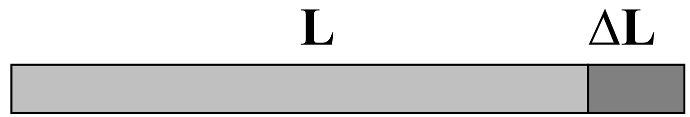
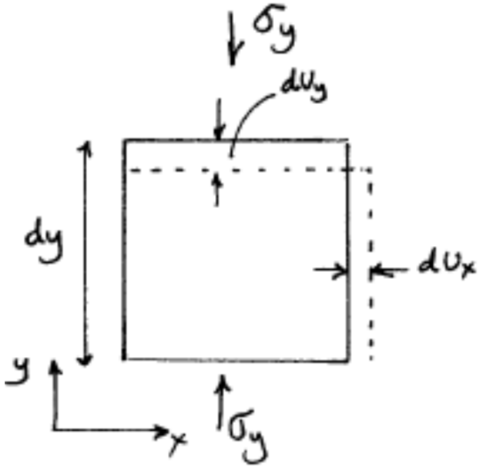
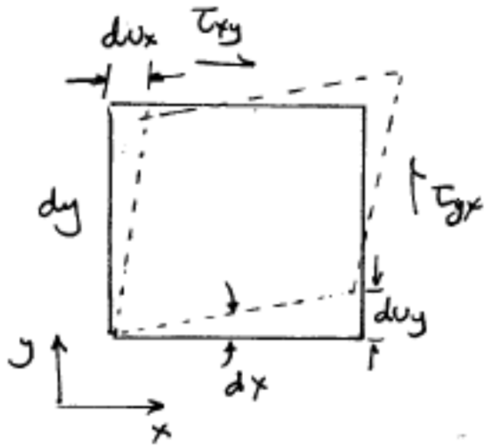
- Geometric expression of deformation caused by stress (dimensionless)

1D $$\varepsilon = \frac{\Delta L}{L} = \frac{du}{dx}$$





$$\varepsilon_y = \frac{\partial u_y}{\partial y} \approx \frac{\Delta u_y}{\Delta y}$$



$$\gamma_{xy} = \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right)$$

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \varepsilon_{yy} = \frac{\partial u_y}{\partial y},$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

2D & 3D elasticity
Strain – 2D & 3D

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$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \varepsilon_{zz} = \frac{\partial u_z}{\partial z},$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)$$

$$\varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)$$

$$\begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix}$$

Tensor form

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{xy} \end{pmatrix} = \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{pmatrix}$$

matrix form **Engineering strain**

Strain is also a 2nd order tensor and symmetric by definition.

2D & 3D elasticity
Equation of motion (Equilibrium equation)

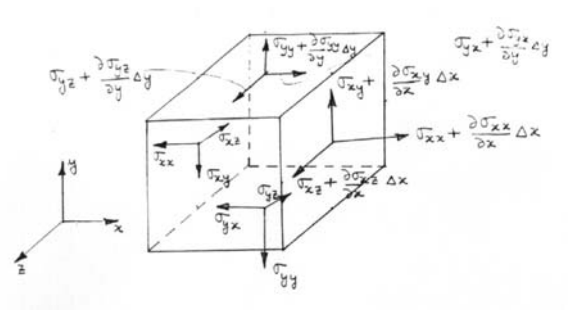
- Sum of traction, body and inertial forces (and moment) are zero

$$\sum F_i = 0 \longrightarrow \sigma_{ji,j} + \rho b_i = \rho \frac{\partial^2 u_i}{\partial t^2}$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + \rho b_x = \rho \frac{\partial^2 u_x}{\partial t^2}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + \rho b_y = \rho \frac{\partial^2 u_y}{\partial t^2}$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho b_z = \rho \frac{\partial^2 u_z}{\partial t^2}$$



www.mcasco.com/eande.html

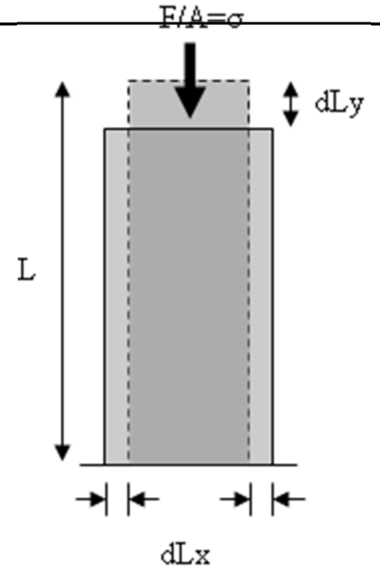
$$\sum M_i = 0 \longrightarrow \sigma_{ij} = \sigma_{ji}$$

2D & 3D elasticity
Stress-strain relationship – 1D & 2D

- Hooke's Law

  1D $\sigma = E\varepsilon$

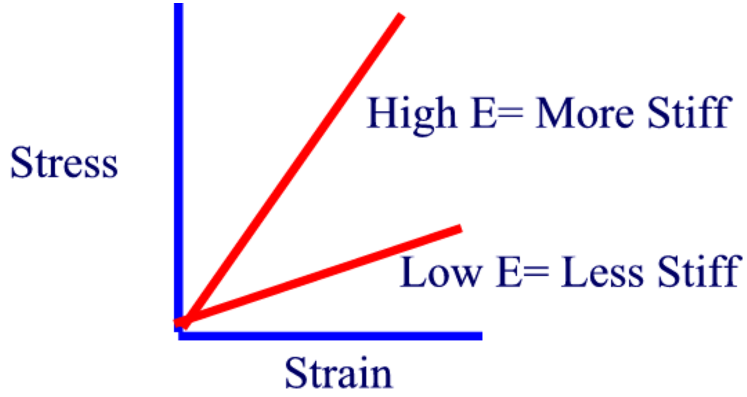
  – Stress is directly proportional to strain



- Stress and Strain

$$\sigma = \frac{F}{A} \qquad \varepsilon = \frac{dL}{L}$$

- Elastic modulus (N/m^2=Pa) $E = \dfrac{\sigma_y}{\varepsilon_y}$



- Poisson's ratio (dimensionless) $\nu = -\dfrac{\varepsilon_x}{\varepsilon_y}$

2D & 3D elasticity
Stress-strain relationship – 1D & 2D

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- Hooke's Law

1D $\sigma = E\varepsilon$

- Shear modulus G

$$\tau_{xy} = G\gamma_{xy}$$

- Generalized Hooke's law (isotropy)
  - 2 independent parameters (E, ic material
  - $G = \dfrac{E}{2(1+\nu)}$
  - E>0, -1<v<0.5

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{pmatrix}$$

2D & 3D elasticity
Stress-strain relationship – 1D & 2D

- Complete anisotropy
  - 21 independent parameters

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{pmatrix}$$

$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl} \qquad \sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$

2D & 3D elasticity
Stress-strain relationship – 1D & 2D

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- Stress and strain in different dimensions are coupled. Therefore, we need a special consideration –plane strain and plane stress

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{pmatrix}$$

- Plane strain
  - 3rd dimensional strain goes zero
  - Stresses around drill hole or 2D tunnel

$$\begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ \varepsilon_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{pmatrix}$$

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \frac{(1-\nu^2)}{E} & -\frac{\nu(1+\nu)}{E} & 0 \\ -\frac{\nu(1+\nu)}{E} & \frac{(1-\nu^2)}{E} & 0 \\ 0 & 0 & \frac{2(1+\nu)}{E} \end{pmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix}$$

2D & 3D elasticity
Stress-strain relationship – 1D & 2D

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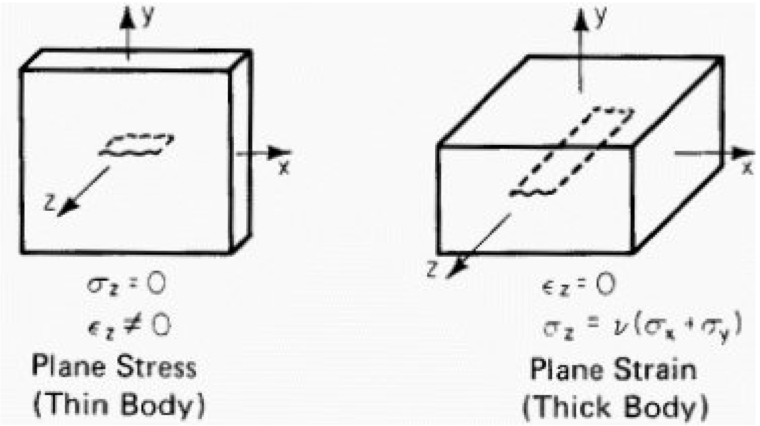
- Plane stress
  - 3rd dimensional stress goes zero
  - Thin plate stressed in its own plane

$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ \varepsilon_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix}$$

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \frac{1}{E} & -\frac{\nu}{E} & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{2(1+\nu)}{E} \end{pmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix}$$



2D & 3D elasticity
Governing equations

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- Strain-displacement relationship (6)
- Stress-strain relationship (6)
- Equation of motion (3)

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$

$$\sigma_{ji,j} + \rho b_i = \rho \frac{\partial^2 u_i}{\partial t^2}$$

- Navier's equation

$$Gu_{i,jj} + (\lambda + G)u_{j,ji} + \rho b_i = 0$$

$$G\nabla^2 \mathbf{u} + (\lambda + G)\nabla\nabla \cdot \mathbf{u} + \rho \mathbf{b} = 0$$

$$G\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right) + (\lambda + G)\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial x \partial y} + \frac{\partial^2 u_z}{\partial x \partial z}\right) + \rho b_x = 0$$

$$G\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2}\right) + (\lambda + G)\left(\frac{\partial^2 u_x}{\partial x \partial y} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_z}{\partial y \partial z}\right) + \rho b_y = 0$$

$$G\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right) + (\lambda + G)\left(\frac{\partial^2 u_x}{\partial x \partial z} + \frac{\partial^2 u_y}{\partial y \partial z} + \frac{\partial^2 u_z}{\partial z^2}\right) + \rho b_z = 0$$

  – Three governing equations for three displacement components

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

2D & 3D elasticity
Comparison with diffusion equation

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- Diffusion equation

$$A\frac{\partial c}{\partial t} + \nabla \cdot (-D\nabla c) = R$$

$$k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) = \rho c \frac{\partial T}{\partial t}$$

  - Time-dependent
  - One parameter k is necessary for steady state behaviour

- Navier's equation

$$G\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right) + (\lambda + G)\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial x \partial y} + \frac{\partial^2 u_z}{\partial x \partial z}\right) + \rho b_x = 0$$

$$G\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2}\right) + (\lambda + G)\left(\frac{\partial^2 u_x}{\partial x \partial y} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_z}{\partial y \partial z}\right) + \rho b_y = 0$$

$$G\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right) + (\lambda + G)\left(\frac{\partial^2 u_x}{\partial x \partial z} + \frac{\partial^2 u_y}{\partial y \partial z} + \frac{\partial^2 u_z}{\partial z^2}\right) + \rho b_z = 0$$

  - Not time-dependent
  - Three coupled equations
  - Two parameters (isotropy)

1D & steady state

1D

$$-\frac{d}{dx}\left(\alpha(x)\frac{dU(x)}{dx}\right) = f(x)$$

Elasticity formulation

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- Here we stop digression and continue to talk about FEM.

1D Boundary Value Problem
Elasticity

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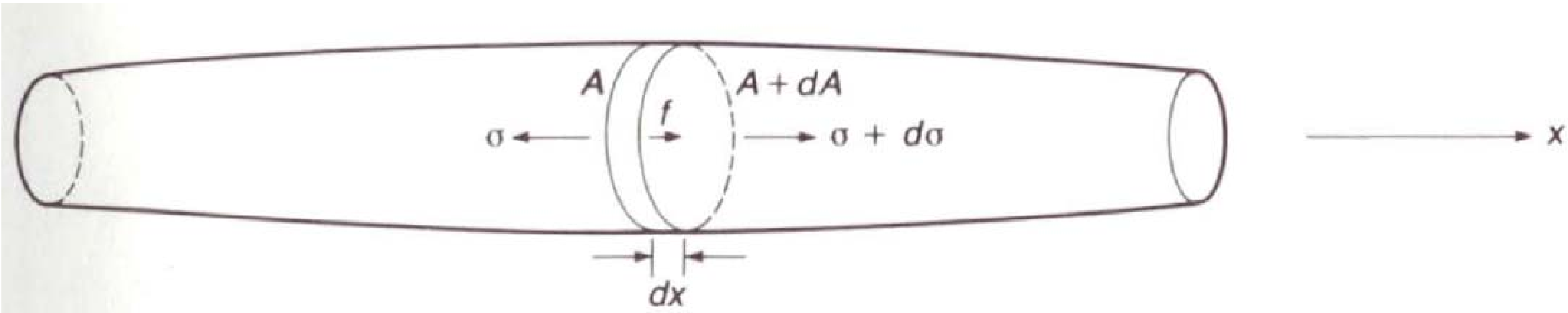

An elastic rod, in approximately a uniaxial stress state

- Balance Principle
  - Equilibrium Equation
- Constitutive Equation
  - Hooke's law
- Governing Equation

1D Boundary Value Problem
General Formulation

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- In case of cylindrical rod, Elasticity Eq becomes;

$$-\frac{d}{dx}\left(E(x)\frac{du(x)}{dx}\right)=f(x)$$

  - At least one of BC must be an essential BC. Why?

1D Boundary Value Problem
General Formulation

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- General formulation

$$-\frac{d}{dx}\left(\alpha(x)\frac{dU(x)}{dx}\right)+\beta(x)U(x)=f(x)$$

  - Elasticity

$$-\frac{d}{dx}\left(E(x)\frac{du(x)}{dx}\right)=f(x)$$

  - Diffusion Equation (steady-state)

$$-\frac{d}{dx}\left(k(x)\frac{dT(x)}{dx}\right)+\frac{hl}{A}T(x)=Q(x)+\frac{hlT_{\infty}}{A}$$

1D Boundary Value Problem
General Formulation

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- Governing equation ← conservation principle + constitutive equation

$$\frac{d\tau(x)}{dx} + \beta(x)U(x) = f(x)$$

Gain of energy to the interior of the system

Loss of energy from the system due to a flow across the boundaries

Loss of energy from the interior of the system, *β(x)* is a physical property

$$\tau(x) = -\alpha(x)\frac{dU(x)}{dx}$$

Flux (heat flux, fluid flux or stress)
For stress, this terminology is not appropriate but conventionally used.

Experimentally measured property

1D Boundary Value Problem
General Formulation

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- Boundary condition
  - Essential (Dirichlet) BC: specify the function U
  - Natural (Neumann) BC: specify the flux τ
- With two end points (x_a, x_b)

$$At\ \ x = x_a \qquad U(x_a) = U_a \qquad or \qquad -\alpha \left.\frac{dU}{dx}\right|_{x=x_a} = \tau_a$$

$$At\ \ x = x_b \qquad U(x_b) = U_b \qquad or \qquad -\alpha \left.\frac{dU}{dx}\right|_{x=x_b} = \tau_b$$

Finite Element Method
Introduction

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- A few analytical solution to the partial differential equation (Navier's equation or diffusion equation) → governing equations can be solved numerically (FEM, FDM, BEM)
- Invention?*
  - Precise moment of invention is not clear
  - Courant (1943) used piecewise linear polynomials for torsion problem
  - Much progress in 1960s (The name finite element appeared in Clough(1960)
- Methodology
  - Continuum is divided into a finite number of parts (elements), the behavior is specified by parameters
  - Solution of complete system as an assembly of its elements
- Mathematically speaking
  - FEM replaces solutions by simple equations such as polynomials
  - FDM replaced derivatives by differences

*30-32p, Burnett DS, Finite Element Analysis – From concepts to applications, 1987, Addison-Wesley Publishing Co.

Finite Element Method
Introduction

- Essential characteristics of FEM is the special form of trial solution.
- Transforms unsolvable calculus problem → approximately equivalent but solve algebra problems.
- Mesh = node + element
- Mesh generation: defining the length, locations of the elements & nodes, assigning numbers to each node and element

- Element trial function = shape function
  - A trial function defined over one element
  - Basis function: piecing together shape functions in each element

Finite Element Method
Principal operations

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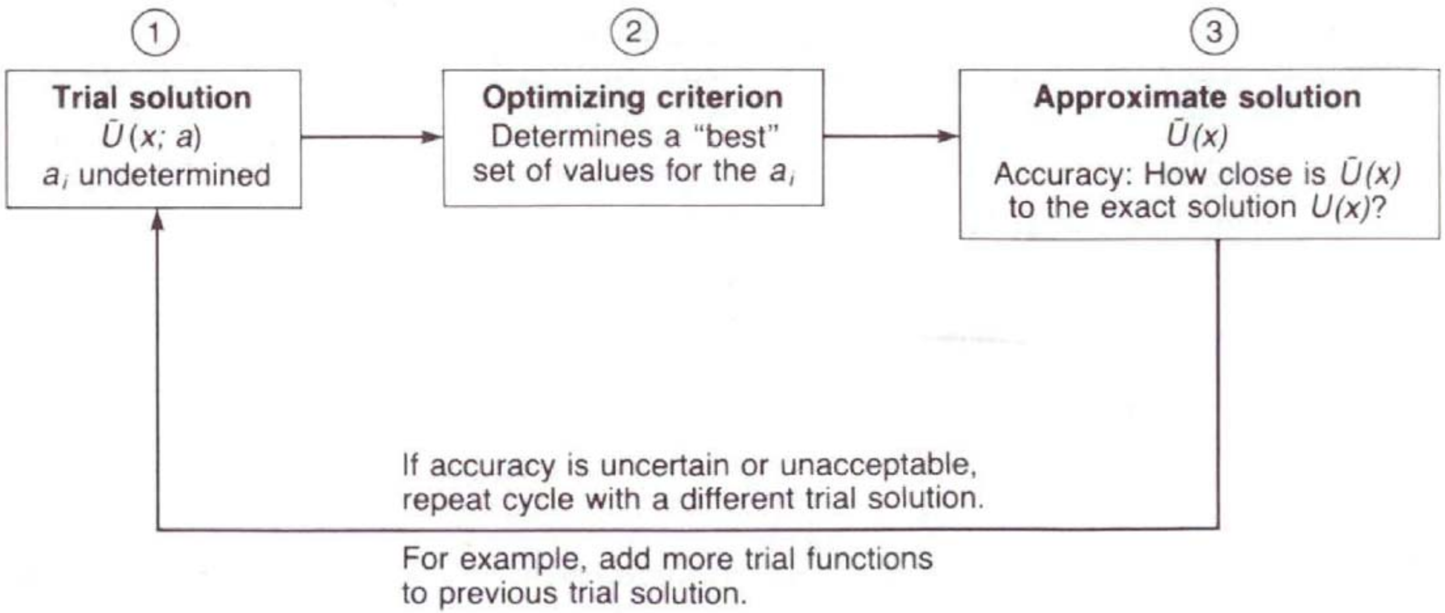
- Construction of trial solution for $\tilde{U}$
- Application of optimizing criterion to $\tilde{U}$
- Estimation of accuracy of $\tilde{U}$



*56p, Burnett DS, Finite Element Analysis – From concepts to applications, 1987, Addison-Wesley Publishing Co.

Finite Element Method
Trial solutioin

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- Trial solution in the form of a finite sum of functions;

$$\tilde{U}(x;a) = a_1\phi_1(x) + a_2\phi_2(x) + \ldots + a_N\phi_N(x)$$

Known!

$\phi_1(x), \phi_2(x), \ldots \phi_N(x)$: trial functions(basis functions)

$a_1, a_2, \ldots, a_N$ are undetermined parameters or degrees of freedom

unknown!

  - N degrees of freedom

- Residual (non-zero results after trial solution were substituted)

$$\frac{d}{dx}\left(x\frac{dU(x)}{dx}\right) = \frac{2}{x^2} \qquad 1 < x < 2$$

$$R(x;a) = \frac{d}{dx}\left(x\frac{d\tilde{U}(x)}{dx}\right) - \frac{2}{x^2} \neq 0 \qquad 1 < x < 2$$

Finite Element Method
Galerkin's Method

- Two types of optimizing criteria
  - Methods of weighted residuals (MWR): differential governing equations
    - The collocation method
    - The subdomain method
    - The least-squares method
    - The Galerkin Method
  - Ritz variational methods (RVM) : variational governing equations
    - Produce identical solution with Galerkin method when trial solutions are the same
    - Minimum of potential energy in solid mechanics

Finite Element Method
Optimizing Criteria - Galerkin's Method

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- The Collocation method
  - For each undetermined parameters ai, choose x_i and force the residual to be zero

$$R(x_1;a)=0$$
$$R(x_2;a)=0$$
$$\cdots$$
$$R(x_N;a)=0$$

- The Subdomain method
  - For each undetermined ai, choose an interval Δxi, and force the average of the residual to be zero

$$\frac{1}{\Delta x_1}\int_{\Delta x_1} R(x;a)dx=0$$
$$\frac{1}{\Delta x_2}\int_{\Delta x_2} R(x;a)dx=0$$
$$\cdots$$
$$\frac{1}{\Delta x_N}\int_{\Delta x_N} R(x;a)dx=0$$

- The Least-Square method
  - For each undetermined ai, minimize the integral over the entire domain of the square of the residual

$$\frac{\partial}{\partial a_1}\int_1^2 R^2(x;a)dx=0$$
$$\frac{\partial}{\partial a_2}\int_1^2 R^2(x;a)dx=0$$
$$\cdots$$
$$\frac{\partial}{\partial a_N}\int_1^2 R^2(x;a)dx=0$$

- Galerkin's Method
  - For each parameter a_i we require that a weighted average of R(x;a) over the entire domain be zero.
  - The weighting functions are the trial functions associated with each a_i.

$$\int_1^2 R(x;a)\phi_1(x)dx = 0$$

$$\int_1^2 R(x;a)\phi_2(x)dx = 0$$

$$\cdots$$

$$\int_1^2 R(x;a)\phi_N(x)dx = 0$$

Finite Element Method
Optimizing Criteria - Galerkin's Method

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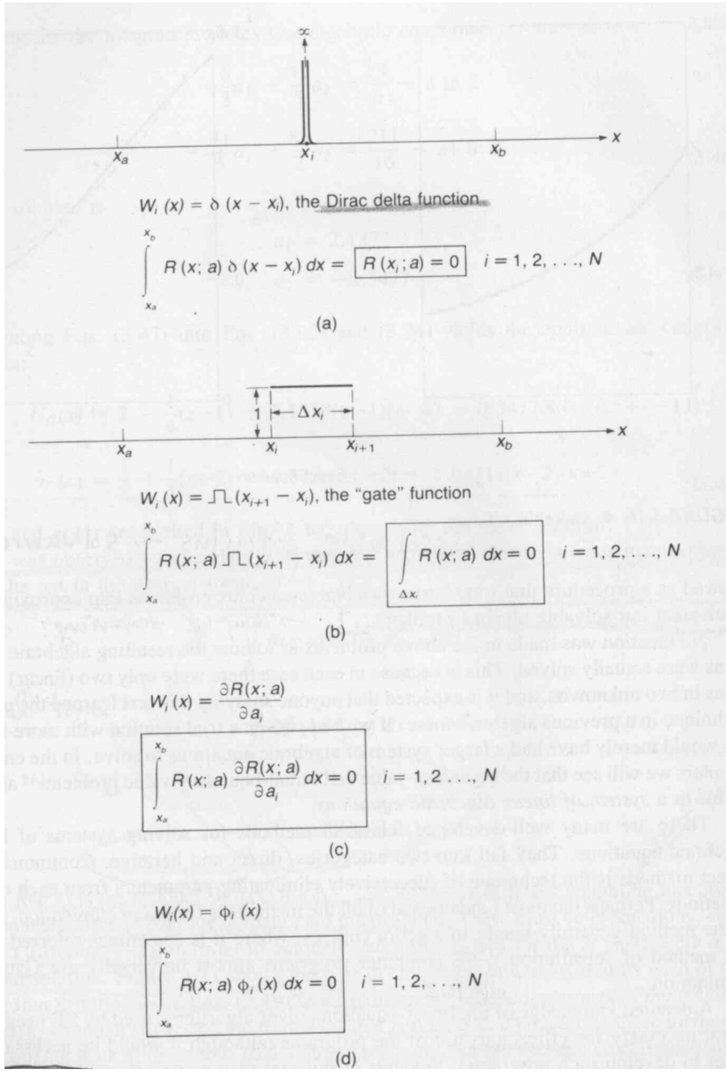
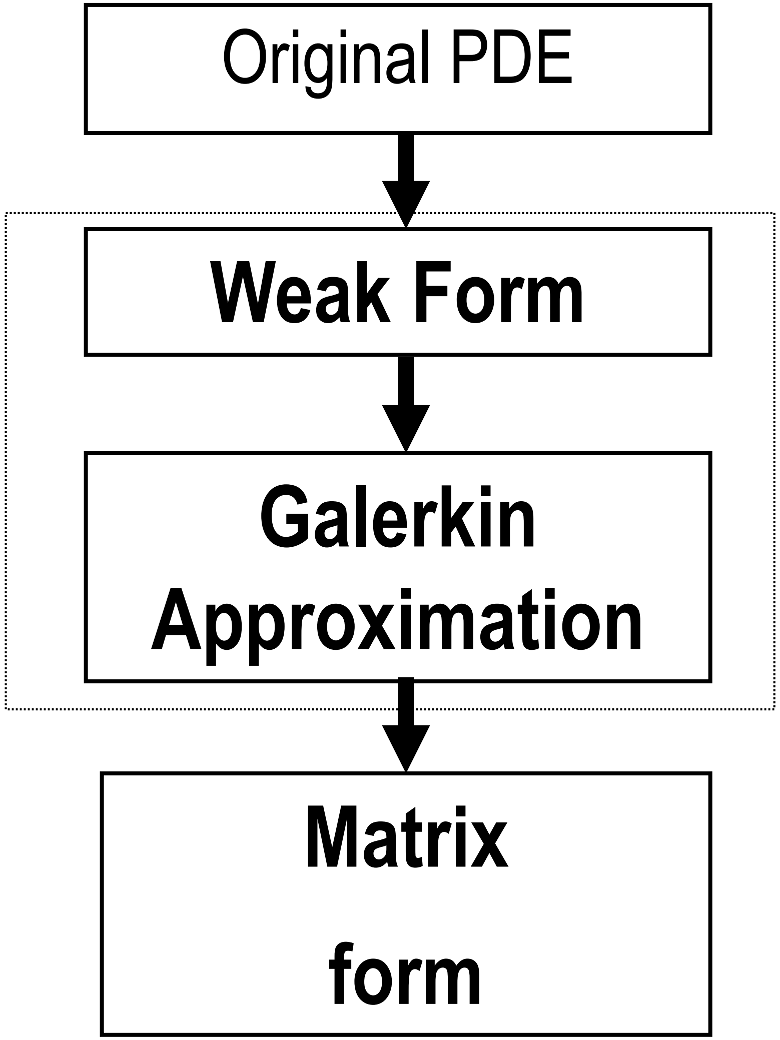
- Different weight functions for weighted residual methods



Finite Element Method
Galerkin's Method



Finite Element Method
Trial solutioin

- Trial solution in the form of a finite sum of functions;

$$\tilde{U}(x;a) = a_1\phi_1(x) + a_2\phi_2(x) + \ldots + a_N\phi_N(x)$$

Known!

$\phi_1(x), \phi_2(x), \ldots \phi_N(x)$: trial functions(basis functions)

$a_1, a_2, \ldots, a_N$ are undetermined parameters or degrees of freedom

unknown!

  – N degrees of freedom

Finite Element Method
Optimization Criteria - Example

$$\frac{d}{dx}\left(x\frac{dU(x)}{dx}\right)=\frac{2}{x^2} \qquad 1<x<2, \quad U(1)=2, \quad \left(-x\frac{dU(x)}{dx}\right)_{x=2}=\frac{1}{2}$$

$$R(x;a)=\frac{d}{dx}\left(x\frac{d\tilde{U}(x)}{dx}\right)-\frac{2}{x^2}\neq 0 \qquad 1<x<2$$

Finite Element Method
Solutioin procedure (12 steps)

- Theoretical development (1- 6 steps) + Numerical Computation(7-12 steps)
- Theoretical Development
  - Step 1: Write the Galerkin residual equations for a typical element
  - Step 2: Integrate by parts
  - Step 3: substitute the element trial solution into integrals (LHS)
  - Step 4: Develop specific expression for the element trial functions (shape function)
  - Step 5: Substitute the shape functions into the element equations
  - Step 6: prepare expression for the flux

Finite Element Method
Solutioin procedure (12 steps)

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- Numerical Computation(7-12 steps)
  - Step 7: Specify numerical data
  - Step 8: Evaluate the interior terms in the element equations for each element, and assemble the terms into system equations
  - Step 9: Apply Boundary Conditions
  - Step 10: Solve the system equations
  - Step 11: Evaluate the flux
  - Step 12: Display the solution and estimate its accuracy

preprocessing

solution

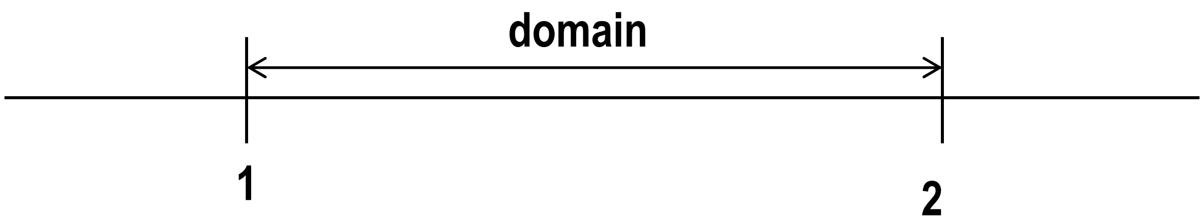
postprocessing

The Element Concept
Illustrative Problem - Problem description

$$\frac{d}{dx}\left(x\frac{dU(x)}{dx}\right)=\frac{2}{x^2} \qquad 1<x<2$$

$$U(1)=2$$

$$-x\frac{dU}{dx}\bigg|_{x=2}=\tau(2)=\frac{1}{2}$$



Foundations of FEM
12 Step trial solution procedure

- Step 1: Write the Galerkin residual equations
  - Trial solution $\tilde{U}(x;a) = a_1\phi_1(x) + a_2\phi_2(x) + \ldots + a_N\phi_N(x)$
  - The residual for the governing equation,

$$R(x;a) = \frac{d}{dx}\left(x\frac{d\tilde{U}(x)}{dx}\right) - \frac{2}{x^2}$$

  - N Galerkin residual equations

$$\int_{x_a}^{x_b} R(x;a)\phi_i(x)dx = 0 \qquad i = 1, 2, \ldots, N$$

$$\int_{x_a}^{x_b}\left[\frac{d}{dx}\left(x\frac{d\tilde{U}(x)}{dx}\right) - \frac{2}{x^2}\right]\phi_i(x)dx = 0 \qquad i = 1, 2, \ldots, N$$

Foundations of FEM
12 Step trial solution procedure

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- Step 2: Integrate by parts the highest derivative term

$$\int_{x_a}^{x_b} \frac{df}{dx} g dx = [fg]_{x_a}^{x_b} - \int_{x_a}^{x_b} f \frac{dg}{dx} dx$$

$$\int_{x_a}^{x_b} \frac{d}{dx}\left(x \frac{d\tilde{U}}{dx} \right) \phi_i dx = -\left[\left(-x \frac{d\tilde{U}}{dx} \right) \phi_i \right]_{x_a}^{x_b} - \int_{x_a}^{x_b} x \frac{d\tilde{U}}{dx} \frac{d\phi_i}{dx} dx$$

- Residual equations become

$$\int_{x_a}^{x_b} x \frac{d\tilde{U}}{dx} \frac{d\phi_i}{dx} dx = -\int_{x_a}^{x_b} \frac{2}{x^2} \phi_i dx - \left[\left(-x \frac{d\tilde{U}}{dx} \right) \phi_i \right]_{x_a}^{x_b} \qquad i = 1, 2, \ldots, N$$

Foundations of FEM
12 Step trial solution procedure

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- Step 3: Substitute the trial solution into interior integral

$$\int_{x_a}^{x_b} x\frac{d\tilde{U}}{dx}\frac{d\phi_i}{dx}dx = -\int_{x_a}^{x_b}\frac{2}{x^2}\phi_i dx - \left[\left(-x\frac{d\tilde{U}}{dx}\right)\phi_i\right]_{x_a}^{x_b} \qquad i = 1, 2, \ldots, N$$

$$\frac{d\tilde{U}}{dx} = \sum_{j=1}^{N} a_j \frac{d\phi_j}{dx}$$

$$\int_{x_a}^{x_b} x\frac{d\tilde{U}}{dx}\frac{d\phi_i}{dx}dx = \int_{x_a}^{x_b} x\left(\sum_{j=1}^{N} a_j\frac{d\phi_j}{dx}\right)\frac{d\phi_i}{dx}dx = \sum_{j=1}^{N}\int_{x_a}^{x_b}\frac{d\phi_i}{dx}x\frac{d\phi_j}{dx}a_j$$

  - Residual equation becomes

$$\sum_{j=1}^{N}\left(\int_{x_a}^{x_b}\frac{d\phi_i}{dx}x\frac{d\phi_j}{dx}dx\right)a_j = -\int_{x_a}^{x_b}\frac{2}{x^2}\phi_i dx - \left[\left(-x\frac{d\tilde{U}}{dx}\right)\phi_i\right]_{x_a}^{x_b} \qquad i = 1, 2, \ldots, N$$

Foundations of FEM
12 Step trial solution procedure

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– Writing the residual equations in full

$$i=1,\quad \left(\int_{x_a}^{x_b}\frac{d\phi_1}{dx}x\frac{d\phi_1}{dx}dx\right)a_1+\left(\int_{x_a}^{x_b}\frac{d\phi_1}{dx}x\frac{d\phi_2}{dx}dx\right)a_2\ldots+\left(\int_{x_a}^{x_b}\frac{d\phi_1}{dx}x\frac{d\phi_N}{dx}dx\right)a_N=-\int_{x_a}^{x_b}\frac{2}{x^2}\phi_1 dx-\left[\left(-x\frac{d\tilde{U}}{dx}\right)\phi_1\right]_{x_a}^{x_b}$$

$$i=2,\quad \left(\int_{x_a}^{x_b}\frac{d\phi_2}{dx}x\frac{d\phi_1}{dx}dx\right)a_1+\left(\int_{x_a}^{x_b}\frac{d\phi_2}{dx}x\frac{d\phi_2}{dx}dx\right)a_2\ldots+\left(\int_{x_a}^{x_b}\frac{d\phi_2}{dx}x\frac{d\phi_N}{dx}dx\right)a_N=-\int_{x_a}^{x_b}\frac{2}{x^2}\phi_2 dx-\left[\left(-x\frac{d\tilde{U}}{dx}\right)\phi_2\right]_{x_a}^{x_b}$$

$$i=N,\quad \left(\int_{x_a}^{x_b}\frac{d\phi_N}{dx}x\frac{d\phi_1}{dx}dx\right)a_1+\left(\int_{x_a}^{x_b}\frac{d\phi_N}{dx}x\frac{d\phi_2}{dx}dx\right)a_2\ldots+\left(\int_{x_a}^{x_b}\frac{d\phi_N}{dx}x\frac{d\phi_N}{dx}dx\right)a_N=-\int_{x_a}^{x_b}\frac{2}{x^2}\phi_N dx-\left[\left(-x\frac{d\tilde{U}}{dx}\right)\phi_N\right]_{x_a}^{x_b}$$

Foundations of FEM
12 Step trial solution procedure

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- In matrix form

$$\begin{bmatrix} \int_{x_a}^{x_b} \frac{d\phi_1}{dx} x \frac{d\phi_1}{dx} dx & \int_{x_a}^{x_b} \frac{d\phi_1}{dx} x \frac{d\phi_2}{dx} dx & \cdot & \cdot & \int_{x_a}^{x_b} \frac{d\phi_1}{dx} x \frac{d\phi_N}{dx} dx \\ \int_{x_a}^{x_b} \frac{d\phi_2}{dx} x \frac{d\phi_1}{dx} dx & \int_{x_a}^{x_b} \frac{d\phi_2}{dx} x \frac{d\phi_2}{dx} dx & \cdot & \cdot & \int_{x_a}^{x_b} \frac{d\phi_2}{dx} x \frac{d\phi_N}{dx} dx \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \int_{x_a}^{x_b} \frac{d\phi_N}{dx} x \frac{d\phi_1}{dx} dx & \int_{x_a}^{x_b} \frac{d\phi_N}{dx} x \frac{d\phi_2}{dx} dx & \cdot & \cdot & \int_{x_a}^{x_b} \frac{d\phi_N}{dx} x \frac{d\phi_N}{dx} dx \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ a_N \end{Bmatrix} = \begin{Bmatrix} -\int_{x_a}^{x_b} \frac{2}{x^2} \phi_1 dx - \left[\left(-x \frac{d\tilde{U}}{dx} \right) \phi_1 \right]_{x_a}^{x_b} \\ -\int_{x_a}^{x_b} \frac{2}{x^2} \phi_2 dx - \left[\left(-x \frac{d\tilde{U}}{dx} \right) \phi_2 \right]_{x_a}^{x_b} \\ \cdot \\ \cdot \\ -\int_{x_a}^{x_b} \frac{2}{x^2} \phi_N dx - \left[\left(-x \frac{d\tilde{U}}{dx} \right) \phi_N \right]_{x_a}^{x_b} \end{Bmatrix}$$

$$\begin{bmatrix} K_{11} & K_{12} & \cdot & \cdot & K_{1N} \\ K_{21} & K_{22} & \cdot & \cdot & K_{2N} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ K_{N1} & K_{N2} & \cdot & \cdot & K_{NN} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ a_N \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ \cdot \\ \cdot \\ F_N \end{Bmatrix}$$

Foundations of FEM
12 Step trial solution procedure

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$$[K]\{a\}=[F]$$

Element equation (defined in an element)

Stiffness matrix

Load vector

- Three benefits of integrating by parts
  - Order of trial functions lowed
  - Stiffness matrix is symmetric
  - A boundary term was created

$$K_{ij}=\int_{x_a}^{x_b}\frac{d\phi_i}{dx}x\frac{d\phi_j}{dx}dx$$

$$F_i=-\int_{x_a}^{x_b}\frac{2}{x^2}\phi_i dx-\left[\left(-x\frac{d\tilde{U}}{dx}\right)\phi_i\right]_{x_a}^{x_b}$$

Foundations of FEM
12 Step trial solution procedure

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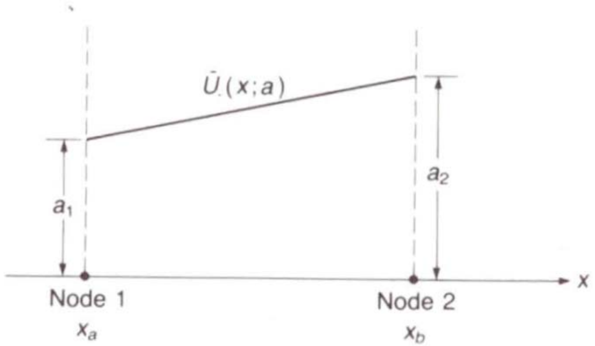
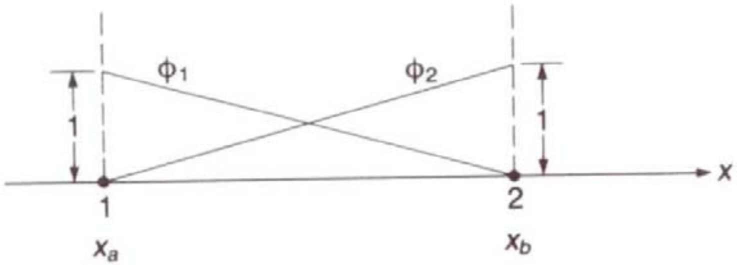
- Step 4: Develop specific expressions for the trial functions
  - Let's consider a linear (interpolation) polynomial $\tilde{U}(x;\alpha) = \alpha_1 + \alpha_2 x$
  - Each parameter a_i must represent the value of the trial solution at a specific point in the element. Each such point is called *node*.





$$\tilde{U}(x_a;a) = a_1$$

$$\tilde{U}(x_b;a) = a_2$$

$$\begin{aligned} \alpha_1 + \alpha_2 x_a &= a_1 \\ \alpha_1 + \alpha_2 x_b &= a_2 \end{aligned} \longrightarrow \begin{aligned} \alpha_1 &= \frac{x_b a_1 - x_a a_2}{x_b - x_a} \\ \alpha_2 &= \frac{a_2 - a_1}{x_b - x_a} \end{aligned}$$

$$\tilde{U}(x;a) = a_1\phi_1(x) + a_2\phi_2(x)$$

Foundations of FEM
12 Step trial solution procedure

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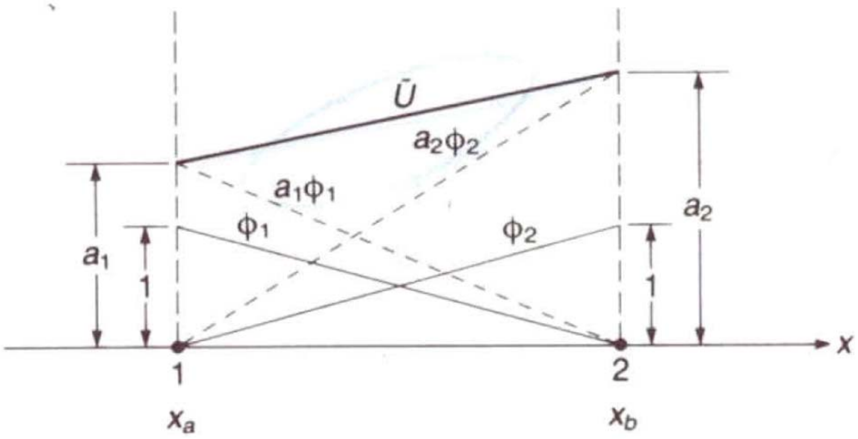
- Trial solution and trial function



$$\tilde{U}(x;a) = a_1\phi_1(x) + a_2\phi_2(x)$$

$$\phi_1 = \frac{x_b - x}{x_b - x_a}$$

$$\phi_2 = \frac{x - x_a}{x_b - x_a}$$

$$\phi_1(x_a) = 1 \qquad \phi_2(x_a) = 0$$

$$\phi_1(x_b) = 0 \qquad \phi_2(x_b) = 1$$

– With x_1 for x_a and x_2 for x_b,

$$\phi_j(x_i) = \delta_{ji}$$

12 Step trial solution procedure

- Step 5: substitute the trial functions into stiffness and load terms, and transform the integrals into a form appropriate for numerical evaluation
  - Because the trial solution contain two trial function, and there is one element

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$K_{ij} = \int_{x_a}^{x_b} \frac{d\phi_i}{dx} x \frac{d\phi_j}{dx} dx \qquad i = 1,2 \quad \text{and} \quad j = 1,2$$

$$F_i = FI_i + FB_i = -\int_{x_a}^{x_b} \frac{2}{x^2} \phi_i dx - \left[\left(-x \frac{d\tilde{U}}{dx} \right) \phi_i \right]_{x_a}^{x_b}$$

$$\phi_1 = \frac{x_b - x}{x_b - x_a} \qquad \phi_2 = \frac{x - x_a}{x_b - x_a} \longrightarrow \frac{d\phi_1}{dx} = -\frac{1}{x_b - x_a} \qquad \frac{d\phi_2}{dx} = \frac{1_a}{x_b - x_a}$$

$$K_{11} = \int_{x_a}^{x_b} \left(-\frac{1}{x_b - x_a} \right) x \left(-\frac{1}{x_b - x_a} \right) dx = \frac{1}{2} \frac{x_b + x_a}{x_b - x_a}$$

$$K_{12} = K_{21} = -K_{11}$$

$$K_{22} = K_{11}$$

Foundations of FEM
12 Step trial solution procedure

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$$FI_1 = -\int_{x_a}^{x_b} \frac{2}{x^2}\frac{x_b - x}{x_b - x_a}\,dx = -\frac{2}{x_a} + \frac{2}{x_b - x_a}\ln\frac{x_b}{x_a}$$

$$FI_2 = \frac{2}{x_b} - \frac{2}{x_b - x_a}\ln\frac{x_b}{x_a}$$

$$FB_1 = -\left(-x\frac{d\tilde{U}}{dx}\right)_{x_b}\phi_1(x_b) + \left(-x\frac{d\tilde{U}}{dx}\right)_{x_a}\phi_1(x_a) = \left(-x\frac{d\tilde{U}}{dx}\right)_{x_a}$$

$$FB_2 = -\left(-x\frac{d\tilde{U}}{dx}\right)_{x_b}$$

Foundations of FEM
12 Step trial solution procedure

- Step 6: Prepare expressions for flux

$$\tau(x) = \text{flux} = -x\frac{d\tilde{U}}{dx} = -x\left(a_1\frac{d\phi_1}{dx} + a_2\frac{d\phi_2}{dx}\right) = \frac{x}{x_b - x_a}(a_1 - a_2)$$

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- From step 1 to step 6

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\begin{bmatrix} \frac{1}{2}\frac{x_b + x_a}{x_b - x_a} & -\frac{1}{2}\frac{x_b + x_a}{x_b - x_a} \\ -\frac{1}{2}\frac{x_b + x_a}{x_b - x_a} & \frac{1}{2}\frac{x_b + x_a}{x_b - x_a} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} -\frac{2}{x_a} + \frac{2}{x_b - x_a}\ln\frac{x_b}{x_a} \\ \frac{2}{x_b} - \frac{2}{x_b - x_a}\ln\frac{x_b}{x_a} \end{Bmatrix} + \begin{Bmatrix} \left(-x\frac{d\tilde{U}}{dx}\right)_{x_a} \\ -\left(-x\frac{d\tilde{U}}{dx}\right)_{x_b} \end{Bmatrix}$$

Residual equations for a single element (linear element) = element equations

Foundations of FEM
12 Step trial solution procedure

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- Step 7: Specify the numerical data for the problem
  - Geometry data $x_a = 1,\ x_b = 2$
  - Physical properties and applied loads ← already given in this example
    $$\frac{d}{dx}\left(x\frac{dU(x)}{dx}\right) = \frac{2}{x^2} \qquad 1 < x < 2$$
- Step 8. Evaluate the interior terms in the system equation

$$\begin{bmatrix} \frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} -2 + 2\ln 2 \\ 1 - 2\ln 2 \end{Bmatrix} + \begin{Bmatrix} \left(-x\frac{d\tilde{U}}{dx}\right)_{x=1} \\ -\left(-x\frac{d\tilde{U}}{dx}\right)_{x=2} \end{Bmatrix}$$

- Step 9. Apply the boundary condition

$$-x\frac{dU}{dx}\bigg|_{x=2} = \frac{1}{2} \longrightarrow \begin{bmatrix} \frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} -2+2\ln 2 \\ 1-2\ln 2 \end{Bmatrix} + \begin{Bmatrix} \left(-x\frac{d\tilde{U}}{dx}\right)_{x=1} \\ -\frac{1}{2} \end{Bmatrix}$$

$$U(1) = 2 \longrightarrow \tilde{U}(1;a) = a_1\phi_1(1) + a_2\phi_2(1) = 2 \longrightarrow a_1 = 2$$

- Step 10: Solve the system equations

$$a_2 = \frac{7}{3} - \frac{4}{3}\ln 2 = 1.409$$

$$\tilde{U}(x;a) = a_1\phi_1(1) + a_2\phi_2(1) = 2(2-x) + 1.409(x-1)$$

Foundations of FEM
12 Step trial solution procedure

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- Step 11: Evaluate the flux

$$\tilde{\tau}(x) = x(a_1 - a_2) = 0.591x$$

- Step 12: Plot the solution and estimate its accuracy

Foundations of FEM
12 Step trial solution procedure (two elements)

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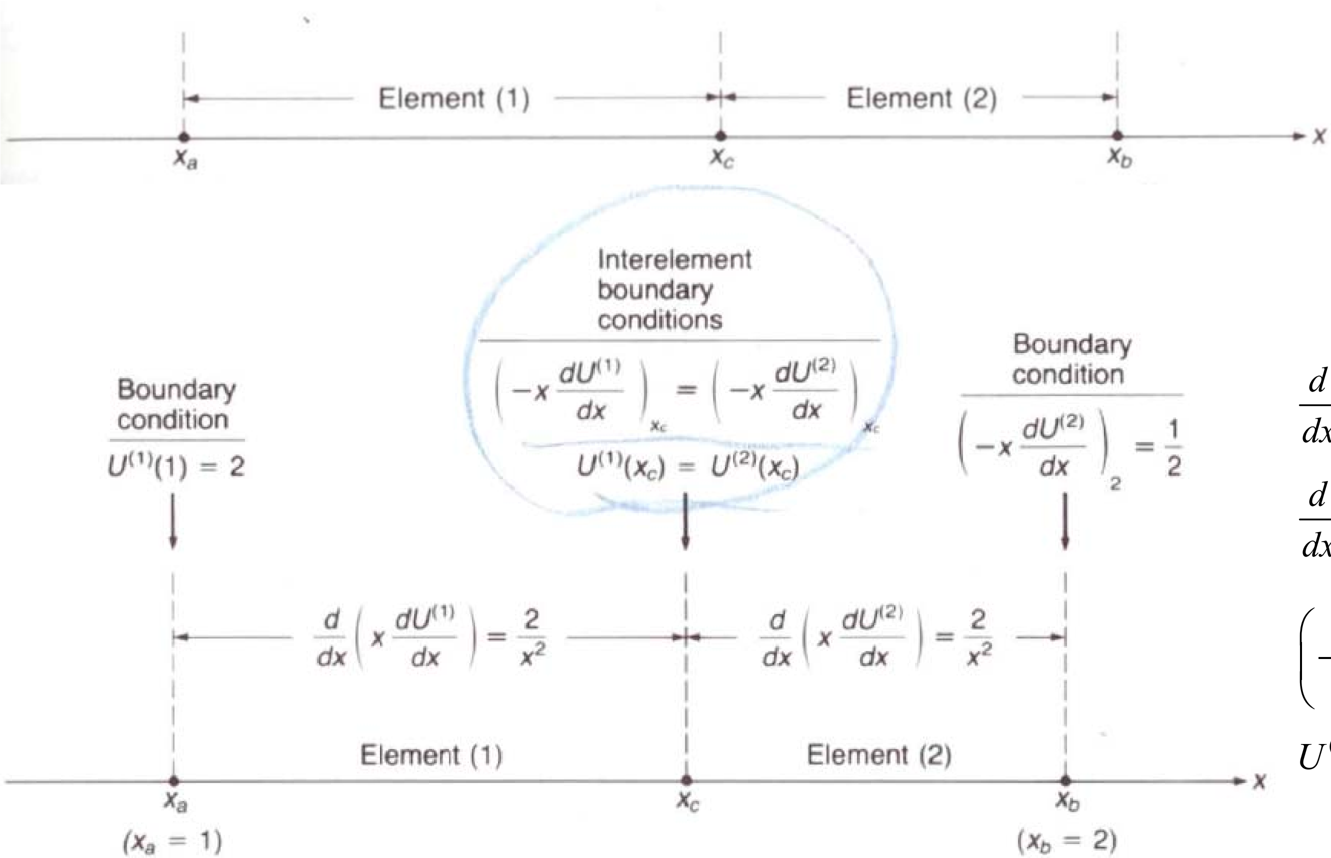
- A Two-Element Solution

$$\frac{d}{dx}\left(x\frac{dU(x)}{dx}\right)=\frac{2}{x^2} \qquad 1<x<2$$



$$\frac{d}{dx}\left(x\frac{dU^{(1)}(x)}{dx}\right)=\frac{2}{x^2} \qquad x_a<x<x_c$$

$$\frac{d}{dx}\left(x\frac{dU^{(2)}(x)}{dx}\right)=\frac{2}{x^2} \qquad x_c<x<x_b$$

$$\left(-x\frac{dU^{(1)}(x)}{dx}\right)_{x_c}=\left(-x\frac{dU^{(2)}(x)}{dx}\right)_{x_c}$$

$$U^{(1)}(x_c)=U^{(2)}(x_c)$$

Foundations of FEM
12 Step trial solution procedure (two elements)

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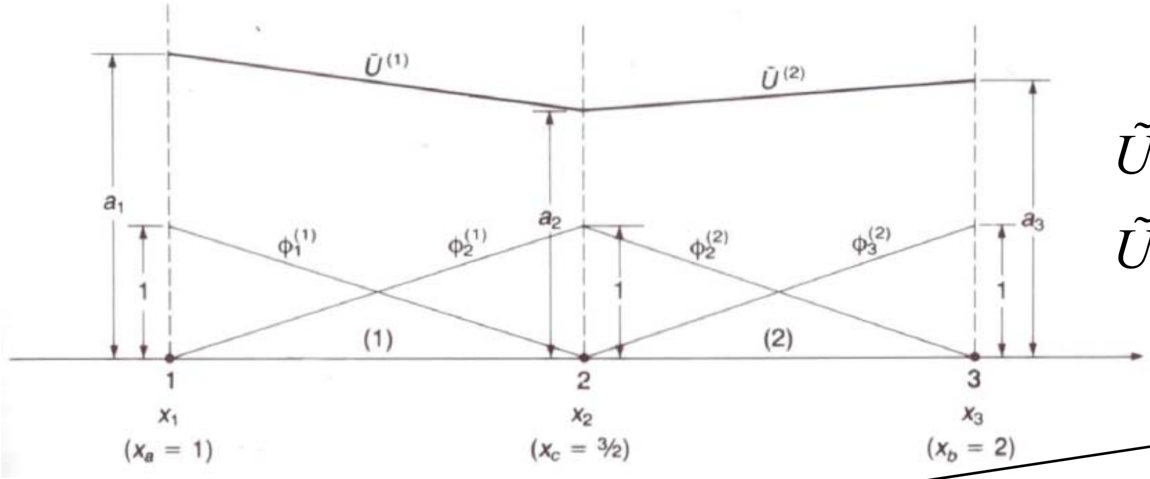
- Trial functions and node



$$\tilde{U}^{(1)}(x;a) = a_1\phi_1^{(1)}(x) + a_2\phi_2^{(1)}(x)$$

$$\tilde{U}^{(2)}(x;a) = a_2\phi_2^{(2)}(x) + a_3\phi_3^{(2)}(x)$$

Element equation

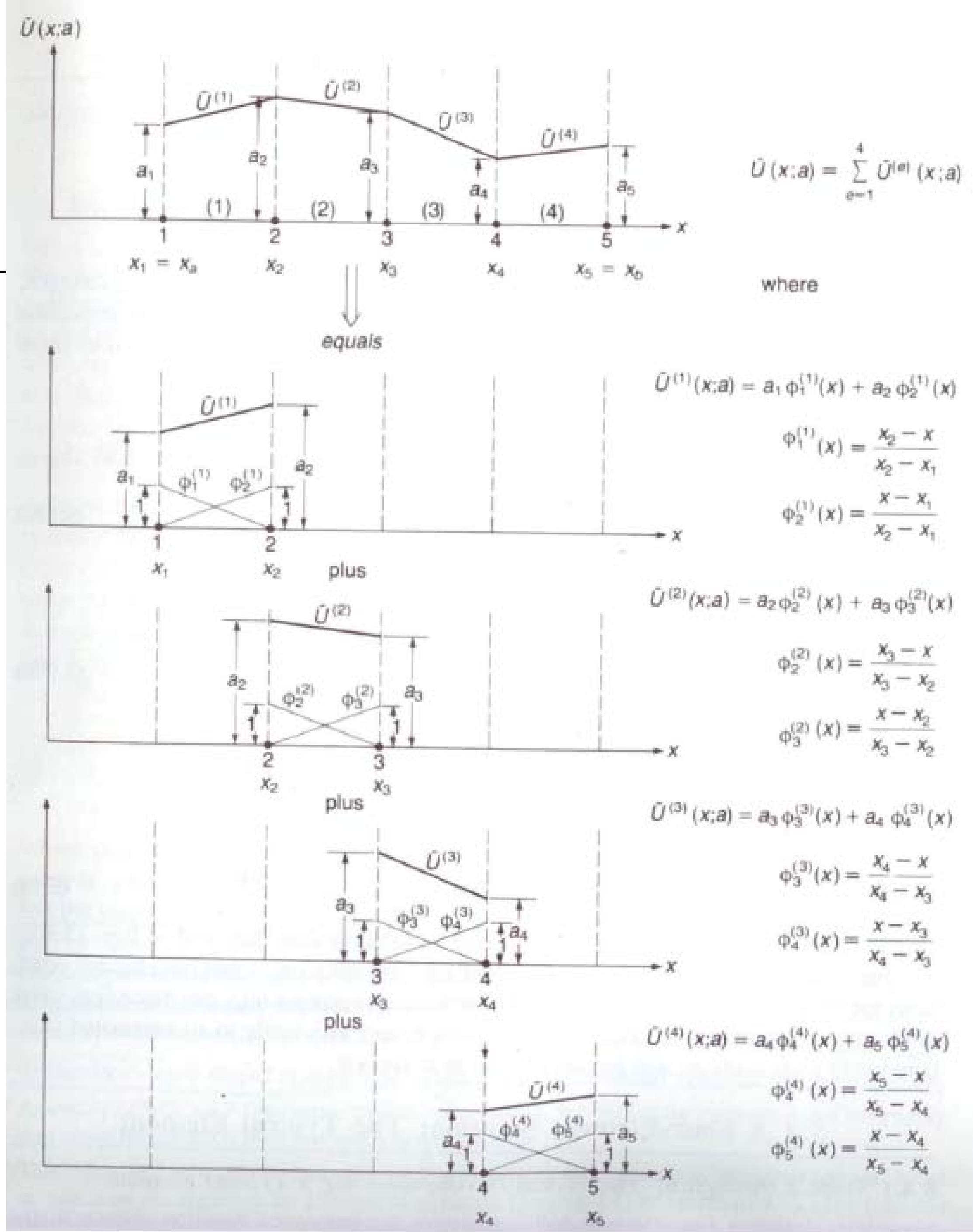
$$\begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} \\ K_{21}^{(1)} & K_{22}^{(1)} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} F_1^{(1)} \\ F_2^{(1)} \end{Bmatrix} \qquad \begin{bmatrix} K_{22}^{(2)} & K_{23}^{(2)} \\ K_{32}^{(2)} & K_{33}^{(2)} \end{bmatrix} \begin{Bmatrix} a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} F_2^{(2)} \\ F_3^{(2)} \end{Bmatrix}$$

System equation

assembling

$$\begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & 0 \\ K_{21}^{(1)} & K_{22}^{(1)} + K_{22}^{(2)} & K_{23}^{(2)} \\ 0 & K_{32}^{(2)} & K_{33}^{(2)} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} F_1^{(1)} \\ F_2^{(1)} + F_2^{(2)} \\ F_3^{(2)} \end{Bmatrix}$$

Foundations of FEM
12 Step trial solution procedure (four element)



Foundations of FEM
12 Step trial solution procedure (four elements)

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- System equations

$$\begin{bmatrix} \frac{9}{2} & -\frac{9}{2} & 0 & 0 & 0 \\ -\frac{9}{2} & 10 & -\frac{11}{2} & 0 & 0 \\ 0 & -\frac{11}{2} & 12 & -\frac{13}{2} & 0 \\ 0 & 0 & -\frac{13}{2} & 14 & -\frac{15}{2} \\ 0 & 0 & 0 & -\frac{15}{2} & \frac{15}{2} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix}$$

$$= \begin{Bmatrix} -2 + 8 \ln \frac{5}{4} \\ 8 \ln \frac{24}{25} \\ 8 \ln \frac{35}{36} \\ 8 \ln \frac{48}{49} \\ 1 - 8 \ln \frac{8}{7} \end{Bmatrix} + \begin{Bmatrix} \left(-x \frac{d\tilde{U}^{(1)}}{dx}\right)_{x=1} \\ 0 \\ 0 \\ 0 \\ -\frac{1}{2} \end{Bmatrix}$$

Foundations of FEM
12 Step trial solution procedure (eight elements)

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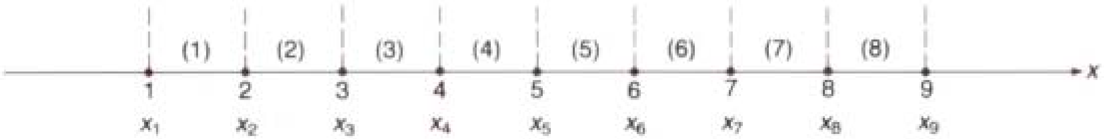

$$
\begin{bmatrix}
K_{11}^{(1)} & K_{12}^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
K_{21}^{(1)} & K_{22}^{(1)}+K_{22}^{(2)} & K_{23}^{(2)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & K_{32}^{(2)} & K_{33}^{(2)}+K_{33}^{(3)} & K_{34}^{(3)} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & K_{43}^{(3)} & K_{44}^{(3)}+K_{44}^{(4)} & K_{45}^{(4)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & K_{54}^{(4)} & K_{55}^{(4)}+K_{55}^{(5)} & K_{56}^{(5)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & K_{65}^{(5)} & K_{66}^{(5)}+K_{66}^{(6)} & K_{67}^{(6)} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & K_{76}^{(6)} & K_{77}^{(6)}+K_{77}^{(7)} & K_{78}^{(7)} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & K_{87}^{(7)} & K_{88}^{(7)}+K_{88}^{(8)} & K_{89}^{(8)} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & K_{98}^{(8)} & K_{99}^{(8)}
\end{bmatrix}
\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{Bmatrix}
=
\begin{Bmatrix}
F_1^{(1)} \\ F_2^{(1)}+F_2^{(2)} \\ F_3^{(2)}+F_3^{(3)} \\ F_4^{(3)}+F_4^{(4)} \\ F_5^{(4)}+F_5^{(5)} \\ F_6^{(5)}+F_6^{(6)} \\ F_7^{(6)}+F_7^{(7)} \\ F_8^{(7)}+F_8^{(8)} \\ F_9^{(8)}
\end{Bmatrix}
$$

Foundations of FEM

12 Step trial solution procedure (eight elements)

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$$
\begin{bmatrix}
K_{11}^{(1)} & K_{12}^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
K_{21}^{(1)} & K_{22}^{(1)}+K_{22}^{(2)} & K_{23}^{(2)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & K_{32}^{(2)} & K_{33}^{(2)}+K_{33}^{(3)} & K_{34}^{(3)} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & K_{43}^{(3)} & K_{44}^{(3)}+K_{44}^{(4)} & K_{45}^{(4)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & K_{54}^{(4)} & K_{55}^{(4)}+K_{55}^{(5)} & K_{56}^{(5)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & K_{65}^{(5)} & K_{66}^{(5)}+K_{66}^{(6)} & K_{67}^{(6)} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & K_{76}^{(6)} & K_{77}^{(6)}+K_{77}^{(7)} & K_{78}^{(7)} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & K_{87}^{(7)} & K_{88}^{(7)}+K_{88}^{(8)} & K_{89}^{(8)} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & K_{98}^{(8)} & K_{99}^{(8)}+K_{99}^{(9)}
\end{bmatrix}
\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{Bmatrix}
=
\begin{Bmatrix}
F_1^{(1)} \\ F_2^{(1)}+F_2^{(2)} \\ F_3^{(2)}+F_3^{(3)} \\ F_4^{(3)}+F_4^{(4)} \\ F_5^{(4)}+F_5^{(5)} \\ F_6^{(5)}+F_6^{(6)} \\ F_7^{(6)}+F_7^{(7)} \\ F_8^{(7)}+F_8^{(8)} \\ F_9^{(8)}
\end{Bmatrix}
$$

Foundations of FEM
12 Step trial solution procedure (eight elements)

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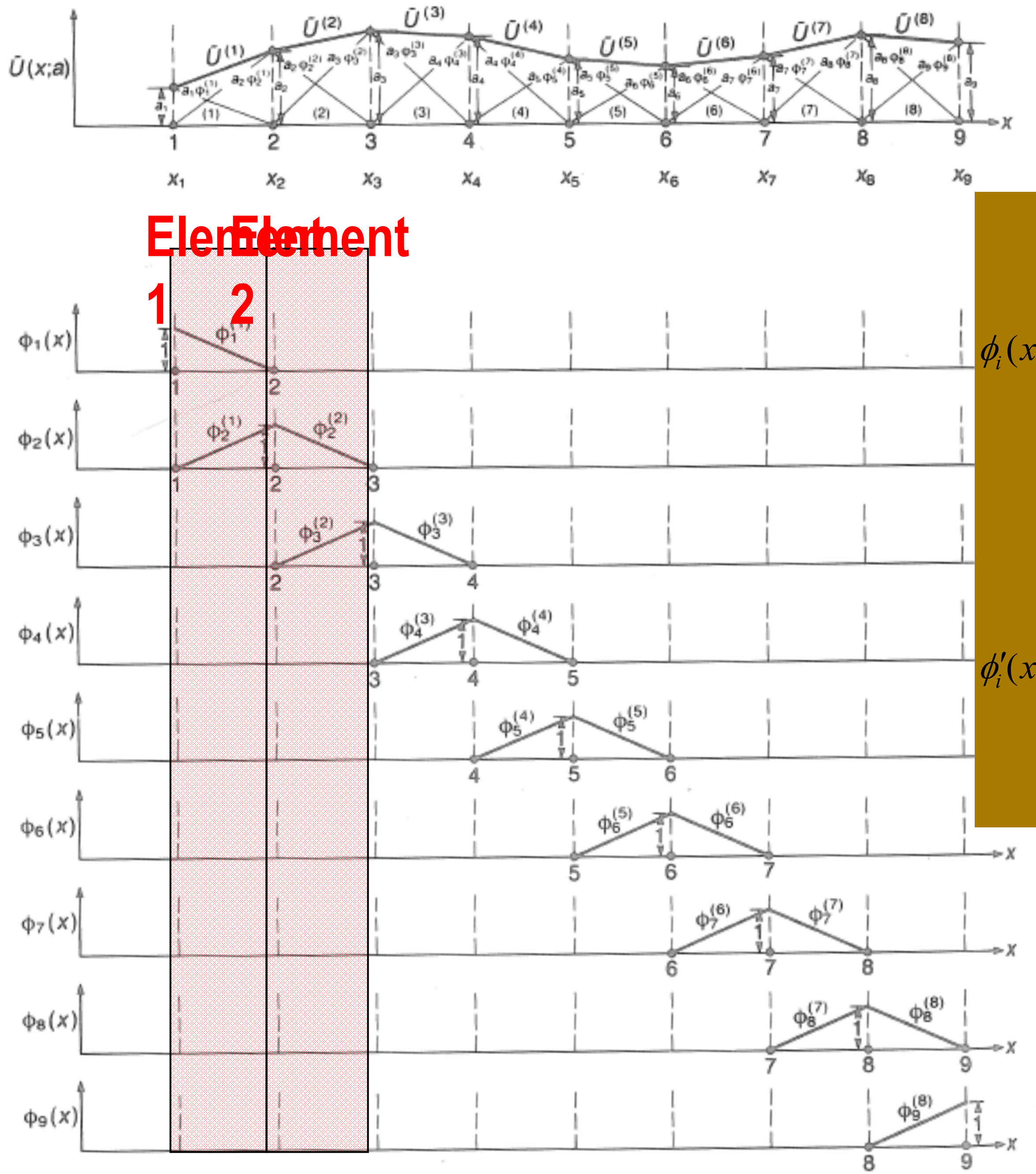
$$
\begin{bmatrix}
\frac{17}{2} & -\frac{17}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{17}{2} & 18 & -\frac{19}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{19}{2} & 20 & -\frac{21}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{21}{2} & 22 & -\frac{23}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{23}{2} & 24 & -\frac{25}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{25}{2} & 26 & -\frac{27}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{27}{2} & 28 & -\frac{29}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{29}{2} & 30 & -\frac{31}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{31}{2} & \frac{31}{2}
\end{bmatrix}
\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{Bmatrix}
=
\begin{Bmatrix}
-2 + 16 \ln \frac{9}{8} \\
16 \ln \frac{80}{81} \\
16 \ln \frac{99}{100} \\
16 \ln \frac{120}{121} \\
16 \ln \frac{143}{144} \\
16 \ln \frac{168}{169} \\
16 \ln \frac{195}{196} \\
16 \ln \frac{224}{225} \\
1 - 16 \ln \frac{16}{\;}
\end{Bmatrix}
+
\begin{Bmatrix}
\left(-x \frac{d\tilde{U}^{(1)}}{dx}\right)_{x=1} \\
\left(-x \frac{d\tilde{U}^{(2)}}{dx}\right)_{x=1.125} - \left(-x \frac{d\tilde{U}^{(1)}}{dx}\right)_{x=1.125} \\
\left(-x \frac{d\tilde{U}^{(3)}}{dx}\right)_{x=1.250} - \left(-x \frac{d\tilde{U}^{(2)}}{dx}\right)_{x=1.250} \\
\left(-x \frac{d\tilde{U}^{(4)}}{dx}\right)_{x=1.375} - \left(-x \frac{d\tilde{U}^{(3)}}{dx}\right)_{x=1.375} \\
\left(-x \frac{d\tilde{U}^{(5)}}{dx}\right)_{x=1.500} - \left(-x \frac{d\tilde{U}^{(4)}}{dx}\right)_{x=1.500} \\
\left(-x \frac{d\tilde{U}^{(6)}}{dx}\right)_{x=1.625} - \left(-x \frac{d\tilde{U}^{(5)}}{dx}\right)_{x=1.625} \\
\left(-x \frac{d\tilde{U}^{(7)}}{dx}\right)_{x=1.750} - \left(-x \frac{d\tilde{U}^{(6)}}{dx}\right)_{x=1.750} \\
\left(-x \frac{d\tilde{U}^{(8)}}{dx}\right)_{x=1.875} - \left(-x \frac{d\tilde{U}^{(7)}}{dx}\right)_{x=1} \\
-\left(-x \frac{d\tilde{U}^{(8)}}{\;}\right)
\end{Bmatrix}
$$



$$\phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & for \quad x_{i-1} \le x \le x_i \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & for \quad x_i \le x \le x_{i+1} \\ 0 & for \quad x \le x_{i-1} \ \& \ x \ge x_{i+1} \end{cases}$$

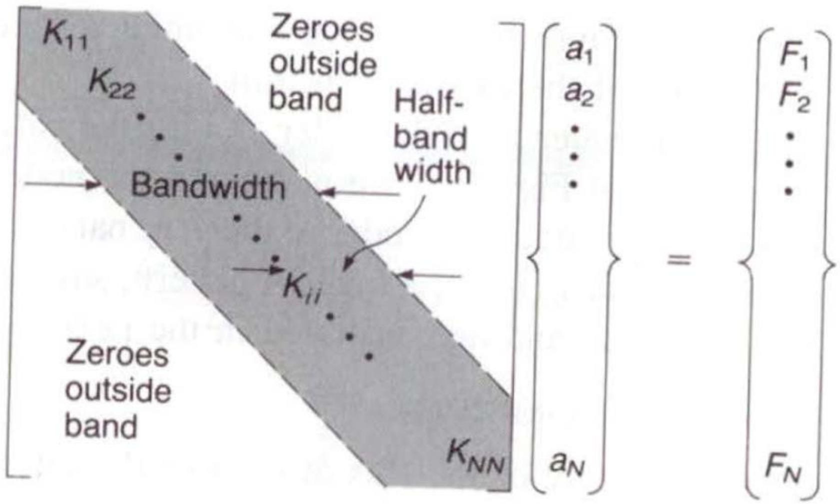
$$\phi_i'(x) = \begin{cases} \dfrac{1}{x_i - x_{i-1}} & for \quad x_{i-1} \le x \le x_i \\ \dfrac{-1}{x_{i+1} - x_i} & for \quad x_i \le x \le x_{i+1} \\ 0 & for \quad x \le x_{i-1} \ \& \ x \ge x_{i+1} \end{cases}$$

Note that basis function is almost zero except a few → the most important trick in FEM

Foundations of FEM
12 Step trial solution procedure (eight elements)

- Global stiffness matrix
  - Most of the terms in global stiffness matrix are zero → sparse matrix
  - Stiffness matrix is banded
  - Bandwidth 3, half-bandwidth 2

$$\begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ K_{21}^{(1)} & K_{22}^{(1)}+K_{22}^{(2)} & K_{23}^{(2)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & K_{32}^{(2)} & K_{33}^{(2)}+K_{33}^{(3)} & K_{34}^{(3)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{43}^{(3)} & K_{44}^{(3)}+K_{44}^{(4)} & K_{45}^{(4)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{54}^{(4)} & K_{55}^{(4)}+K_{55}^{(5)} & K_{56}^{(5)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & K_{65}^{(5)} & K_{66}^{(5)}+K_{66}^{(6)} & K_{67}^{(6)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & K_{76}^{(6)} & K_{77}^{(6)}+K_{77}^{(7)} & K_{78}^{(7)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & K_{87}^{(7)} & K_{88}^{(7)}+K_{88}^{(8)} & K_{89}^{(8)} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & K_{98}^{(8)} & K_{99}^{(8)}+K_{99}^{(9)} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{Bmatrix} = \begin{Bmatrix} F_1^{(1)} \\ F_2^{(1)}+F_2^{(2)} \\ F_3^{(2)}+F_3^{(3)} \\ F_4^{(3)}+F_4^{(4)} \\ F_5^{(4)}+F_5^{(5)} \\ F_6^{(5)}+F_6^{(6)} \\ F_7^{(6)}+F_7^{(7)} \\ F_8^{(7)}+F_8^{(8)} \\ F_9^{(8)} \end{Bmatrix}$$



Foundations of FEM
12 Step trial solution procedure (N elements)

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$$\begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & 0 & \cdots & 0 & 0 & 0 \\ K_{21}^{(1)} & K_{22}^{(1)}+K_{22}^{(2)} & K_{23}^{(2)} & \cdots & 0 & 0 & 0 \\ 0 & K_{32}^{(2)} & K_{33}^{(2)}+K_{33}^{(3)} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & K_{N-1,N-1}^{(N-2)}+K_{N-1,N-1}^{(N-1)} & K_{N-1,N}^{(N-1)} & 0 \\ 0 & 0 & 0 & \cdots & K_{N,N-1}^{(N-1)} & K_{N,N}^{(N-1)}+K_{N,N}^{(N)} & K_{N,N+1}^{(N)} \\ 0 & 0 & 0 & \cdots & 0 & K_{N+1,N}^{(N)} & K_{N+1,N+1}^{(N)} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{N-1} \\ u_N \\ u_{N+1} \end{Bmatrix} = \begin{Bmatrix} F_1^{(1)} \\ F_2^{(1)}+F_2^{(2)} \\ F_3^{(2)}+F_3^{(3)} \\ \vdots \\ F_{N-1}^{(N-2)}+F_{N-1}^{(N-1)} \\ F_N^{(N-1)}+F_N^{(N)} \\ F_{N+1}^{(N)} \end{Bmatrix}$$

- Depending on the boundary condition one or two of node displacement is automatically computed.

Foundations of FEM
General 1D problem

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- Governing Equation

$$-\frac{d}{dx}\left(\alpha(x)\frac{dU(x)}{dx}\right)+\beta(x)U(x)=f(x) \qquad x_a < x < x_b$$

  – Boundary condition

$$At\ \ x=x_a \qquad U(x_a)=U_a \qquad or \qquad -\alpha\left.\frac{dU}{dx}\right|_{x=x_a}=\tau_a$$

$$At\ \ x=x_b \qquad U(x_b)=U_b \qquad or \qquad -\alpha\left.\frac{dU}{dx}\right|_{x=x_b}=\tau_b$$

Foundations of FEM
General 1D problem

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- Element trial solution

$$\tilde{U}^{(e)}(x;a)=\sum_{j=1}^{n}a_j\phi_j^{(e)}(x)$$

Linear element

$$\tilde{U}^{(e)}(x;a)=\sum_{j=1}^{2}a_j\phi_j^{(e)}(x)$$

- Step1: Write the Galerkin Residual equation for a typical element

$$R(x;a)=-\frac{d}{dx}\left(\alpha(x)\frac{d\tilde{U}^{(e)}(x;a)}{dx}\right)+\beta(x)\tilde{U}^{(e)}(x;a)-f(x)$$

$$\int_{(e)}R(x;a)\phi_i^{(e)}(x)dx=0 \qquad i=1,2,\ldots,n$$

$$\int_{(e)}\left[-\frac{d}{dx}\left(\alpha(x)\frac{d\tilde{U}^{(e)}(x;a)}{dx}\right)+\beta(x)\tilde{U}^{(e)}(x;a)-f(x)\right]\phi_i^{(e)}(x)dx=0 \qquad i=1,2,\ldots,n$$

Foundations of FEM
General 1D problem

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- Step2: Integrate by parts

$$\int_{(e)}\left[-\frac{d}{dx}\left(\alpha(x)\frac{d\tilde{U}^{(e)}(x;a)}{dx}\right)\right]\phi_i^{(e)}(x)dx=\left[-\alpha(x)\frac{d\tilde{U}^{(e)}}{dx}\phi_i^{(e)}(x)\right]^{(e)}+\int_{(e)}\alpha(x)\frac{d\tilde{U}^{(e)}(x;a)}{dx}\frac{d\phi_i^{(e)}(x)}{dx}dx$$

$$\int_{(e)}\left[\alpha(x)\frac{d\tilde{U}^{(e)}(x;a)}{dx}\frac{d\phi^{(e)}{}_i(x)}{dx}+\beta(x)d\tilde{U}^{(e)}(x;a)\phi^{(e)}{}_i(x)\right]dx$$

$$=\int_{(e)}f(x)\phi^{(e)}{}_i(x)dx-\left[\left(-\alpha(x)\frac{d\tilde{U}^{(e)}}{dx}\right)\phi^{(e)}{}_i(x)\right]^{(e)}\qquad i=1,2,\ldots,n$$

- Step 3: Substitute the element trial solution into integrals on the LHS

$$\frac{d\tilde{U}^{(e)}(x;a)}{dx}=\sum_{j=1}^{n}a_j\frac{d\phi_j^{(e)}(x)}{dx}$$

Foundations of FEM
General 1D problem

$$\sum_{j=1}^{n}\left(\int_{(e)}\frac{d\phi_i}{dx}\alpha(x)\frac{d\phi_j}{dx}dx+\int_{(e)}\phi^{(e)}{}_i(x)\beta(x)\phi^{(e)}{}_j(x)dx\right)a_j=\int_{(e)}f(x)\phi^{(e)}{}_i(x)dx-\left[\left(-\alpha(x)\frac{d\tilde{U}^{(e)}}{dx}\right)\phi^{(e)}{}_i(x)\right]^{(e)}\qquad i=1,2,\ldots,n$$

$$\begin{bmatrix} K_{11}{}^{(e)} & K_{12}{}^{(e)} & \cdot & \cdot & K_{1n}{}^{(e)} \\ K_{21}{}^{(e)} & K_{22}{}^{(e)} & \cdot & \cdot & K_{2n}{}^{(e)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ K_{n1}{}^{(e)} & K_{n2}{}^{(e)} & \cdot & \cdot & K_{nn}{}^{(e)} \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ a_n \end{Bmatrix}=\begin{Bmatrix} F_1{}^{(e)} \\ F_2{}^{(e)} \\ \cdot \\ \cdot \\ F_n{}^{(e)} \end{Bmatrix}$$

$$K_{ij}^{(e)}=\int_{(e)}\frac{d\phi_i}{dx}\alpha(x)\frac{d\phi_j}{dx}dx+\int_{(e)}\phi^{(e)}{}_i(x)\beta(x)\phi^{(e)}{}_j(x)dx$$

$$F_i^{(e)}=\int_{(e)}f(x)\phi^{(e)}{}_i(x)dx-\left[\left(-\alpha(x)\frac{d\tilde{U}^{(e)}}{dx}\right)\phi^{(e)}{}_i(x)\right]^{(e)}$$

Foundations of FEM
General 1D problem

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- Step 4: Develop specific expressions for the shape function (element trial function)
  - Linear element (n=2)

$$\tilde{U}^{(e)}(x;a) = \sum_{j=1}^{2} a_j \phi_j^{(e)}(x)$$

$$\phi_1^{(e)}(x) = \frac{x_2 - x}{x_2 - x_1}$$

$$\phi_2^{(e)}(x) = \frac{x - x_1}{x_2 - x_1}$$

  - ……
  - Follow the similar steps as before

Foundations of FEM
Quadratic element

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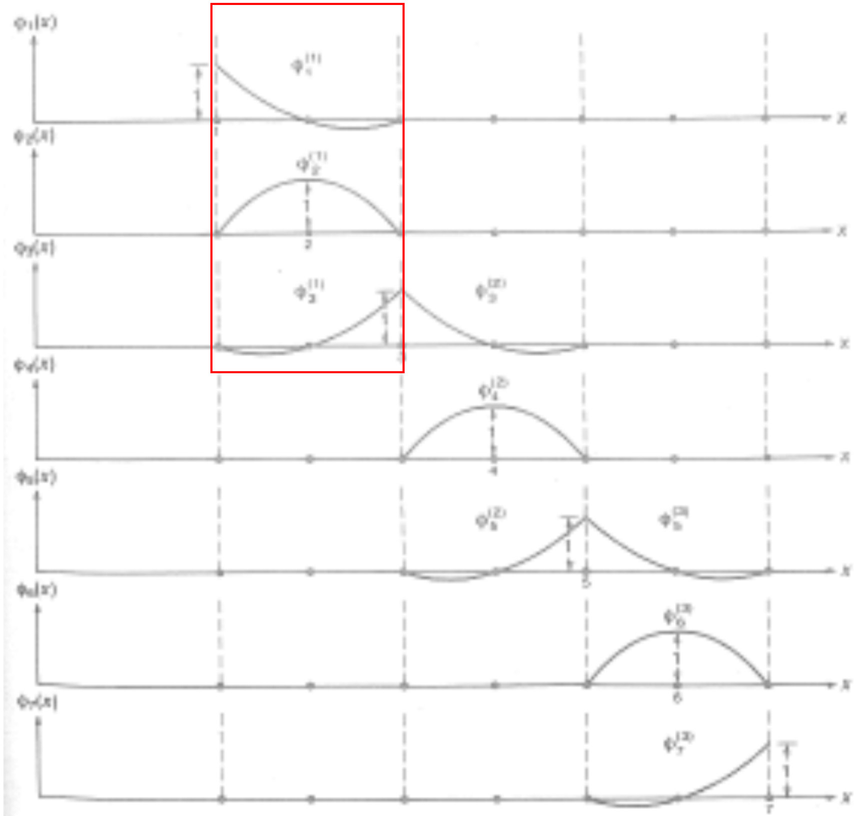

$$\phi^{(e)}{}_1(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}$$

$$\phi^{(e)}{}_2(x) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}$$

$$\phi^{(e)}{}_3(x) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}$$

$$\begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & & & & \\ K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} & & & & \\ K_{31}^{(1)} & K_{32}^{(1)} & K_{33}^{(1)}+K_{33}^{(2)} & K_{34}^{(2)} & K_{35}^{(2)} & & \\ & & K_{43}^{(2)} & K_{44}^{(2)} & K_{45}^{(2)} & & \\ & & K_{53}^{(2)} & K_{54}^{(2)} & K_{55}^{(2)}+K_{55}^{(3)} & K_{56}^{(3)} & K_{57}^{(3)} \\ & & & & K_{65}^{(3)} & K_{66}^{(3)} & K_{67}^{(3)} \\ & & & & K_{75}^{(3)} & K_{76}^{(3)} & K_{77}^{(3)} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{Bmatrix} = \begin{Bmatrix} F_1^{(1)} \\ F_2^{(1)} \\ F_3^{(1)}+F_3^{(2)} \\ F_4^{(2)} \\ F_5^{(2)}+F_5^{(3)} \\ F_6^{(3)} \\ F_7^{(3)} \end{Bmatrix}.$$

Quadratic element has quadratic basis function but it essentially end up with similar sparse matrix.

Foundations of FEM
1D elasticity example

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$$-E\frac{\partial^2 u_x}{\partial x^2} = \rho b_x$$

Strong form

$$-E\frac{\partial^2 u_x}{\partial x^2} - \rho b_x = 0$$

Residual *R* =0

$$\int_0^L \left(E\frac{\partial^2 u_x}{\partial x^2} + \rho b_x \right) v dx = 0$$

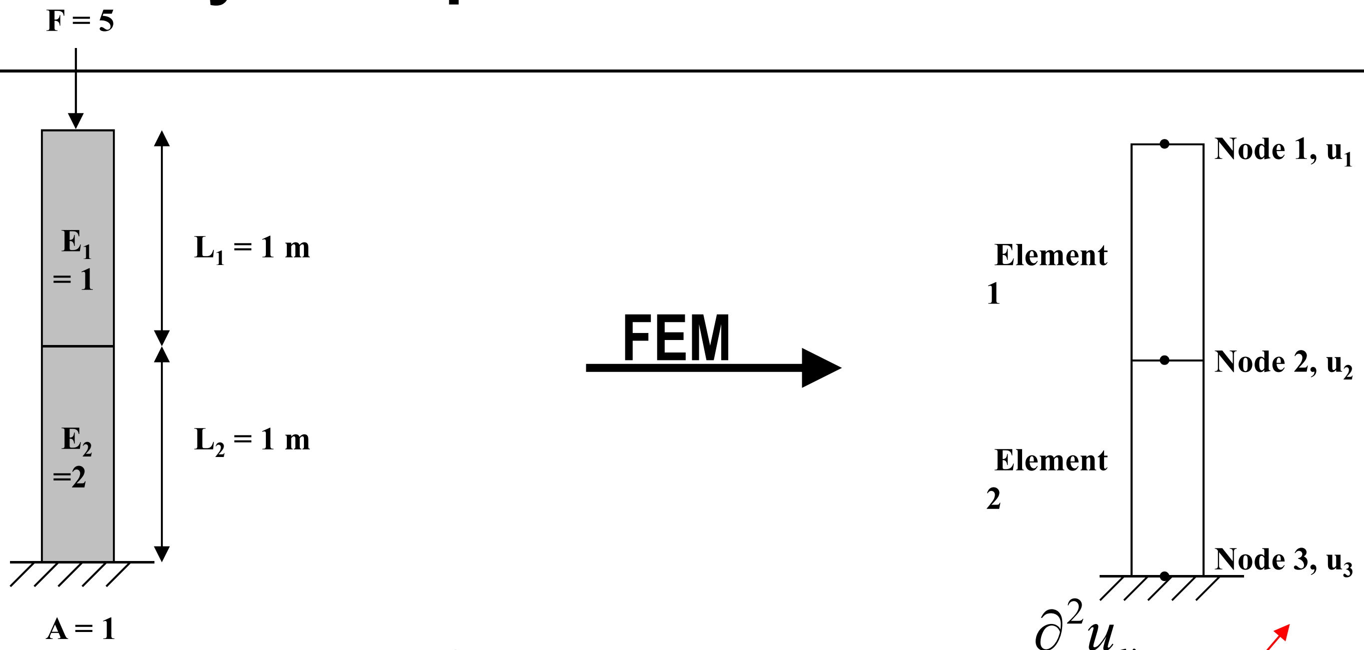
Weak form (weighted residual)

$$\int_0^L \frac{d^2 u}{dx^2} v = -\int_0^L \frac{du}{dx}\frac{dv}{dx} + \frac{du}{dx} v \Big|_{x=0}^{x=L}$$

$$\int_0^L \frac{du}{dx} E \frac{dv}{dx} dx = \int_0^L \rho b_x v dx$$

v is test function. We chose test function that is same as element trial function (shape function).

Foundations of FEM
1D elasticity example



- Find the displacement and stress?

$$E\frac{\partial^2 u_y}{\partial y^2} + \cancel{\rho b_y} = 0$$

BC: at y=0, u = 0 (Dirichlet)

at y=2, $EA\frac{\partial u_y}{\partial y} = -5$ (Neumann)

- Find the u_1, u_2, u_3 & F

Foundations of FEM
1D elasticity example

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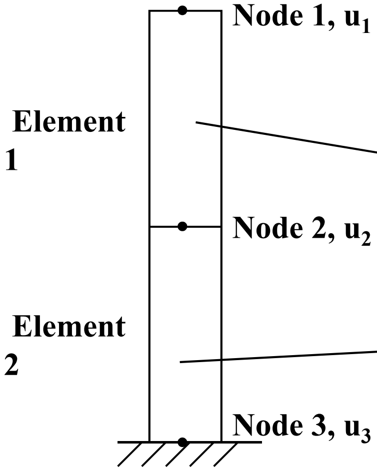

$$\begin{bmatrix} K^1_{11} & K^1_{12} & 0 \\ K^1_{21} & K^1_{22}+K^2_{22} & K^2_{23} \\ 0 & K^2_{32} & K^2_{33}\end{bmatrix}\begin{Bmatrix}u_1\\u_2\\u_3\end{Bmatrix} = \begin{Bmatrix}F^{(1)}_1 \\ F^{(1)}_2 + F^{(2)}_2 \\ F^{(2)}_3\end{Bmatrix}$$

– Global K & F matrix is formed through summation of element K and F matrix

– Same for whatever number of elements

Initial Boundary Value problem
pure initial value problem

- "pure" initial value problem (with only one unknown fn, U(t))

$$c\frac{dU(t)}{dt} + kU(t) = f(t) \qquad t > t_0$$

$$IC:\ U(t_0) = U_0$$

$$\downarrow$$

$$U(t) = \frac{f}{k} + \left(U_0 - \frac{f}{k}\right)e^{-(k/c)(t-t_0)}$$

Semi-infinite domain characteristic of an IVP

- System of coupled 'n' ordinary differential equations, in matrix form (with unknowns, $U_1(t)$, $U_2(t)$, ,,,$U_n(t)$)

$$[c]\left\{\frac{dU(t)}{dt}\right\} + [k]\{U(t)\} = \{f(t)\} \qquad t > t_0$$

$$IC:\ \{U(t_0)\} = \{U_0\}$$

Initial Boundary Value problem
Problem statement (1D)

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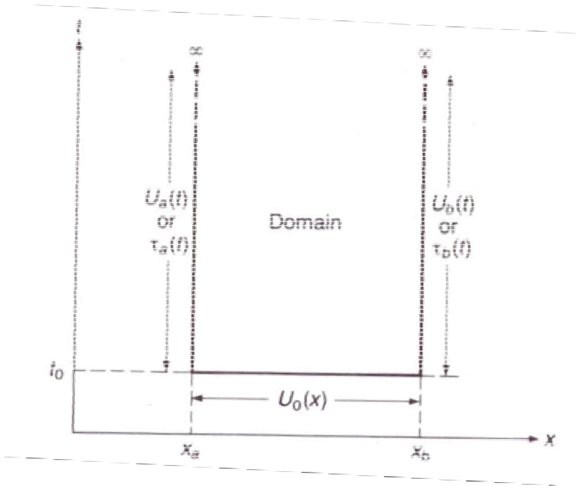
$$\mu(x)\frac{\partial U(x,t)}{\partial t}-\frac{\partial}{\partial x}\left(\alpha(x)\frac{\partial U(x,t)}{\partial x}\right)+\beta(x)U(x,t)=f(x,t)$$

domain : $x_a < x < x_b \qquad t > t_0$

BC at $x_a : U(x_a,t)=U_a(t)$ *or* $\left(-\alpha(x)\frac{\partial U(x,t)}{\partial x}\right)_{x_a} = \tau_a(t)$

at $x_b : U(x_b,t)=U_b(t)$ *or* $\left(-\alpha(x)\frac{\partial U(x,t)}{\partial x}\right)_{x_b} = \tau_b(t)$

IC at $t_0 \quad U(x,t_0)=U_0(x)$



$$\rho(x)c(x)\frac{\partial T(x,t)}{\partial t}-\frac{d}{dx}\left(k(x)\frac{dT(x,t)}{dx}\right)+\frac{hl}{A}T(x,t)=Q(x,t)+\frac{hlT_\infty}{A}$$

Initial Boundary Value problem
Trial functions (1D)

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- Unknown U is a function of x and t

$$\tilde{U}^{(e)}(x,t;a)=\sum_{j=1}^{n} a_j \phi_j^{(e)}(x,t) \qquad \tilde{U}^{(e)}(x,t;a)=\sum_{j=1}^{n} a_j(t) \phi_j^{(e)}(x)$$

  – Parameters a_j is functions of time
  – Assembled system equations will be ordinary differential equations in time (not algebraic equation)
  – Initial boundary value problem → pure initial value problem
  – Can be solved by time-stepping technique

Initial Boundary Value problem
12-step Procedure

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- Step1: Write the Galerkin residual equations for a typical element

$$\int_{(e)}\left[\mu(x)\frac{\partial \tilde{U}^{(e)}(x,t;a)}{\partial t}-\frac{\partial}{\partial x}\left(\alpha(x)\frac{\partial \tilde{U}^{(e)}(x,t;a)}{\partial x}\right)+\beta(x)\tilde{U}^{(e)}(x,t;a)-f(x,t)\right]\phi_i^{(e)}(x)dx=0$$

$$i=1,2,\ldots,n$$

- Step2: Integrate by parts

$$\int_{(e)}\phi^{(e)}{}_i(x)\mu(x)\frac{\partial \tilde{U}^{(e)}(x,t;a)}{\partial t}+\int_{(e)}\frac{\partial \phi^{(e)}{}_i(x)}{\partial x}\alpha(x)\frac{\partial \tilde{U}^{(e)}(x,t;a)}{\partial x}dx+\int_{(e)}\phi^{(e)}{}_i(x)\beta(x)d\tilde{U}^{(e)}(x,t;a)dx$$

$$=\int_{(e)}f(x,t)\phi^{(e)}{}_i(x)dx-\left[\left(-\alpha(x)\frac{\partial \tilde{U}^{(e)}(x,t;a)}{\partial x}\right)\phi^{(e)}{}_i(x)\right]_{x_1}^{x_n} \qquad i=1,2,\ldots,n$$

$$\left[\left(-\alpha(x)\frac{\partial \tilde{U}^{(e)}(x,t;a)}{\partial x}\right)\phi^{(e)}{}_i(x)\right]_{x_1}^{x_n}=\left[\tilde{\tau}^{(e)}(x,t;a)\phi^{(e)}{}_i(x)\right]_{x_1}^{x_n}$$

Initial Boundary Value problem
12-step Procedure

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- Step3: Substitute the general form of the element trial solution into interior integrals

$$\frac{\partial \tilde{U}^{(e)}(x,t;a)}{\partial x} = \sum_{j=1}^{n} a_j(t) \frac{d\phi_j^{(e)}(x)}{dx} \quad \longleftrightarrow \quad \frac{\partial \tilde{U}^{(e)}(x,t;a)}{\partial t} = \sum_{j=1}^{n} \frac{da_j(t)}{dt} \phi_j^{(e)}(x)$$

$$\sum_{j=1}^{n}\left(\int_{(e)} \phi^{(e)}{}_i(x)\mu(x)\phi_j^{(e)}(x)dx \right)\frac{da_j(t)}{dt} + \sum_{j=1}^{n}\left(\int_{(e)} \frac{d\phi_i^{(e)}}{dx}\alpha(x)\frac{d\phi_j^{(e)}}{dx}dx \right)a_j(t) + \sum_{j=1}^{n}\left(\int_{(e)} \phi^{(e)}{}_i(x)\beta(x)\phi^{(e)}{}_j(x)dx \right)a_j(t)$$

$$= \int_{(e)} f(x,t)\phi^{(e)}{}_i(x)dx - \left[\tilde{\tau}^{(e)}(x,t;a)\phi^{(e)}{}_i(x) \right]_{x_1}^{x_n} \qquad i = 1,2,\ldots,n$$

Initial Boundary Value problem
12-step Procedure

Capacity matrix

$$[c]\left\{\frac{da(t)}{dt}\right\}+[K]\{a(t)\}=\{F(t)\} \qquad\qquad t>t_0$$

(heat) Capacity integral $IC:\ \{U(t_0)\}=\{U_0\}$

$$C_{ij}^{(e)}=\int_{(e)}\phi^{(e)}{}_i(x)\mu(x)\phi_j^{(e)}(x)dx$$

$$K_{ij}^{(e)}=K\alpha_{ij}^{(e)}+K\beta_{ij}^{(e)}=\int_{(e)}\frac{d\phi^{(e)}{}_i}{dx}\alpha(x)\frac{d\phi_j^{(e)}}{dx}dx+\int_{(e)}\phi^{(e)}{}_i(x)\beta(x)\phi^{(e)}{}_j(x)dx$$

$$F_i^{(e)}(t)=Ff_i^{(e)}(t)+F\tau_i^{(e)}(t)=\int_{(e)}f(x,t)\phi^{(e)}{}_i(x)dx-\left[\tilde{\tau}^{(e)}(x,t;a)\phi^{(e)}{}_i(x)\right]_{x_1}^{x_n}$$

Initial Boundary Value problem
12-step Procedure

- Step 4: Develop specific expressions for the shape functions

- Step 5: Substitute the shape functions into the element equations, and transform the integrals into a form appropriate for numerical evaluation
  - Linear element

$$\phi_1^{(e)}(x) = \frac{x_2 - x}{x_2 - x_1}$$

$$\phi_2^{(e)}(x) = \frac{x - x_1}{x_2 - x_1}$$

$$C_{11}^{(e)} = \int_{(e)} \frac{x_2 - x}{x_2 - x_1} \mu(x) \frac{x_2 - x}{x_2 - x_1} dx = \frac{1}{3} \mu^{(e)} L$$

$$C_{12}^{(e)} = \int_{(e)} \frac{x_2 - x}{x_2 - x_1} \mu(x) \frac{x - x_1}{x_2 - x_1} dx = \frac{1}{6} \mu^{(e)} L$$

Initial Boundary Value problem
12-step Procedure

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- Step 6: Derive expression for the flux

$$\tilde{\tau}^{(e)}(x,t) = -\alpha(x)\frac{a_2(t)-a_1(t)}{x_2-x_1}$$

$$[c]\left\{\frac{dU(t)}{dt}\right\} + [K]\{U(t)\} = \{f(t)\} \qquad t > t_0$$

$$IC: \ \{U(t_0)\} = \{U_0\}$$

  – e.g., when there is two trial functions. A system of coupled ordinary differential equations.

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{Bmatrix} \dfrac{da_1(t)}{dt} \\ \dfrac{da_2(t)}{dt} \end{Bmatrix} + \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} a_1(t) \\ a_2(t) \end{Bmatrix} = \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix}$$

1D Boundary Value Problem
Heat Conduction (diffusion eq.) – steady state

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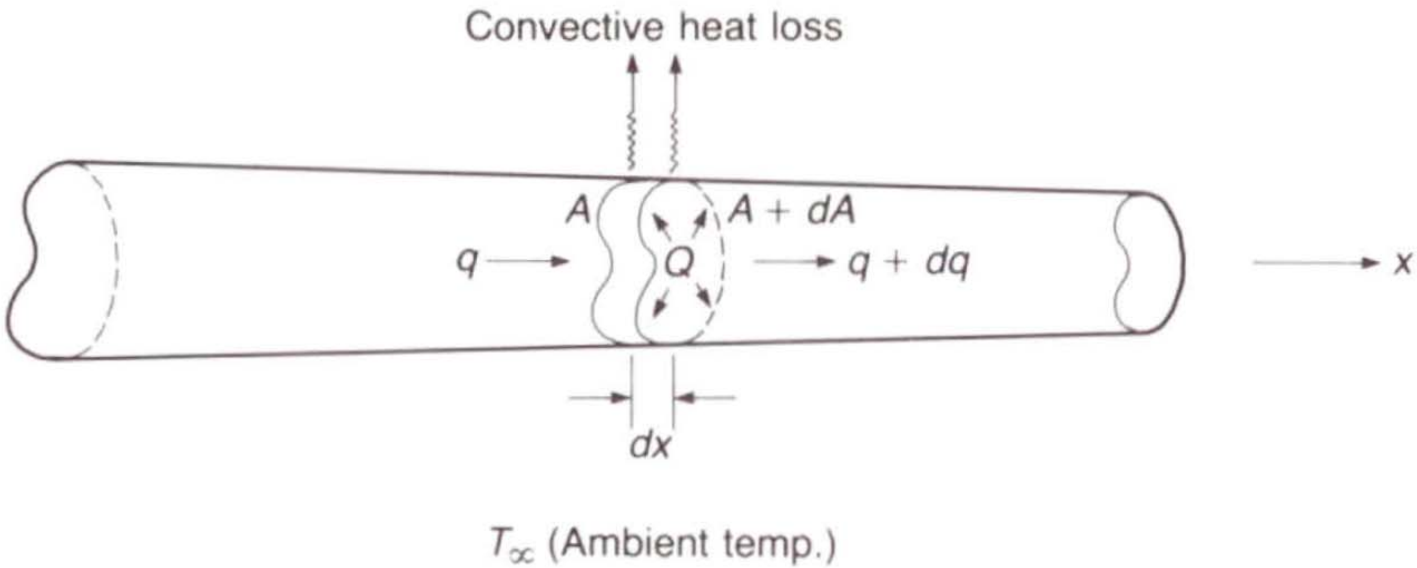

Heating conducting rod with convection from its lateral surface

Burnett, 1987

1D Boundary Value Problem
Heat Conduction (diffusion eq.) – steady state

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Rate of heat energy added	–	Rate of heat energy lost	=	Rate of increase of internal heat energy

0

$$qA + QAdx - (q + dq)A - hldx(T - T_\infty) = 0$$

$$-\frac{\partial}{\partial x}\left(k(x)\frac{\partial T(x)}{\partial x}\right) + \frac{hl}{A}T(x) = Q(x) + \frac{hl}{A}T_\infty$$

Initial Boundary Value problem
Time-stepping method

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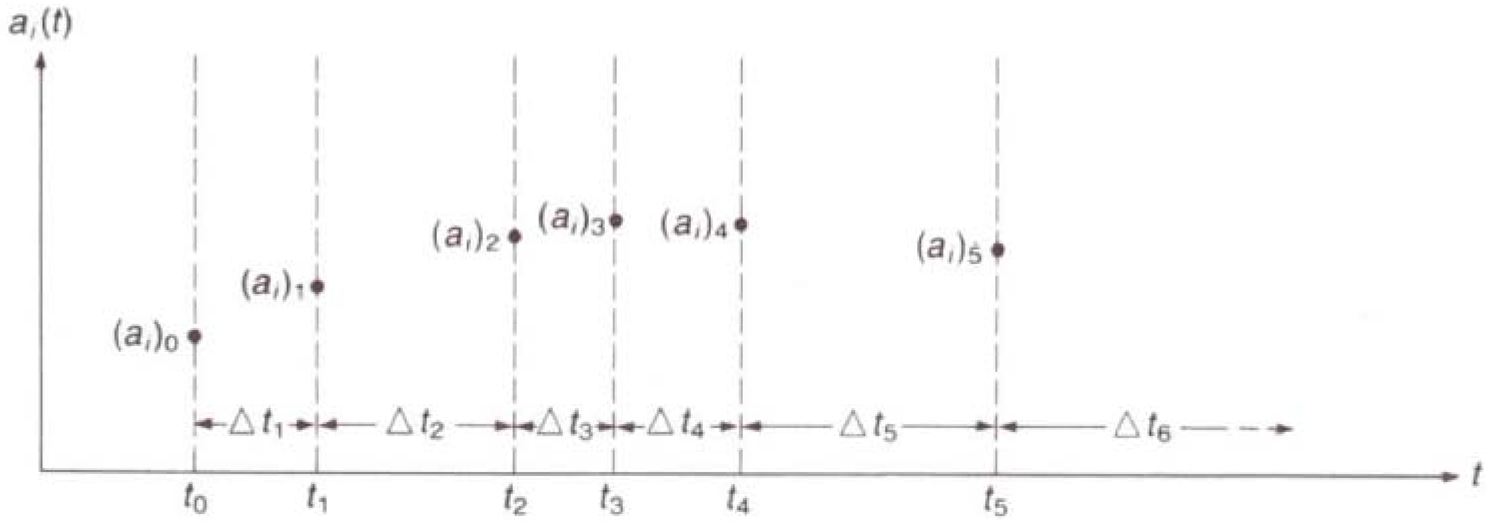
- Time-stepping method
  - Time axis is divided into a succession of time steps Δt_i
  - Discrete a(t) at the end of each step
  - $\{a\}_0$ at time t_0, $\{a\}_1$ at time t_1, ….$\{a\}_n$ at time t_n



Initial Boundary Value problem
Time-stepping method

$$[c]\left\{\frac{da(t)}{dt}\right\}+[K]\{a(t)\}=\{F(t)\} \qquad t>t_0$$

$$IC:\ \{U(t_0)\}=\{U_0\}$$

- Time stepping method
  - Time-stepping, time-marching, ….

$$[P]\{a\}_n+[Q]\{a\}_{n-1}=p\{F\}_n+q\{F\}_{n-1}$$

$$[P]\{a\}_1=p\{F\}_1+q\{F\}_0-[Q]\{a\}_0$$

$$[P]\{a\}_2=p\{F\}_2+q\{F\}_1-[Q]\{a\}_1$$

– 1) Backward difference method

– 2) Mid-difference method (central difference method)

– 3) Forward difference method

– 4) The θ method

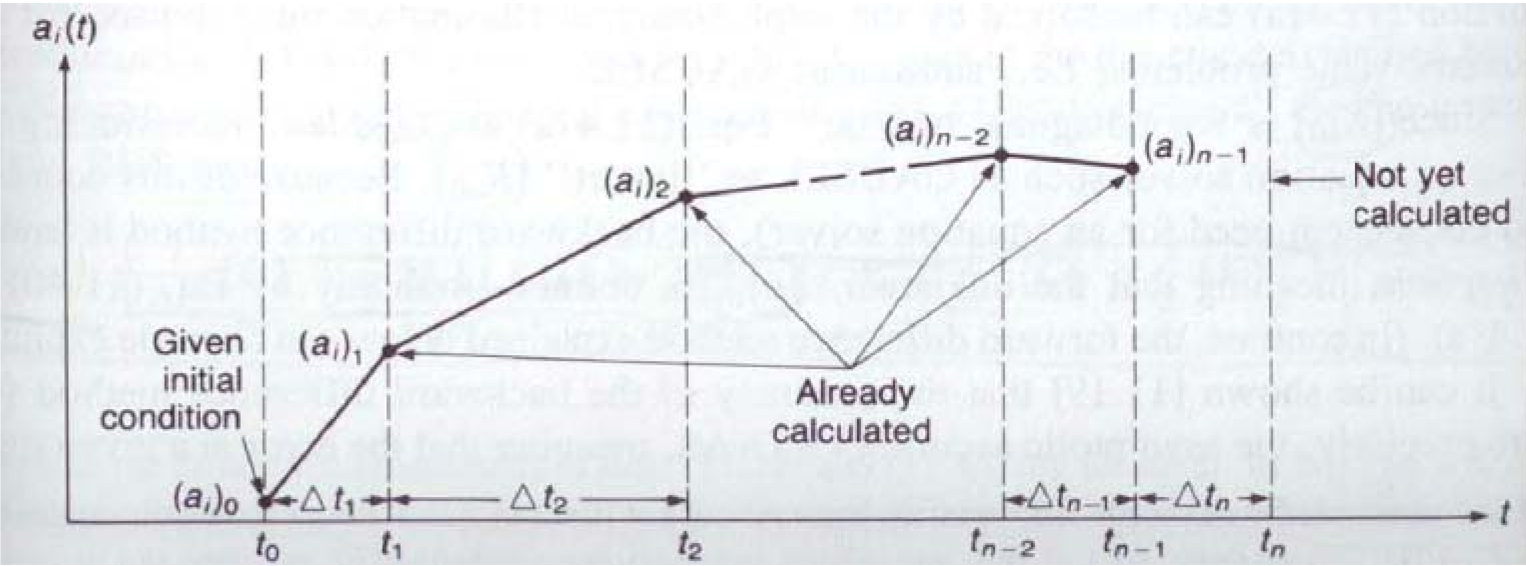
Time-stepping method
Backward difference method

$$[c]\left\{\frac{da}{dt}\right\}_n + [K]\{a\}_n = \{F\}_n$$

$$\left\{\frac{da}{dt}\right\}_n = \frac{\{a\}_n - \{a\}_{n-1}}{\Delta t_n}$$

$$\Delta t_n = t_n - t_{n-1}$$

$$\left(\frac{da_i}{dt}\right)_n = \frac{(a_i)_n - (a_i)_{n-1}}{\Delta t_n} \qquad i = 1, 2, \ldots, N$$



Time-stepping method
Backward difference method

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- A system of algebraic equation

$$\left(\frac{1}{\Delta t_n}[c]+[K]\right)\{a\}_n=\{F\}_n+\frac{1}{\Delta t_n}[c]\{a\}_{n-1}$$

Non-diagonal matrix

$$\left[K_{eff}\right]=\frac{1}{\Delta t_n}[c]+[K]$$

$$\left[K_{eff}\right]\{a\}_n=\left\{F_{eff}\right\}_n \longleftarrow$$

$$\left[F_{eff}\right]=\{F\}_n+\frac{1}{\Delta t_n}[c]\{a\}_{n-1}$$

  - Use the same Gaussian elimination
  - Equations are coupled, need matrix solver
  - Unknown {a} is defined implicitly → backward difference is implicit
  - Accuracy : $O(\Delta t)$ →asymptotic rate of convergence is dt. E.g., error decrease by one half if dt is one half.

Time-stepping method
Mid-difference method

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– Evaluate at the center of time step

$$[c]\left\{\frac{da}{dt}\right\}_{n-1/2}+[K]\{a\}_{n-1/2}=\{F\}_{n-1/2}$$

$$\left\{\frac{da}{dt}\right\}_{n-1/2}=\frac{\{a\}_n-\{a\}_{n-1}}{\Delta t_n} \qquad \{a\}_{n-1/2}=\frac{\{a\}_{n-1}+\{a\}_n}{2} \qquad \Delta t_n=t_n-t_{n-1}$$

$$\{a\}=(1-\theta)\{a\}_{n-1}+\theta\{a\}_n \qquad \theta=\frac{t-t_{n-1}}{\Delta t_n}$$

Time-stepping method
Mid-difference method

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$$\left(\frac{1}{\Delta t_n}[c]+\frac{1}{2}[K]\right)\{a\}_n=\{F\}_{n-1/2}+\left(\frac{1}{\Delta t_n}[c]-\frac{1}{2}[K]\right)\{a\}_{n-1} \qquad \{F\}_{n-1/2}=\frac{\{F\}_{n-1}+\{F\}_n}{2}$$

$$\left[K_{eff}\right]=\frac{1}{\Delta t_n}[c]+\frac{1}{2}[K]$$

$$\left[F_{eff}\right]=\{F\}_{n-1/2}+\left(\frac{1}{\Delta t_n}[c]-\frac{1}{2}[K]\right)\{a\}_{n-1}$$

- Accuracy: $O(\Delta t^2)$
  - asymptotic rate of convergence is dt^2
  - Frequent oscillations with typical time-step

Time-stepping method
Forward-difference method

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- Evaluated at the backward end of the time step, t_{n-1}

$$[c]\left\{\frac{da}{dt}\right\}_{n-1}+[K]\{a\}_{n-1}=\{F\}_{n-1}$$

$$\Delta t_n = t_n - t_{n-1}$$

$$\left\{\frac{da}{dt}\right\}_{n-1}=\frac{\{a\}_n-\{a\}_{n-1}}{\Delta t_n}$$

$$\frac{1}{\Delta t_n}[c]\{a\}_n=\{F\}_{n-1}+\left(\frac{1}{\Delta t_n}[c]-[K]\right)\{a\}_{n-1}$$

This can be diagonalized

$$\left[K_{eff}\right]\{a\}_n=\left\{F_{eff}\right\}_n$$

$$\left[K_{eff}\right]=\frac{1}{\Delta t_n}[c]$$

$$\left[F_{eff}\right]=\{F\}_{n-1}+\left(\frac{1}{\Delta t_n}[c]-[K]\right)\{a\}_{n-1}$$

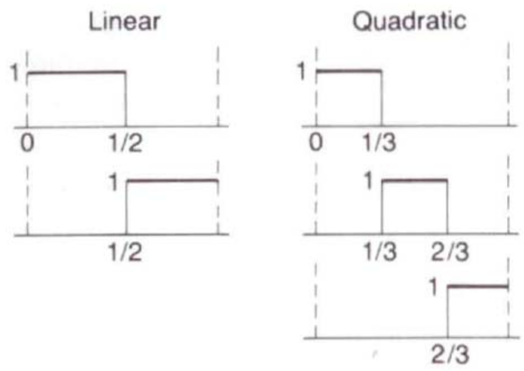
Time-stepping method
Forward-difference method

- Lumping
  - Techniques for diagonalizing [C]

$$CL_{ii}^{e} = \sum_{j=1}^{n} C_{ij}^{e} \quad n = 1, 2, \ldots, n$$

$$CL_{ij}^{e} = 0$$

  - consistent capacity matrix: $[C]^e$ (Not lumped)
  - Lumped capacity matrix : $[CL]^e$
  - Lumping can be interpreted as using a different set of shape functions for just the capacity integrals



Time-stepping method
Forward-difference method

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- Lumped capacity matrix

$$[CL] = \begin{bmatrix} CL_{11} & .. & 0 \\ .. & CL_{22} & .. \\ 0 & .. & CL_{33} \end{bmatrix}$$

$$\{a\}_n = \{a\}_{n-1} + \Delta t_n [CL]^{-1} \left(\{F\}_{n-1} - [K]\{a\}_{n-1} \right)$$

  – Matrix inversion is unnecessary and {a}n can be evaluated explicitly.

  – Much faster than backward or mid-difference method

  – Accuracy: $O(\Delta t)$ asymptotic rate of convergence is dt

  – Potentially unstable

Time-stepping method
θ- method

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- Generalization of previous three methods – evaluate at a general location

$$[c]\left\{\frac{da}{dt}\right\}_{\theta} + [K]\{a\}_{\theta} = \{F\}_{\theta}$$

$$\theta = \frac{t - t_{n-1}}{\Delta t_n}$$

$$\{a\} = (1-\theta)\{a\}_{n-1} + \theta\{a\}_n$$

$$\left\{\frac{da}{dt}\right\}_{\theta} = \frac{1}{\Delta t_n}\frac{d\{a\}_{\theta}}{d\theta} = \frac{\{a\}_n - \{a\}_{n-1}}{\Delta t_n}$$

Time-stepping method
θ- method

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$$\left(\frac{1}{\Delta t_n}[c]+\theta[K]\right)\{a\}_n=(1-\theta)\{F\}_{n-1}+\theta\{F\}_n+\left(\frac{1}{\Delta t_n}[c]-(1-\theta)[K]\right)\{a\}_{n-1}$$

$$\left[K_{eff}\right]=\frac{1}{\Delta t_n}[c]+\theta[K]$$

$$\left[F_{eff}\right]=(1-\theta)\{F\}_{n-1}+\theta\{F\}_n+\left(\frac{1}{\Delta t_n}[c]-(1-\theta)[K]\right)\{a\}_{n-1}$$

– θ= 0: forward difference

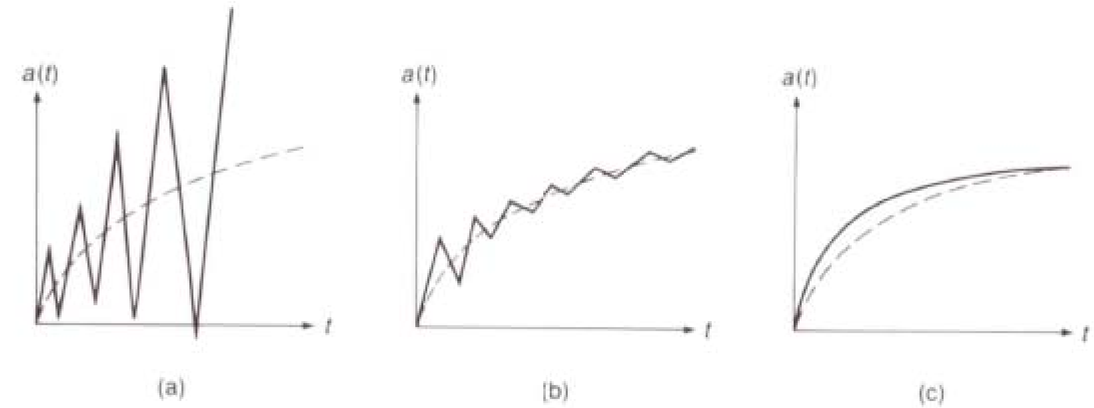
– θ= 1/2: mid-difference

– θ= 1: backward difference

– We may choose something else which might perform better

Time-stepping method
Comparison of performances

- We can only accept the stable solution



Unstable – oscillatory divergence

stable – oscillatory decay

stable – monotonic decay

Time-stepping method
Comparison of performances

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- A single equation

$$C\frac{da(t)}{dt}+Ka(t)=F(t)$$

- For free response (when applied load F(t) vanishes)

$$C\frac{da(t)}{dt}+Ka(t)=0$$

- Exact solution

$$a(t)=Ae^{-\lambda t} \qquad \text{eigenvalue, } \lambda = K/C$$

$$\left(\frac{1}{\Delta t_n}C+\theta K\left[K\right]\right)a_n=(1-\theta)F_{n-1}+\theta F_n+\left(\frac{1}{\Delta t_n}C-(1-\theta)K\right)a_{n-1}$$

$\downarrow$ **Free response**

$$\left(\frac{1}{\Delta t_n}C+\theta K\left[K\right]\right)a_n=\left(\frac{1}{\Delta t_n}C-(1-\theta)K\right)a_{n-1}$$

Time-stepping method
Comparison of performances

- Single DOF

$$\frac{a_n}{a_{n-1}} = \frac{1-(1-\theta)\lambda\Delta t}{1+\theta\lambda\Delta t}$$

- Multi DOF system

$$\frac{\left(A_i\right)_n}{\left(A_i\right)_{n-1}} = \frac{1-(1-\theta)\lambda_i\Delta t}{1+\theta\lambda_i\Delta t} \qquad i = 1, 2, \ldots, N$$

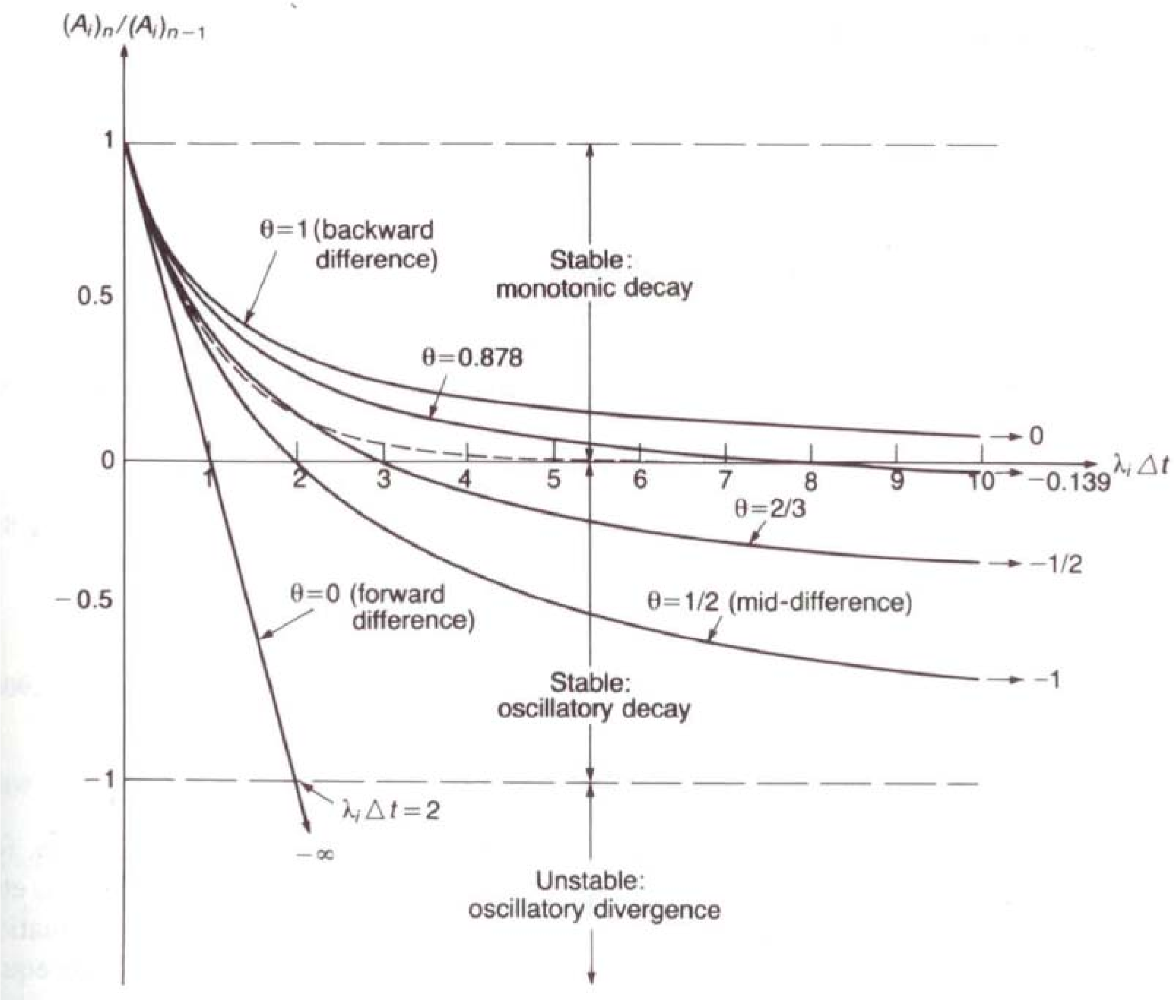
- Condition for stability

$$\left|\frac{\left(A_i\right)_n}{\left(A_i\right)_{n-1}}\right| < 1 \qquad i = 1, 2, \ldots, N$$

$$0 \le \theta < 1/2 : \lambda_i\Delta t < \frac{2}{1-2\theta} \qquad i = 1, 2, \ldots, N \longrightarrow$$ **Conditionally stable**

$$\theta \ge 1/2 : \lambda_i\Delta t > \frac{-2}{2\theta-1} \qquad i = 1, 2, \ldots, N \longrightarrow$$ **Unconditionally stable**

Time-stepping method
Comparison of performances



Time-stepping method
Comparison of performances

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- Critical time step: smallest time step of a system

$$\Delta t_{crit} \approx \frac{2}{d(1-2\theta)\pi^2}\left((\mu/\alpha)\delta^2\right)^e_{\min} \quad 0 \le \theta < 1/2$$

Approximated from 1D eigenproblem

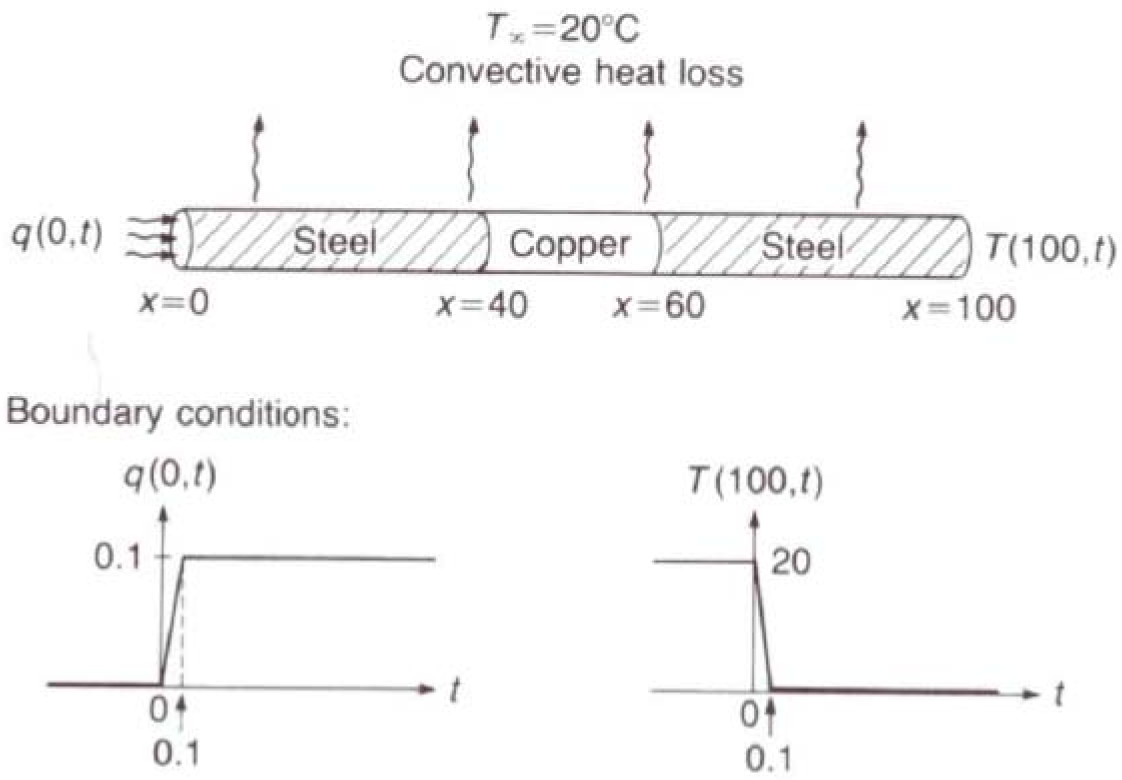
$$\Delta t_{crit} \approx \frac{2}{d\pi^2}(\mu/\alpha)\delta_{\min}{}^2 \quad \theta = 0,\ \alpha, \mu \text{ are constants}$$

  – Conservative low estimation - within a factor of 5 of the exact value (Burnett, 1987)

δ : distance between two adjacent nodes in the element

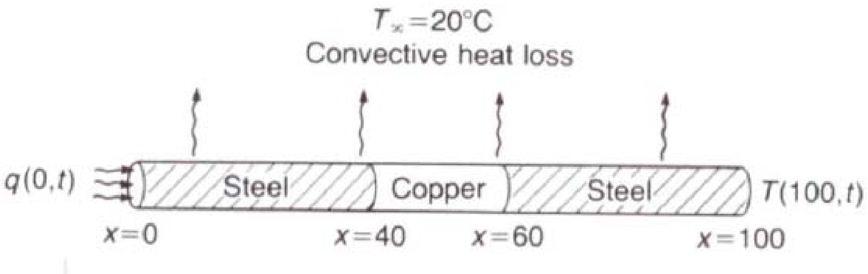
$$\alpha \frac{\partial^2 U(x,t)}{\partial x^2} - \mu \frac{\partial U(x,t)}{\partial t} = 0$$

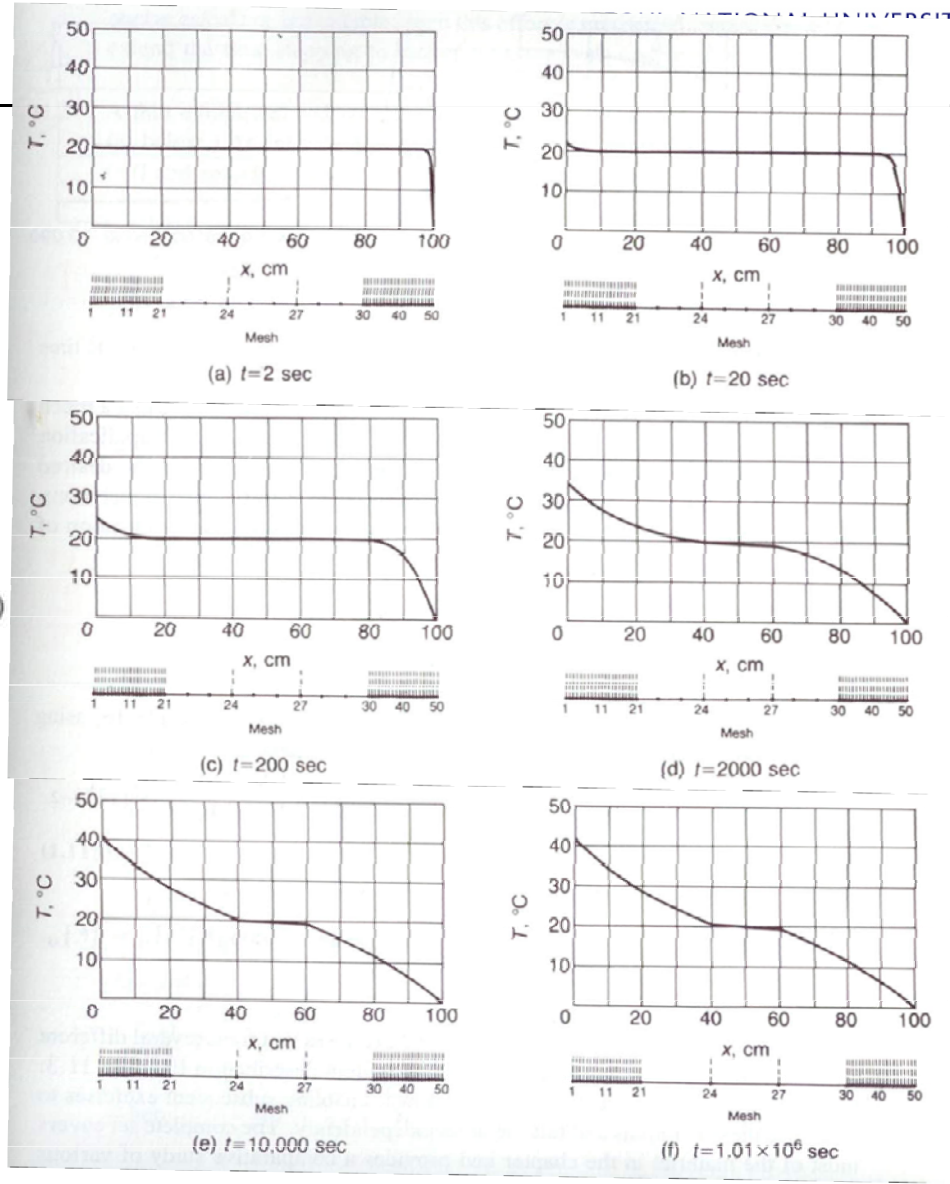
Initial Boundary Value problem
Example



Initial conditions: $T(x,0) = 20°C \qquad 0 \leq x \leq 100$

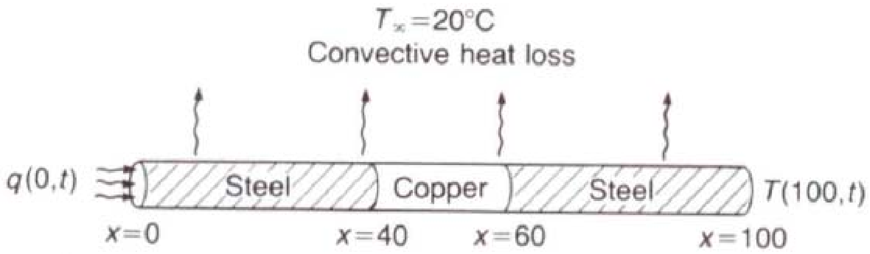
Initial Boundary Value problem
Example

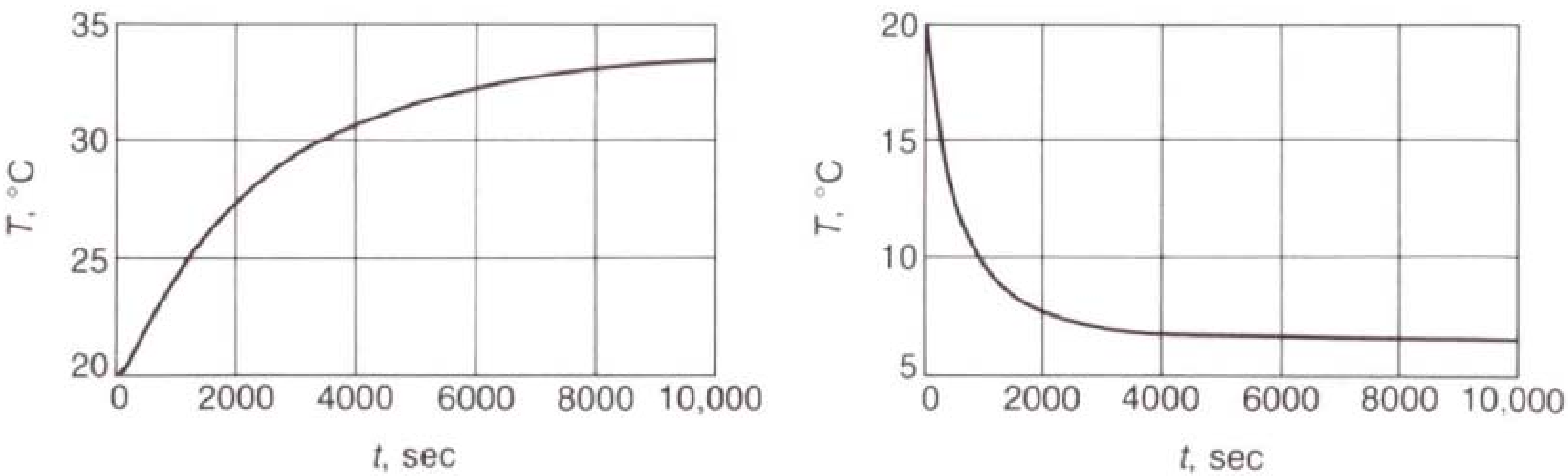




(a) $t=2$ sec
(b) $t=20$ sec
(c) $t=200$ sec
(d) $t=2000$ sec
(e) $t=10{,}000$ sec
(f) $t=1.01\times10^6$ sec

Initial Boundary Value problem
Example





(a) $x=10$ cm (node 11)

(b) $x=90$ cm (node 40)

Initial Boundary Value problem
1D Example

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Interval	θ	Δt, sec	Number of steps, n_s	Time span, sec
I	0	0.05	2	0 – 0.1
II	0	0.05	38	0.1 – 2
III	2/3	1	18	2 – 20
IV	2/3	10	18	20 – 200
V	2/3	100	18	200 – 2000
VI	2/3	500	16	2000 – 10,000
VII	1	10^6	1	10,000 – 1.01×10^6
VIII	1	10^6	1	1.01×10^6 – 2.01×10^6

- Guidelines
  - Use several time steps
  - Very small time step during shock response
  - Increase time step during transition stage
  - At near steady state, use single very large time step.

Foundations of FEM
2D formulation

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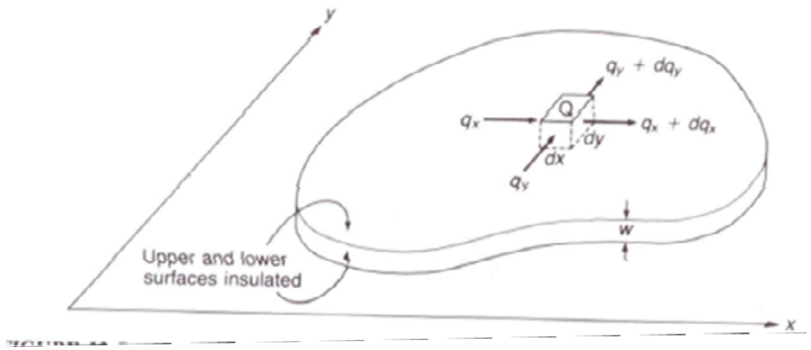
$$-\frac{d}{dx}\left(\alpha(x)\frac{dU(x)}{dx}\right)+\beta(x)U(x)=f(x)$$

$$\Big\downarrow \ \textbf{2D}$$

$$-\frac{\partial}{\partial x}\left(\alpha_x(x,y)\frac{\partial U(x,y)}{\partial x}\right)-\frac{\partial}{\partial y}\left(\alpha_y(x,y)\frac{\partial U(x,y)}{\partial y}\right)+\beta(x,y)U(x,y)=f(x,y)$$

$$\tilde{U}^{(e)}(x;a)=\sum_{j=1}^{n}a_j\phi_j^{(e)}(x)\ \xrightarrow{\textbf{2D}}\ \tilde{U}^{(e)}(x,y;a)=\sum_{j=1}^{n}a_j\phi_j^{(e)}(x,y)$$



Foundations of FEM
2D formualtion : 12-step procedures

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- Step 1: Write the Galerkin residual equations for a typical element

$$R(x,y;a) = -\frac{\partial}{\partial x}\left(\alpha_x(x,y)\frac{\partial U(x,y)}{\partial x}\right) - \frac{\partial}{\partial y}\left(\alpha_y(x,y)\frac{\partial U(x,y)}{\partial y}\right) + \beta(x,y)U(x,y) - f(x,y)$$

$$\iint_{(e)} R(x,y;a)\phi_i^{(e)}(x,y)dxdy = 0 \qquad i = 1,2,\ldots,n$$ ← **Integrate over the area of element**

$$\iint_{(e)}\left[-\frac{\partial}{\partial x}\left(\alpha_x(x,y)\frac{\partial U(x,y)}{\partial x}\right) - \frac{\partial}{\partial y}\left(\alpha_y(x,y)\frac{\partial U(x,y)}{\partial y}\right) + \beta(x,y)U(x,y) - f(x,y)\right]\phi_i^{(e)}(x,y)dxdy = 0$$

$$i = 1,2,\ldots,n$$

Foundations of FEM
2D formualtion : 12-step procedures

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- Step 2: Integrate by parts

$$\frac{\partial}{\partial x}\left(\alpha_x \frac{\partial \tilde{U}^{(e)}}{\partial x}\right)\phi_i^{(e)} = \frac{\partial}{\partial x}\left(\alpha_x \frac{\partial \tilde{U}^{(e)}}{\partial x}\phi_i^{(e)}\right) - \left(\alpha_y \frac{\partial \tilde{U}^{(e)}}{\partial x}\right)\frac{\partial \phi_i^{(e)}}{\partial x}$$

$$\frac{\partial}{\partial y}\left(\alpha_y \frac{\partial \tilde{U}^{(e)}}{\partial y}\right)\phi_i^{(e)} = \frac{\partial}{\partial y}\left(\alpha_y \frac{\partial \tilde{U}^{(e)}}{\partial y}\phi_i^{(e)}\right) - \left(\alpha_y \frac{\partial \tilde{U}^{(e)}}{\partial y}\right)\frac{\partial \phi_i^{(e)}}{\partial y}$$

$$-\iint_{(e)}\left[\frac{\partial}{\partial x}\left(\alpha_x \frac{\partial \tilde{U}^{(e)}}{\partial x}\phi_i^{(e)}\right) + \frac{\partial}{\partial y}\left(\alpha_y \frac{\partial \tilde{U}^{(e)}}{\partial y}\phi_i^{(e)}\right)\right]dxdy$$

$$+\iint_{(e)}\left[\left(\alpha_x \frac{\partial \tilde{U}^{(e)}}{\partial x}\right)\frac{\partial \phi_i^{(e)}}{\partial x} + \left(\alpha_y \frac{\partial \tilde{U}^{(e)}}{\partial y}\right)\frac{\partial \phi_i^{(e)}}{\partial y} + \beta \tilde{U}^{(e)}\phi_i^{(e)} - f\phi_i^{(e)}\right]dxdy = 0 \qquad i = 1, 2, \ldots, n$$

$$\iint_{(e)}\left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial x}\right)dxdy = \oint_{(e)}\left(Fn_x^{(e)} + Gn_y^{(e)}\right)ds$$

Divergence theorem

$$-\iint_{(e)}\left[\frac{\partial}{\partial x}\left(\alpha_x \frac{\partial \tilde{U}^{(e)}}{\partial x}\phi_i^{(e)}\right) + \frac{\partial}{\partial y}\left(\alpha_y \frac{\partial \tilde{U}^{(e)}}{\partial y}\phi_i^{(e)}\right)\right]dxdy = -\oint_{(e)}\left(\alpha_x \frac{\partial \tilde{U}^{(e)}}{\partial x}\phi_i^{(e)}n_x^{(e)} + \alpha_y \frac{\partial \tilde{U}^{(e)}}{\partial y}\phi_i^{(e)}n_y^{(e)}\right)ds$$

Foundations of FEM
2D formualtion : 12-step procedures

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$$\tilde{\tau}_x^{(e)} = -\alpha_x \frac{\partial \tilde{U}^{(e)}}{\partial x}$$

$$\tilde{\tau}_y^{(e)} = -\alpha_y \frac{\partial \tilde{U}^{(e)}}{\partial y}$$

$$\tilde{\tau}_n^{(e)} = \tilde{\tau}_x^{(e)} n_x^{(e)} + \tilde{\tau}_y^{(e)} n_y^{(e)}$$

$$\oint_{(e)} \left(\alpha_x \frac{\partial \tilde{U}^{(e)}}{\partial x} \phi_i^{(e)} n_x^{(e)} + \alpha_y \frac{\partial \tilde{U}^{(e)}}{\partial y} \phi_i^{(e)} n_y^{(e)} \right) ds = \oint_{(e)} \tilde{\tau}_n^{(e)} \phi_i^{(e)} ds$$

- Finally, residual equations becomes

$$\iint_{(e)} \left[\left(\alpha_x \frac{\partial \tilde{U}^{(e)}}{\partial x} \right) \frac{\partial \phi_i^{(e)}}{\partial x} + \left(\alpha_y \frac{\partial \tilde{U}^{(e)}}{\partial y} \right) \frac{\partial \phi_i^{(e)}}{\partial y} + \beta \tilde{U}^{(e)} \phi_i^{(e)} \right] dxdy = \iint_{e} f \phi_i^{(e)} dxdy - \oint_{(e)} \tilde{\tau}_n^{(e)} \phi_i^{(e)} ds$$

$$i = 1, 2, \ldots, n$$

Foundations of FEM
2D formualtion : 12-step procedures

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- Step 3: substitute the general form of the element trial solution into interior integrals in residual equations

$$\sum_{j=1}^{n}\left(\iint_{(e)}\frac{\partial\phi_i^{(e)}}{\partial x}\alpha_x\frac{\partial\phi_j^{(e)}}{\partial x}dxdy+\iint_{(e)}\frac{\partial\phi_i^{(e)}}{\partial y}\alpha_y\frac{\partial\phi_j^{(e)}}{\partial y}dxdy+\iint_{(e)}\phi_i^{(e)}\beta\phi_j^{(e)}dxdy\right)a_j$$

$$=\iint_{e} f\phi_i^{(e)}dxdy+\oint_{(e)}\tilde{\tau}_{-n}^{(e)}\phi_i^{(e)}ds \qquad i=1,2,\ldots,n$$

$$\tilde{\tau}_{-n}^{(e)}=-\tilde{\tau}_{n}^{(e)}$$

$$\begin{bmatrix} K_{11}^{(e)} & K_{12}^{(e)} & \cdot & \cdot & K_{1n}^{(e)} \\ K_{21}^{(e)} & K_{22}^{(e)} & \cdot & \cdot & K_{2n}^{(e)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ K_{n1}^{(e)} & K_{n2}^{(e)} & \cdot & \cdot & K_{nn}^{(e)} \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ a_n \end{Bmatrix}=\begin{Bmatrix} F_1^{(e)} \\ F_2^{(e)} \\ \cdot \\ \cdot \\ F_n^{(e)} \end{Bmatrix}$$

e.g., Linear element

$$\begin{bmatrix} K_{11}^{(e)} & K_{12}^{(e)} & K_{13}^{(e)} \\ K_{21}^{(e)} & K_{22}^{(e)} & K_{23}^{(e)} \\ K_{31}^{(e)} & K_{32}^{(e)} & K_{33}^{(e)} \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}=\begin{Bmatrix} F_1^{(e)} \\ F_2^{(e)} \\ F_3^{(e)} \end{Bmatrix}$$

$$K_{ij}^{(e)}=\iint_{(e)}\frac{\partial\phi^{(e)}{}_i}{\partial x}\alpha_x\frac{\partial\phi_j^{(e)}}{\partial x}dxdy+\iint_{(e)}\frac{\partial\phi^{(e)}{}_i}{\partial y}\alpha_y\frac{\partial\phi_j^{(e)}}{\partial y}dxdy+\iint_{(e)}\phi^{(e)}{}_i\beta(x)\phi^{(e)}{}_j dxdy$$

$$F_i^{(e)}=\iint_{(e)}f\phi^{(e)}{}_i dxdy+\oint_{(e)}\tilde{\tau}_{-n}^{(e)}\phi_i^{(e)}ds$$

Foundations of FEM
2D formualtion : 12-step procedures

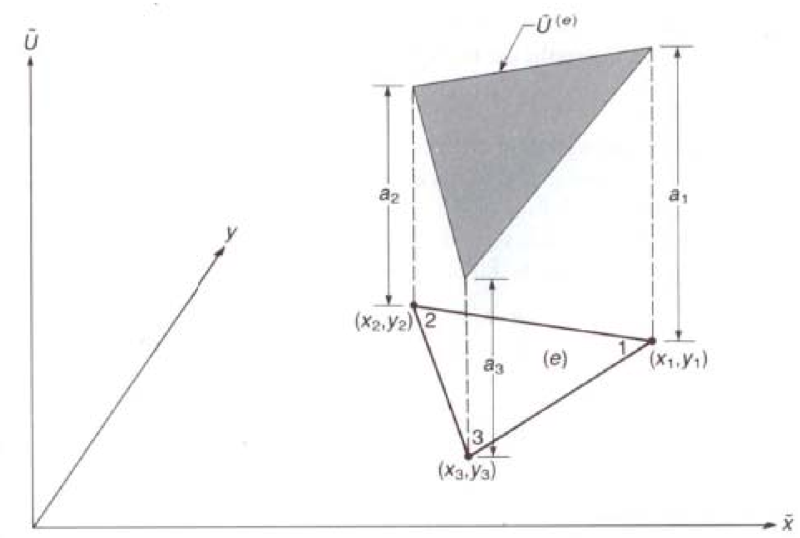
- Step 4: Develop specific expressions for the shape functions
  - Complete linear polynomial

$$\tilde{U}^{(e)} = \mathbf{a} + \mathbf{b}x + \mathbf{c}y$$

$$\tilde{U}^{(e)}(x_i, y_i; a) = a_i$$

$$\mathbf{a} + \mathbf{b}x_1 + \mathbf{c}y_2 = a_1$$
$$\mathbf{a} + \mathbf{b}x_2 + \mathbf{c}y_2 = a_2$$
$$\mathbf{a} + \mathbf{b}x_3 + \mathbf{c}y_3 = a_3$$

$$\tilde{U}^{(e)}(x, y; a) = \sum_{j=1}^{3} a_j \phi_j^{(e)}(x, y)$$



Foundations of FEM
2D formualtion : 12-step procedures

$$
\begin{array}{ccccccccccc|l}
 & & & & & 1 & & & & & & p=0 \\
 & & & & x & & y & & & & & p=1 \\
 & & & x^2 & & xy & & y^2 & & & & p=2 \\
 & & x^3 & & x^2y & & xy^2 & & y^3 & & & p=3 \\
 & x^4 & & x^3y & & x^2y^2 & & xy^3 & & y^4 & & p=4 \\
x^5 & & x^4y & & x^3y^2 & & x^2y^3 & & xy^4 & & y^5 & p=5 \\
\vdots & & & & & & & & & & \ddots &
\end{array}
$$

Foundations of FEM
2D formualtion : 12-step procedures

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$$\phi_j^{(e)}(x,y) = \frac{\mathbf{a}_j + \mathbf{b}_j x + \mathbf{c}_j y}{2\Delta} \qquad j = 1,2,3$$

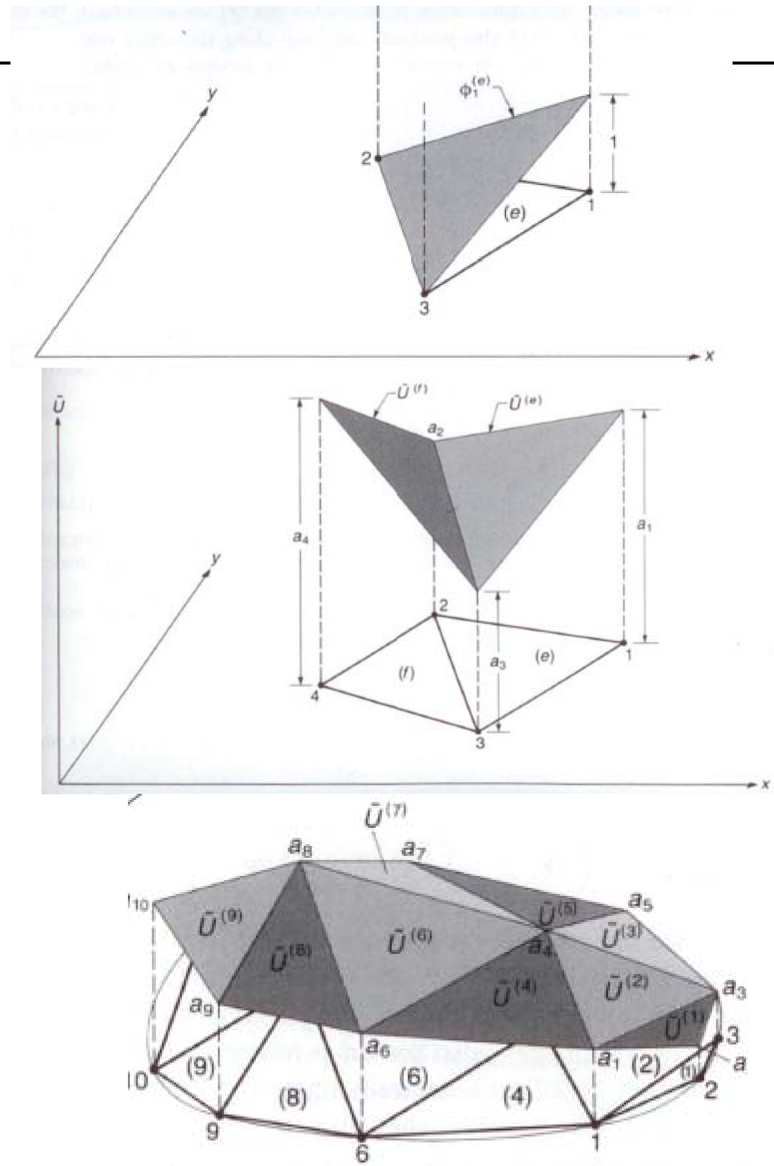
$$\mathbf{a}_j = x_k y_l - x_l y_k$$

$$\mathbf{b}_j = y_k - y_l$$

$$\mathbf{c}_j = x_l - x_k$$

$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \text{area of element}$$

$$\phi_1^{(e)}(x,y) = \frac{x_2 y_3 - x_3 y_2 + (y_2 - y_3)x + (x_3 - x_2)y}{2\Delta}$$



Foundations of FEM
2D formualtion : 12-step procedures

- Step 5: Substitute the shape functions into the element equations, and transform the integrals into a form appropriate for numerical evaluation

$$\frac{\partial \phi_j^{(e)}(x,y)}{\partial x} = \frac{\mathbf{b}_j}{2\Delta}$$

$$\frac{\partial \phi_j^{(e)}(x,y)}{\partial y} = \frac{\mathbf{c}_j}{2\Delta}$$

Foundations of FEM
2D formualtion : 12-step procedures

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$$K\alpha_{ij}^{(e)} = \iint_{(e)} \frac{\partial \phi^{(e)}{}_i}{\partial x} \alpha_x \frac{\partial \phi_j{}^{(e)}}{\partial x} dxdy + \iint_{(e)} \frac{\partial \phi^{(e)}{}_i}{\partial y} \alpha_y \frac{\partial \phi_j{}^{(e)}}{\partial y} dxdy$$

$$\square \frac{\mathbf{b}_i}{2\Delta} \alpha^{(e)}{}_x \frac{\mathbf{b}_j}{2\Delta} \iint_{(e)} dxdy + \frac{\mathbf{c}_i}{2\Delta} \alpha^{(e)}{}_y \frac{\mathbf{c}_j}{2\Delta} \iint_{(e)} dxdy$$

$$= \frac{\alpha^{(e)}{}_x}{4\Delta} \mathbf{b}_i \mathbf{b}_j + \frac{\alpha^{(e)}{}_y}{4\Delta} \mathbf{c}_i \mathbf{c}_j$$

$$\varsigma_1 + \varsigma_2 + \varsigma_3 = 1$$

$$\iint_{(e)} \varsigma_1^l \varsigma_2^m \varsigma_3^n dxdy = \frac{l!m!n!}{(l+m+n+2)!} 2\Delta$$

Triangle integration rule

$$K\beta_{ij}^{(e)} = \iint_{(e)} \phi^{(e)}{}_i \beta(x) \phi^{(e)}{}_j dxdy \square \beta^{(e)} \iint_{(e)} \phi^{(e)}{}_i \phi^{(e)}{}_j dxdy = \begin{cases} \dfrac{\beta^{(e)}\Delta}{6} & i = j \\ \dfrac{\beta^{(e)}\Delta}{12} & i \neq j \end{cases}$$

$$Ff_i^{(e)} = f^{(e)} \iint_{(e)} \phi^{(e)}{}_i dxdy = \frac{f^{(e)}\Delta}{3}$$

$$F\tau_i^{(e)} = \oint_{(e)} \tilde{\tau}_{-n}^{(e)} \phi_i^{(e)} ds = \begin{Bmatrix} \int_1^2 \tilde{\tau}_{-n}^{(e)} \phi_1^{(e)} ds + \int_2^3 \tilde{\tau}_{-n}^{(e)} \phi_1^{(e)} ds + \int_3^1 \tilde{\tau}_{-n}^{(e)} \phi_1^{(e)} ds \\ \int_1^2 \tilde{\tau}_{-n}^{(e)} \phi_2^{(e)} ds + \int_2^3 \tilde{\tau}_{-n}^{(e)} \phi_2^{(e)} ds + \int_3^1 \tilde{\tau}_{-n}^{(e)} \phi_2^{(e)} ds \\ \int_1^2 \tilde{\tau}_{-n}^{(e)} \phi_3^{(e)} ds + \int_2^3 \tilde{\tau}_{-n}^{(e)} \phi_3^{(e)} ds + \int_3^1 \tilde{\tau}_{-n}^{(e)} \phi_3^{(e)} ds \end{Bmatrix}$$

Foundations of FEM
2D formualtion : 12-step procedures

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- Step 6: Prepare expressions for the flux

$$\tilde{\tau}_x^{(e)}(x,y) = -\alpha_x(x,y)\frac{\partial \tilde{U}^{(e)}(x,y;a)}{\partial x}$$

$$\tilde{\tau}_y^{(e)}(x,y) = -\alpha_y(x,y)\frac{\partial \tilde{U}^{(e)}(x,y;a)}{\partial y}$$

$$\frac{\partial \tilde{U}^{(e)}(x,y;a)}{\partial x} = \sum_{j=1}^{3} a_j \frac{\partial \phi_j^{(e)}(x,y)}{\partial x} = \sum_{j=1}^{3} a_j \frac{\mathbf{b}_j}{2\Delta}$$

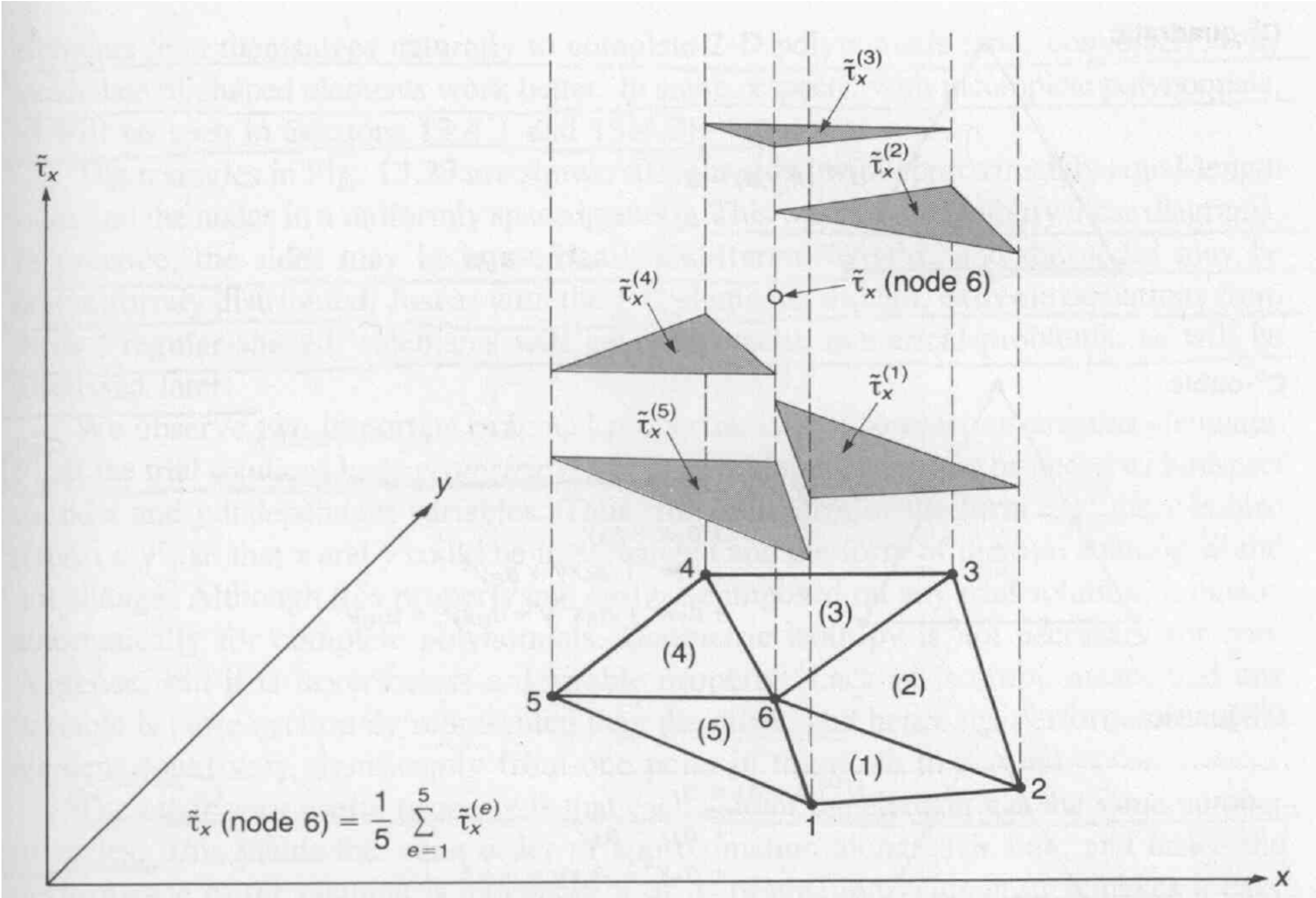
$$\frac{\partial \tilde{U}^{(e)}(x,y;a)}{\partial y} = \sum_{j=1}^{3} a_j \frac{\partial \phi_j^{(e)}(x,y)}{\partial y} = \sum_{j=1}^{3} a_j \frac{\mathbf{c}_j}{2\Delta}$$

$$\tilde{\tau}_x^{(e)}(x,y) = -\alpha_x(x,y)\sum_{j=1}^{3} a_j \frac{\mathbf{b}_j}{2\Delta}$$

$$\tilde{\tau}_y^{(e)}(x,y) = -\alpha_y(x,y)\sum_{j=1}^{3} a_j \frac{\mathbf{c}_j}{2\Delta}$$

Foundations of FEM
2D formualtion : 12-step procedures

- Nodal flux



Foundations of FEM
2D formualtion : 12-step procedures

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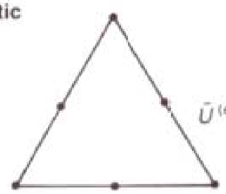
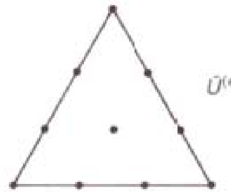
- Higher order triangular element
  - 3, 6, 10, 15,…
  - 4, 5 are possible but…Sides may be curved
  - Nodes nonuniformly distributedued

C^0-quadratic



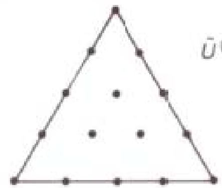
$$\tilde{U}^{(e)}(x,y;a) = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2$$

C^0-cubic



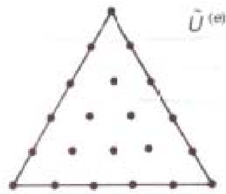
$$\tilde{U}^{(e)}(x,y;a) = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 + a_7x^3 + a_8x^2y + a_9xy^2 + a_{10}y^3$$

C^0-quartic



$$\tilde{U}^{(e)}(x,y;a) = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 + a_7x^3 + a_8x^2y + a_9xy^2 + a_{10}y^3 + a_{11}x^4 + a_{12}x^3y + a_{13}x^2y^2 + a_{14}xy^3 + a_{15}y^4$$

C^0-quintic



$$\tilde{U}^{(e)}(x,y;a) = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 + a_7x^3 + a_8x^2y + a_9xy^2 + a_{10}y^3 + a_{11}x^4 + a_{12}x^3y + a_{13}x^2y^2 + a_{14}xy^3 + a_{15}y^4 + a_{16}x^5 + a_{17}x^4y + a_{18}x^3y^2 + a_{19}x^2y^3 + a_{20}xy^4 + a_{21}y^5$$

Foundations of FEM
12-step procedures: higher order element

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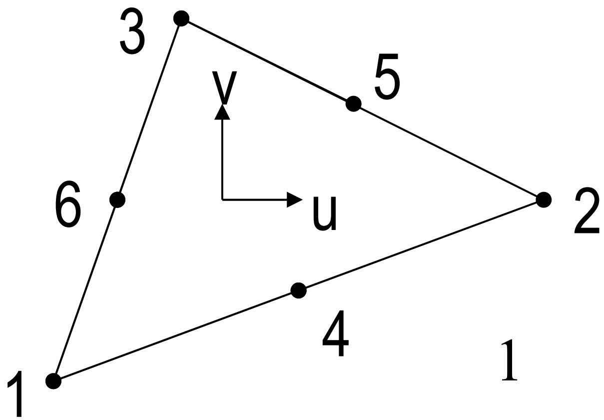

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2$$
$$v = \alpha_7 + \alpha_8 x + \alpha_9 y + \alpha_{10} x^2 + \alpha_{11} xy + \alpha_{12} y^2$$

$$\begin{array}{ccccccccccccl}
 & & & & & 1 & & & & & & & -\text{constant}\,(0+1=1) \\
 & & & & x & & y & & & & & & -\text{linear}\,(1+2=3) \\
 & & & x^2 & & xy & & y^2 & & & & & -\text{quardatic}\,(3+3=6) \\
 & & x^3 & & x^2y & & xy^2 & & y^3 & & & & -\text{cubic}\,(6+4=10) \\
 & x^4 & & x^3y & & x^2y^2 & & xy^3 & & y^4 & & & -\text{qurtic}\,(10+5=15) \\
x^5 & & \ldots\ldots & & \ldots\ldots & & \ldots\ldots & & \ldots\ldots & & y^5 & & -\text{quintic}\,(15+6=21)
\end{array}$$

Pascal's Triangle

Foundation of FEM
12-step procedures: higher order element

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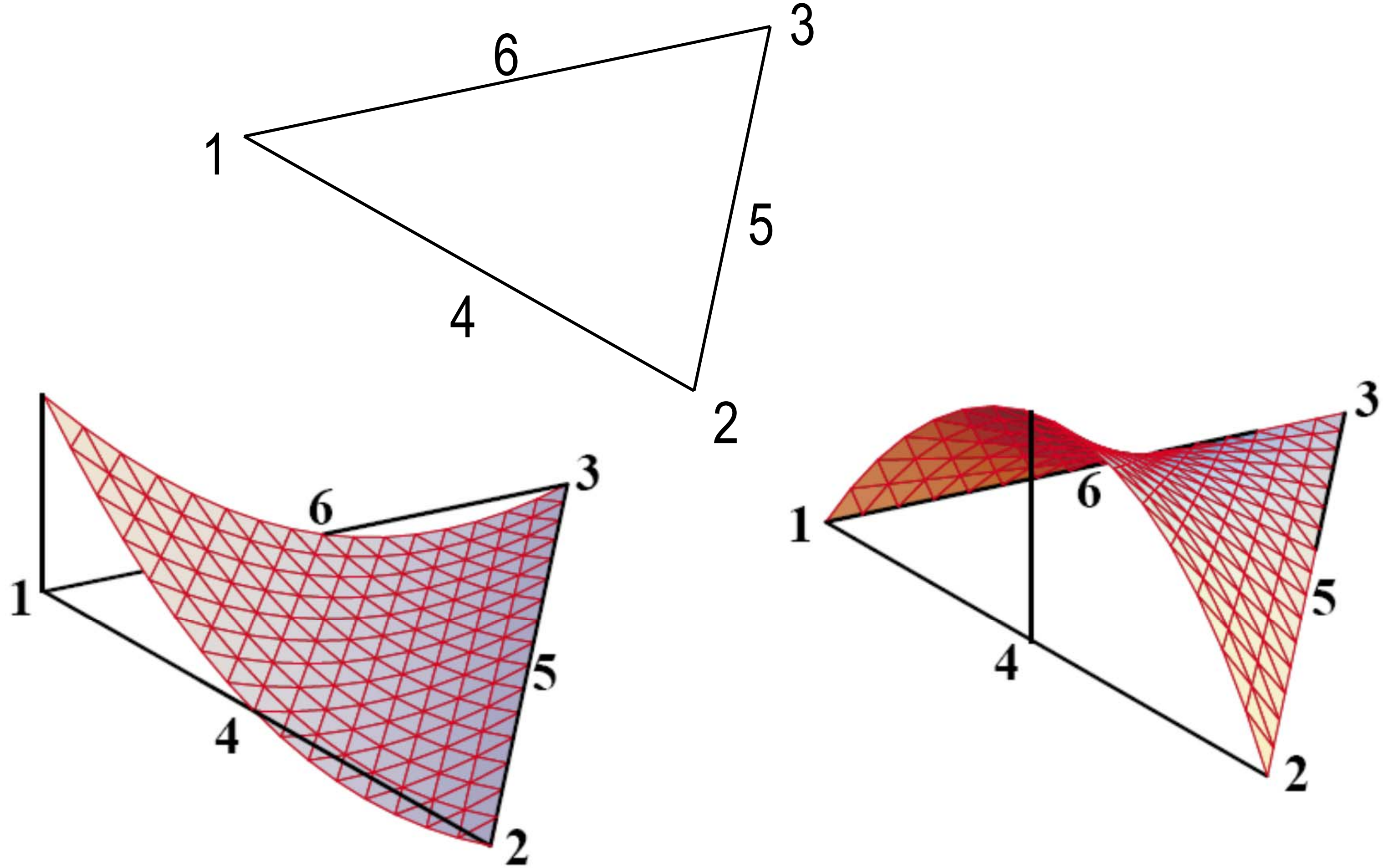

2D Initial boundary value problem

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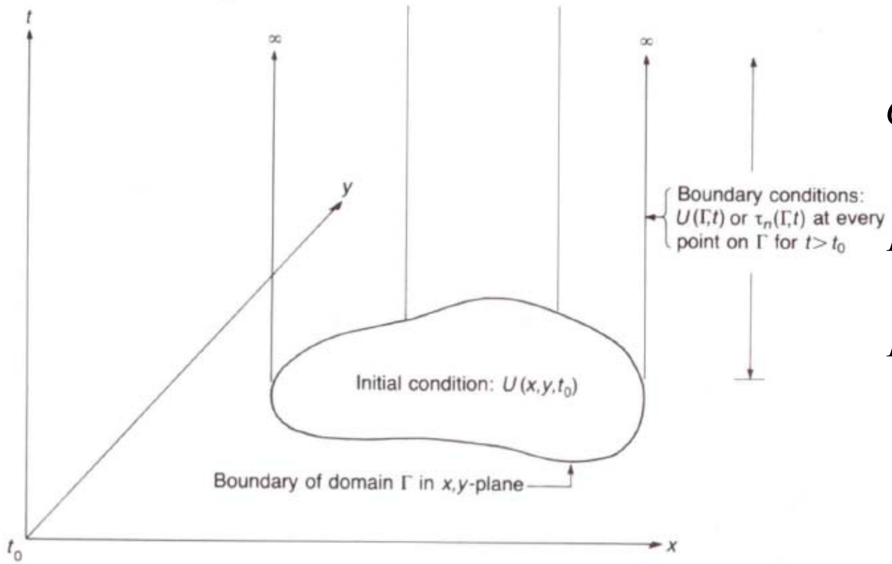
$$\mu(x)\frac{\partial U(x,y;t)}{\partial t}-\frac{\partial}{\partial x}\left(\alpha_x(x,y)\frac{\partial U(x,y,t)}{\partial x}\right)-\frac{\partial}{\partial y}\left(\alpha_y(x,y)\frac{\partial U(x,y,t)}{\partial y}\right)+\beta(x,y)U(x,y,t)=f(x,y,t)$$

$$[c]^{(e)}\left\{\frac{da(t)}{dt}\right\}+[K]^{(e)}\{a(t)\}=\{F(t)\}^{(e)}$$



$$C_{ij}^{(e)}=\iint_{(e)}\phi^{(e)}{}_i\mu\phi_j{}^{(e)}dxdy$$

$$K_{ij}^{(e)}=\iint_{(e)}\frac{\partial\phi_i}{\partial x}\alpha_x\frac{\partial\phi_j}{\partial x}dxdy+\iint_{(e)}\frac{\partial\phi_i}{\partial y}\alpha_y\frac{\partial\phi_j}{\partial y}dxdy+\iint_{(e)}\phi^{(e)}{}_i\beta\phi^{(e)}{}_j dxdy$$

$$F_i^{(e)}=\iint_{(e)}f\phi^{(e)}{}_i dxdy+\oint_{(e)}\tilde{\tau}_{-n}^{(e)}\phi_i^{(e)}ds$$

Time-stepping method
Comparison of performances

- Critical time step: smallest time step of a system (p.476-479, Burnett, 1987)

$$\Delta t_{crit} = \frac{2}{d(1-2\theta)\pi^2}\left((\mu/\alpha)\delta^2\right)^e_{\min} \quad 0 \le \theta < 1/2$$

Approximated from 1D eigenproblem

$$\Delta t_{crit} = \frac{2}{d\pi^2}(\mu/\alpha)\delta_{\min}{}^2 \quad \theta = 0,\ \alpha, \mu \text{ are constants}$$

  – Conservative low estimation - within a factor of 5 of the exact value (Burnett, 1987)

δ : distance between two adjacent nodes in the element

$$\alpha\frac{\partial^2 U(x,t)}{\partial x^2} - \mu\frac{\partial U(x,t)}{\partial t} = 0$$

FEM - 2D Linear Elasticity
Governing equations

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$$\frac{E}{1-\nu^2}\frac{\partial^2 u}{\partial x^2}+\frac{E}{2(1-\nu)}\frac{\partial^2 v}{\partial x \partial y}+\frac{E}{2(1+\nu)}\frac{\partial^2 u}{\partial y^2}=-f_x$$

$$\frac{E}{1-\nu^2}\frac{\partial^2 v}{\partial y^2}+\frac{E}{2(1-\nu)}\frac{\partial^2 u}{\partial x \partial y}+\frac{E}{2(1+\nu)}\frac{\partial^2 v}{\partial x^2}=-f_y$$

FEM - 2D Linear Elasticity
Constitutive equations

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$$\{\sigma\} = [C]\big(\underbrace{\{\varepsilon\}}_{\text{Total strain}} - \underbrace{\{\varepsilon_T\}}_{\text{thermal strain}}\big)$$

- 2D plane stress (isotropic)

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \left(\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} - \begin{Bmatrix} \alpha\Delta T \\ \alpha\Delta T \\ 0 \end{Bmatrix} \right)$$

- 2D Plane stress (isotropic)

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}-\nu \end{bmatrix} \left(\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} - \begin{Bmatrix} (1+\nu)\alpha\Delta T \\ (1+\nu)\alpha\Delta T \\ 0 \end{Bmatrix} \right)$$

FEM - 2D Linear Elasticity
Trial solutions

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$$\tilde{u}^{(e)}(x,y;a)=\sum_{j=1}^{n}u_j\phi_j^{(e)}(x,y)$$

$$\tilde{v}^{(e)}(x,y;a)=\sum_{j=1}^{n}v_j\phi_j^{(e)}(x,y)$$

$$\left\{\tilde{U}\right\}^{(e)}=\left[\Phi\right]^{(e)}\left\{a\right\}$$

$$\underset{2\times 1}{\left\{\tilde{U}\right\}^{(e)}}=\begin{Bmatrix}\tilde{u}^{(e)}\\ \tilde{v}^{(e)}\end{Bmatrix}$$

$$\underset{2\times 2n}{\left[\Phi\right]^{(e)}}=\begin{bmatrix}\phi_1^{(e)} & 0 & \phi_2^{(e)} & 0 & \dots & \phi_n^{(e)} & 0\\ 0 & \phi_1^{(e)} & 0 & \phi_2^{(e)} & \dots & 0 & \phi_n^{(e)}\end{bmatrix}$$

$$\underset{2n\times 1}{\left\{a\right\}}=\begin{Bmatrix}u_1\\ v_1\\ u_2\\ v_2\\ .\\ .\\ .\\ u_n\\ v_n\end{Bmatrix}=\begin{Bmatrix}a_1\\ a_2\\ a_3\\ a_4\\ .\\ .\\ .\\ a_{2n-1}\\ a_{2n}\end{Bmatrix}$$

FEM - 2D Linear Elasticity
Trial solutions

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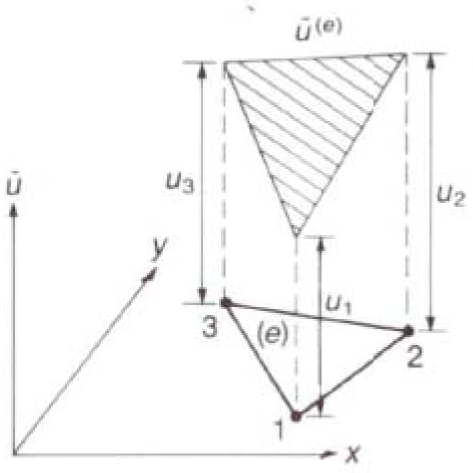

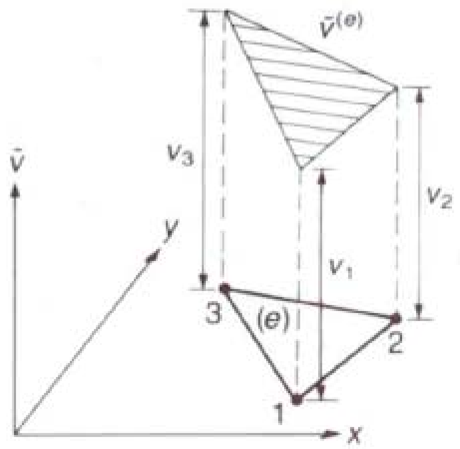

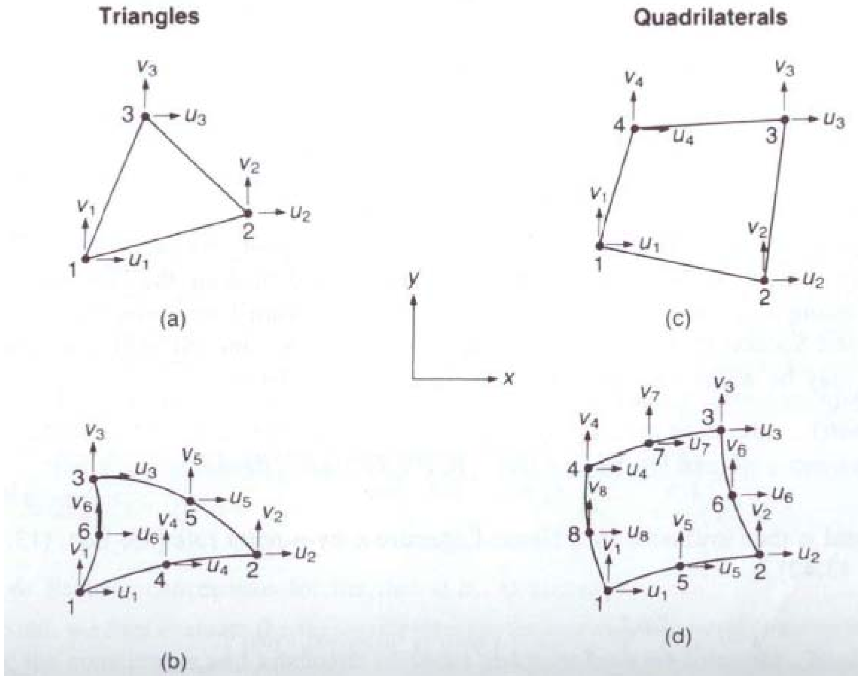

FEM - 2D Linear Elasticity
12-step procedures

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- Step 1: Write the Galerkin residual equations for a typical element

$$R_x = \frac{\partial \tilde{\sigma}_x^{(e)}}{\partial x} + \frac{\partial \tilde{\tau}_{xy}^{(e)}}{\partial y} + f_x$$

$$R_y = \frac{\partial \tilde{\tau}_{xy}^{(e)}}{\partial x} + \frac{\partial \tilde{\sigma}_y^{(e)}}{\partial y} + f_y$$

$$\iint_{(e)} R_x \phi_i^{(e)} dxdy = 0 \qquad i = 1, 2, \ldots, n$$

$$\iint_{(e)} R_y \phi_i^{(e)} dxdy = 0 \qquad i = 1, 2, \ldots, n$$

$$\iint_{(e)} \left[\frac{\partial \tilde{\sigma}_x^{(e)}}{\partial x} + \frac{\partial \tilde{\tau}_{xy}^{(e)}}{\partial y} + f_x \right] \phi_i^{(e)} dxdy = 0 \qquad i = 1, 2, \ldots, n$$

$$\iint_{(e)} \left[\frac{\partial \tilde{\tau}_{xy}^{(e)}}{\partial x} + \frac{\partial \tilde{\sigma}_y^{(e)}}{\partial y} + f_y \right] \phi_i^{(e)} dxdy = 0 \qquad i = 1, 2, \ldots, n$$

FEM - 2D Linear Elasticity
12-step procedures

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- Step 2: Integrate by parts

$$\iint_{(e)} \left[\tilde{\sigma}_x^{(e)} \frac{\partial \phi_i^{(e)}}{\partial x} + \tilde{\tau}_{xy}^{(e)} \frac{\partial \phi_i^{(e)}}{\partial y} \right] dxdy = \iint_{(e)} \left[\frac{\partial}{\partial x} \left(\tilde{\sigma}_x^{(e)} \phi_i^{(e)} \right) + \frac{\partial}{\partial y} \left(\tilde{\tau}_{xy}^{(e)} \phi_i^{(e)} \right) \right] dxdy + \iint_{(e)} f_x \phi_i^{(e)} dxdy \qquad i = 1, 2, \ldots, n$$

$$\iint_{(e)} \left[\tilde{\tau}_{xy}^{(e)} \frac{\partial \phi_i^{(e)}}{\partial x} + \tilde{\sigma}_y^{(e)} \frac{\partial \phi_i^{(e)}}{\partial y} \right] dxdy = \iint_{(e)} \left[\frac{\partial}{\partial x} \left(\tilde{\tau}_{xy}^{(e)} \phi_i^{(e)} \right) + \frac{\partial}{\partial y} \left(\tilde{\sigma}_y^{(e)} \phi_i^{(e)} \right) \right] dxdy + \iint_{(e)} f_y \phi_i^{(e)} dxdy \qquad i = 1, 2, \ldots, n$$

Divergence theorem $\longrightarrow$

$$\iint_{(e)} \left[\tilde{\sigma}_x^{(e)} \frac{\partial \phi_i^{(e)}}{\partial x} + \tilde{\tau}_{xy}^{(e)} \frac{\partial \phi_i^{(e)}}{\partial y} \right] dxdy = \oint_{(e)} \left(\tilde{\sigma}_x^{(e)} n_x^{(e)} + \tilde{\tau}_{xy}^{(e)} n_y^{(e)} \right) \phi_i^{(e)} ds + \iint_{(e)} f_x \phi_i^{(e)} dxdy \qquad i = 1, 2, \ldots, n$$

$$\iint_{(e)} \left[\tilde{\tau}_{xy}^{(e)} \frac{\partial \phi_i^{(e)}}{\partial x} + \tilde{\sigma}_y^{(e)} \frac{\partial \phi_i^{(e)}}{\partial y} \right] dxdy = \oint_{(e)} \left(\tilde{\tau}_{xy}^{(e)} n_x^{(e)} + \tilde{\sigma}_y^{(e)} n_y^{(e)} \right) \phi_i^{(e)} ds + \iint_{(e)} f_y \phi_i^{(e)} dxdy \qquad i = 1, 2, \ldots, n$$

FEM - 2D Linear Elasticity
12-step procedures

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$$\iint_{(e)} \left[\tilde{\sigma}_x^{(e)} \frac{\partial \phi_i^{(e)}}{\partial x} + \tilde{\tau}_{xy}^{(e)} \frac{\partial \phi_i^{(e)}}{\partial y} \right] dxdy = \int_{(e)}^{(n)} \tau_x \phi_i^{(e)} ds + \iint_{(e)} f_x \phi_i^{(e)} dxdy \qquad i = 1, 2, \ldots, n$$

$$\iint_{(e)} \left[\tilde{\tau}_{xy}^{(e)} \frac{\partial \phi_i^{(e)}}{\partial x} + \tilde{\sigma}_y^{(e)} \frac{\partial \phi_i^{(e)}}{\partial y} \right] dxdy = \int_{(e)}^{(n)} \tau_y \phi_i^{(e)} ds + \iint_{(e)} f_y \phi_i^{(e)} dxdy \qquad i = 1, 2, \ldots, n$$

- Step 3: Substitute the general form of the element trial solution into interior integrals in residual equations

$$\tilde{\varepsilon}_x^{(e)} = \frac{\partial \tilde{u}^{(e)}}{\partial x} = \sum_{j=1}^{n} u_j \frac{\partial \phi_j^{(e)}}{\partial x}$$

$$\tilde{\varepsilon}_y^{(e)} = \frac{\partial \tilde{v}^{(e)}}{\partial y} = \sum_{j=1}^{n} v_j \frac{\partial \phi_j^{(e)}}{\partial y}$$

$$\tilde{\gamma}_{xy}^{(e)} = \frac{\partial \tilde{u}^{(e)}}{\partial y} + \frac{\partial \tilde{v}^{(e)}}{\partial x} = \sum_{j=1}^{n} \left(u_j \frac{\partial \phi_j^{(e)}}{\partial y} + v_j \frac{\partial \phi_j^{(e)}}{\partial x} \right)$$

$$\{\tilde{\varepsilon}\}^{(e)} = [B]^{(e)} \{a\}$$

$$\{\tilde{\varepsilon}\}^{(e)} = \begin{Bmatrix} \tilde{\varepsilon}_x^{(e)} \\ \tilde{\varepsilon}_y^{(e)} \\ \tilde{\gamma}_{xy}^{(e)} \end{Bmatrix}$$

$$\underset{3\times 2n}{[B]^{(e)}} = \begin{bmatrix} \frac{\partial \phi_1^{(e)}}{\partial x} & 0 & \frac{\partial \phi_2^{(e)}}{\partial x} & 0 & \cdots & \frac{\partial \phi_n^{(e)}}{\partial x} & 0 \\ 0 & \frac{\partial \phi_1^{(e)}}{\partial y} & 0 & \frac{\partial \phi_2^{(e)}}{\partial y} & \cdots & 0 & \frac{\partial \phi_n^{(e)}}{\partial y} \\ \frac{\partial \phi_1^{(e)}}{\partial y} & \frac{\partial \phi_1^{(e)}}{\partial x} & \frac{\partial \phi_2^{(e)}}{\partial y} & \frac{\partial \phi_2^{(e)}}{\partial x} & \cdots & \frac{\partial \phi_n^{(e)}}{\partial y} & \frac{\partial \phi_n^{(e)}}{\partial x} \end{bmatrix}$$

$$\{\tilde{\sigma}\}^{(e)} = [C]^{(e)} \left([B]^{(e)} \{a\} - \{\alpha\}^{(e)} \Delta T \right)$$

FEM - 2D Linear Elasticity
12-step procedures

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$$\iint_{(e)}\left[\tilde{\sigma}_x^{(e)}\frac{\partial\phi_i^{(e)}}{\partial x}+\tilde{\tau}_{xy}^{(e)}\frac{\partial\phi_i^{(e)}}{\partial y}\right]dxdy=\int_{(e)}\overset{(n)}{\tau}_x\phi_i^{(e)}ds+\iint_{(e)}f_x\phi_i^{(e)}dxdy \qquad i=1,2,\dots,n$$

$$\iint_{(e)}\left[\tilde{\tau}_{xy}^{(e)}\frac{\partial\phi_i^{(e)}}{\partial x}+\tilde{\sigma}_y^{(e)}\frac{\partial\phi_i^{(e)}}{\partial y}\right]dxdy=\int_{(e)}\overset{(n)}{\tau}_y\phi_i^{(e)}ds+\iint_{(e)}f_y\phi_i^{(e)}dxdy \qquad i=1,2,\dots,n$$

$$\begin{Bmatrix}
\tilde{\sigma}_x^{(e)}\frac{\partial\phi_1^{(e)}}{\partial x}+\tilde{\tau}_{xy}^{(e)}\frac{\partial\phi_1^{(e)}}{\partial y}\\
\tilde{\tau}_{xy}^{(e)}\frac{\partial\phi_1^{(e)}}{\partial x}+\tilde{\sigma}_y^{(e)}\frac{\partial\phi_1^{(e)}}{\partial y}\\
\tilde{\sigma}_x^{(e)}\frac{\partial\phi_2^{(e)}}{\partial x}+\tilde{\tau}_{xy}^{(e)}\frac{\partial\phi_2^{(e)}}{\partial y}\\
\tilde{\tau}_{xy}^{(e)}\frac{\partial\phi_2^{(e)}}{\partial x}+\tilde{\sigma}_y^{(e)}\frac{\partial\phi_2^{(e)}}{\partial y}\\
.\\
.\\
.\\
\tilde{\sigma}_x^{(e)}\frac{\partial\phi_n^{(e)}}{\partial x}+\tilde{\tau}_{xy}^{(e)}\frac{\partial\phi_n^{(e)}}{\partial y}\\
\tilde{\tau}_{xy}^{(e)}\frac{\partial\phi_n^{(e)}}{\partial x}+\tilde{\sigma}_y^{(e)}\frac{\partial\phi_n^{(e)}}{\partial y}
\end{Bmatrix}
=
\begin{bmatrix}
\frac{\partial\phi_1^{(e)}}{\partial x} & 0 & \frac{\partial\phi_1^{(e)}}{\partial y}\\
0 & \frac{\partial\phi_1^{(e)}}{\partial y} & \frac{\partial\phi_1^{(e)}}{\partial x}\\
\frac{\partial\phi_2^{(e)}}{\partial x} & 0 & \frac{\partial\phi_2^{(e)}}{\partial y}\\
0 & \frac{\partial\phi_2^{(e)}}{\partial y} & \frac{\partial\phi_2^{(e)}}{\partial x}\\
. & . & .\\
. & . & .\\
. & . & .\\
\frac{\partial\phi_n^{(e)}}{\partial x} & 0 & \frac{\partial\phi_n^{(e)}}{\partial y}\\
0 & \frac{\partial\phi_n^{(e)}}{\partial y} & \frac{\partial\phi_n^{(e)}}{\partial x}
\end{bmatrix}
\begin{Bmatrix}
\tilde{\sigma}_x^{(e)}\\
\tilde{\sigma}_y^{(e)}\\
\tilde{\tau}_{xy}^{(e)}
\end{Bmatrix}
=[B]^{(e)^T}\{\tilde{\sigma}\}^{(e)}$$

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$$\iint_{(e)}\left[\tilde{\sigma}_x^{(e)}\frac{\partial\phi_i^{(e)}}{\partial x}+\tilde{\tau}_{xy}^{(e)}\frac{\partial\phi_i^{(e)}}{\partial y}\right]dxdy=\underline{\int_{(e)}\overset{(n)}{\tau}_x\phi_i^{(e)}ds}+\underline{\iint_{(e)}f_x\phi_i^{(e)}dxdy}\qquad i=1,2,\ldots,n$$

$$\iint_{(e)}\left[\tilde{\tau}_{xy}^{(e)}\frac{\partial\phi_i^{(e)}}{\partial x}+\tilde{\sigma}_y^{(e)}\frac{\partial\phi_i^{(e)}}{\partial y}\right]dxdy=\underline{\int_{(e)}\overset{(n)}{\tau}_y\phi_i^{(e)}ds}+\underline{\iint_{(e)}f_y\phi_i^{(e)}dxdy}\qquad i=1,2,\ldots,n$$

$$\begin{Bmatrix}\overset{(n)}{\tau}_x\phi_1^{(e)}\\ \overset{(n)}{\tau}_y\phi_1^{(e)}\\ \overset{(n)}{\tau}_x\phi_2^{(e)}\\ \overset{(n)}{\tau}_y\phi_2^{(e)}\\ \cdot\\ \cdot\\ \cdot\\ \overset{(n)}{\tau}_x\phi_n^{(e)}\\ \overset{(n)}{\tau}_y\phi_n^{(e)}\end{Bmatrix}=\begin{bmatrix}\phi_1^{(e)} & 0\\ 0 & \phi_1^{(e)}\\ \phi_2^{(e)} & 0\\ 0 & \phi_2^{(e)}\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \phi_n^{(e)} & 0\\ 0 & \phi_n^{(e)}\end{bmatrix}\begin{Bmatrix}\overset{(n)}{\tau}_x\\ \overset{(n)}{\tau}_y\end{Bmatrix}=[\Phi]^{(e)^T}\left\{\overset{(n)}{\tau}\right\}$$

$$\begin{Bmatrix}f_x\phi_1^{(e)}\\ f_y\phi_1^{(e)}\\ f_x\phi_2^{(e)}\\ f_y\phi_2^{(e)}\\ \cdot\\ \cdot\\ \cdot\\ f_x\phi_n^{(e)}\\ f_y\phi_n^{(e)}\end{Bmatrix}=\begin{bmatrix}\phi_1^{(e)} & 0\\ 0 & \phi_1^{(e)}\\ \phi_2^{(e)} & 0\\ 0 & \phi_2^{(e)}\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \phi_n^{(e)} & 0\\ 0 & \phi_n^{(e)}\end{bmatrix}\begin{Bmatrix}f_x\\ f_y\end{Bmatrix}=[\Phi]^{(e)^T}\{f\}$$

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$$\iint_{(e)} [B]^{(e)^T} \{\tilde{\sigma}\}^{(e)} \, dxdy = \oint_{(e)} [\Phi]^{(e)^T} \left\{ \overset{(n)}{\tau} \right\} ds + \iint_{(e)} [\Phi]^{(e)^T} \{f\} \, dxdy$$

$$\{\tilde{\sigma}\}^{(e)} = [C]^{(e)} \left([B]^{(e)} \{a\} - \{\alpha\}^{(e)} \Delta T \right)$$

$$\iint_{(e)} [B]^{(e)^T} [C]^{(e)} [B]^{(e)} \{a\} \, dxdy = \oint_{(e)} [\Phi]^{(e)^T} \left\{ \overset{(n)}{\tau} \right\} ds + \iint_{(e)} [\Phi]^{(e)^T} \{f\} \, dxdy + \iint_{(e)} [B]^{(e)^T} [C]^{(e)} \{\alpha\}^{(e)} \Delta T dxdy$$

$$[K]^{(e)} \{a\} = \{F\}^{(e)}$$

$$[K]^{(e)} = \iint_{(e)} [B]^{(e)^T} [C]^{(e)} [B]^{(e)} \, dxdy$$

$$\{F\}^{(e)} = \{F_\tau\}^{(e)} + \{F_f\}^{(e)} + \{F_T\}^{(e)}$$

$$\{F_\tau\}^{(e)} = \oint_{(e)} [\Phi]^{(e)^T} \left\{ \overset{(n)}{\tau} \right\} ds$$

$$\{F_f\}^{(e)} = \iint_{(e)} [\Phi]^{(e)^T} \{f\} \, dxdy$$

$$\{F_T\}^{(e)} = \iint_{(e)} [B]^{(e)^T} [C]^{(e)} \{\alpha\}^{(e)} \Delta T dxdy$$

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- Step 4: Develop specific expressions for the shape functions
- Step 5: substitute the shape functions into the element equations, and transform the integrals into a form appropriate for numerical evaluation

$$\underset{3\times 6}{\left[B\right]}^{(e)} = \begin{bmatrix} \frac{\mathbf{b}_1}{2\Delta} & 0 & \frac{\mathbf{b}_2}{2\Delta} & 0 & \frac{\mathbf{b}_3}{2\Delta} & 0 \\ 0 & \frac{\mathbf{c}_1}{2\Delta} & 0 & \frac{\mathbf{c}_2}{2\Delta} & 0 & \frac{\mathbf{c}_3}{2\Delta} \\ \frac{\mathbf{c}_1}{2\Delta} & \frac{\mathbf{b}_1}{2\Delta} & \frac{\mathbf{c}_2}{2\Delta} & \frac{\mathbf{b}_2}{2\Delta} & \frac{\mathbf{c}_3}{2\Delta} & \frac{\mathbf{b}_3}{2\Delta} \end{bmatrix}$$

$$\underset{6\times 6}{\left[K\right]}^{(e)} = \iint_{(e)} \underset{6\times 3}{\left[B\right]}^{(e)^T} \underset{3\times 3}{\left[C\right]}^{(e)} \underset{3\times 6}{\left[B\right]}^{(e)} dxdy$$

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12-step procedures: Final element K matrix

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$$\underset{6\times6}{[K]^{(e)}} = \iint_{(e)} \underset{6\times3}{[B]^{(e)^T}} \underset{3\times3}{[C]^{(e)}} \underset{3\times6}{[B]^{(e)}} \, dxdy$$

$$\begin{bmatrix} \frac{\partial \phi_1^{(e)}}{\partial x} & 0 & \frac{\partial \phi_1^{(e)}}{\partial y} \\ 0 & \frac{\partial \phi_1^{(e)}}{\partial y} & \frac{\partial \phi_1^{(e)}}{\partial x} \\ \frac{\partial \phi_2^{(e)}}{\partial x} & 0 & \frac{\partial \phi_2^{(e)}}{\partial y} \\ 0 & \frac{\partial \phi_2^{(e)}}{\partial y} & \frac{\partial \phi_2^{(e)}}{\partial x} \\ \frac{\partial \phi_3^{(e)}}{\partial x} & 0 & \frac{\partial \phi_3^{(e)}}{\partial y} \\ 0 & \frac{\partial \phi_3^{(e)}}{\partial y} & \frac{\partial \phi_3^{(e)}}{\partial x} \end{bmatrix} \frac{E}{(1-\nu^2)} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{pmatrix} \begin{bmatrix} \frac{\partial \phi_1^{(e)}}{\partial x} & 0 & \frac{\partial \phi_2^{(e)}}{\partial x} & 0 & \frac{\partial \phi_3^{(e)}}{\partial x} & 0 \\ 0 & \frac{\partial \phi_1^{(e)}}{\partial y} & 0 & \frac{\partial \phi_2^{(e)}}{\partial y} & 0 & \frac{\partial \phi_3^{(e)}}{\partial y} \\ \frac{\partial \phi_1^{(e)}}{\partial y} & \frac{\partial \phi_1^{(e)}}{\partial x} & \frac{\partial \phi_2^{(e)}}{\partial y} & \frac{\partial \phi_2^{(e)}}{\partial x} & \frac{\partial \phi_3^{(e)}}{\partial y} & \frac{\partial \phi_3^{(e)}}{\partial x} \end{bmatrix}$$

Final K in an element is a 6x6 matrix
Global K matrix is formed by summation of all elemental K

Structure of elemental 'K' matrix (linear & triangular element)

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1D elasticity & diffusion	$\underset{2\times2}{[K]^{(e)}} = \int_{(e)} \underset{2\times1}{[B]^{(e)^T}} \underset{1\times1}{[C]^{(e)}} \underset{1\times2}{[B]^{(e)}} dx$	Number of column = 2 node x 1 DOF = 2
2D diffusion	$\underset{3\times3}{[K]^{(e)}} = \iint_{(e)} \underset{3\times2}{[B]^{(e)^T}} \underset{2\times2}{[C]^{(e)}} \underset{2\times3}{[B]^{(e)}} dxdy$	3 x 1= 3
2D elasticity	$\underset{6\times6}{[K]^{(e)}} = \iint_{(e)} \underset{6\times3}{[B]^{(e)^T}} \underset{3\times3}{[C]^{(e)}} \underset{3\times6}{[B]^{(e)}} dxdy$	3 x 2= 6
3D diffusion	$\underset{4\times4}{[K]^{(e)}} = \iiint_{(e)} \underset{4\times3}{[B]^{(e)^T}} \underset{3\times3}{[C]^{(e)}} \underset{3\times4}{[B]^{(e)}} dxdydz$	4 x 1= 4
3D elasticity	$\underset{12\times12}{[K]^{(e)}} = \iiint_{(e)} \underset{12\times6}{[B]^{(e)^T}} \underset{6\times6}{[C]^{(e)}} \underset{6\times12}{[B]^{(e)}} dxdydz$	4 x 3= 12

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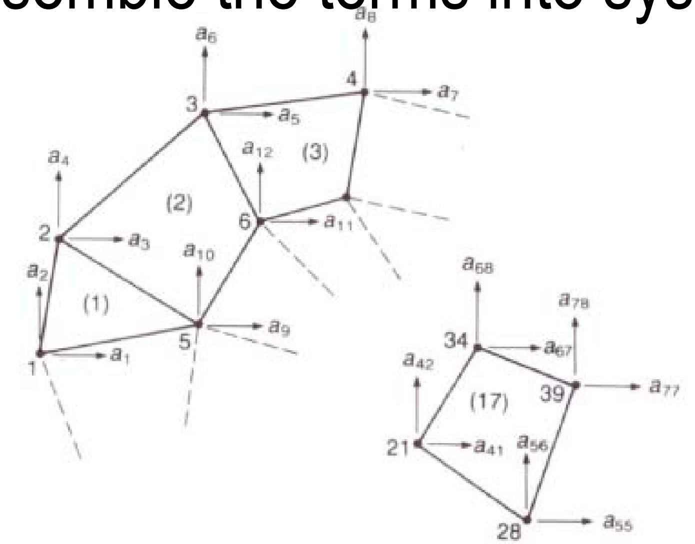
- Step 6: Prepare expressions for the flux (i.e., stresses)

$$\underset{3\times1}{\{\tilde{\sigma}\}^{(e)}} = \underset{3\times3}{[C]^{(e)}}\left(\underset{3\times2n}{[B]^{(e)}}\underset{2n\times1}{\{a\}} - \underset{3\times1}{\{\alpha\}^{(e)}}\Delta T\right)$$

- Step 7: Specify numerical data for a particular problem

- Step 8: Evaluate the interior terms in the element equations for each element, and assemble the terms into system equations



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12-step procedures

$$
\begin{bmatrix}
K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & K_{14}^{(1)} & & & & & K_{19}^{(1)} & K_{1,10}^{(1)} & & \\
 & K_{22}^{(1)} & K_{23}^{(1)} & K_{24}^{(1)} & & & & & K_{29}^{(1)} & K_{2,10}^{(1)} & & \\
 & & K_{33}^{(1)}+K_{33}^{(2)} & K_{34}^{(1)}+K_{34}^{(2)} & K_{35}^{(2)} & K_{36}^{(2)} & & & K_{39}^{(1)}+K_{39}^{(2)} & K_{3,10}^{(1)}+K_{3,10}^{(2)} & K_{3,11}^{(2)} & K_{3,12}^{(2)} \\
 & & & K_{44}^{(1)}+K_{44}^{(2)} & K_{45}^{(2)} & K_{46}^{(2)} & & & K_{49}^{(1)}+K_{49}^{(2)} & K_{4,10}^{(1)}+K_{4,10}^{(2)} & K_{4,11}^{(2)} & K_{4,12}^{(2)} \\
 & & & & K_{55}^{(2)} & K_{56}^{(2)} & & & K_{59}^{(2)} & K_{5,10}^{(2)} & K_{5,11}^{(2)} & K_{5,12}^{(2)} \\
 & & & & & K_{66}^{(2)} & & & K_{69}^{(2)} & K_{6,10}^{(2)} & K_{6,11}^{(2)} & K_{6,12}^{(2)} \\
 & & & & & & & & & & & \\
 & & & & & & & & & & & \\
 & & & & & & & & K_{99}^{(1)}+K_{99}^{(2)} & K_{9,10}^{(1)}+K_{9,10}^{(2)} & K_{9,11}^{(2)} & K_{9,12}^{(2)} \\
 & & & \text{Symmetric} & & & & & & K_{10,10}^{(1)}+K_{10,10}^{(2)} & K_{10,11}^{(2)} & K_{10,12}^{(2)} \\
 & & & & & & & & & & K_{11,11}^{(2)} & K_{11,12}^{(2)} \\
 & & & & & & & & & & & K_{12,12}^{(2)}
\end{bmatrix}
\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \\ a_{11} \\ a_{12} \\ \vdots \end{Bmatrix}
=
\begin{Bmatrix} F_1^{(1)} \\ F_2^{(1)} \\ F_3^{(1)}+F_3^{(2)} \\ F_4^{(1)}+F_4^{(2)} \\ F_5^{(2)} \\ F_6^{(2)} \\ \\ \\ F_9^{(1)}+F_9^{(2)} \\ F_{10}^{(1)}+F_{10}^{(2)} \\ F_{11}^{(2)} \\ F_{12}^{(2)} \\ \vdots \end{Bmatrix}
$$

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12-step procedures

- Step 9: Apply the BCs, including the natural interelement BCs, to the system equations
- Step 10: Solve the system equations
- Step 11: Evaluate the flux
- Step 12: Display the solution and estimate its accuracy
  – In order to improve accuracy we could either use finer mesh (h-refinement) or use higher-order element (p-refinement)

References

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- Becker EB et al., Finite Elements – An introduction, Vol.I, 1981, Prentice-Hall