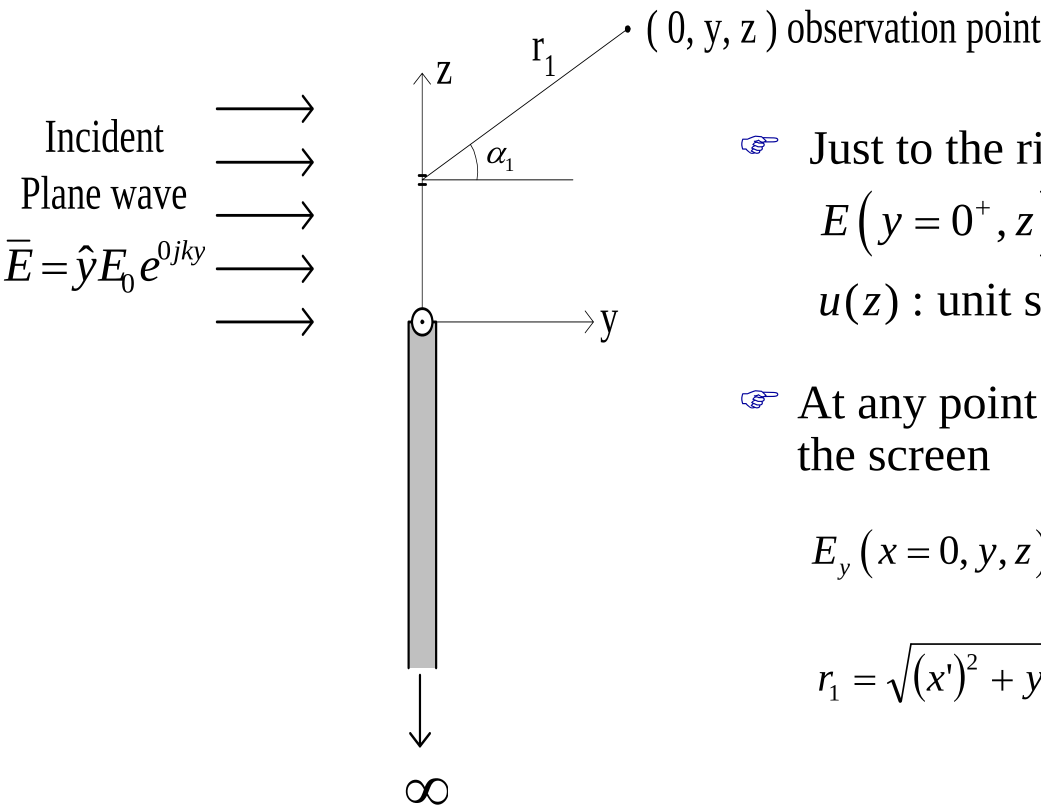
Plane Wave Diffraction

- Consider a plane wave propagation along y and incident on an absorbing half-screen



- Just to the right of the screen,

$$E\left(y = 0^{+}, z\right) = E_0\, u\left(z\right)$$

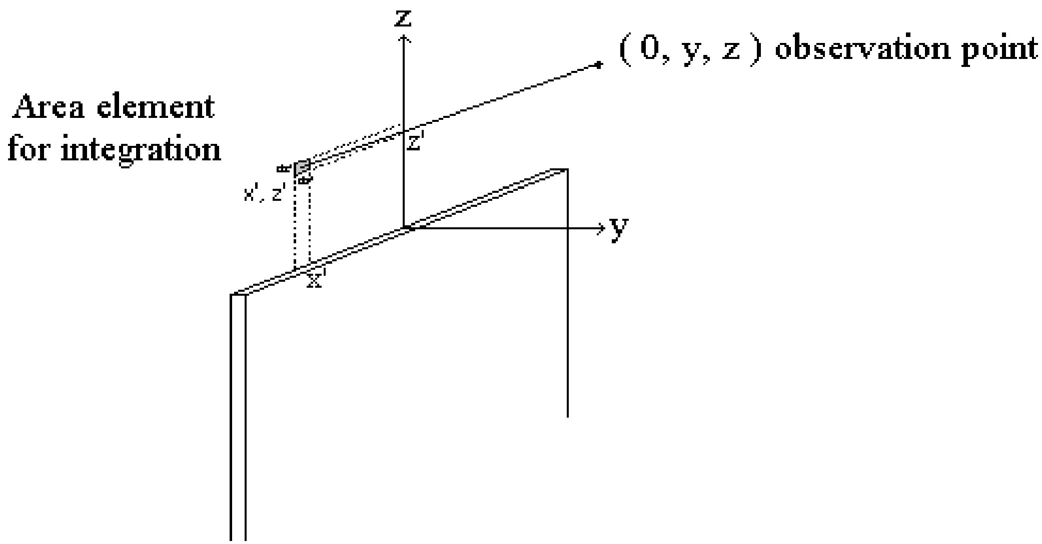
$u(z)$: unit step function

- At any point in the y-z plane to the right of the screen

$$E_y\left(x = 0, y, z\right) = \frac{jkE_0}{4\pi} \int_0^{\infty} \int_{-\infty}^{\infty} \left(1 + \cos\alpha_1\right) \frac{e^{-jkr_1}}{r_1}\, dx'\, dz'$$

$$r_1 = \sqrt{(x')^2 + y^2 + (z - z')^2}$$

Plane Wave Diffraction



☞ In order to carry out the x’ integration, recognize that for y>>λ, the area giving a significant contribution will have width along x that is small compared to y.

$$r_1 \approx \rho_1 + \frac{1}{2}\frac{(x')^2}{\rho_1} \quad \textit{where} \quad \rho_1 = \sqrt{y^2 + (z - z')^2}$$

Plane Wave Diffraction

$$E_z(0,y,z)=\frac{jkE_0}{4\pi}\int_0^{\infty}\left[\frac{1+\cos\alpha_1}{\rho_1}e^{-jk\rho_1}\int_{-\infty}^{\infty}e^{-j\frac{k}{2}\frac{(x')^2}{\rho_1}}dx'\right]dz'$$

$$\int_{-\infty}^{+\infty}e^{-\frac{jk}{2\rho}x'^2}dx'=2\int_0^{+\infty}e^{-\frac{jk}{2\rho}x'^2}dx'$$

$$=2\left[\int_0^{\infty}\cos\left(\frac{k}{2\rho}x'^2\right)dx'-j\int_0^{\infty}\sin\left(\frac{k}{2\rho}x'^2\right)dx'\right]$$

$$=2\sqrt{\frac{2\rho}{k}}\frac{1}{2}\sqrt{\frac{\pi}{2}}(1-j)$$

$$=\sqrt{\frac{2\pi\rho}{k}}e^{-j\frac{\pi}{4}}$$

$$E_z(0,y,z)=\frac{je^{-j\frac{\pi}{4}}}{2\sqrt{2\pi}}\sqrt{k}E_0\int_0^{\infty}\frac{1+\cos\alpha_1}{\sqrt{\rho_1}}e^{-jk\rho_1}dz'$$

$<cf>$

$$\sqrt{(-j)}=\left(e^{-j\frac{\pi}{2}}\right)^{1/2}=e^{-j\frac{\pi}{4}}$$

$$\int_0^{\infty}\cos x^2dx=\int_0^{\infty}\sin x^2dx=\frac{1}{2}\sqrt{\frac{\pi}{2}}$$

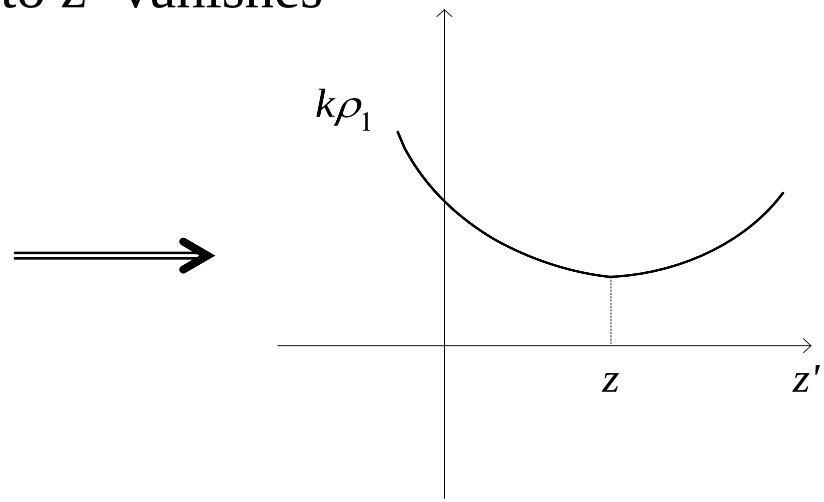
Plane Wave Diffraction

- In order to carry out the integration in the above equation, we observe that $\rho_1 = \sqrt{y^2 + (z - z')^2}$ will have a stationary point when the derivative of ρ_1 with respect to z' vanishes

$$0 = \frac{d\rho_1}{dz'} = \frac{-(z-z')}{\sqrt{y^2 + (z-z')^2}}$$

$$at \quad z = z'$$

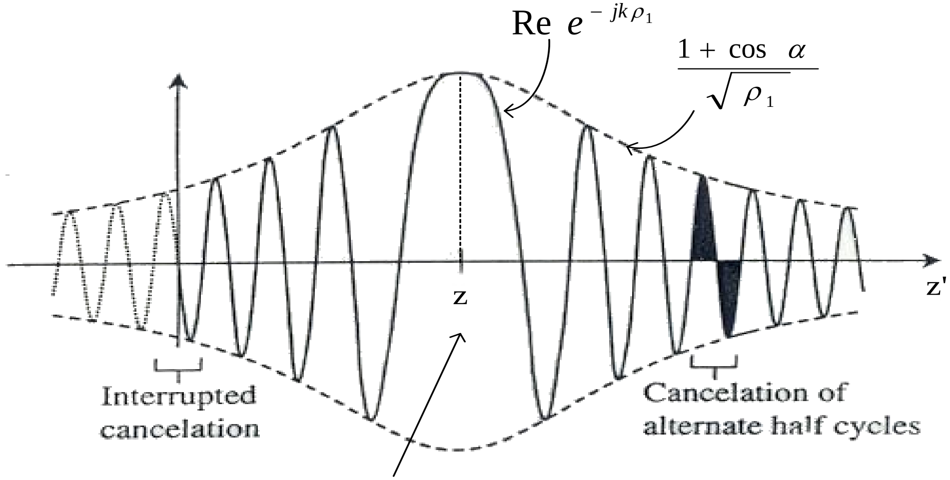
$$\rho_1 \approx y + \frac{1}{2y}(z - z')^2$$



- We identify three cases
  (1) The stationary point is well above the end point z'=0 of the integral
  (2) The stationary point is well below the end point
  (3) The stationary point is near the end point

Plane Wave Diffraction

- Case 1. z > 0 by a large amount



☞ The imaginary part of the integrand will have a similar variation. To account for me interrupted cancellation of alternate ½ cycle at z'=0., we will com compute an end point contribution in the following way.

$$\int_0^{\infty}\frac{(1+\cos\alpha_1)}{\sqrt{\rho_1}}e^{-jk\rho_1}dz' = \int_{-\infty}^{\infty}\frac{1+\cos\alpha_1}{\sqrt{\rho_1}}e^{-jk\rho_1}dz' - \int_{-\infty}^{0}\frac{1+\cos\alpha_1}{\sqrt{\rho_1}}e^{-jk\rho_1}dz'$$

First term   Second term

Plane Wave Diffraction

- The first integral gives the stationary point contribution, which can be find using the approximation

- At the stationary point, $\alpha_1 = 0 \quad and \quad \sqrt{\rho_1} = \sqrt{y}$ in the denominator.

$$\int_{-\infty}^{\infty} \frac{1+\cos\alpha_1}{\sqrt{\rho_1}} e^{-jk\rho_1} dz' = \frac{2}{\sqrt{y}} e^{-jky} \int_{-\infty}^{\infty} e^{-j\frac{k}{2y}(z-z')^2} dz'$$

$$= \left(\frac{2}{\sqrt{y}} e^{-jky}\right)\left(e^{-j\frac{\pi}{4}} \sqrt{\frac{2\pi y}{k}}\right)$$

Plane Wave Diffraction

- The second integral comprises the end point contribution.

$$\rho_1 = \sqrt{y^2 + z^2 - 2zz' + (z')^2} \qquad (z')^2 \to 0$$

$$\approx \sqrt{(y^2 + z^2) - 2zz'} = \sqrt{(y^2 + z^2)} - \frac{zz'}{\sqrt{y^2 + z^2}}$$

$$let\ \ \rho \equiv \sqrt{y^2 + z^2},\ \cos\alpha \equiv \frac{y}{\rho},\ \ \sin\alpha \equiv \frac{z}{\rho}$$

and then

$$-\int_{-\infty}^{0} \frac{1+\cos\alpha_1}{\sqrt{\rho_1}} e^{-jk\rho_1} dz' \approx -\frac{1+\cos\alpha}{\sqrt{\rho}} e^{-jk\rho} \int_{-\infty}^{0} e^{\frac{jkz}{\rho}z'} dz'$$

$$= -\frac{1+\cos\alpha}{\sqrt{\rho}} e^{-jk\rho} \left. \frac{e^{j\frac{kz}{\rho}z'}}{jkz/\rho} \right|_{-\infty}^{0}$$

$$= -\frac{1+\cos\alpha}{\sqrt{\rho}} e^{-jk\rho} \frac{1}{jkz/\rho} = j\frac{(1+\cos\alpha) e^{-jk\rho}}{k\sqrt{\rho}\sin\alpha}$$

< *cf* >

$$\int_{-\infty}^{\infty} e^{j\alpha x} dx = 2\pi\delta(x)$$

$$\int_{-\infty}^{0} e^{j\alpha x} dx = \pi\delta(x) + P\frac{1}{j\alpha}$$

P : excluding singular point i.e. x=0

Plane Wave Diffraction

- Stationary + end point contribution

$$= \int_0^\infty \frac{1+\cos\alpha}{\sqrt{\rho_1}} e^{-jk\rho_1} dz' \approx \frac{2\sqrt{2\pi}}{\sqrt{k}} e^{-j\pi/4} e^{-jky} + j\frac{1+\cos\alpha}{k\sin\alpha}\frac{e^{-jk\rho}}{\sqrt{\rho}}$$

End point contribution

Stationary point contribution

$$for \quad z > 0$$

$$E_z(0,y,z) = \frac{je^{-j\pi/4}}{2\sqrt{2\pi}}\sqrt{k}E_0\left[\frac{2\sqrt{2\pi}}{\sqrt{k}} e^{-j\pi/4} e^{-jky} + j\frac{1+\cos\alpha}{k\sin\alpha}\frac{e^{-jk\rho}}{\sqrt{\rho}}\right]$$

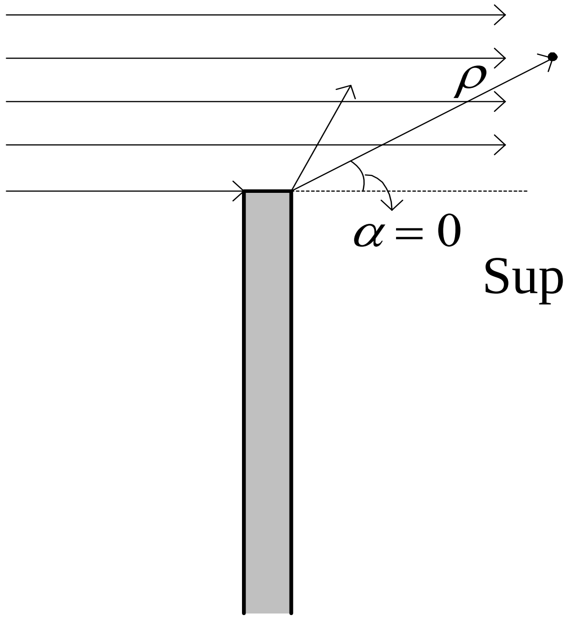
$$= E_0 e^{-jky} - E_0 \frac{e^{-j\pi/4}}{\sqrt{2\pi k}} \frac{1+\cos\alpha}{2\sin\alpha}\frac{e^{-jk\rho}}{\sqrt{\rho}}$$

$\frac{e^{-jk\rho}}{\sqrt{\rho}}$: cylindrical wave form

$\frac{1+\cos\alpha}{2\sin\alpha}$: singular at α=0 (w.r.t z=z_0) i.e. on shadow boundary

Plane Wave Diffraction

- First term : incident plane wave
- Second term : cylindrical wave emitting from the edge

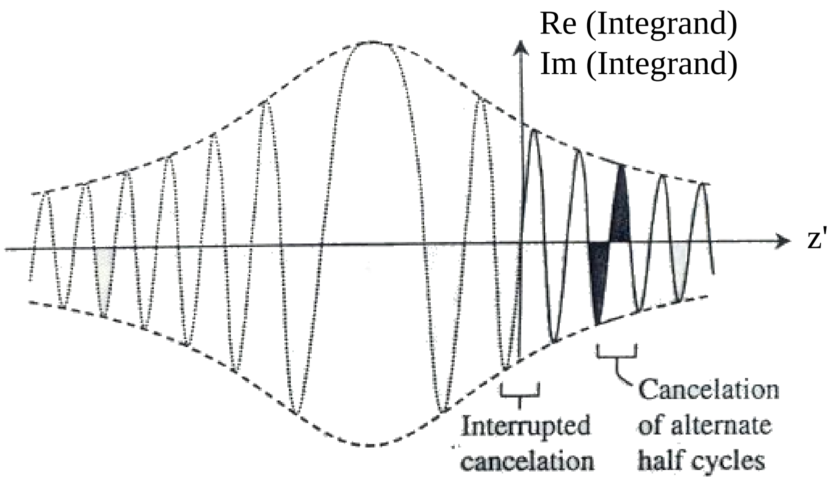


Superposition of incident and diffracted wave

Plane Wave Diffraction

- Case 2. z < 0 by a large amount

$$E_z(0,y,z)=\frac{je^{-j\pi/4}}{2\sqrt{2\pi}}\sqrt{k}E_0\int_0^{\infty}\frac{1+\cos\alpha_1}{\sqrt{\rho_1}}e^{-jk\rho_1}dz'$$



In this case, we have only the end point contribution.

Plane Wave Diffraction

$$\rho_1 \approx \sqrt{y^2 + z^2} - \frac{zz'}{\sqrt{y^2 + z^2}}$$

$$\int_0^\infty \frac{1+\cos\alpha_1}{\sqrt{\rho_1}} e^{-jk\rho_1} dz' \approx \frac{1+\cos\alpha}{\sqrt{\rho}} e^{-jk\rho} \int_0^\infty e^{j\frac{kz}{\rho}z'} dz'$$

$$= \frac{1+\cos\alpha}{\sqrt{\rho}} e^{-jk\rho} \cdot \left. \frac{e^{j\frac{kz}{\rho}z'}}{j\,kz/\rho} \right|_0^\infty$$

$$= \frac{1+\cos\alpha}{\sqrt{\rho}} e^{-jk\rho} \frac{(-1)}{j\,kz/\rho} = j\frac{1+\cos\alpha}{k\sin\alpha}\frac{e^{-jk\rho}}{\sqrt{\rho}} \quad \left(\because \sin\alpha = \frac{z}{\rho}\right)$$

$$\rho \equiv \sqrt{y^2 + z^2} \; \cos\alpha \equiv \frac{y}{\rho} \sin y \equiv \frac{z}{\rho}$$

$$\therefore E_z \approx -E_0 \frac{e^{-j\pi/4}}{\sqrt{2\pi k}} \frac{1+\cos\alpha}{2\sin\alpha} \frac{e^{-jk\rho}}{\sqrt{\rho}}$$

- ☞ Cylindrical wave emanating from the edge.
- ☞ The plane wave is blocked for z<0.

Plane Wave Diffraction

- Case 3. z near 0

For this case, z and z' are both small compared to y so that $\alpha = 0$ on $\sqrt{\rho} \cong \sqrt{y}$

$$\rho_1 \cong y + \frac{1}{2y}(z - z')^2$$

$$E_z = \frac{e^{j\pi/4}}{\sqrt{2\pi}} E_0 \sqrt{\frac{k}{y}} e^{-jky} \int_0^\infty e^{-j\frac{k}{2y}(z-z')^2} dz'$$

$$t = -\sqrt{\frac{k}{2y}(z - z')^2}$$

$$E_z = \frac{e^{j\pi/4}}{\sqrt{\pi}} E_0 e^{-jky} \int_{-\sqrt{\frac{k}{2y}}z}^{\infty} e^{-jt^2} dt$$

(1) For $z < 0$,

$$\int_{-\sqrt{\frac{k}{2y}}z}^{\infty} e^{-jt^2} dt = \int_{\sqrt{\frac{k}{2y}}|z|}^{\infty} e^{-jt^2} dt$$

(2) For $z > 0$,

$$\int_{-\sqrt{\frac{k}{2y}}z}^{\infty} e^{-jt^2} dt = \int_{-\infty}^{\infty} e^{-jt^2} dt - \int_{-\infty}^{-\sqrt{\frac{k}{2y}}|z|} e^{-jt^2} dt$$

$$= \sqrt{\pi} e^{-j\pi/4} - \int_{\sqrt{\frac{k}{2y}}z}^{\infty} e^{-ju^2} du \qquad (u = -t)$$

$$E_z = E_0 e^{-jky} u(z) - E_0 e^{-jky} f\left(\sqrt{\frac{k}{2y}}|z|\right) \operatorname{sgn}(z)$$

$$\operatorname{sgn}(z) = \begin{cases} +1 & for\ z > 0 \\ -1 & for\ z < 0 \end{cases}$$

$$f(x) = \frac{e^{j\pi/4}}{\sqrt{\pi}} \int_x^\infty e^{-ju^2} du$$

Plane Wave Diffraction

For small z,

$$\rho = \sqrt{y^2 + z^2} \approx y \pm \frac{1}{2}\frac{z^2}{y}$$

$$E_z = E_0 e^{-jky} U(z) - E_0 e^{-jk\rho}\left[e^{j\frac{k}{2y}z^2} f\left(\sqrt{\frac{k}{2y}}|z|\right)\right]\operatorname{sgn}(z)$$

$$Let \;\; \xi = \frac{1}{2}\sqrt{\frac{k}{y}}|z|$$

$$F(\xi) = 2e^{j\frac{k}{2y}z^2} f\left(\sqrt{\frac{k}{2y}}|z|\right) = \frac{2}{\sqrt{\pi}} e^{j2\xi^2} e^{j\pi/4}\int_{\sqrt{2}\xi}^{\infty} e^{-ju^2}du$$

$$\therefore E_z = E_0 e^{-jky}U(z) - E_0 e^{-jk\rho}\frac{1}{2}F(\xi)\operatorname{sgn}(z)$$

Fresnel Integral

cf)

$$\int_x^{\infty} e^{-ju^2}du$$

$$= \int_0^{\infty} e^{-ju^2}du - \int_0^{x} e^{-ju^2}du$$

$$= \frac{\sqrt{\pi}}{2}e^{-j\pi/4} - \left[\int_0^x \cos u^2 du - \int_0^x \sin u^2 du\right]$$

$$= \frac{\sqrt{\pi}}{2}e^{-j\pi/4} - \frac{\sqrt{\pi}}{2}\left[C_1(x) - jS_1(x)\right]$$

$$where \begin{cases} C_1(x) = \sqrt{\frac{2}{\pi}}\int_o^x \cos t^2 dt \\ S_1(x) = \sqrt{\frac{2}{\pi}}\int_o^x \sin t^2 dt \end{cases}$$

Fresnel Integral

- The above equation applies for $|z|$ within some region about the shadow boundary y>0, z=0.
- We anticipate that this region will be the Fresnel zone. $|z| = \sqrt{y\lambda}$ in width.

Plane Wave Diffraction

$$|z| = \sqrt{y\lambda}$$

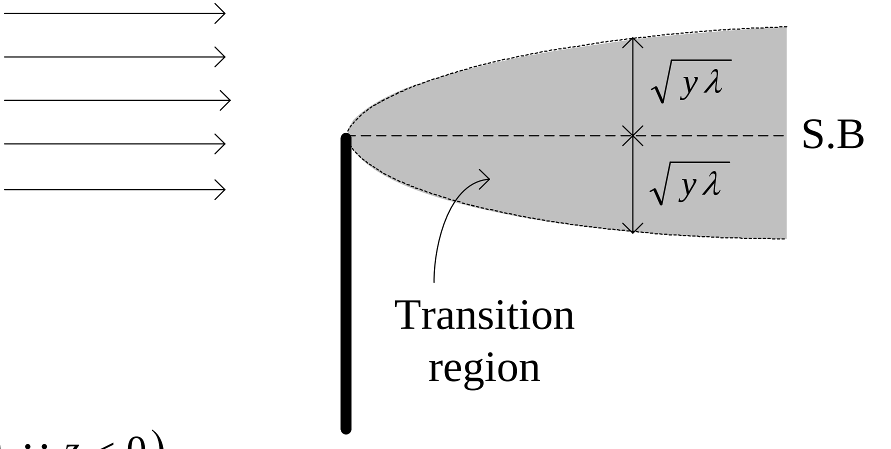
$$\xi = \frac{1}{2}\sqrt{\frac{k}{y}}|z| = \frac{1}{2}\sqrt{\frac{k}{y}}\sqrt{y\lambda} = \sqrt{\frac{\pi}{2}}$$

Outside the transition,

$$E_z = E_0 e^{-jky} - E_0 \frac{e^{-j\pi/4}}{\sqrt{2\pi k}} \frac{1+\cos\alpha}{2\sin\alpha} \frac{e^{-jk\rho}}{\sqrt{\rho}}$$

$$or\ \ E_z = -E_0 \frac{e^{-j\pi/4}}{\sqrt{2\pi k}} \frac{1+\cos\alpha}{2\sin\alpha} \frac{e^{-jk\rho}}{\sqrt{\rho}} \quad (\sin\alpha < 0 \ \because\ z < 0)$$

$$\bar{E}_z = E_0 e^{-jky} u(z) - E_0 \frac{e^{-j\pi/4}}{\sqrt{2\pi k}} \frac{1+\cos\alpha}{2\sin\alpha} \frac{e^{-jk\rho}}{\sqrt{\rho}}$$



- The first term : the incident field E_{in} in the illuminated region z>0, and is zero in the shadow region z<0, as would be given by geometric optics.
- The second term : the cylindrical wave emanating from the edge with pattern function $\frac{1+\cos\alpha}{2\sin\alpha}$

Diffraction coefficient

- Diffraction coefficient

$$D(\alpha) = -\frac{1}{\sqrt{2\pi k}}\frac{1+\cos\alpha}{2\sin\alpha}$$

$$E = E_{in}U(z) + E_0\frac{e^{-j\pi/4}e^{-jk\rho}}{\sqrt{\rho}}D(\alpha)$$

- Inside the transition region

$$E = E_{in}U(z) + E_0\frac{e^{-j\pi/4}e^{-jk\rho}}{\sqrt{\rho}}D_T(\alpha)$$

$$D_T(\alpha) = -\frac{1}{2}e^{j\pi/4}\sqrt{\rho}F(\xi)\operatorname{sgn}(z)$$

$$F(\xi) = \frac{2}{\sqrt{\pi}}e^{j2\xi^2}e^{j\pi/4}\int_{\sqrt{2}\xi}^{\infty}e^{-ju^2}du$$

$$\therefore D_T(\alpha) = -\frac{-j\sqrt{\rho}}{\sqrt{\pi}}e^{j2\xi^2}\left[\int_{\sqrt{2}\xi}^{\infty}e^{-ju^2}du\right]\operatorname{sgn}(z)$$

Diffraction coefficient

- Outside transition region

$$u^2 = 2\xi^2 + v \ , \ u = \sqrt{2\xi^2 + v}$$

$$du = \frac{1/2}{\sqrt{2\xi^2 + v}} dv$$

$$\int_{2\xi}^{\infty} e^{-ju^2} du = \int_0^{\infty} e^{-j2\xi^2} e^{-jv} \frac{dv}{2\sqrt{2\xi^2 + v}}$$

$$\frac{1}{\sqrt{2\xi^2 + v}} = \frac{1}{\sqrt{2}\xi} - \frac{1}{2\left(\sqrt{2\xi^2}\right)^3} v + \cdots$$

By taylor series

$$\therefore \int_{2\xi}^{\infty} e^{-ju^2} du = \frac{1}{2\sqrt{2}\xi} e^{-j2\xi^2} \left[\int_0^{\infty} e^{-jv} dv - \frac{1}{4\xi^2} \int_0^{\infty} v e^{-jv} dv + \cdots \right]$$

$$= \frac{1}{2\sqrt{2}\xi} e^{-j2\xi^2} \left[\frac{1}{j} + \frac{1}{4\xi^2} + \cdots \right]$$

$$at \ |z| = \sqrt{\lambda y}, \ \xi = \sqrt{\frac{\pi}{2}} \ and \ \frac{1}{4\xi^2} = \frac{1}{\pi}$$

- So that outside the Fresnel zone the term and higher decreases rapidly leaving only the first term in the series.

Diffraction coefficient

- For large $|z|$,

$$D_T(\alpha) \cong -\frac{j\sqrt{\rho}}{\sqrt{\pi}} e^{j2\xi^2} \cdot \frac{1}{j2\sqrt{2}\xi} e^{-j2\xi^2} \operatorname{sgn}(z)$$

$$\xi = \frac{1}{2}\sqrt{\frac{k}{y}}|z|$$

$$D_T(\alpha) \cong -\frac{\sqrt{\rho y}\operatorname{sgn}(z)}{\sqrt{\pi}\cdot 2\sqrt{2}} \cdot \frac{2}{\sqrt{k}|z|} = -\frac{1}{\sqrt{2\pi k}} \frac{\operatorname{sgn}(z)}{|z|/\sqrt{\rho y}}$$

- For small $|z|$,

$$y \cong \rho \ \ and$$

$$\frac{\operatorname{sgn}(z)}{|z|/\sqrt{\rho y}} \cong \frac{\operatorname{sgn}(z)}{|z|/\rho} = \frac{1}{\sin\alpha} \Rightarrow \frac{1+\cos\alpha}{2\sin\alpha} \cong \sin\alpha$$

Diffraction coefficient

$$D(\alpha) = -\frac{1}{\sqrt{2\pi k}}\frac{1+\cos\alpha}{2\sin\alpha} \Leftrightarrow D_T(\alpha) = -\frac{1}{\sqrt{2\pi k}}\frac{\operatorname{sgn}(z)}{|z|/\sqrt{\rho y}} \qquad for\ |z|\ small$$

$$D_T(\alpha) = -\frac{1}{2}e^{j\pi/4}\sqrt{\rho}F(\xi)\operatorname{sgn}(z) \Leftrightarrow D_T(\alpha) = -\frac{1}{\sqrt{2\pi k}}\frac{\operatorname{sgn}(z)}{|z|/\sqrt{\rho y}} \qquad for\ |z|\ large$$

$\hat{D}(\alpha)$: new diffraction coefficient that is uniformly valid for all z

$$\hat{D}(\alpha) = D(\alpha) + D_T(\alpha) + \frac{1}{\sqrt{2\pi k}}\frac{\operatorname{sgn}(z)}{|z|/\sqrt{\rho y}}$$

For z (or α) large : the last tow term cancel $\Rightarrow \hat{D}(\alpha) = D(\alpha)$

For z (or α) small : the first and last cancel $\Rightarrow \hat{D}(\alpha) = D_T(\alpha)$

$$\therefore \bar{E} = E_{in}U(z) + E_0\frac{e^{-j\pi/4}e^{-jk\rho}}{\sqrt{\rho}}\hat{D}(\alpha)$$

Diffraction coefficient

- More rigorous treatments of diffraction lead to the same result as the above equation with somewhat different value of D(α).
- A useful solution for an absorbing screen given by Felson is

$$D(\alpha) = \frac{-1}{\sqrt{2\pi k}}\left[\frac{1}{\alpha} + \frac{1}{2\pi - \alpha}\right]$$

$$D(\alpha) = \frac{-1}{\sqrt{2\pi k}}\left[\frac{1}{\pi - |\Phi - \Phi'|} + \frac{1}{\pi + |\Phi - \Phi'|}\right]$$

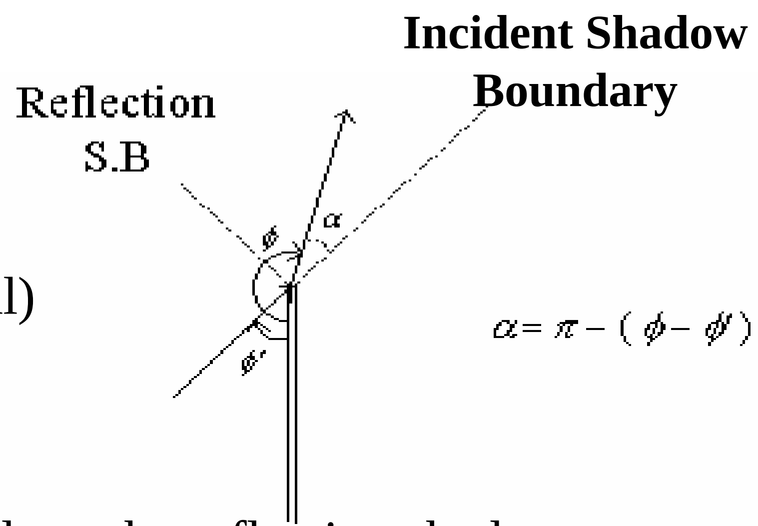
Diffraction coefficient

- For a conducting screen, there is a reflected wave that has its own shadow boundary. The diffraction coefficient in this case is different for two vertical and horizontal polarizations and is given by

$$D(\alpha)=\frac{-1}{\sqrt{2\pi k}}\frac{1}{2}\left[\frac{1}{\cos\frac{\Phi-\Phi'}{2}}\pm\frac{1}{\cos\frac{\Phi+\Phi'}{2}}\right]$$

where + is for E // edge axis (horizontal)

- is for H // edge axis (vertical)



- In this case the second term blows up along the reflection shadow boundary $\Phi+\Phi'=\pi$. To cancel this infinity, two more terms are required.

Reference

- Advanced Engineering Electromagnetics by Balanis chapter 13.
- R.F. Luebbers, “Finite Conductivity Uniform GTD versus knife Edge Diffraction in Prediction of propagation path loss”IEEE, Tras. AP., vol Ap-32,no.1 pp70~76. Jan.1984