Capacity of Wireless Channels

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Introduction

- Channel capacity limit
  - The maximum channel rates that can be transmitted over the wireless channel with asymptotically small error probability, assuming no constraints on the delay or complexity of the encoder/decoder
- Scope of this chapter
  - Capacity of a single-user wireless channel where the transmitter and/or receiver has a single antenna
    - a time-invariant additive white Gaussian Noise (AWGN) channel
    - a flat fading channel
    - a frequency selective fading channel

Capacity of AWGN Channel

Capacity in AWGN

- Shannon Capacity
  - $$\boxed{C = B\log_2(1+\gamma)}$$
  - $\gamma = P / N_0 B$
    - Received signal-to-noise ratio (SNR)
    - P : the transmitted signal power
    - Nose power: 2 x two-sided noise PSD (N_0/2) x B or one-sided PSD (N_0) x B
  - Upper bound on the data rates that can be achieved under the real system constraints
  - On AWGN radio channel, turbo codes have come within a fraction of a decibel of Shannon capacity limit

Capacity of discrete time-invariant channel

- Mutual information
  - The average amount of information received over the channel per symbol
  - $\mathrm{I(X;Y) = H(X) - H(X \mid Y)}$
    - H(X): the average amount of information transmitted per symbol (entropy)
    - H(X|Y): the average uncertainty about a transmitted symbol when a symbol is received, and the average amount of information lost over noisy channel per symbol
    - $\mathrm{H(X)} = \sum_{x \in S_X} p(x) \log \frac{1}{p(x)}, \quad \mathrm{H(X \mid Y)} = \sum_{x \in S_X, y \in S_Y} p(x,y) \log \frac{1}{p(x \mid y)}$
  - $\mathrm{I(X;Y)} = \sum_{x \in S_X} p(x) \log \frac{1}{p(x)} - \sum_{x \in S_X, y \in S_Y} p(x,y) \log \frac{1}{p(x \mid y)}$

Channel Capacity of a Continuous Channel

- Entropy of X: $\mathrm{H(X)} = \int_{-\infty}^{\infty} p(x) \log \dfrac{1}{p(x)} \, dx$
- Mutual Information I(X;Y)

$$\mathrm{I(X;Y)} = \int_{-\infty}^{\infty} p(x) \log \frac{1}{p(x)} \, dx - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x,y) \log \frac{1}{p(x|y)} \, dxdy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x,y) \left(\log \frac{1}{p(x)} - \log \frac{1}{p(x|y)} \right) dxdy \;\; = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x,y) \log \frac{p(x|y)}{p(x)} \, dxdy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x,y) \log \frac{p(x,y)}{p(x)\,p(y)} \, dxdy \;\; = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x,y) \log \frac{p(y|x)}{p(y)} \, dydx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x,y) \left(\log \frac{1}{p(y)} - \log \frac{1}{p(y|x)} \right) dxdy$$

$$= \int_{-\infty}^{\infty} p(y) \log \frac{1}{p(y)} \, dy - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x,y) \log \frac{1}{p(y|x)} \, dxdy$$

$$= \mathrm{H(Y)} - \mathrm{H(Y \mid X)}$$

Channel Capacity of a Continuous Channel

- Entropy of Z: $\mathrm{H(Z)} = \int_{-\infty}^{\infty} p(z) \log \frac{1}{p(z)} dz$
- Maximum entropy of Z, for a given $E[Z^2]$
  - The maximum entropy is obtained when the distribution of Z is Gaussian
    - $p(z) = \frac{1}{\sigma\sqrt{2\pi}} e^{-z^2/2\sigma^2}$, where $\sigma^2 = \int_{-\infty}^{\infty} z^2 p(z) dz$
    - $\mathrm{H}_{\max}(\mathrm{Z}) = \frac{1}{2}\log(2\pi e\sigma^2)$

Capacity of a Band-limited AWGN Channel (1)

- Channel capacity
  - Maximum amount of mutual information I(X;Y) per second
  - Two steps
    - the maximum mutual information per sample
    - 2B samples (Nyquist's sampling theory)
- Maximum mutual information per sample
  - x, n, y: samples of the transmitted signal, noise, and received signal
  - H(y|x)
    - $$\mathrm{H}(\mathrm{y} \mid \mathrm{x}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x, y) \log \frac{1}{p(y|x)} \, dxdy$$
      $$= \int_{-\infty}^{\infty} p(x) \int_{-\infty}^{\infty} p(y \mid x) \log \frac{1}{p(y|x)} \, dydx$$
    - Because y=x+n, for a given x, y is equal to n plus a constant. The distribution of y is identical to that of n except for a translation by x
    - $p(y \mid x) = p_n(y - x),$ where $p_n(\cdot)$ is the PDF of noise sample

Capacity of a Band-limited AWGN Channel (2)

– $\int_{-\infty}^{\infty} p(y \mid x) \log \frac{1}{p(y|x)} dy = \int_{-\infty}^{\infty} p_n(y-x) \log \frac{1}{p_n(y-x)} dy$

$$= \int_{-\infty}^{\infty} p_n(u) \log \frac{1}{p_n(u)} du = H(\mathrm{n})$$

– $\mathrm{H(y \mid x)} = \int_{-\infty}^{\infty} p(x) \int_{-\infty}^{\infty} p(y \mid x) \log \frac{1}{p(y|x)} dydx$

$$= \int_{-\infty}^{\infty} \mathrm{H(n)} p(x) dx = \mathrm{H(n)} \int_{-\infty}^{\infty} p(x) dx = \mathrm{H(n)}$$

– I(x;y)=H(y) - H(n)

- Entropy of a band-limited white Gaussian noise with PSD $N_0/2$
  – Noise power: $N = N_0 B$
  – $\mathrm{H(n)} = \frac{1}{2} \log(2\pi e N)$
- When the signal power is S and the noise power is N, and the signal $s(t)$ and noise $n(t)$ are independent, the mean square value of y is $E[\mathrm{y}^2] = S + N$

Capacity of a Band-limited AWGN Channel (3)

- H(y) will be maximum if y is Gaussian
- $H_{max}(y) = \frac{1}{2}\log[2\pi e(S+N)]$

- $I_{max}(x;y) = H_{max}(y) - H(n)$

$$= \frac{1}{2}\log[2\pi e(S+N)] - \frac{1}{2}\log(2\pi e N)$$

$$= \frac{1}{2}\log(1 + S/N)$$

- Channel capacity: $2 \times B \times I_{max}(x;y)$

$$\boxed{C = B\log(1 + \frac{S}{N})}$$

- Reference :B. P. Lathi, *Modern Digital and Analog Communication System*, 3rd Ed., Oxford. (Chapter 15)

Capacity of Flat-Fading Channels

Capacity of Flat-Fading Channels

- The channel capacity depends on the information about $g[i]$
  - Channel distribution information (CDI) of $g[i]$ known to the transmitter and receiver (**) and Channel side information (the value of $g[i]$) known to the receiver
  - CSI known to the receiver and transmitter, and (**)

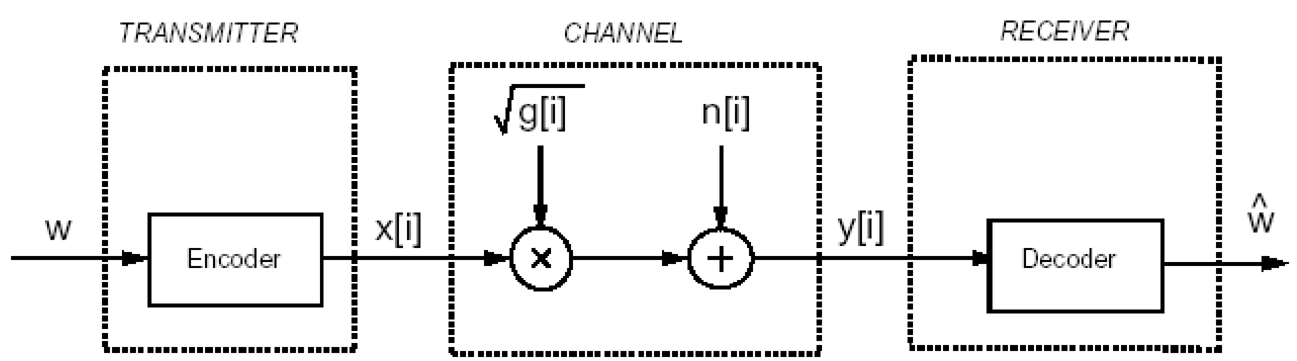


Figure 4.1: Flat-Fading Channel and System Model.

CSI at Receiver (1)

- The rate transmitted over the channel is constant (The transmitter cannot adapt its transmission strategy relative to the CSI)
- Shannon (ergodic) capacity
  - $C = \int_0^\infty B \log_2(1+\gamma)\, p(\gamma)\, d\gamma$
  - Shannon capacity for AWGN channel averaged over the distribution of γ
  - $\int_0^\infty B \log_2(1+\gamma)\, p(\gamma)\, d\gamma = \mathrm{E}\left[B \log_2(1+\gamma)\right] \le B \log_2(1+\mathrm{E}[\gamma])$

    Shannon capacity of AWGN channel with the same average SNR
  - Capacity-achieving code must be sufficiently long that a received codeword is affected by all possible fading states => This can result in significant delay
  - If the receiver CSI is not perfect, capacity can be significantly decreased

CSI at Receiver (2)

- Capacity with outage
  - The transmitter fixes a minimum received SNR $\gamma_{\min}$ and encodes for a fixed data rate $C = B\log_2(1+\gamma_{\min})$
  - For the received SNR below $\gamma_{\min}$, the received bits cannot be decoded correctly (outage)
  - Outage probability: $P_{out} = p(\gamma < \gamma_{\min})$
  - Capacity: $C_{out} = (1-P_{out})B\log_2(1+\gamma_{\min})$
  - The outage probability is a design parameter

CSI at Receiver (3)

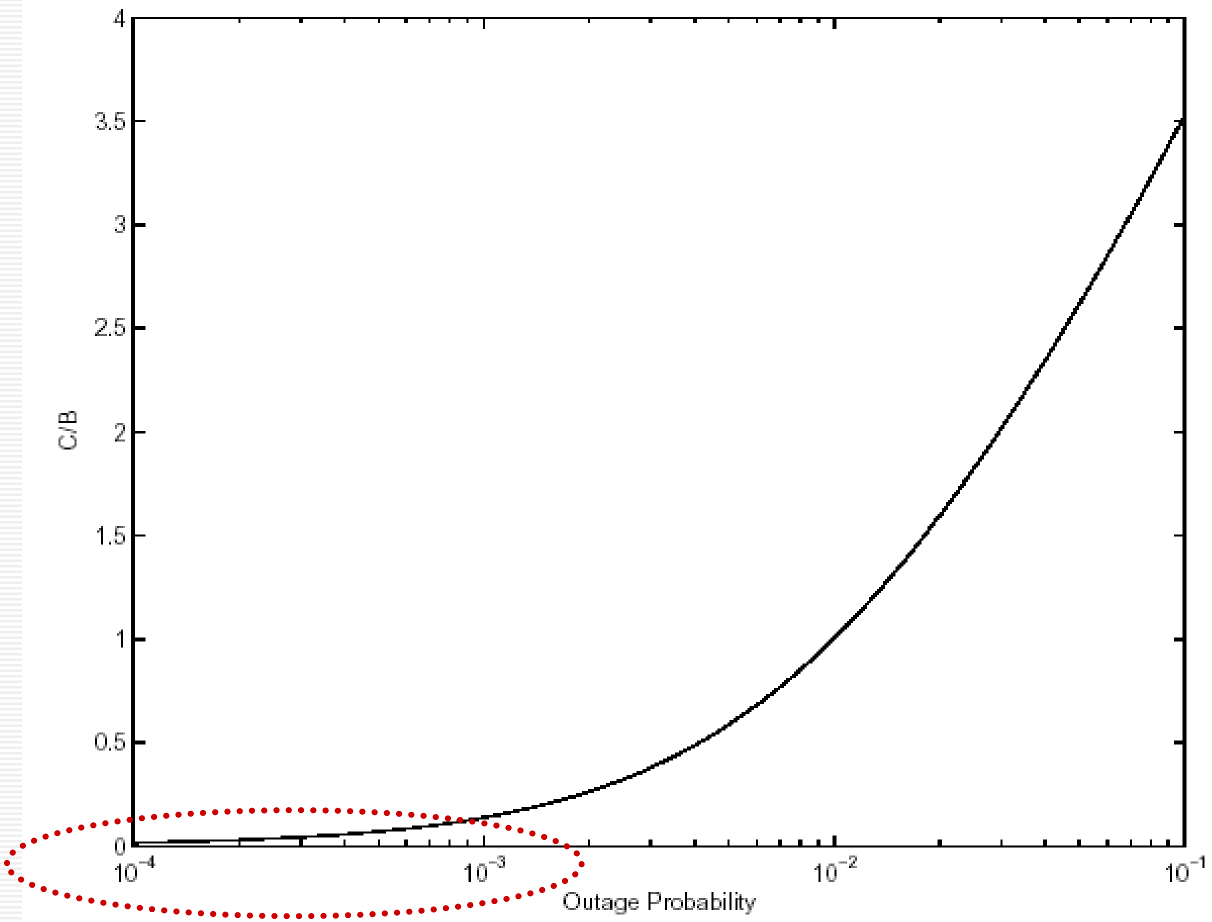


Figure 4.2: Normalized Capacity (C/B) versus Outage Probability.

Rayleigh fading channel with $E[\gamma] = 20\,\text{dB}$

CSI at Transmitter and Receiver (1)

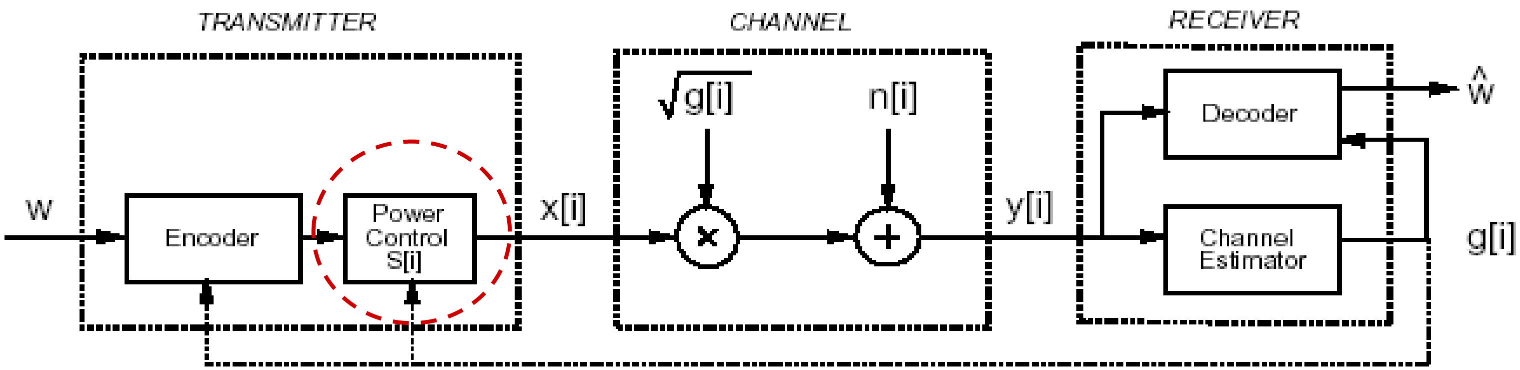


Figure 4.3: System Model with Transmitter and Receiver CSI.

- For fixed transmission power, the same capacity as when only receiver knows fading

$$C = \int_0^\infty B \log_2(1+\gamma)\, p(\gamma)\, d\gamma$$

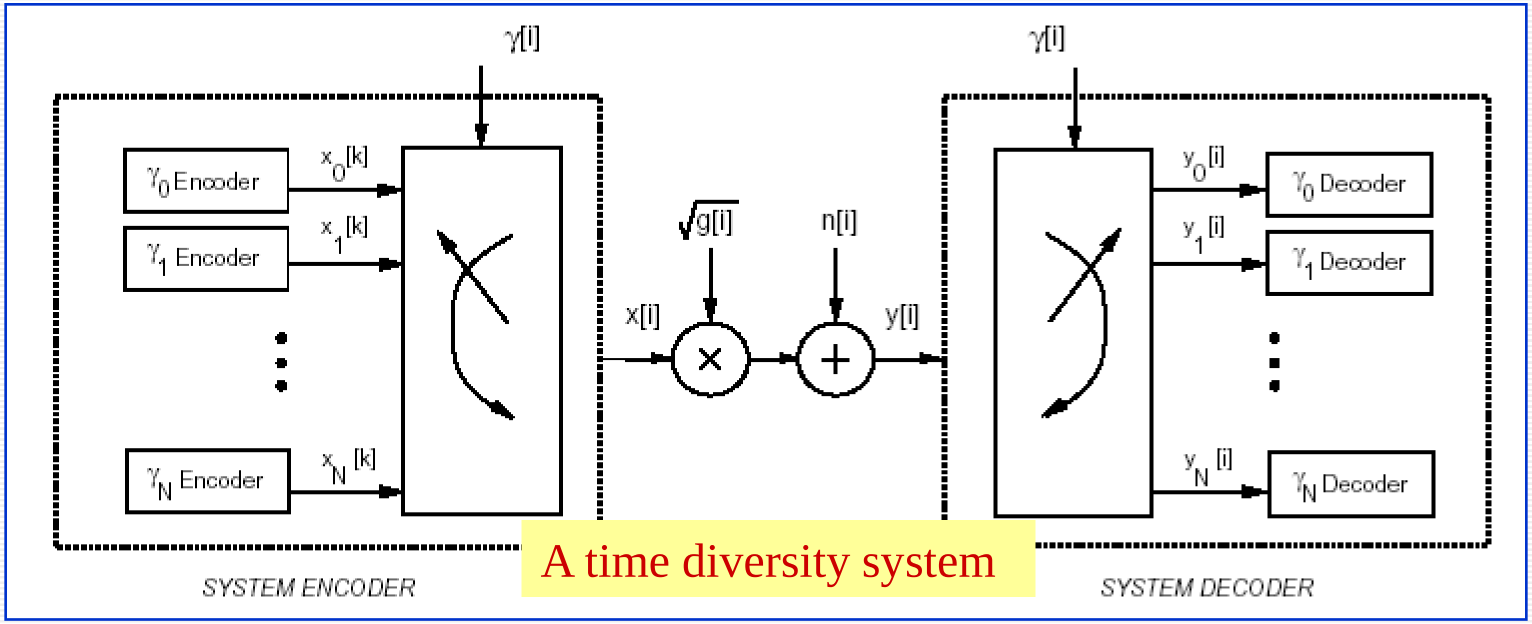
CSI at Transmitter and Receiver (2)

- Transmission power as well as rate can be adapted.
- Adaptation of transmission power $P_t(\gamma)$ to the received SNR γ subject to an average power constraint Φ
- average power constraint: $\int_0^\infty P_t(\gamma)\, p(\gamma)\, d\gamma \le \Phi$
- The fading channel capacity with average power constraint

$$C = \max_{P_t(\gamma): \int P_t(\gamma)\, p(\gamma)\, d\gamma = \Phi} \int_0^\infty B \log_2\left(1 + \frac{P_t(\gamma)\gamma}{\Phi}\right) p(\gamma)\, d\gamma$$

* γ: the received SNR at transmission power Φ

CSI at Transmitter and Receiver (3)



- The range of fading values is quantized to a finite set $\{\gamma_j : 1 \leq j \leq N\}$
- For each γ_j, an encoder-decoder pair for an AWGN channel with SNR γ_j
- The input x_j for encoder has average power $P_t(\gamma_j)$ and data rate C_j where C_j is the capacity of time-invariant AWGN channel with received SNR $P_t(\gamma_j)\gamma_j / \Phi$

CSI at Transmitter and Receiver (4)

- Optimal power allocation
  - Lagrangian

$$J(P_t(\gamma),\lambda) = \int_0^\infty B\log_2\left(1+\frac{P_t(\gamma)\gamma}{\Phi}\right)p(\gamma)\,d\gamma + \lambda\left[\Phi - \int_0^\infty P_t(\gamma)\,p(\gamma)\,d\gamma\right]$$

  - Differentiate the Lagrangian and set the derivate to zero

$$\frac{\partial J(P_t(\gamma),\lambda)}{\partial P_t(\gamma)} = \left[\left(\frac{B/\ln 2}{1+\gamma\,P_t(\gamma)/\Phi}\right)\frac{\gamma}{\Phi} - \lambda\right]p(\gamma) = 0$$

  - Solve for $P_t(\gamma)$ with the constraint that $P_t(\gamma) > 0$

$$\boxed{\frac{P_t(\gamma)}{\Phi} = \begin{cases} 1/\gamma_0 - 1/\gamma & \gamma \geq \gamma_0 \\ 0 & \gamma < \gamma_0 \end{cases}}$$

CSI at Transmitter and Receiver (5)

- Capacity

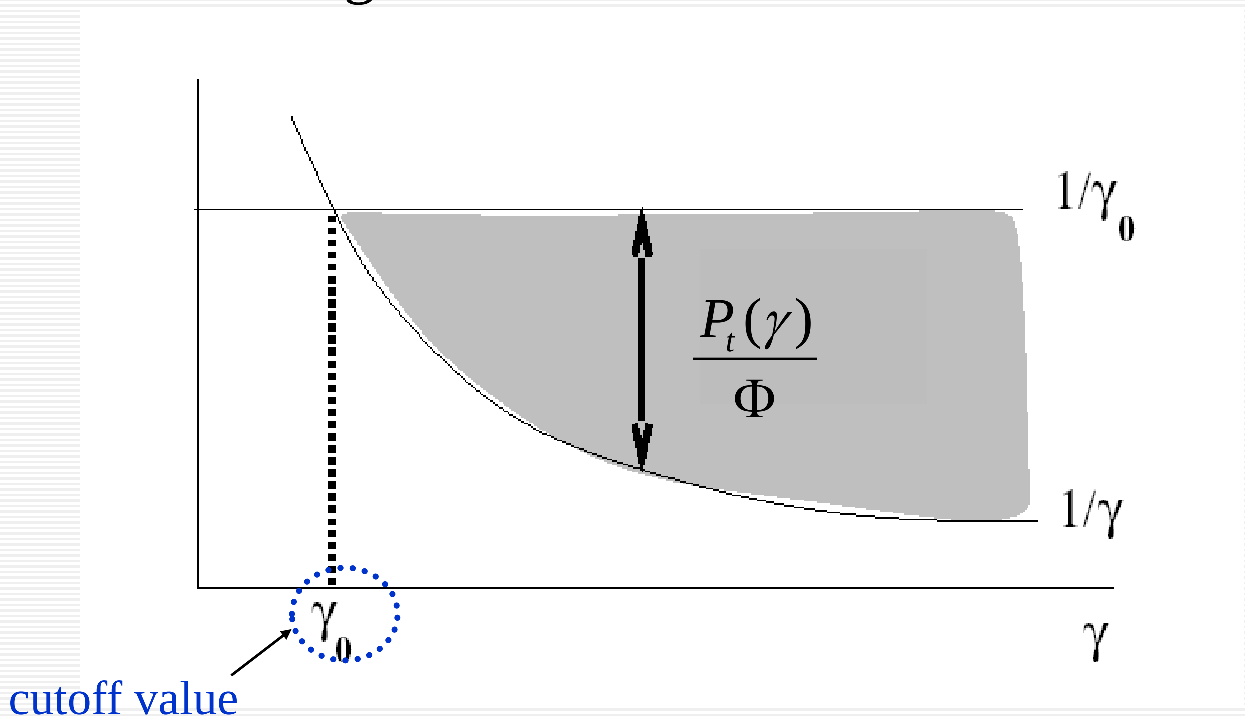
$$\boxed{C = \int_{\gamma_0}^{\infty} B \log_2\left(\frac{\gamma}{\gamma_0}\right) p(\gamma)\, d\gamma}$$

  - Time-varying data rate : the rate corresponding to the instantaneous SNR γ is $B \log_2(\gamma/\gamma_0)$
  - Transmission power adaption
    - Optimal power allocation (Water filling)

$$\frac{P_t(\gamma)}{\Phi} = \begin{cases} 1/\gamma_0 - 1/\gamma & \gamma \geq \gamma_0 \\ 0 & \gamma < \gamma_0 \end{cases}$$

CSI at Transmitter and Receiver (6)

- Water filling



$$\boxed{\gamma_0 \text{ such that } \int_{\gamma_0}^{\infty}\left(\frac{1}{\gamma_0}-\frac{1}{\gamma}\right)p(\gamma)\,d\gamma = 1}$$

the better channel, the more power and the higher data rate

CSI at Transmitter and Receiver (7)

- Channel inversion and zero outage
  - The transmitter controls the transmission power using CSI so as to maintain a constant received power (inverts the channel fading)
  - The channel appears to the encoder and decoder as a time-invariant AWGN channel
  - transmission power: $P_t(\gamma)/\Phi = \sigma/\gamma$
    - $\sigma = 1\big/\mathrm{E}[1/\gamma]$ from $\int_0^\infty (\sigma/\gamma)\, p(\gamma)\, d\gamma = 1$
  - Fading channel capacity with channel inversion is equal to the AWGN channel capacity with SNR σ

$$C = B\log_2(1+\sigma) = B\log_2\left(1+\frac{1}{\mathrm{E}[1/\gamma]}\right)$$

CSI at Transmitter and Receiver (8)

- Channel inversion and zero outage
  - A fixed data rate regardless of channel condition
  - Encoder and decoder are designed for an AWGN channel with SNR σ => the simplest scheme to implement
  - zero outage:
    - Should maintain a constant data rate in all fading states
    - Zero outage capacity is significantly smaller than Shannon capacity on fading channel
      - In Rayleigh fading, the zero outage capacity is zero
  - Channel inversion is common in spread-spectrum system with near-far interference imbalances

CSI at Transmitter and Receiver (9)

- Truncated channel inversion
  - Suspending transmission in bad fading states
  - Truncated channel inversion
    - Power adaptation policy that compensates only for fading above a cutoff γ_0
    - $\dfrac{P_t(\gamma)}{\Phi} = \begin{cases} \sigma/\gamma & \gamma \geq \gamma_0 \\ 0 & \gamma < \gamma_0 \end{cases}$ where $\sigma = \left(\mathrm{E}_{\gamma_0}[1/\gamma]\right)^{-1} = \left(\int_{\gamma_0}^{\infty} \frac{1}{\gamma} p(\gamma) d\gamma\right)^{-1}$
    - Outage probability $P_{out} = p(\gamma < \gamma_0)$
    - Outage capacity for a given P_{out} and corresponding cutoff γ_0
      - $C(P_{out}) = B\log_2\left(1 + \dfrac{1}{\mathrm{E}_{\gamma_0}[1/\gamma]}\right) p(\gamma \geq \gamma_0)$
    - Maximum outage capacity
      - $C = \max\limits_{\gamma_0} B\log_2\left(1 + \dfrac{1}{\mathrm{E}_{\gamma_0}[1/\gamma]}\right) p(\gamma \geq \gamma_0)$

Capacity Comparison

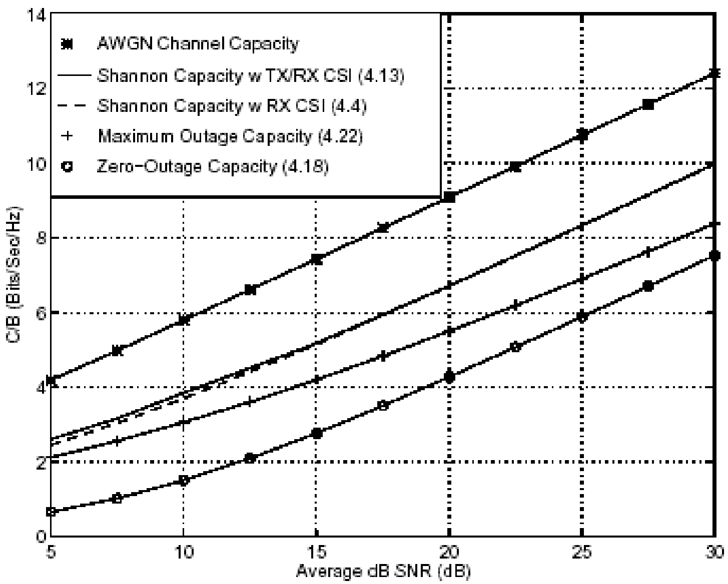


Figure 4.6: Capacity in Log-Normal Shadowing.

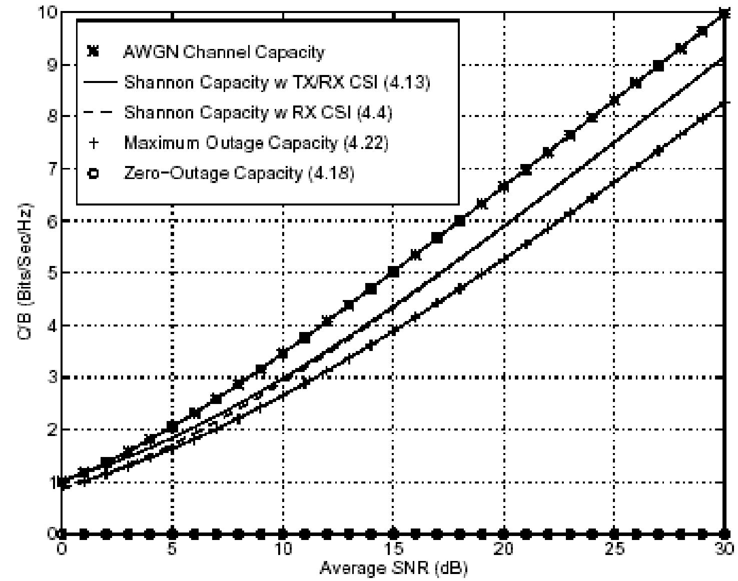
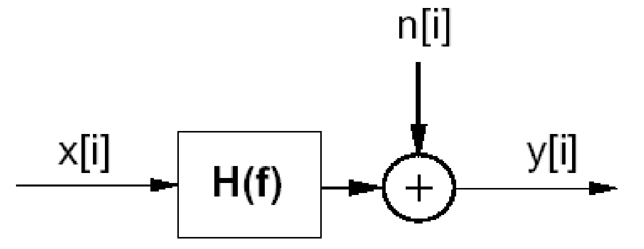


Figure 4.7: Capacity in Rayleigh Fading.

Capacity of frequency-selective fading channel

Time-invariant Channel (1)

- Analysis is like that of flat fading channel with frequency axis
- Total power constraint: P
- A time-invariant channel with frequency response $H(f)$ that is known to both transmitter and receiver
- Block fading
  - Frequency is divided into subchannels of bandwidth B with constant frequency response H_j over each subchannel
  - P_j: Tx power on the jth subchannel
  - A set of AWGN channels in parallel with SNR $(|H_j|^2 P_j / N_0 B)$ on the jth channel
  - Power constraint: $\sum_j P_j \leq P$



Time-Invariant Frequency-Selective Fading Channel.

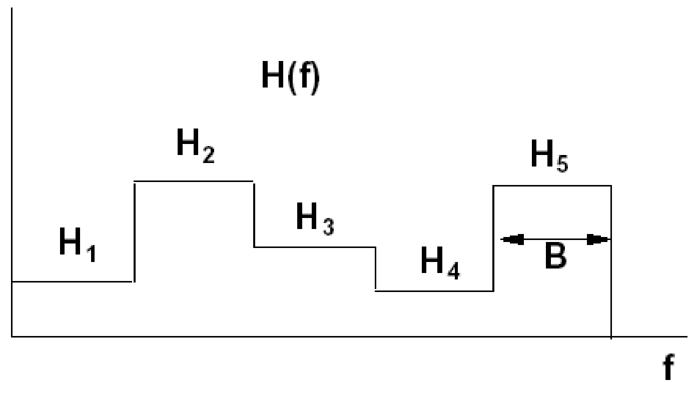
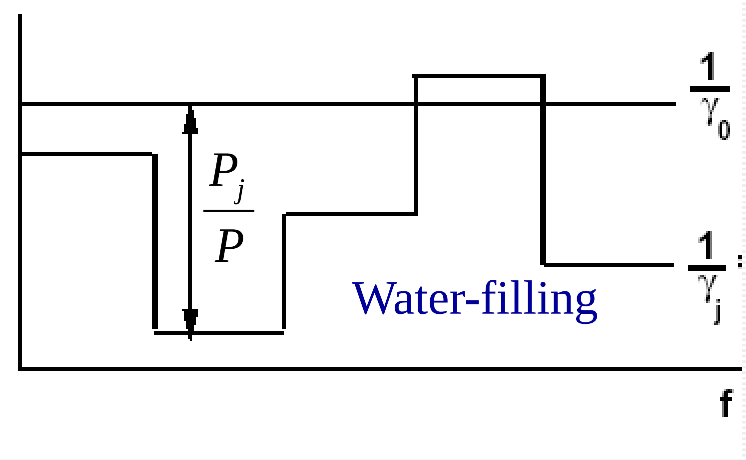


Figure 4.10: Block Frequency-Selective Fading

Time-invariant Channel (2)

- Capacity under block fading
  - $$C = \max_{P_j : \sum_j P_j \le P} \sum B \log_2\left(1 + \frac{|H_j|^2 P_j}{N_0 B}\right)$$
  - $$\frac{P_j}{P} = \begin{cases} 1/\gamma_0 - 1/\gamma_j & \gamma_j \ge \gamma_0 \\ 0 & \gamma_j < \gamma_0 \end{cases}$$
  - $$\gamma_j = \frac{|H_j|^2 P}{N_0 B}$$
  - $$\gamma_0 \text{ such that } \sum_j \left({1}/{\gamma_0} - {1}/{\gamma_j} \right) = 1$$
  - $$\boxed{C = \sum_{j:\gamma_j \ge \gamma_0} B \log_2\left(\frac{\gamma_j}{\gamma_0}\right)}$$

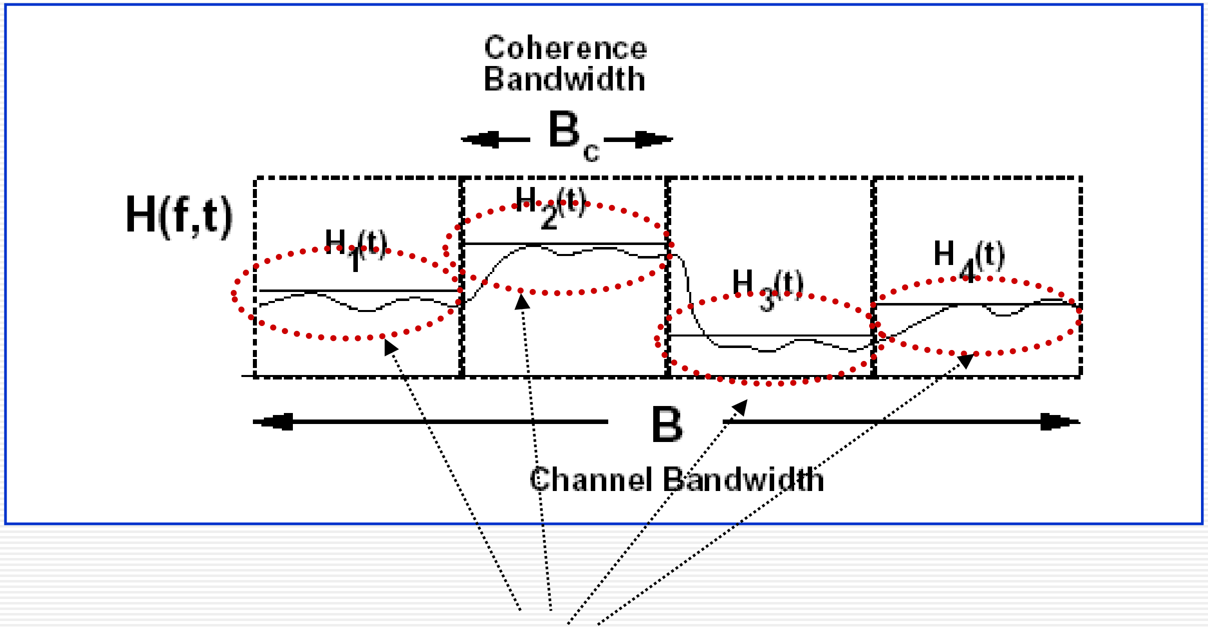


Time-invariant Channel (3)

- Continuous *H*(*f*)
  - $$C = \max_{P(f):\int P(f)df \le P} \int \log_2\left(1 + \frac{|H(f)|^2 P(f)}{N_0}\right) df$$
  - $$\frac{P(f)}{P} = \begin{cases} 1/\gamma_0 - 1/\gamma(f) & \gamma(f) \ge \gamma_0 \\ 0 & \gamma(f) < \gamma_0 \end{cases}$$
  - $$\gamma(f) = \frac{|H(f)|^2 P}{N_0}$$
  - γ_0 such that $$\int_{f:\gamma(f)\ge\gamma_0} \left(\frac{1}{\gamma_0} - \frac{1}{\gamma(f)} \right) df = 1$$
  - $$\boxed{C = \int_{f:\gamma(f)\ge\gamma_0} \log_2\left(\frac{\gamma(f)}{\gamma_0}\right) df}$$

Time-varying Channel (1)

Channel division



Time-varying flat fading channel

Time-varying Channel (2)

- $$C = \max_{P_j(\gamma_j):\sum_j \int P_j(\gamma_j)p(\gamma_j)d\gamma_j \le \Phi} \sum_j \int_0^\infty B_c \log_2\left(1 + \frac{P_j(\gamma_j)\gamma_j}{\Phi}\right) p(\gamma_j)\, d\gamma_j$$
- $$\frac{P_j(\gamma_j)}{\Phi} = \begin{cases} 1/\gamma_0 - 1/\gamma_j & \gamma_j \ge \gamma_0 \\ 0 & \gamma_j < \gamma_0 \end{cases}$$
- $\gamma_j = \dfrac{|H_j|^2 \Phi}{N_0 B_c}$: the instantaneous SNR on the jth subchannel assuming the total power Φ is allocated to the frequency
- γ_0 such that $\sum_j \int_{\gamma_0}^\infty \left(1/\gamma_0 - 1/\gamma_j \right) p(\gamma_j) d\gamma_j = 1$
- $$\boxed{C = \sum_j \int_{\gamma_0}^\infty B_c \log_2\left(\frac{\gamma_j}{\gamma_0}\right) p(\gamma_j) d\gamma_j}$$