

System Control

2. The Laplace Transform

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Laplace Transformation

· Definition: $\mathcal{L} [f(t)] = \int_0^{\infty} f(t)e^{-st} dt = F(s)$

$\mathcal{L}: f(t) \Rightarrow F(s)$, $s = \sigma + j\omega$ (complex variable)

$f(t)$: a time function such that $f(t)=0$ for $t < 0$

· Inverse Laplace Transformation

$$\mathcal{L}^{-1}[F(s)] = f(t)$$

Existence of Laplace Transformation

- **f(t) Laplace – transformable**

if i) **f(t) piecewise–continuous**

ii) **f(t) of exponential order as t approaches infinity**

– $e^{\alpha t} |f(t)|$ **bounded, α exist.**

– **or $e^{-\sigma t} |f(t)|$ approaches zero as t approaches infinity.**

• If $\lim_{t \rightarrow \infty} e^{-\sigma t} |f(t)| = \begin{cases} 0 & \text{for } \sigma > \sigma_c \\ \infty & \text{for } \sigma < \sigma_c \end{cases}$

the σ_c : **the abscissa of convergence**

Existence of Laplace Transformation

example : 1) $t, \sin \omega t, t \sin \omega t \dots$

$$\lim_{t \rightarrow \infty} e^{-\sigma t} |t \sin \omega t| = \begin{cases} 0 & \text{if } \sigma > 0 \\ \infty & \text{if } \sigma < 0 \end{cases} \quad \text{the abscissa of convergence } \sigma_c = 0$$

2) $e^{-ct}, te^{-ct}, e^{-ct} \sin \omega t, \quad c = \text{const.}$

$$\lim_{t \rightarrow \infty} e^{-\sigma t} |te^{-ct}| = \begin{cases} 0 & \text{if } \sigma > -c \\ \infty & \text{if } \sigma < -c \end{cases} \quad \text{the abscissa of convergence } \sigma_c = -c$$

• e^{t^2}, te^{t^2} does not possess L. T.

• $f(t) = \begin{cases} e^{t^2} & \text{for } 0 \leq t \leq T < \infty \\ 0 & \text{for } t < 0, T < t \end{cases} \quad \text{L}[f(t)] \text{ exists.}$

– The signals that can be physically generated always have corresponding Laplace transforms

Laplace Transformation of simple function

- Exponential function

$$f(t) = \begin{cases} 0 & \text{for } t < 0 \\ Ae^{-\alpha t} & \text{for } t \geq 0 \end{cases}$$

A, α : constants

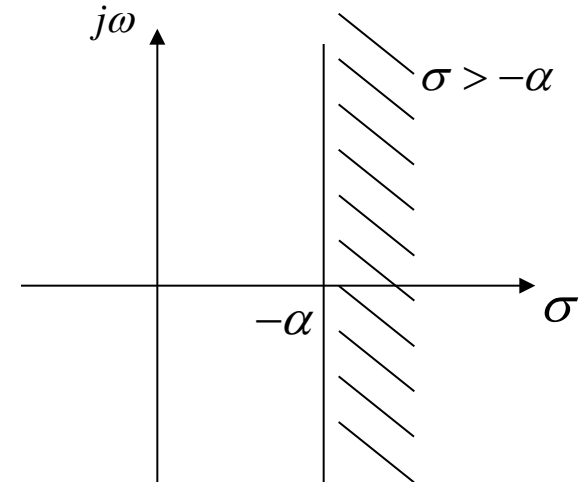
$$\mathcal{L}[Ae^{-\alpha t}] = \int_0^{\infty} Ae^{-\alpha t} \cdot e^{-st} dt$$

$$= \int_0^{\infty} Ae^{-(\alpha+s)t} dt$$

$$= A \left[-\frac{1}{s+\alpha} e^{-(s+\alpha)\cdot\infty} + \frac{1}{s+\alpha} e^0 \right]$$

$$= A \left[-\frac{1}{s+\alpha} e^{-(\sigma+\alpha)\cdot\infty - j\omega\cdot\infty} + \frac{1}{s+\alpha} \right]$$

$$= A \left[0 + \frac{1}{s+\alpha} \right] = \frac{A}{s+\alpha}$$



$(s = \sigma + j\omega)$

Laplace Transformation of simple function

- **step function**

$$f(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\begin{aligned} \mathcal{L}[f(t)] &= \int_0^{\infty} A e^{-st} dt = A \left[-\frac{1}{s} e^{-s \cdot \infty} + \frac{1}{s} e^{-s \cdot 0} \right] \\ &= A \left[0 + \frac{1}{s} \right] \quad \text{if } \operatorname{Re}[s] > 0 \end{aligned}$$

- **unit step input function** $1(t-t_0) = \begin{cases} 1 & t \geq t_0 \\ 0 & t < t_0 \end{cases}$

$$1(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\mathcal{L}[1(t)] = \frac{1}{s}$$

Laplace Transformation of simple function

- Sinusoidal Functions

$$f(t) = \begin{cases} A \sin \omega t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\cos \omega t = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t})$$

$$\sin \omega t = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$$

$$e^{j\omega t} = \cos \omega t + j \sin \omega t$$

$$e^{-j\omega t} = \cos \omega t - j \sin \omega t$$

$$\mathcal{L}[A \sin \omega t] = \int_0^{\infty} \frac{A}{2j} (e^{j\omega t} - e^{-j\omega t}) e^{-st} dt$$

$$= \frac{1}{2j} \left(\frac{1}{s - j\omega} - \frac{1}{s + j\omega} \right)$$

$$= \frac{A\omega}{s^2 + \omega^2}$$

$$\mathcal{L}[A \cos \omega t] = \frac{As}{s^2 + \omega^2}$$

- Ramp function

$$f(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\mathcal{L}[At] = A \int_0^{\infty} t e^{-st} dt$$

$$= A \left(t \frac{e^{-st}}{-s} \Big|_0^{\infty} - \int_0^{\infty} \frac{e^{-st}}{-s} dt \right)$$

$$= A \cdot \frac{1}{s} \int_0^{\infty} e^{-st} dt = A \cdot \frac{1}{s^2}$$

Laplace Transformation of simple function

- Pulse function

$$f(t) = \begin{cases} \frac{A}{t_0} & \text{for } 0 \leq t \leq t_0 \\ 0 & \text{for } t < 0, t_0 < t \end{cases}$$

$$f(t) = \frac{A}{t_0} 1(t) - \frac{A}{t_0} 1(t - t_0)$$

$$\begin{aligned} \mathcal{L}[f(t)] &= \mathcal{L}\left[\frac{A}{t_0} 1(t)\right] - \mathcal{L}\left[\frac{A}{t_0} 1(t - t_0)\right] \\ &= \frac{A}{t_0} \frac{1}{s} (1 - e^{-t_0 s}) \end{aligned}$$

- Impulse function

$$f(t) = \begin{cases} \lim_{t_0 \rightarrow \infty} \frac{A}{t_0} & \text{for } 0 \leq t < t_0 \\ 0 & \text{for } t < 0, t_0 < t \end{cases}$$

$$\mathcal{L}[f(t)] = \lim_{t_0 \rightarrow 0} \left[\frac{A}{t_0} \frac{1}{s} (1 - e^{-t_0 s}) \right]$$

$$= \lim_{t_0 \rightarrow 0} \frac{\frac{d}{dt_0} [A(1 - e^{-t_0 s})]}{\frac{d}{dt_0} (t_0 s)} = \frac{A \cdot s}{s} = A$$

Laplace Transformation of simple function

- Unit impulse function ; impulse function of magnitude 1

$$f(t) = \begin{cases} \lim_{t_0 \rightarrow \infty} \frac{1}{t_0} & \text{for } 0 \leq t < t_0 \\ 0 & \text{for } t < 0, t_0 < t \end{cases}$$

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases} \quad \mathcal{L}[\delta(t)] = \int_0^{\infty} \delta(t) e^{-st} dt = \int_{0^-}^{0^+} \delta(t) dt = 1$$

The unit-impulse function occurring at $t = t_0$

$$\delta(t) = \begin{cases} \infty & t = t_0 \\ 0 & t \neq t_0 \end{cases} \quad \mathcal{L}[\delta(t - t_0)] = \int_0^{\infty} \delta(t - t_0) e^{-st} dt = \int_{t_0^-}^{t_0^+} \delta(t - t_0) e^{-st_0} dt = e^{-t_0 s} \cdot 1$$

$$\delta(t - t_0) = \frac{d}{dt} 1(t - t_0)$$

Useful Theorems

Theorem 1. Linearity

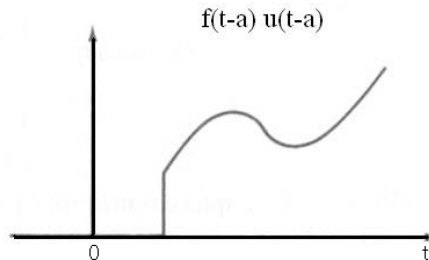
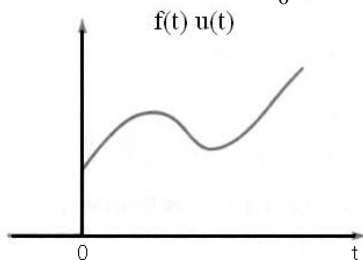
$$\mathcal{L}[af(t)] = aF(s)$$

Theorem 2. Superposition

$$\mathcal{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$$

Theorem 3. Translation in time.

$$\begin{aligned}\mathcal{L}[f(t-a)u(t-a)] &= \int_0^{\infty} f(t-a)u(t-a)e^{-st} dt \quad (a > 0) \\ &= \int_{-a}^{\infty} f(\tau)u(\tau)e^{-s(\tau+a)} d\tau \quad (\text{let } t-a = \tau) \\ &= \int_0^{\infty} f(\tau)u(\tau)e^{-s\tau} e^{-sa} d\tau = e^{-sa} F(s) \quad (\because f(\tau)u(\tau) = 0 \text{ for } \tau < 0)\end{aligned}$$



Useful Theorems

Theorem 4. Complex differentiations

$$\mathcal{L}[tf(t)] = -\frac{d}{ds}F(s)$$

$$\mathcal{L}[1] = \frac{1}{s} \quad \mathcal{L}[t \cdot 1] = -\frac{d}{ds}F(s) = \frac{1}{s^2}$$

$$\mathcal{L}[t^2] = \mathcal{L}[t \cdot t] = -\frac{d}{ds}\left(\frac{1}{s^2}\right) = \frac{2}{s^3} \quad \mathcal{L}[tf(t)] = -\frac{d}{ds}F(s)$$

proof) let. $F(s) = \int_0^{\infty} f(t)e^{-st} dt$

$$\frac{d}{ds}F(s) = \int_0^{\infty} \frac{\partial}{\partial s} [f(t)e^{-st}] dt = -\int_0^{\infty} tf(t)e^{-st} dt = -\mathcal{L}[tf(t)]$$

similarly, $\mathcal{L}[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2} F(s) \quad \mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$

Theorem 5. Translation in the s-domain

$$\mathcal{L}[e^{at} f(t)] = F(s-a) \quad \mathcal{L}[e^{at} \cos \omega t] = \frac{(s-a)}{(s-a)^2 + \omega^2}$$

Useful Theorems

Theorem 6. Real Differentiation

$$Df(t) = \frac{d}{dt} f(t)$$

$$\begin{aligned} \text{proof) let. } \mathcal{L}\left[\frac{d}{dt} f(t)\right] &= \int_0^{\infty} \frac{d}{dt} f(t) e^{-st} dt \\ &= f(t) e^{-st} \Big|_0^{\infty} - \int_0^{\infty} f(t) e^{-st} dt (-s) \\ &= -f(0) + s \cdot F(s) \\ &= s \cdot F(s) - f(0) \end{aligned}$$

$$\begin{aligned} \text{similarly, } \mathcal{L}[D^2 f(t)] &= \mathcal{L}[D \cdot Df(t)] = \mathcal{L}[Df'(t)] \\ &= s \{s \cdot F(s) - f(0)\} = s^2 \cdot F(s) - s \cdot f(0) - f'(0) \end{aligned}$$

$$\begin{aligned} \mathcal{L}\left[\frac{d^n}{dt^n} f(t)\right] &= s^n \cdot F(s) - s^{n-1} \cdot f(0) - s^{n-2} \cdot f'(0) \\ &\quad \dots - f^{(n-1)}(0) \end{aligned}$$

Useful Theorems

Theorem 7. Real Integration

$$\int_0^t f(t)dt = D^{-1}f(t) - D^{-1}f(0)$$

$$\begin{aligned}\mathcal{L}\left[\int_0^t f(t)dt\right] &= \int_0^\infty \int_0^t f(\tau)d\tau \cdot e^{-st} dt \\ &= -\frac{1}{s}e^{-st} \int_0^t f(\tau)d\tau \Big|_0^\infty - \int_0^\infty -\frac{1}{s}e^{-st} f(t)dt \\ &= \frac{1}{s} \int_0^t f(\tau)d\tau \Big|_{t=0} + \frac{1}{s} \int_0^\infty e^{-st} f(t)dt = \frac{F(s)}{s} + \frac{D^{-1}f(0)}{s}\end{aligned}$$

Theorem 8. Final value Theorem

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$$

Theorem 9. Initial value Theorem

$$\lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} sF(s)$$

Theorem 10. Complex Integration

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)ds$$

Inverse Laplace Transformation

$$\mathcal{L}: f(t) \rightarrow F(s)$$

$$\mathcal{L}^{-1}: F(s) \rightarrow f(t)$$

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j} \int_{c-j\omega}^{c+j\omega} F(s)e^{st} ds \quad (\text{c : real constant})$$

* Inverse Laplace Transformation by Partial Fraction Method

$$F(s) = \frac{P(s)}{Q(s)} = \frac{a_m s^m + a_{m-1} s^{m-1} + \cdots + a_0}{s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0} \quad (n \geq m)$$

$s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0 \Rightarrow$ real complex conjugate, a, b : real num.

$$= (s - c_1)(s - c_2) \cdots (s^2 + d_1 s + d_2)$$

$$\Rightarrow F(s) = \frac{\alpha_1}{s - c_1} + \frac{\alpha_2}{s - c_2} + \cdots + \frac{\beta_1 s + \beta_2}{s^2 + d_1 s + d_2} + \cdots$$

Examples of Inverse Laplace Transformation

$$\text{ex)} \quad F(s) = \frac{1}{(s+2)^2(s+3)} = \frac{a}{s+2} + \frac{b}{(s+2)^2} + \frac{c}{s+3}$$

$$a(s+2)(s+3) + b(s+3) + c(s+2)^2 = 1$$

let $s = -2$, then $b = 1$, let $s = -3$, then $c = 1$

$$a(s+2) + b + c \cdot \frac{s+2}{s+3} = \frac{1}{s+3} \quad \xrightarrow{\frac{d}{ds}} \quad a + c \cdot \frac{(s+3) - (s+2)}{s+3} = \frac{-1}{(s+3)^2}$$

$$\therefore a = -1, b = 1, c = 1, \quad F(s) = \frac{-1}{s+2} + \frac{1}{(s+2)^2} + \frac{1}{s+3}$$

=> Inverse Laplace Transformation

$$f(t) = -e^{-2t} + te^{-2t} + e^{-3t} \quad \text{(partial fraction method)}$$

$$\text{ex)} \quad F(s) = \frac{10}{s^2 + 6s + 25} = \frac{10 \times \frac{1}{4} \times 4}{(s+3)^2 + 4^2} = \frac{10}{4} \frac{4}{(s+3)^2 + 4^2}$$

$$\therefore f(t) = \frac{10}{4} \sin 4te^{-3t}$$

Solution of Differential Equation by Laplace Transformation

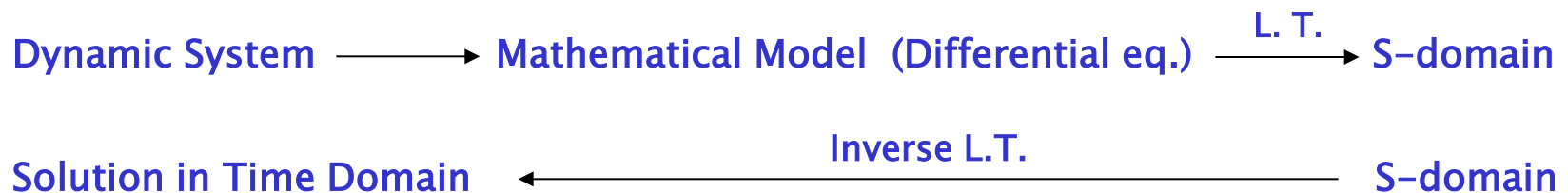
$$y'' + 2y' + 4y = 1 \quad y(0) = 0, \quad y'(0) = 2$$

$$\text{L.T: } s^2Y(s) - sy(0) - y'(0) + 2\{sY(s) - y(0)\} + 4Y(s) = \frac{1}{s}$$

$$(s^2 + 2s + 4)Y(s) = \frac{1}{s} + 2 = \frac{2s + 1}{s}$$

$$Y(s) = \frac{2s + 1}{s(s^2 + 2s + 4)} = \frac{1}{4s} - \frac{1}{4} \frac{s + 1 - 1}{(s + 1)^2 + (\sqrt{3})^2}$$

$$\therefore y(t) = \frac{1}{4} - \frac{1}{4} \cos \sqrt{3}te^{-t} + \frac{1}{4\sqrt{3}} \sin \sqrt{3}te^{-t}$$



End of lecture note 2