System Control

6. Root Locus Analysis

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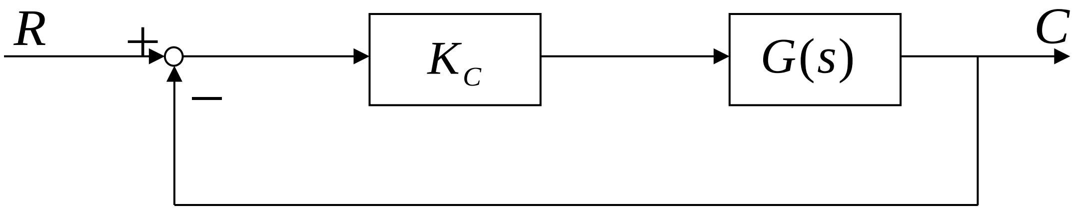
Characteristic Equation



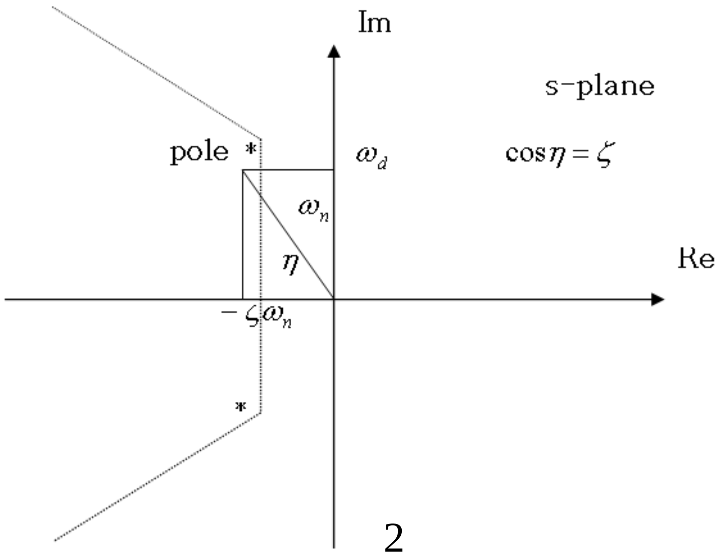
$$G(s) = \frac{B(s)}{A(s)}$$

$$\frac{C}{R} = \frac{K_C G(s)}{1 + K_C G(s)} = \frac{K_C B(s)}{A(s) + K_C B(s)}$$

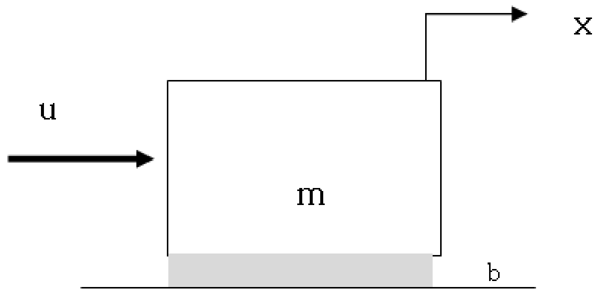
Characteristic Equation : $A(s) + K_C B(s) = 0$

⇨ Characteristic roots, poles

⇨ Transient response



Example: Position control



$$G(s) = \frac{1}{s(ms+b)}$$

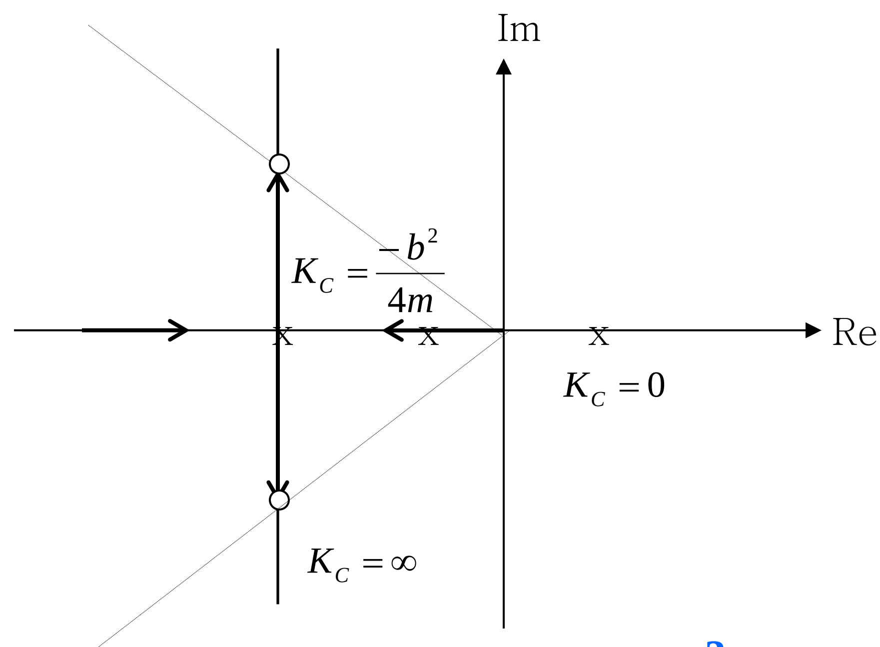
Characteristic Equation : $ms^2 + bs + K_C = 0$

$$s = \frac{-b \pm \sqrt{b^2 - 4mK_C}}{2m}$$

$K_C = 0$ $\qquad s = \frac{-b}{m}, 0$

$K_C = -\frac{b^2}{4m}$ $\qquad s = \frac{-b}{2m}$ double roots

$K_C \rangle -\frac{b^2}{4m}$ $\qquad s = \frac{-b}{2m} \pm \frac{\sqrt{4mK_C - b^2}}{2m} j$



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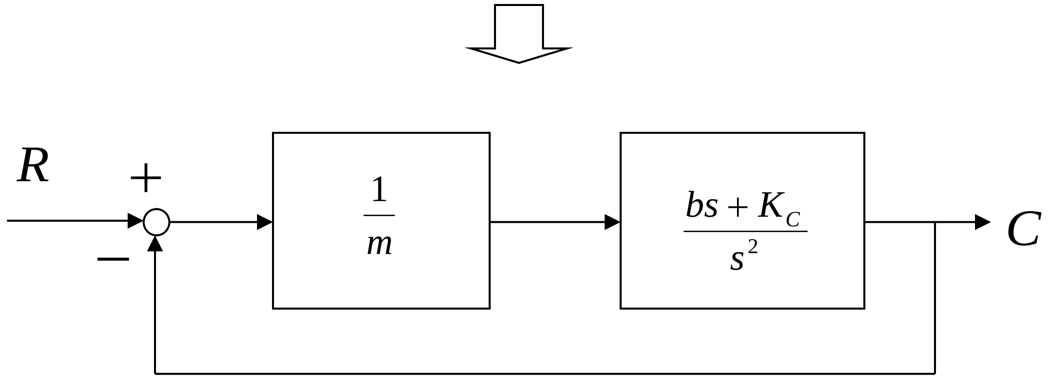
Characteristic roots

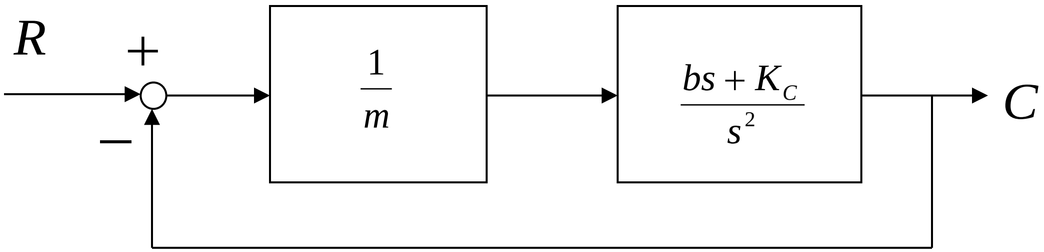
Example: Effect of Mass Variation

$$ms^2 + bs + K_C = 0$$

$$1 + \frac{1}{m}\frac{bs + K_C}{s^2} = 0$$



Characteristic roots



$$\frac{1}{m} \approx 0 \qquad 1 + \frac{1}{m}\frac{B(s)}{A(s)} = 0$$

$$A(s) + \frac{1}{m}B(s) = 0$$

$$A(s) = 0$$

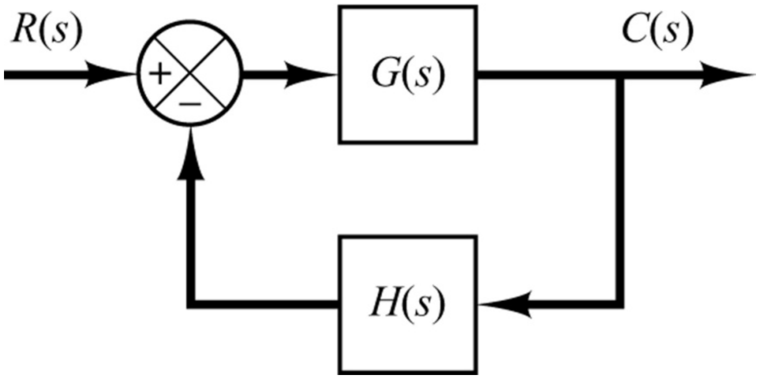
$s = 0$; double roots

$$\frac{1}{m} \approx \infty \qquad A(s) + \frac{1}{m}B(s) = 0$$

$$B(s) = 0$$

$$s = -\frac{K_b}{b}$$

Root-Locus Plots



Closed-loop transfer function

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$G(s)H(s) = K_C \frac{B(s)}{A(s)} = K_C \frac{\prod_{j=1}^{m}(s - z_i)}{\prod_{i=1}^{n}(s - p_i)} = 0 \qquad n \geq m$$

Characteristic equation

$$1 + K_C \frac{B(s)}{A(s)} = 0$$

$$A(s) + K_C B(s) = 0$$

$$\prod_{i=1}^{n}(s - p_i) + K_C \prod_{j=1}^{m}(s - z_i) = 0$$

Characteristic Roots

Characteristic equation

$$\prod_{i=1}^{n}(s-p_i)+K_C\prod_{j=1}^{m}(s-z_i)=0$$

Openloop poles   openloop zeros

Closed-loop poles

$K_C=0$ $\prod_{i=1}^{n}(s-p_i)=0$ closed loop poles=open loop poles

$K_C \rangle\rangle 1$ $\prod_{j=1}^{m}(s-z_i)=0$ closed loop poles=open loop zeros

Characteristic Roots

Characteristic equation

$$\prod_{i=1}^{n}(s-p_i)+K_C\prod_{j=1}^{m}(s-z_i)=0$$

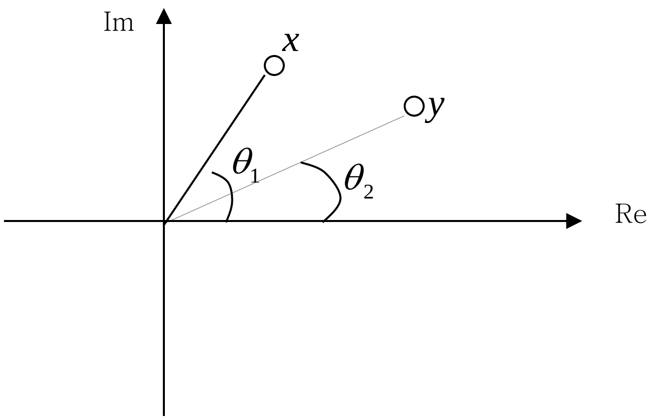
Openloop poles　　openloop zeros

$$\frac{B(s)}{A(s)}=-\frac{1}{K_C}=\frac{\prod_{j=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)}$$

real, negative

s：complex number

Characteristic roots



$$x = r_1 e^{j\theta_1} = |x| e^{j\angle x}$$

$$y = r_2 e^{j\theta_2} = |y| e^{j\angle y}$$

$$\frac{x}{y} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}$$

$$x \cdot y = r_1 \cdot r_2 e^{j(\theta_1 \cdot \theta_2)}$$

$$|x \cdot y| = |x| \cdot |y|$$

$$\angle(x \cdot y) = \angle x \cdot \angle y$$

$$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$$

$$\angle \frac{x}{y} = \angle x - \angle y$$

$$\angle \frac{\prod_{j=1}^{m}(s - z_i)}{\prod_{i=1}^{n}(s - p_i)} = \sum_{j=1}^{m} \angle(s - z_i) - \sum_{i=1}^{n} \angle(s - p_i) = \pm 180 \pm 360^\circ n$$

Characteristic roots

$K_C \rangle\rangle 1$

$$A(s)+K_C B(s)=0$$

$$\left(\frac{A(s)}{B(s)}+K_c\right)B(s)=0$$

$$B(s)=0 \Rightarrow m-roots$$

$$\frac{A(s)}{B(s)}+K_c=0 \Rightarrow n-m \quad roots$$

$$\frac{\prod_{j=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)}+K_c=0$$

Characteristic roots

$$B(s) = 0 \Rightarrow m - roots$$

$$\frac{A(s)}{B(s)} + K_c = 0 \Rightarrow n - m \quad roots$$

n-m roots

$$\frac{\prod_{j=1}^{m}(s - z_i)}{\prod_{i=1}^{n}(s - p_i)} + K_c = 0$$

$$\frac{\left(s^n - \sum_{i=1}^{n} p_i s^{n-1} + \cdots\right)}{\left(s^m - \sum_{j=1}^{m} z_i s^{m-1} + \cdots\right)} + K_c = 0$$

$$s^{n-m} - \left(\sum_{i=1}^{n} p_i - \sum_{j=1}^{m} z_i\right) s^{n-m-1} + \cdots + K_c = 0$$

$$\approx \left\{ s - \left(\frac{\sum_{i=1}^{n} p_i - \sum_{j=1}^{m} z_i}{n - m} \right) \right\}^{n-m} + K_c = 0$$

Characteristic roots

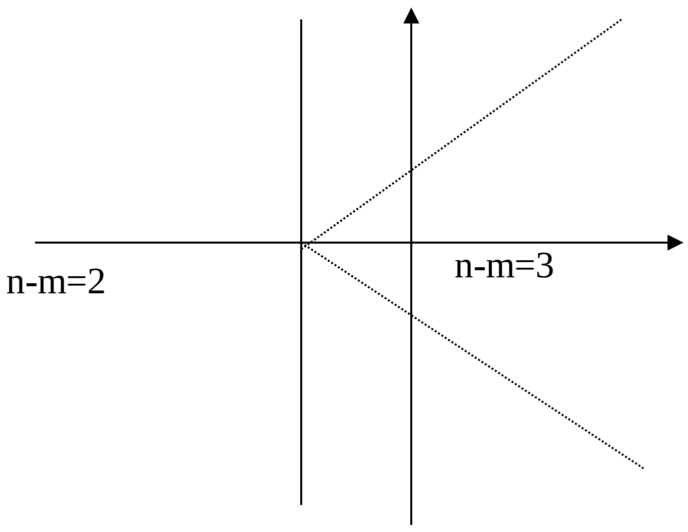
n-m roots

$$\left\{ s - \left(\frac{\sum_{i=1}^{n} p_i - \sum_{j=1}^{m} z_i}{n-m} \right) \right\}^{n-m} + K_c = 0$$

$$\Rightarrow s = (-1)^{\frac{1}{n-m}} \cdot K_c^{\frac{1}{n-m}} + \frac{\sum_{i=1}^{n} p_i - \sum_{j=1}^{m} z_i}{n-m}$$

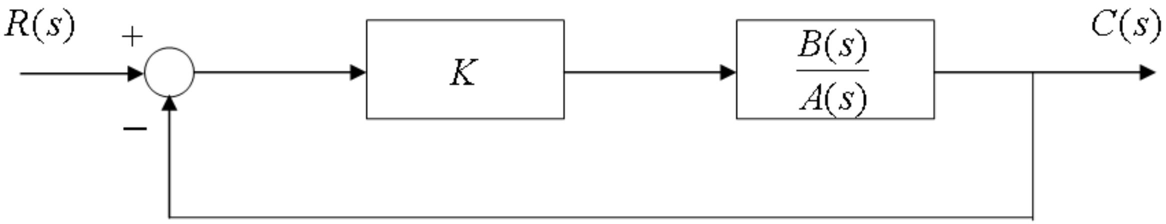
complex real

$$(-1)^{\frac{1}{n-m}} = 1 \cdot e^{j\left(\frac{1}{n-m}(\pi \pm 2n\pi)\right)}$$



Root – Locus Method

- Graphically obtains closed-loop poles as function of a parameter in feedback system.



$$\frac{R}{C} = \frac{K \cdot \frac{B(s)}{A(s)}}{1 + K \cdot \frac{B(s)}{A(s)}} = \frac{K \cdot B(s)}{A(s) + K \cdot B(s)}$$

closed-loop poles : (characteristic roots)

$$A(s) + K \cdot B(s) = 0$$

$A(s) = 0$; open-loop poles p_i

$B(s) = 0$; open-loop zeros z_i

$$A(s) = \prod_{i=1}^{n}(s - p_i) = 0$$

$$B(s) = \prod_{i=1}^{m}(s - z_i) = 0$$ Where, $n \geq m$

$$\prod_{i=1}^{n}(s - p_i) + K\prod_{i=1}^{m}(s - z_i) = 0$$

Root – Locus Method

$$\prod_{i=1}^{n}(s-p_i)+K\prod_{i=1}^{m}(s-z_i)=0$$

1) K=0

: closed-loop poles = open-loop poles

closed-loop characteristic eq. ;

$\Rightarrow$ closed-loop poles are asymptotic to open-loop poles for small K

2) K>>1

$$A(s)+K\cdot B(s)=0$$

$$\underbrace{\left(\frac{A(s)}{B(s)}+K\right)}_{\text{(n-m)-poles}}\cdot\underbrace{B(s)}_{\text{m-poles}}=0$$

(n-m)-poles m-poles

$\Rightarrow$ m closed-loop poles are asymptotic to open-loop zeros n-m poles

Root – Locus Method

n-m Poles

$$\frac{\prod_{i=1}^{n}(s-p_i)}{\prod_{i=1}^{m}(s-z_i)}+K=0 \qquad \frac{s^n-\left(\sum_{i=1}^{n}p_i\right)s^{n-1}+\cdots}{s^m-\left(\sum_{i=1}^{m}p_i\right)s^{m-1}+\cdots}+K=0$$

$$s^{n-m}-\left(\sum_{i=1}^{n}p_i-\sum_{i=1}^{m}z_i\right)s^{n-m-1}+\cdots+K=0$$

(approximation) $\cong \left[s-\frac{\left(\sum_{i=1}^{n}p_i-\sum_{i=1}^{m}z_i\right)}{n-m}\right]^{n-m}+K=0$

$$s=\frac{\left(\sum_{i=1}^{n}p_i-\sum_{i=1}^{m}z_i\right)}{n-m}+(-K)^{\frac{1}{n-m}}$$

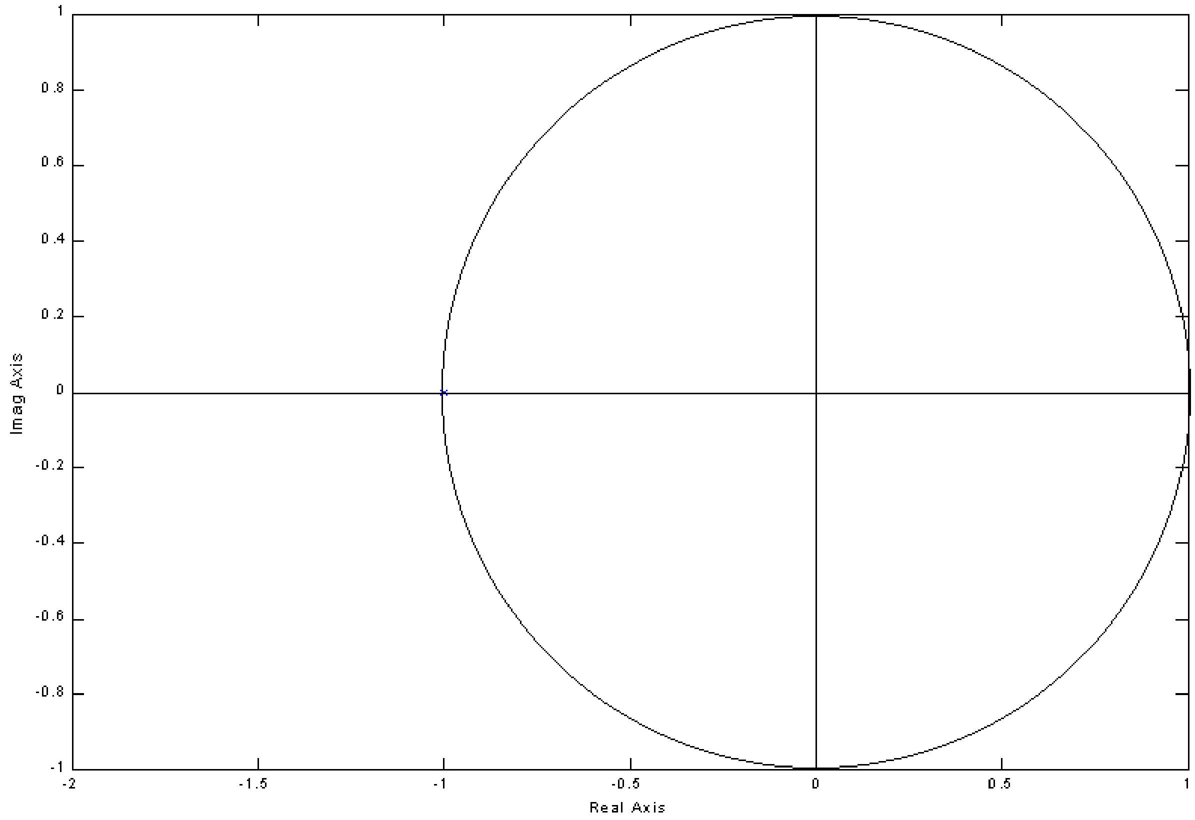
$$=\underbrace{\frac{\left(\sum_{i=1}^{n}p_i-\sum_{i=1}^{m}z_i\right)}{n-m}}_{\text{real}}+\underbrace{(-1)^{\frac{1}{n-m}}\cdot K^{\frac{1}{n-m}}}_{\text{image}}$$

$$-1=e^{\pm i(1+2N)\pi} \qquad N=1,2,3\cdots$$

$$(-1)^{\frac{1}{n-m}}=e^{\pm i(1+2N)\pi/n-m}$$

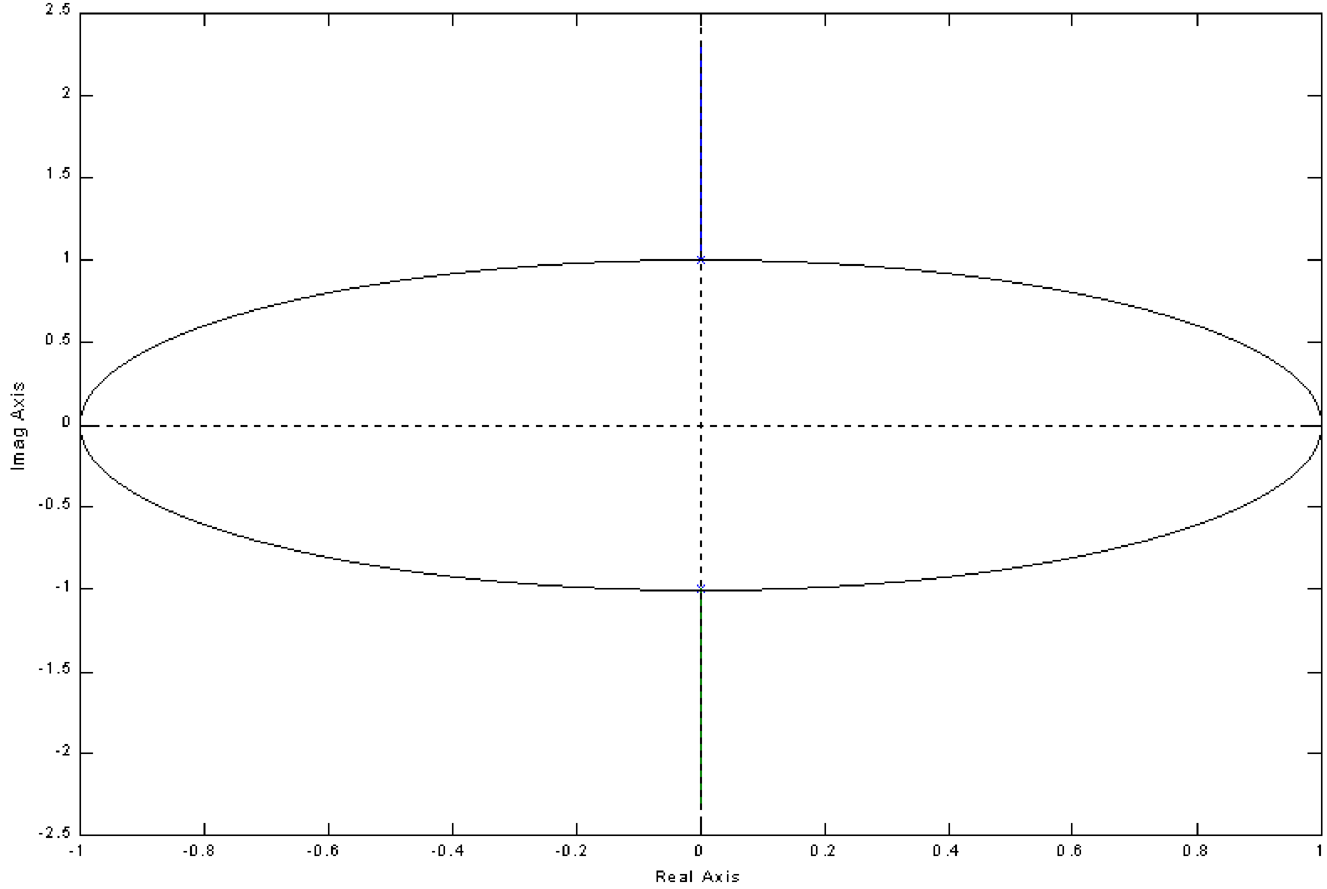
Root – Locus Method

$$n - m = 1 \qquad (-1)^{\frac{1}{1}} = -1$$



Root – Locus Method

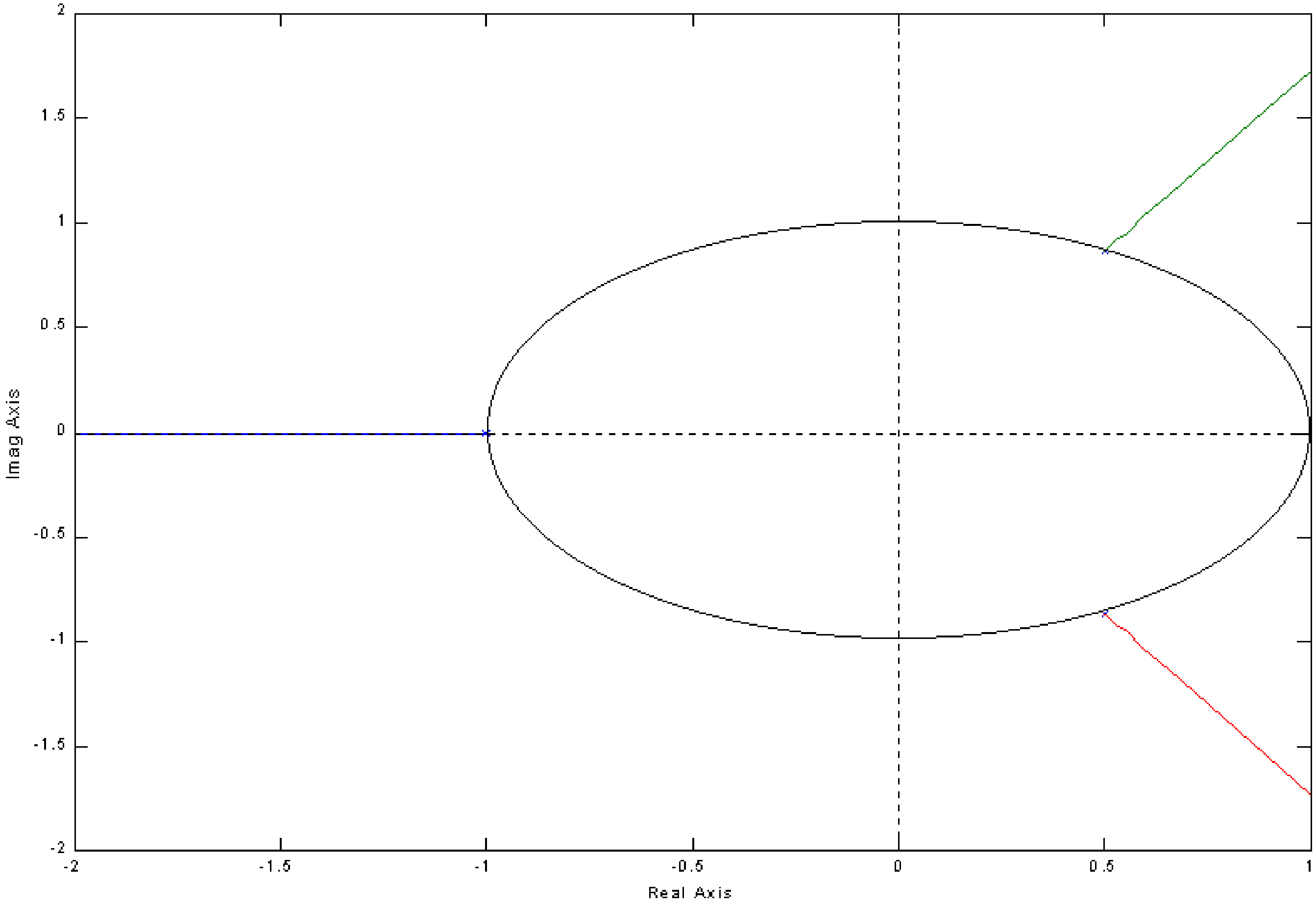
$$n - m = 2 \qquad (-1)^{\frac{1}{2}} = e^{\pm i(1+2N)\pi/2} = e^{\pm i\left(\frac{\pi}{2}+N\pi\right)} = e^{\pm\frac{\pi}{2}i}$$



Root – Locus Method

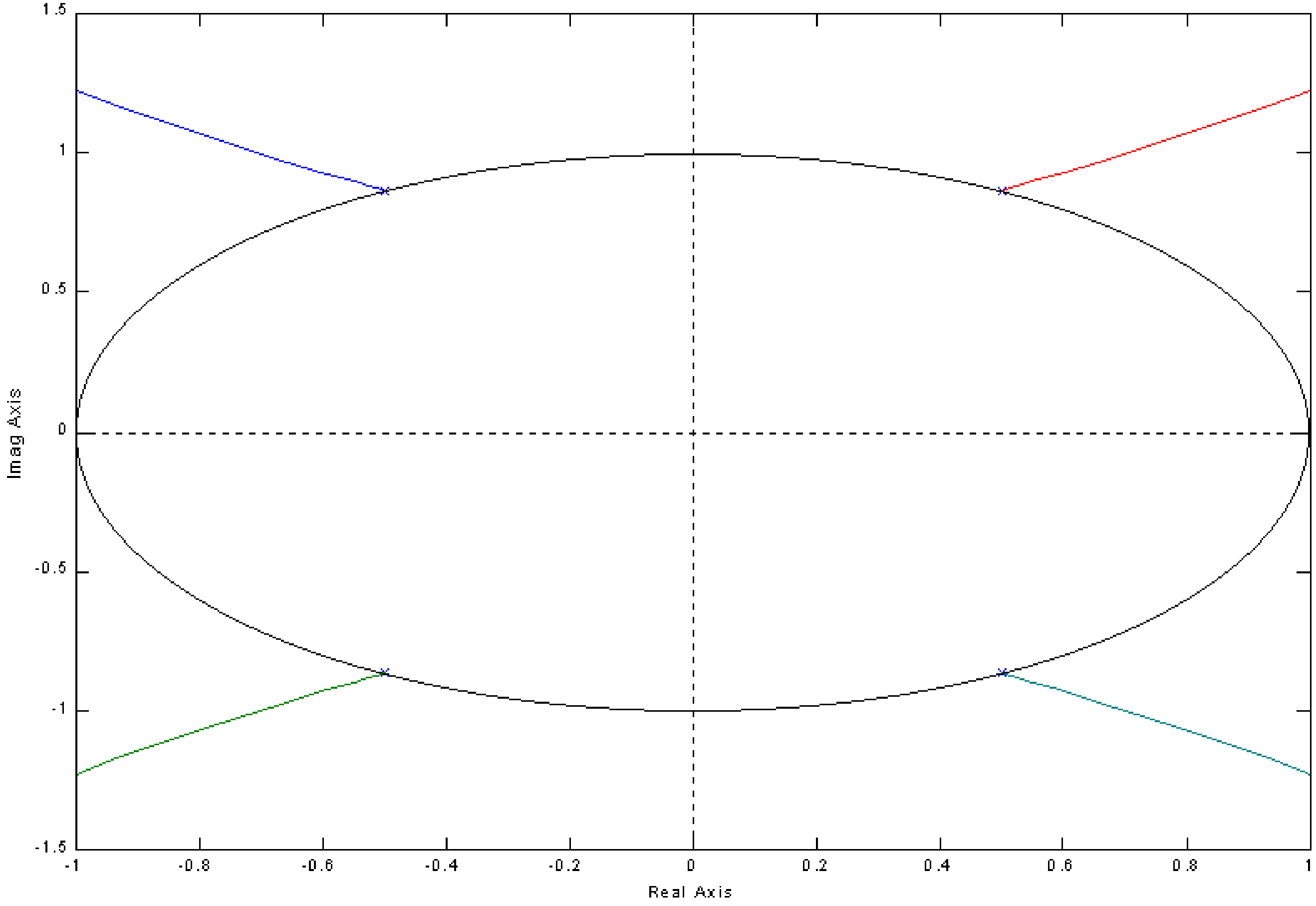
$$n - m = 3$$

$$(-1)^{\frac{1}{3}} = e^{\pm i(1+2N)\pi/3} = e^{\pm\frac{\pi}{3}i}, e^{i\pi}$$



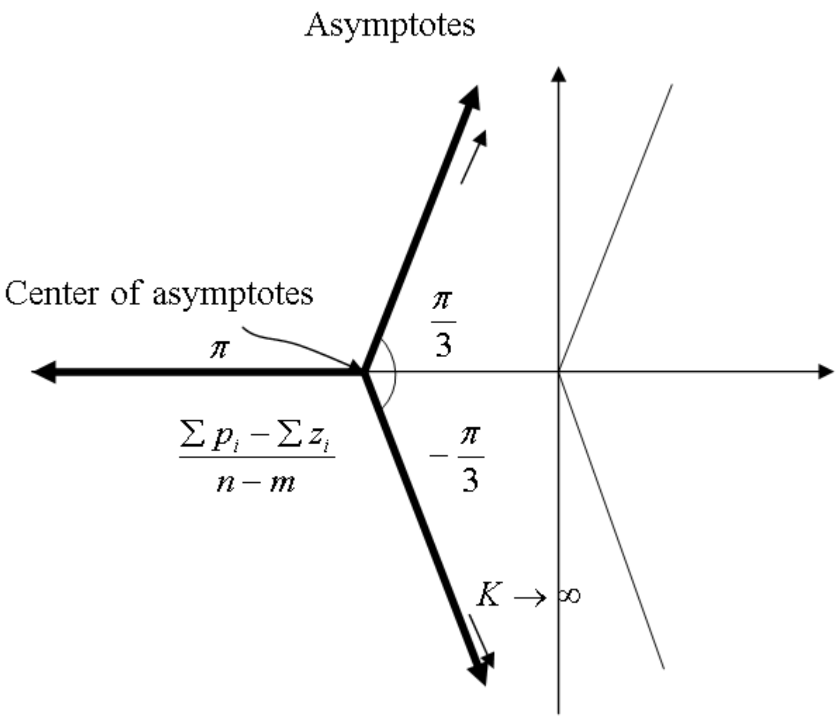
Root – Locus Method

$$n - m = 4 \qquad (-1)^{\frac{1}{4}} = e^{\pm\frac{\pi}{4}i}, e^{\pm\frac{3\pi}{4}i}$$



Root – Locus Method

Example) $n-m=3$ $n-m \ \ poles$



Angle condition

$$A(s)+K\cdot B(s)=0$$

$$\prod_{i=1}^{n}(s-p_i)+K\prod_{i=1}^{m}(s-z_i)=0$$

$$1+K\cdot\frac{B(s)}{A(s)}=0$$

$$\frac{B(s)}{A(s)}=-\frac{1}{K}$$

$$\frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)}=-\frac{1}{K}$$

real negative

$$\measuredangle\frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)}=\measuredangle\prod_{i=1}^{m}(s-z_i)-\measuredangle\prod_{i=1}^{n}(s-p_i)$$

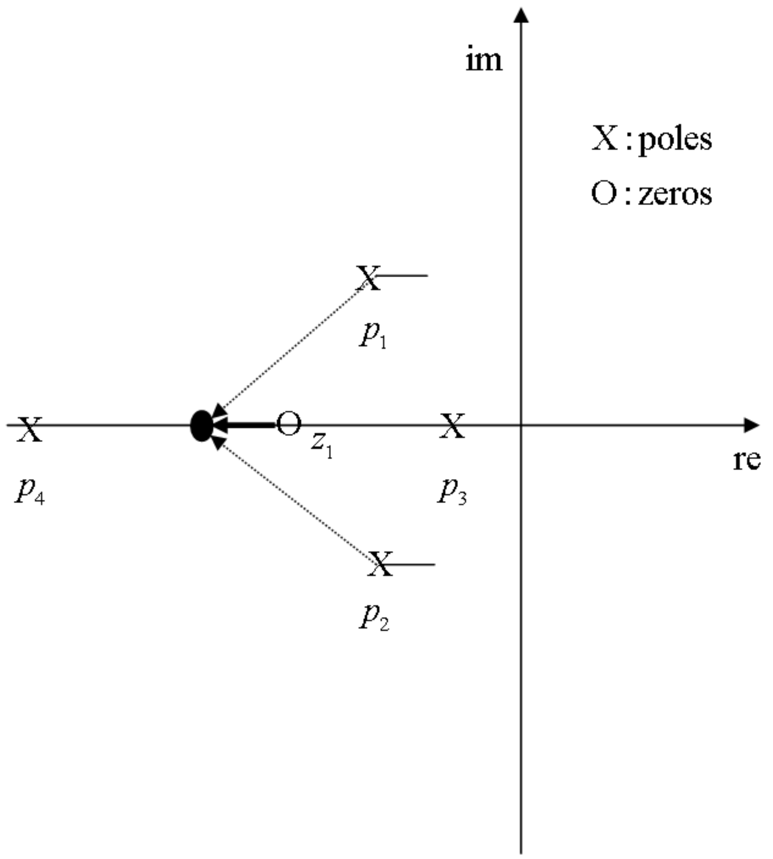
$$=\sum_{i=1}^{m}\measuredangle(s-z_i)-\sum_{i=1}^{n}\measuredangle(s-p_i)$$

$$=180\pm360\pm360\cdots$$

$$=\pm180(1+2N)\qquad N=1,2,3\cdots$$

3) Real axis: angle condition

$$\frac{B(s)}{A(s)} = -\frac{1}{K}, \qquad \measuredangle \frac{B(s)}{A(s)} = \sum_{i=1}^{n} \measuredangle (s - Z_i) - \sum_{i=1}^{n} \measuredangle (s - P_i) = \pm 100(1 + 2N)$$



Example)

4 open-loop poles, 1 open loop zero

$$\angle s - z_1 = 180^\circ$$

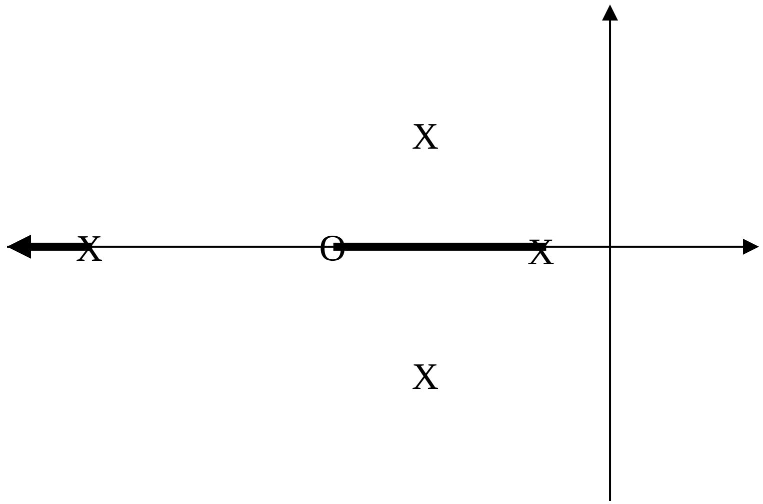
$$\angle s - p_1 + \angle s - p_2 = 360^\circ$$

$$\angle s - p_3 = 180^\circ$$

$$\angle s - p_4 = 0^\circ$$

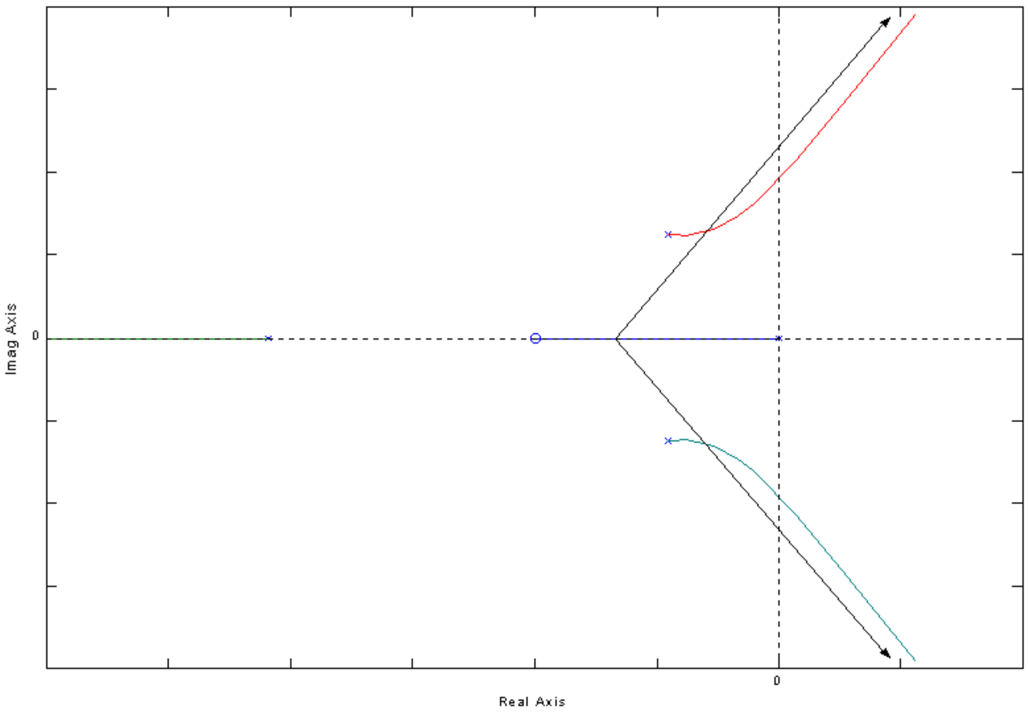
$$= \angle(s - z_i) - \sum_{i=1}^{4}(s - p_i) = -360^\circ$$

Root Roci in the real axis



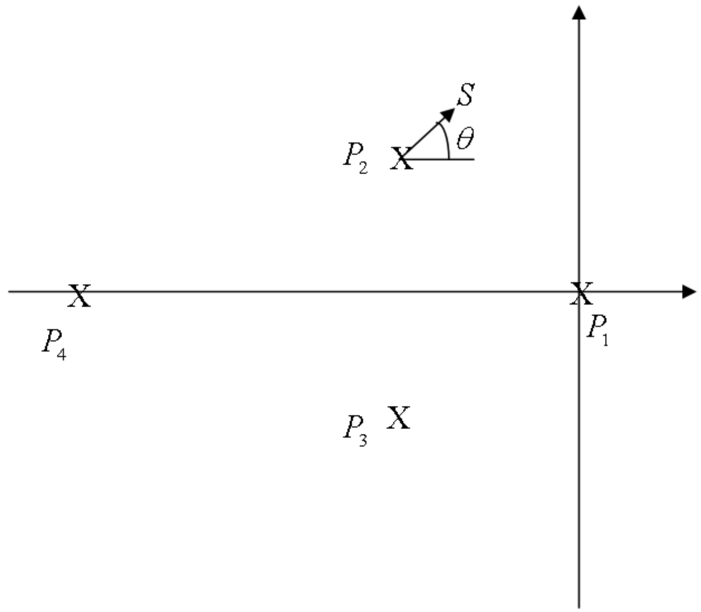
; Segments which have an odd number of open loop poles & zeros lying to the right on the real axis become portions of the root locus

Portions of the root locus



n-m=3

4) Angle of departure



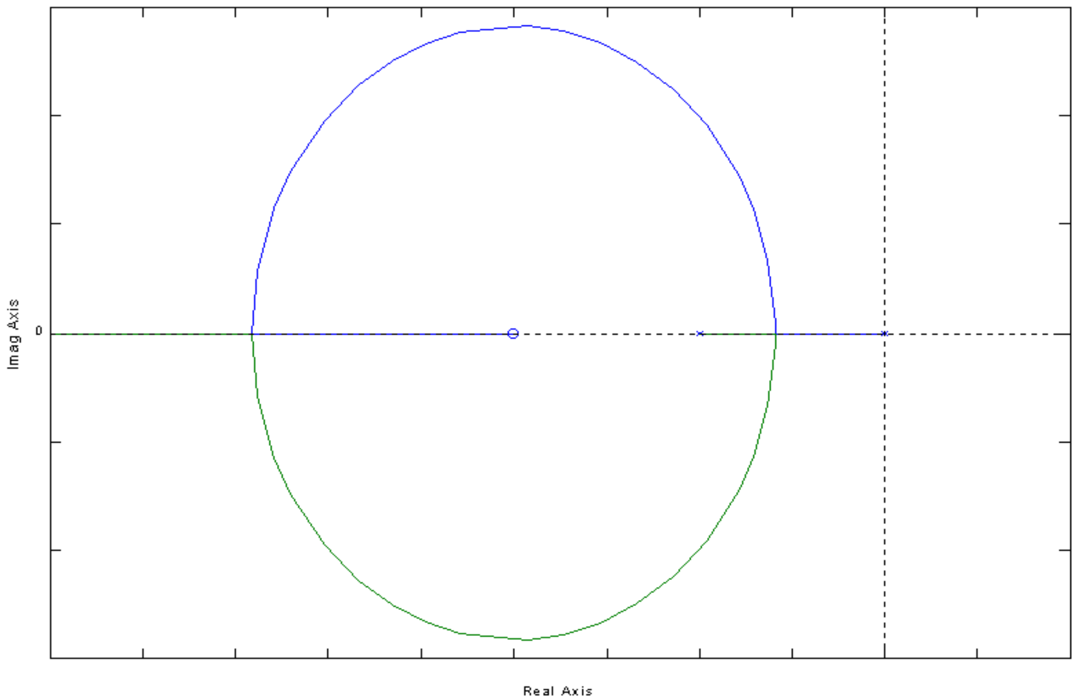
$$\measuredangle \frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)} = \sum_{i=1}^{m}\measuredangle(s-z_i) - \sum_{i=1}^{n}\measuredangle(s-p_i)$$

$$= \pm 180(1+2N) \qquad N = 1,2,3\cdots$$

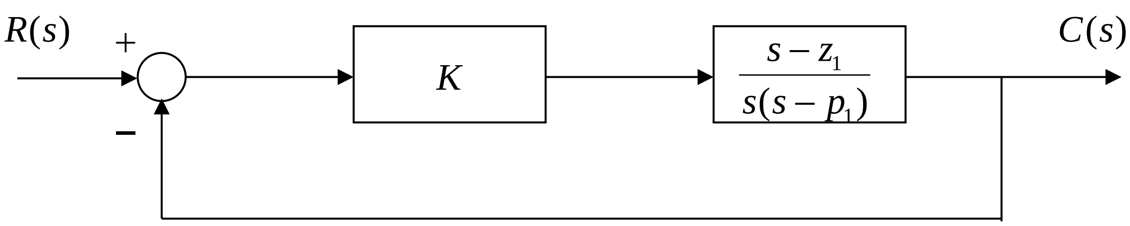
$$\angle(p_2 - z_1) - \angle(p_2 - p_1) - \angle(s - p_2) - \angle(p_2 - p_3) - \angle(p_2 - p_4) = \pm 180(1+2N)$$

$$\angle(s - p_2) = \pm 180(1+2N) - \angle(p_2 - z_1) + \angle(p_2 - p_1) + \angle(p_2 - p_3) + \angle(p_2 - p_4)$$

Root – Locus Method



n=2, m=1



$$1+K\frac{s-z_1}{s(s-p_1)}$$

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5) Double root **s=b**

Characteristic eq. $(s-b)^2(\quad)=0$

$$\frac{d}{ds}[\qquad]_{s=b}=0$$

At double root

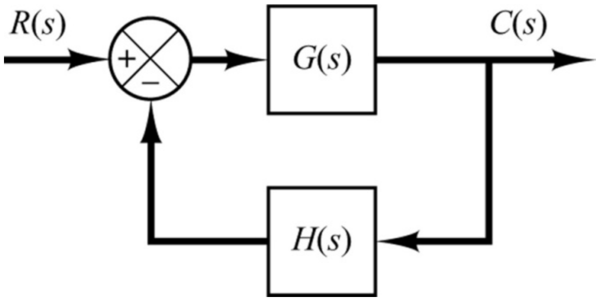
$$A(s)+K\cdot B(s)=0\Rightarrow \quad K=-\frac{A(s)}{B(s)}$$

$$\frac{dA(s)}{ds}-K\frac{dB(s)}{ds}=0$$

$$\frac{dA(s)}{ds}-\frac{A(s)}{B(s)}\frac{dB(s)}{ds}=0$$

$$B(s)\frac{dA(s)}{ds}-A(s)\frac{dB(s)}{ds}=0$$

Root Locus: Summary



Closed-loop transfer function

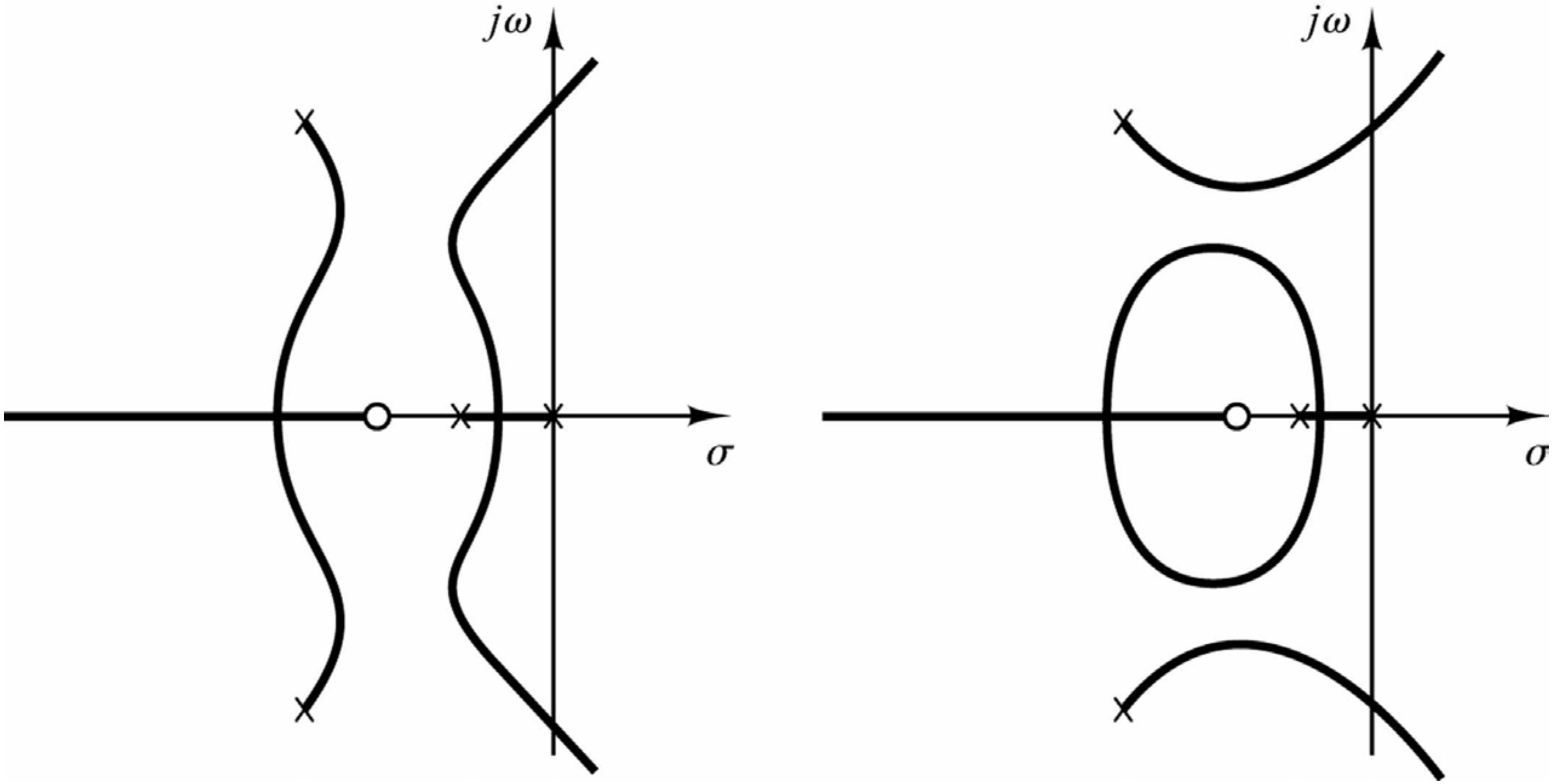
$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

Characteristic equation

$$1 + K_C \frac{B(s)}{A(s)} = 0$$

1. K=0
2. K>>1
3. Real axis
4. Angle of departure
5. Double roots

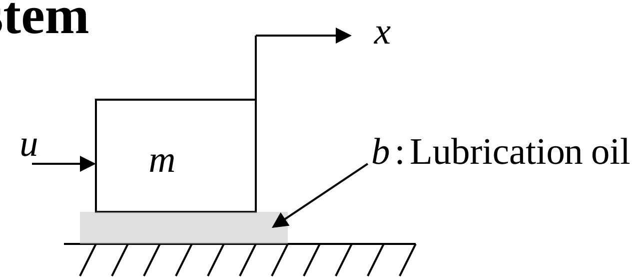
System with 4 poles and 1 zero: two possible root loci



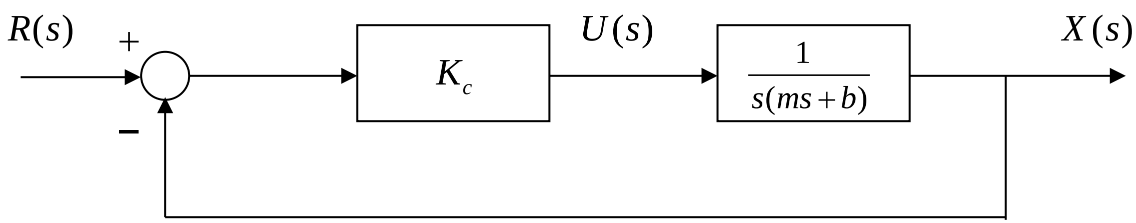
a slight change in the pole–zero configuration may cause significant changes in the root-locus configurations

Root – Locus Method

- **Positioning system**



P control

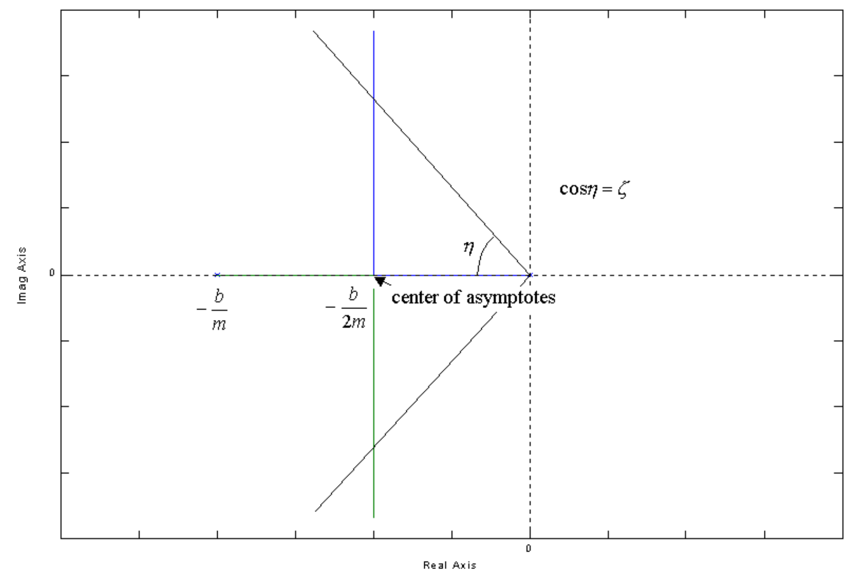


$$1 + K_c \frac{1}{s(ms+b)} = 0$$

$$\underline{A(s) = s(ms+b)} \qquad \underline{B(s) = 1}$$

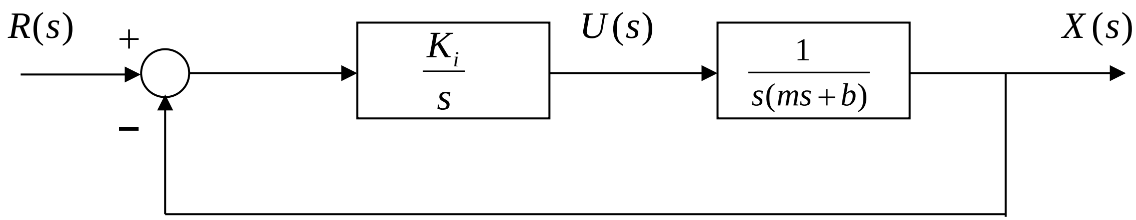
$$n=2 \qquad m=0$$

Open loop poles : 0, $-\frac{b}{m}$



Root – Locus Method

I control

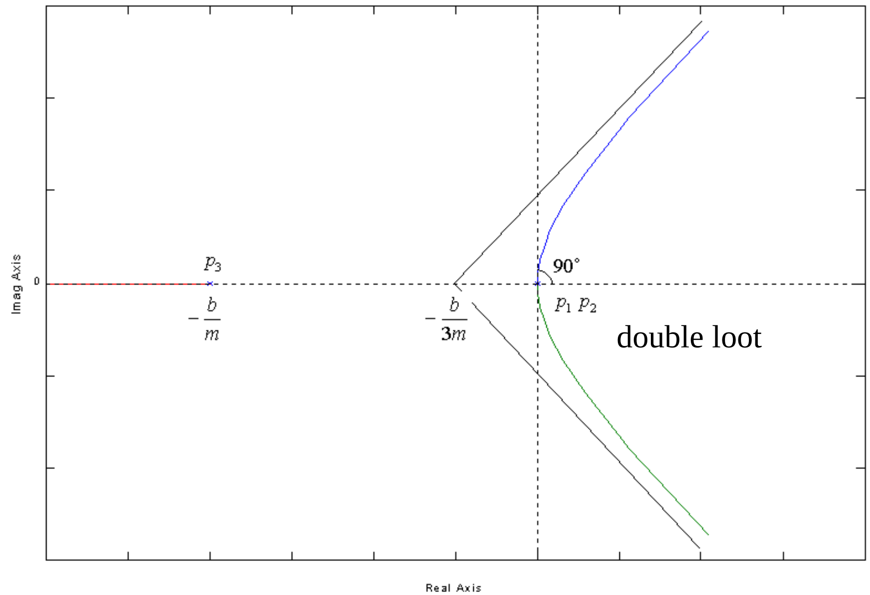


$$1+\frac{K_i}{s}\frac{1}{s(ms+b)}=0$$

$$1+K_i\frac{1}{s^2(ms+b)}=0$$

n=3 m=0

Open loop poles : $0,0-\dfrac{b}{m}$



angle of departure

$$\sum_{i=1}^{m}(s-z_i)-\sum_{i=1}^{n}(s-p_i)=\pm 180(1+2N)$$

$$-\angle(s-p_1)-\angle(s-p_2)-\angle(s-p_3)=\pm 180(1+2N)$$

$$\angle(s-p_1)=90^\circ$$

$$-2\angle(s-p_1)=\pm 180$$

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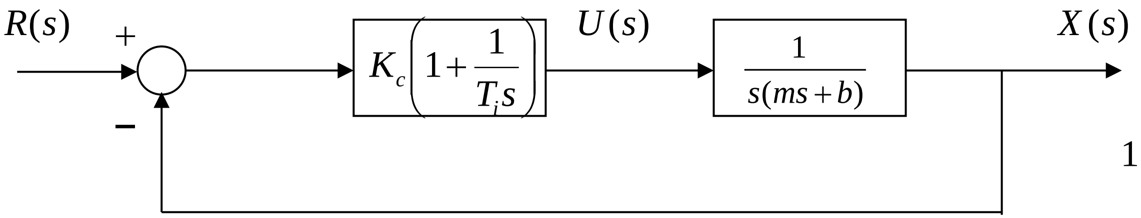
PI control

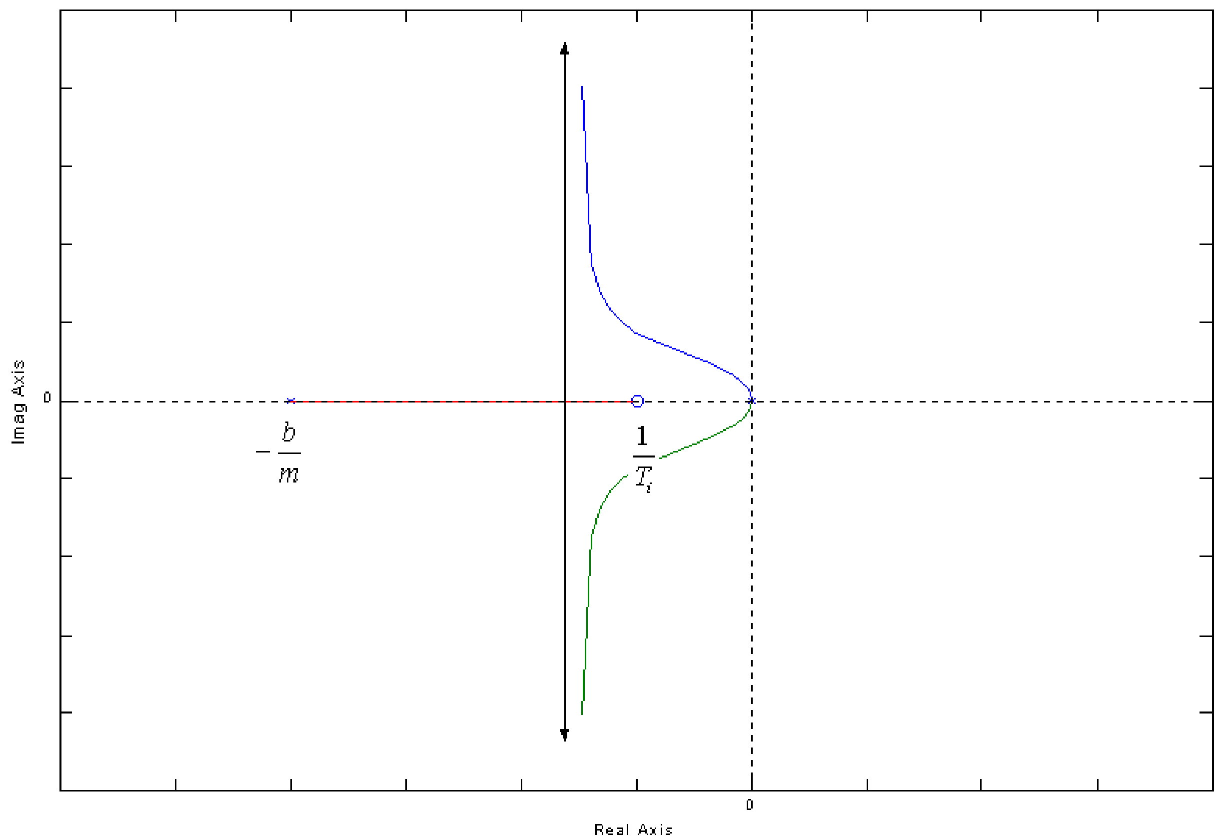


$$1+K_c\frac{T_i s+1}{T_i s}\frac{1}{s(ms+b)}=0$$

n=3 m=1 n-m=2

Open loop zero : $\frac{1}{T_i}$

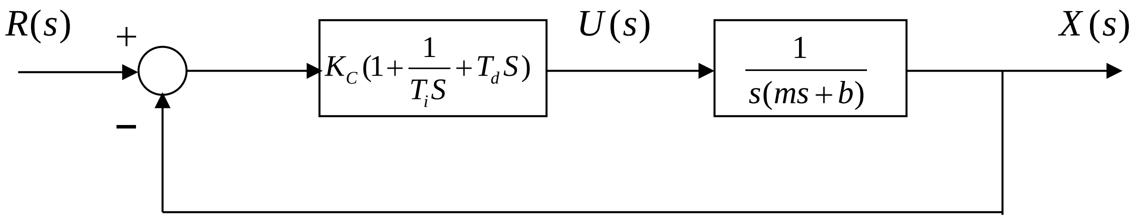
Open loop pole : $0, 0, -\frac{b}{m}$



Root – Locus Method

PID control

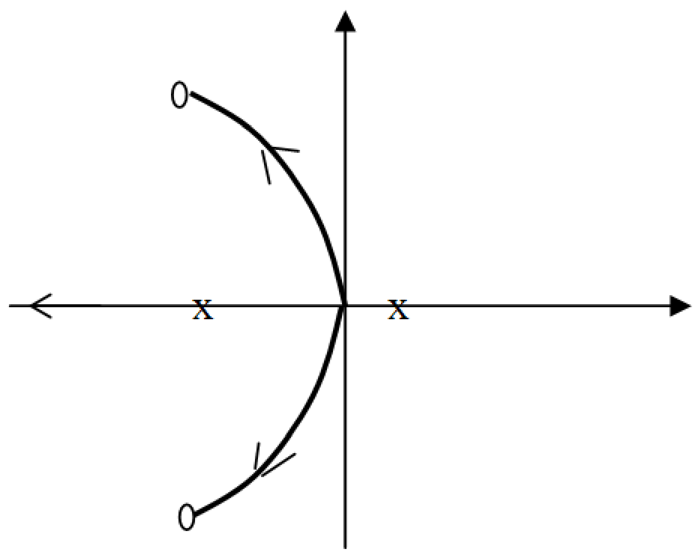
- Mass change, - PI, I gain variation



$$1+K_C\frac{T_i s+1+T_d s^2}{T_i s}\frac{1}{s(ms+b)}=0$$

2 Open loop zeros

3 Open loop poles : $0,0,-\frac{b}{m}$



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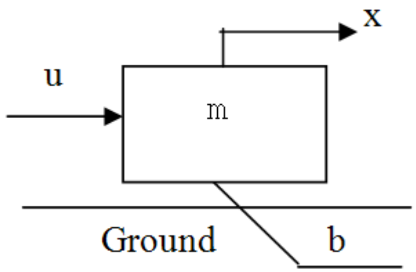
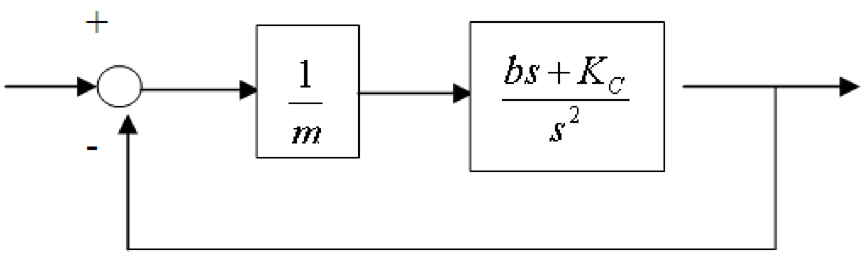
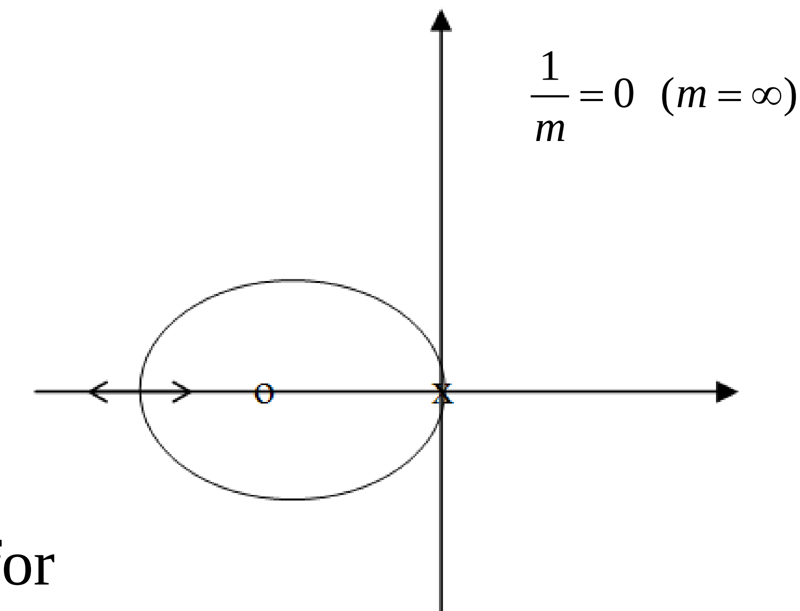
Example)



$$ms^2 + bs + K = 0$$

$$1 + \frac{1}{m}\frac{bs + K_C}{s^2} = 0$$





Effect of increasing mass, m, for fixed gain Kc

$$\frac{1}{m} = 0;\ \text{s=0, 0}\ \ closed\ \ loop\ poles$$

$$\frac{1}{m} = \infty;\ s = -\frac{K_c}{b} \quad s = -\infty$$

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Plotting Root Loci with MATLAB

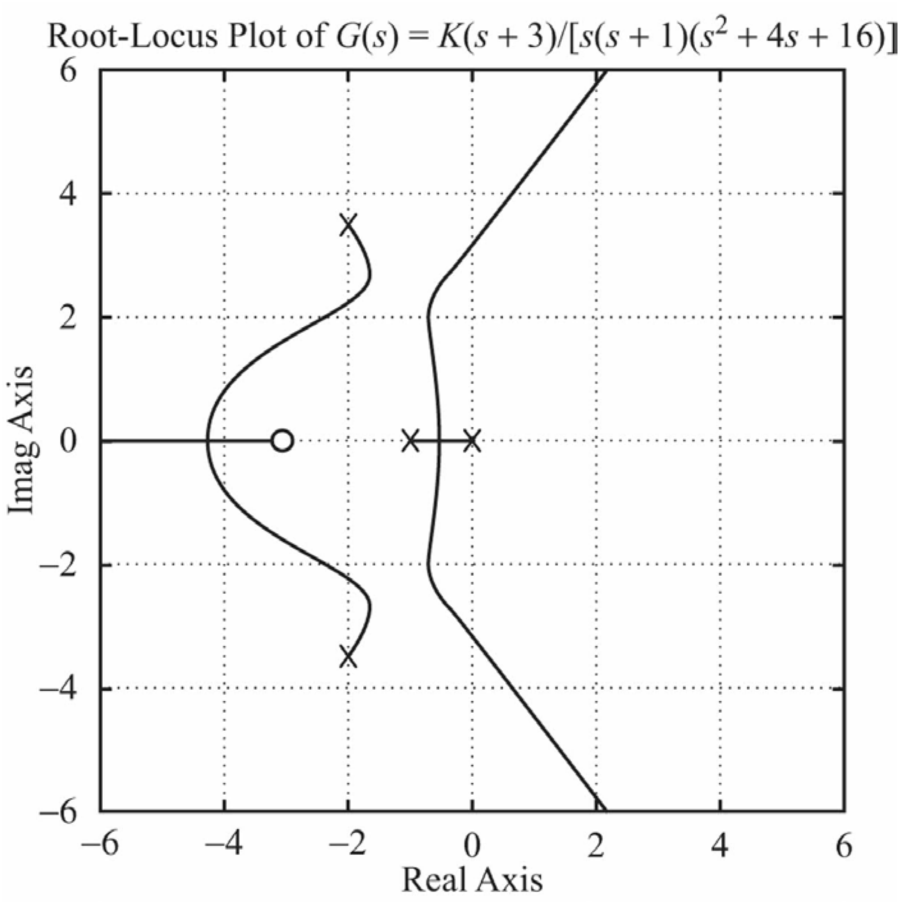
$$1 + K\frac{num}{den} = 0$$

```
rlocus(num, den)

r=rlocus(num,den)
Plot(r, ‘o’)
```

MATLAB Program

```
% --------- Root-locus plot ---------
num = [1 3];
den = [1 5 20 16 0];
rlocus(num,den)
v = [-6 6 -6 6];
axis(v); axis('square')
grid;
title ('Root-Locus Plot of G(s) =
K(s + 3)/[s(s + 1)(s^2 + 4s + 16)]')
```



End of root locus

Root – Locus Method

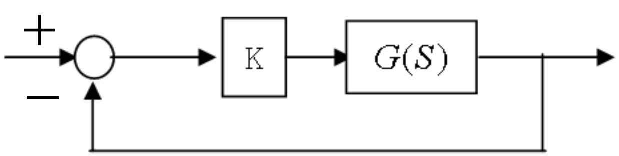
Minimum phase systems:

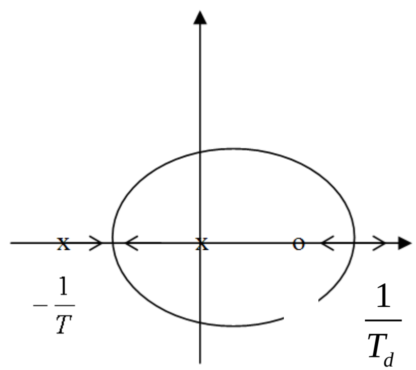
시스템의 모든 pole과 zero가 LHP에 있는 경우

Non minimum phase systems:

적어도 시스템의 한 개의 pole이나 zero가 평면의 RHP에 있는 경우

$$G(s) = \frac{(1 - T_d s)}{s(Ts + 1)}$$





$$1 + K\frac{(1 - T_d s)}{s(Ts + 1)} = 0$$

$$\frac{T_d s - 1}{s(Ts + 1)} = \frac{1}{K}$$

$\angle (T_d s - 1)$	$- \angle s$	$- \angle (Ts + 1)$	=	360N
0	0	0	=	0
180	0	0	=	180

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